Chapter 22
Symplectic Manifolds

In this final chapter we introduce a new kind of geometric structure on manifolds, called a *symplectic structure*, which is superficially similar to a Riemannian metric but turns out to have profoundly different properties. It is simply a choice of a closed, nondegenerate 2-form. The motivation for the definition may not be evident at first, but it will emerge gradually as we see how these properties are used. For now, suffice it to say that nondegeneracy is important because it yields a tangent-cotangent isomorphism like that provided by a Riemannian metric, and closedness is important because it leads to a deep relationship between smooth functions and flows (see the discussions of Hamiltonian vector fields and Poisson brackets later in this chapter). Symplectic structures have surprisingly varied applications in mathematics and physics, including partial differential equations, differential topology, and classical mechanics, among many other fields.

In this chapter, we can give only a quick overview of the subject of symplectic geometry. We begin with a discussion of the algebra of nondegenerate alternating 2-tensors on a finite-dimensional vector space, and then turn our attention to symplectic structures on manifolds. The most important example is a canonically defined symplectic structure on the cotangent bundle of each smooth manifold. We give a proof of the important *Darboux theorem*, which shows that every symplectic form can be put into canonical form locally by a choice of smooth coordinates, so, unlike the situation for Riemannian metrics, there is no local obstruction to "flatness" of symplectic structures.

Then we explore one of the most important applications of symplectic structures. Any smooth real-valued function on a symplectic manifold gives rise to a canonical system of ordinary differential equations called a *Hamiltonian system*. These systems are central to the study of classical mechanics.

After treating Hamiltonian systems, we give a brief introduction to an odd-dimensional analogue of symplectic structures, called *contact structures*. Then at the end of the chapter, we show how symplectic and contact geometry can be used to construct solutions to first-order partial differential equations.

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Symplectic Tensors

We begin with linear algebra. A 2-covector ω on a finite-dimensional vector space V is said to be ***nondegenerate*** if the linear map $\widehat{\omega}\colon V \to V^*$ defined by $\widehat{\omega}(v) = v \lrcorner \omega$ is invertible.

► **Exercise 22.1.** Show that the following are equivalent for 2-covector ω on a finite-dimensional vector space V:

(a) ω is nondegenerate.
(b) For each nonzero $v \in V$, there exists $w \in V$ such that $\omega(v, w) \neq 0$.
(c) In terms of some (hence every) basis, the matrix (ω_{ij}) representing ω is nonsingular.

A nondegenerate 2-covector is called a ***symplectic tensor***. A vector space V endowed with a specific symplectic tensor is called a ***symplectic vector space***. (A symplectic tensor is also often called a "symplectic form," because it is in particular a bilinear form. But to avoid confusion, we reserve that name for something slightly different, to be defined below.)

Example 22.2. Let V be a vector space of dimension $2n$, with a basis denoted by $(A_1, B_1, \dots, A_n, B_n)$. Let $\left(\alpha^1, \beta^1, \dots, \alpha^n, \beta^n\right)$ denote the corresponding dual basis for V^*, and let $\omega \in \Lambda^2(V^*)$ be the 2-covector defined by

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i. \tag{22.1}$$

Note that the action of ω on basis vectors is given by

$$\omega(A_i, A_j) = \omega(B_i, B_j) = 0, \qquad \omega(A_i, B_j) = -\omega(B_j, A_i) = \delta_{ij}. \tag{22.2}$$

Suppose $v = a^i A_i + b^i B_i \in V$ satisfies $\omega(v, w) = 0$ for all $w \in V$. Then $0 = \omega(v, B_i) = a^i$ and $0 = \omega(v, A_i) = -b^i$, which implies that $v = 0$. Thus ω is nondegenerate, and so is a symplectic tensor. //

It is interesting to consider the special case in which $\dim V = 2$. In this case, using the notation of the preceding example, $\alpha^1 \wedge \beta^1$ is nondegenerate, and every 2-covector is a multiple of $\alpha^1 \wedge \beta^1$. Thus every nonzero 2-covector on a 2-dimensional vector space is symplectic.

If (V, ω) is a symplectic vector space and $S \subseteq V$ is any linear subspace, we define the ***symplectic complement of S***, denoted by $S^\perp$, to be the subspace

$$S^\perp = \{v \in V : \omega(v, w) = 0 \text{ for all } w \in S\}.$$

As the notation suggests, the symplectic complement is analogous to the orthogonal complement in an inner product space. Just as in the inner product case, the dimension of $S^\perp$ is the codimension of S, as the next lemma shows.

Lemma 22.3. *Let (V, ω) be a symplectic vector space. For any linear subspace $S \subseteq V$, we have* $\dim S + \dim S^\perp = \dim V$.

Proof. Let $S \subseteq V$ be a subspace, and define a linear map $\Phi\colon V \to S^*$ by $\Phi(v) = (v \lrcorner \omega)|_S$, or equivalently

$$\Phi(v)(w) = \omega(v, w) \quad \text{for } v \in V,\ w \in S.$$

Suppose φ is an arbitrary element of S^*, and let $\widetilde{\varphi} \in V^*$ be any extension of φ to a linear functional on all of V. Since the map $\widehat{\omega}\colon V \to V^*$ defined by $v \mapsto v \lrcorner \omega$ is an isomorphism, there exists $v \in V$ such that $v \lrcorner \omega = \widetilde{\varphi}$. It follows that $\Phi(v) = \varphi$, and therefore Φ is surjective. By the rank–nullity law, $S^\perp = \operatorname{Ker} \Phi$ has dimension equal to $\dim V - \dim S^* = \dim V - \dim S$. □

► **Exercise 22.4.** Let (V, ω) be a symplectic vector space and $S \subseteq V$ be a linear subspace. Show that $(S^\perp)^\perp = S$.

Symplectic complements differ from orthogonal complements in one important respect: although it is always true that $S \cap S^\perp = \{0\}$ in an inner product space, this need not be true in a symplectic vector space. Indeed, if S is 1-dimensional, the fact that ω is alternating forces $\omega(v, v) = 0$ for every $v \in S$, so $S \subseteq S^\perp$. Carrying this idea a little further, a linear subspace $S \subseteq V$ is said to be

- ***symplectic*** if $S \cap S^\perp = \{0\}$;
- ***isotropic*** if $S \subseteq S^\perp$;
- ***coisotropic*** if $S \supseteq S^\perp$;
- ***Lagrangian*** if $S = S^\perp$.

Proposition 22.5. *Let (V, ω) be a symplectic vector space, and let $S \subseteq V$ be a linear subspace.*

(a) *S is symplectic if and only if $S^\perp$ is symplectic.*
(b) *S is symplectic if and only if $\omega|_S$ is nondegenerate.*
(c) *S is isotropic if and only if $\omega|_S = 0$.*
(d) *S is coisotropic if and only if $S^\perp$ is isotropic.*
(e) *S is Lagrangian if and only if $\omega|_S = 0$ and $\dim S = \frac{1}{2} \dim V$.*

Proof. Problem 22-1. □

► **Exercise 22.6.** Let (V, ω) be the symplectic vector space of dimension $2n$ described in Example 22.2, and let k be an integer such that $0 \le k \le n$.

(a) Show that $\operatorname{span}(A_1, B_1, \dots, A_k, B_k)$ is symplectic.
(b) Show that $\operatorname{span}(A_1, \dots, A_k)$ is isotropic.
(c) Show that $\operatorname{span}(A_1, \dots, A_n, B_1, \dots, B_k)$ is coisotropic.
(d) Show that $\operatorname{span}(A_1, \dots, A_n)$ is Lagrangian.
(e) If $n \ge 3$, which of the four kinds of subspace, if any, is $\operatorname{span}(A_1, A_2, B_1)$?

The symplectic tensor ω defined in Example 22.2 turns out to be the prototype of all symplectic tensors, as the next proposition shows. This can be viewed as a symplectic version of the Gram–Schmidt algorithm.

Proposition 22.7 (Canonical Form for a Symplectic Tensor). *Let ω be a symplectic tensor on an m-dimensional vector space V. Then V has even dimension $m = 2n$, and there exists a basis for V in which ω has the form (22.1).*

Proof. The tensor ω has the form (22.1) with respect to a basis $(A_1, B_1, \dots, A_n, B_n)$ if and only if its action on basis vectors is given by (22.2). We prove the theorem by induction on $m = \dim V$ by showing that there is a basis with this property.

For $m = 0$ there is nothing to prove. Suppose (V, ω) is a symplectic vector space of dimension $m \geq 1$, and assume that the proposition is true for all symplectic vector spaces of dimension less than m. Let A_1 be any nonzero vector in V. Since ω is nondegenerate, there exists $B_1 \in V$ such that $\omega(A_1, B_1) \neq 0$. Multiplying B_1 by a constant if necessary, we may assume that $\omega(A_1, B_1) = 1$. Because ω is alternating, B_1 cannot be a multiple of A_1, so the set $\{A_1, B_1\}$ is linearly independent, and hence $\dim V \geq 2$.

Let $S \subseteq V$ be the span of $\{A_1, B_1\}$. Then $\dim S^\perp = m - 2$ by Lemma 22.3. Since $\omega|_S$ is nondegenerate, by Proposition 22.5 it follows that S is symplectic, and thus $S^\perp$ is also symplectic. By induction, $S^\perp$ is even-dimensional and there is a basis $(A_2, B_2, \dots, A_n, B_n)$ for $S^\perp$ such that (22.2) is satisfied for $2 \leq i, j \leq n$. It follows easily that $(A_1, B_1, A_2, B_2, \dots, A_n, B_n)$ is the required basis for V. □

Because of this proposition, if (V, ω) is a symplectic vector space, a basis $(A_1, B_1, \dots, A_n, B_n)$ for V is called a ***symplectic basis*** if (22.2) holds, which is equivalent to ω being given by (22.1) in terms of the dual basis. The proposition then says that every symplectic vector space has a symplectic basis.

This leads to another useful criterion for 2-covector to be nondegenerate. For an alternating tensor ω, the notation ω^k denotes the k-fold wedge product $\omega \wedge \cdots \wedge \omega$.

Proposition 22.8. *Suppose V is a $2n$-dimensional vector space and $\omega \in \Lambda^2(V^*)$. Then ω is a symplectic tensor if and only if $\omega^n \neq 0$.*

Proof. Suppose first that ω is a symplectic tensor. Let (A_i, B_i) be a symplectic basis for V, and write $\omega = \sum_i \alpha^i \wedge \beta^i$ in terms of the dual coframe. Then $\omega^n = \sum_I \alpha^{i_1} \wedge \beta^{i_1} \wedge \cdots \wedge \alpha^{i_n} \wedge \beta^{i_n}$, where $I = (i_1, \dots, i_n)$ ranges over all multi-indices of length n. Any term in this sum for which I has a repeated index is zero because $\alpha^i \wedge \alpha^i = 0$. The surviving terms are those for which I is a permutation of $(1, \dots, n)$, and these terms are all equal to each other because 2-forms commute with each other under wedge product. Thus

$$\omega^n = n!\big(\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n\big) \neq 0.$$

Conversely, suppose ω is degenerate. Then there is a nonzero vector $v \in V$ such that $v \lrcorner \omega = \widehat{\omega}(v) = 0$. Since interior multiplication by v is an antiderivation, this implies $v \lrcorner (\omega^n) = n(v \lrcorner \omega) \wedge \omega^{n-1} = 0$. We can extend v to a basis $(E_1, E_2, \dots, E_{2n})$ for V with $E_1 = v$, and then $\omega^n(E_1, \dots, E_{2n}) = 0$, which implies $\omega^n = 0$. □

Symplectic Structures on Manifolds

Now let us turn to a smooth manifold M. A ***nondegenerate 2-form*** on M is a 2-form ω such that ω_p is a nondegenerate 2-covector for each $p \in M$. A ***symplectic***

form on M is a closed nondegenerate 2-form. A smooth manifold endowed with a specific choice of symplectic form is called a ***symplectic manifold***. A choice of symplectic form is also sometimes called a ***symplectic structure***.

Proposition 22.7 implies that a symplectic manifold must be even-dimensional. However, not all even-dimensional smooth manifolds admit symplectic structures. For example, Proposition 22.8 shows that if ω is a symplectic form on a $2n$-manifold, then ω^n is a nonvanishing $2n$-form, so every symplectic manifold is orientable. In addition, a necessary homological condition is described in Problem 22-5. It implies, in particular, that $\mathbb{S}^2$ is the only sphere that admits a symplectic structure.

If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, a diffeomorphism $F\colon M_1 \to M_2$ satisfying $F^*\omega_2 = \omega_1$ is called a ***symplectomorphism***. The study of properties of symplectic manifolds that are invariant under symplectomorphisms is known as ***symplectic geometry*** or ***symplectic topology***.

Example 22.9 (Symplectic Manifolds).

(a) With standard coordinates on $\mathbb{R}^{2n}$ denoted by $\left(x^1, \dots, x^n, y^1, \dots, x^n\right)$, the 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

is symplectic: it is obviously closed, and it is nondegenerate because its value at each point is the symplectic tensor of Example 22.2. This is called the ***standard symplectic form on*** $\mathbb{R}^{2n}$. (In formulas involving the standard symplectic form, like those involving the Euclidean inner product, it is usually necessary to insert explicit summation signs, because the summation index i appears twice in the upper position.)

(b) Suppose Σ is any orientable smooth 2-manifold and ω is a nonvanishing smooth 2-form on Σ. Then ω is closed because $d\omega$ is a 3-form on a 2-manifold. Moreover, as we observed just after Example 22.2, in two dimensions every nonvanishing 2-form is nondegenerate, so (Σ, ω) is a symplectic manifold. //

Suppose (M, ω) is a symplectic manifold. An (immersed or embedded) submanifold $S \subseteq M$ is said to be a ***symplectic***, ***isotropic***, ***coisotropic***, or ***Lagrangian submanifold*** if T_pS (thought of as a subspace of T_pM) has the corresponding property at each point $p \in S$. More generally, a smooth immersion (or embedding) $F\colon N \to M$ is said to have one of these properties if the subspace $dF_p(T_pN) \subseteq T_{F(p)}M$ has the corresponding property for every $p \in N$. Thus a submanifold is symplectic (isotropic, etc.) if and only if its inclusion map has the same property.

► **Exercise 22.10.** Suppose (M, ω) is a symplectic manifold and $F\colon N \to M$ is a smooth immersion. Show that F is isotropic if and only if $F^*\omega = 0$, and F is symplectic if and only if $F^*\omega$ is a symplectic form.

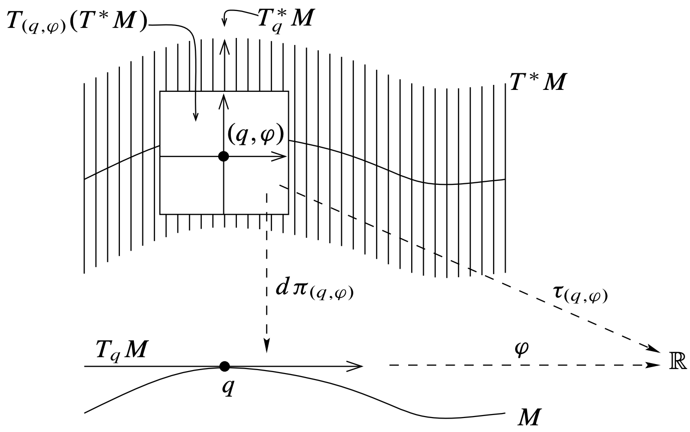

Fig. 22.1 The tautological 1-form on T^*M

The Canonical Symplectic Form on the Cotangent Bundle

The most important symplectic manifolds are total spaces of cotangent bundles, which carry canonical symplectic structures that we now define. First, there is a natural 1-form τ on the total space of T^*M, called the ***tautological 1-form***, defined as follows. A point in T^*M is a covector $\varphi \in T_q^*M$ for some $q \in M$; we denote such a point by the notation (q,φ). The natural projection $\pi\colon T^*M \to M$ is then just $\pi(q,\varphi) = q$, and its pointwise pullback at q is a linear map $d\pi^*_{(q,\varphi)}\colon T_q^*M \to T^*_{(q,\varphi)}(T^*M)$. We define $\tau \in \Omega^1(T^*M)$ (a 1-form on the total space of T^*M) by

$$\tau_{(q,\varphi)} = d\pi^*_{(q,\varphi)}\varphi. \tag{22.3}$$

(See Fig. 22.1.) In other words, the value of τ at $(q,\varphi) \in T^*M$ is the pullback with respect to π of the covector φ itself. If v is a tangent vector in $T_{(q,\varphi)}(T^*M)$, then

$$\tau_{(q,\varphi)}(v) = \varphi\big(d\pi_{(q,\varphi)}(v)\big).$$

Proposition 22.11. *Let M be a smooth manifold. The tautological 1-form τ is smooth, and $\omega = -d\tau$ is a symplectic form on the total space of T^*M.*

Proof. Let $\left(x^i\right)$ be smooth coordinates on M, and let $\left(x^i,\xi_i\right)$ denote the corresponding natural coordinates on T^*M as defined on p. 277. Recall that the coordinates of $(q,\varphi) \in T^*M$ are defined to be $\left(x^i,\xi_i\right)$, where $\left(x^i\right)$ is the coordinate representation of q, and $\xi_i\, dx^i$ is the coordinate representation of φ. In terms of these coordinates, the projection $\pi\colon T^*M \to M$ has the coordinate expression $\pi(x,\xi) = x$. This implies that $d\pi^*(dx^i) = dx^i$, so the coordinate expression for τ is

$$\tau_{(x,\xi)} = d\pi^*_{(x,\xi)}\left(\xi_i\, dx^i\right) = \xi_i\, dx^i. \tag{22.4}$$

It follows immediately that τ is smooth, because its component functions in these coordinates are linear.

Let $\omega = -d\tau \in \Omega^2(T^*M)$. Clearly, ω is closed, because it is exact. Moreover, in natural coordinates, (22.4) yields

$$\omega = \sum_i dx^i \wedge d\xi_i.$$

Under the identification of an open subset of T^*M with an open subset of $\mathbb{R}^{2n}$ by means of these coordinates, ω corresponds to the standard symplectic form on $\mathbb{R}^{2n}$ (with ξ_i substituted for y^i). It follows that ω is symplectic. □

The symplectic form defined in this proposition is called the ***canonical symplectic form on T^*M***. One of its many uses is in giving the following somewhat more "geometric" interpretation of what it means for a 1-form to be closed.

Proposition 22.12. *Let M be a smooth manifold, and let σ be a smooth 1-form on M. Thought of as a smooth map from M to T^*M, σ is a smooth embedding, and σ is closed if and only if its image $\sigma(M)$ is a Lagrangian submanifold of T^*M.*

Proof. Throughout this proof we need to remember that $\sigma\colon M \to T^*M$ is playing two roles: on the one hand, it is a 1-form on M, and on the other hand, it is a smooth map between manifolds. Since they are literally the same map, we do not use different notations to distinguish between them; but you should be careful to think about which role σ is playing at each step of the argument.

In terms of smooth local coordinates (x^i) for M and corresponding natural coordinates (x^i, ξ_i) for T^*M, the map $\sigma\colon M \to T^*M$ has the coordinate representation

$$\sigma\left(x^1, \dots, x^n\right) = \left(x^1, \dots, x^n, \sigma_1(x), \dots, \sigma_n(x)\right),$$

where $\sigma_i\, dx^i$ is the coordinate representation of σ as a 1-form. It follows immediately that σ is a smooth immersion, and it is injective because $\pi \circ \sigma = \mathrm{Id}_M$. To show that it is an embedding, it suffices by Proposition 4.22 to show that it is a proper map. This in turn follows from the fact that π is a left inverse for σ, by Proposition A.53.

Because $\sigma(M)$ is n-dimensional, it is Lagrangian if and only if it is isotropic, which is the case if and only if $\sigma^*\omega = 0$. The pullback of the tautological form τ under σ is

$$\sigma^*\tau = \sigma^*\left(\xi_i\, dx^i\right) = \sigma_i\, dx^i = \sigma.$$

This can also be seen somewhat more invariantly from the computation

$$(\sigma^*\tau)_p(v) = \tau_{\sigma(p)}\big(d\sigma_p(v)\big) = \sigma_p\big(d\pi_{\sigma(p)} \circ d\sigma_p(v)\big) = \sigma_p(v),$$

which follows from the definition of τ and the fact that $\pi \circ \sigma = \mathrm{Id}_M$. Therefore,

$$\sigma^*\omega = -\sigma^* d\tau = -d(\sigma^*\tau) = -d\sigma.$$

It follows that σ is a Lagrangian embedding if and only if $d\sigma = 0$. □

The Darboux Theorem

Our next theorem is one of the most fundamental results in symplectic geometry. It is a nonlinear analogue of the canonical form for a symplectic tensor given in Proposition 22.7. It illustrates the most dramatic difference between symplectic structures and Riemannian metrics: unlike the Riemannian case, there is no local obstruction to a symplectic structure being locally equivalent to the standard flat model.

Theorem 22.13 (Darboux). *Let (M,ω) be a $2n$-dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $\left(x^1,\dots,x^n,y^1,\dots,y^n\right)$ centered at p in which ω has the coordinate representation*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{22.5}$$

We will prove the theorem below. Any coordinates satisfying (22.5) theorem are called ***Darboux coordinates***, ***symplectic coordinates***, or ***canonical coordinates***. Obviously, the standard coordinates $\left(x^1,\dots,x^n,y^1,\dots,y^n\right)$ on $\mathbb{R}^{2n}$ are Darboux coordinates. The proof of Proposition 22.11 showed that the natural coordinates $\left(x^i,\xi_i\right)$ are Darboux coordinates for T^*M with its canonical symplectic structure.

The Darboux theorem was first proved (in a slightly different form) by Gaston Darboux in 1882, in connection with his work on ordinary differential equations arising in classical mechanics. The proof we give was discovered in 1971 by Alan Weinstein [Wei71], based on a technique due to Jürgen Moser [Mos65]. A more elementary—but less elegant—proof is outlined in Problem 22-19.

Weinstein's proof of the Darboux theorem is based on the theory of time-dependent flows (see Theorem 9.48). Before we carry out the proof, we need some preliminary results regarding such flows.

First, recall that Proposition 12.36 shows how to use Lie derivatives to compute the derivative of a tensor field under a flow. We need the following generalization of that result to the case of time-dependent flows.

Proposition 22.14. *Let M be a smooth manifold. Suppose $V\colon J \times M \to TM$ is a smooth time-dependent vector field and $\psi\colon \mathcal{E} \to M$ is its time-dependent flow. For any smooth covariant tensor field $A \in \mathcal{T}^k(M)$ and any $(t_1,t_0,p) \in \mathcal{E}$,*

$$\left.\frac{d}{dt}\right|_{t=t_1} \left(\psi_{t,t_0}^* A\right)_p = \left(\psi_{t_1,t_0}^*\left(\mathcal{L}_{V_{t_1}} A\right)\right)_p. \tag{22.6}$$

Proof. First, assume $t_1 = t_0$. In this case, ψ_{t_0,t_0} is the identity map of M, so we need to prove

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\psi_{t,t_0}^* A\right)_p = \left(\mathcal{L}_{V_{t_0}} A\right)_p. \tag{22.7}$$

We begin with the special case in which $A = f$ is a smooth 0-tensor field:

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\psi_{t,t_0}^* f\right)(p) = \left.\frac{\partial}{\partial t}\right|_{t=t_0} f\left(\psi(t,t_0,p)\right) = V\left(t_0,\psi(t_0,t_0,p)\right) f$$

$$= \big(\mathcal{L}_{V_{t_0}} f\big)(p).$$

Next consider an exact 1-form $A = df$. In any smooth local coordinates $\big(x^i\big)$, the function $\psi^*_{t,t_0} f(x) = f\big(\psi(t, t_0, x)\big)$ depends smoothly on all $n + 1$ variables $\big(t, x^1, \dots, x^n\big)$. Thus, the operator d/dt (which is more properly written as $\partial/\partial t$ in this situation) commutes with each of the partial derivatives $\partial/\partial x^i$ when applied to $\psi^*_{t,t_0} f$. In particular, this means that the exterior derivative operator d commutes with $\partial/\partial t$, and so

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_0} \big(\psi^*_{t,t_0} df\big)_p &= \frac{\partial}{\partial t}\bigg|_{t=t_0} d\left(\psi^*_{t,t_0} f\right)_p = d\left(\frac{\partial}{\partial t}\bigg|_{t=t_0} \big(\psi^*_{t,t_0} f\big) \right)_p \\ &= d\big(\mathcal{L}_{V_{t_0}} f\big)_p = \big(\mathcal{L}_{V_{t_0}} df\big)_p. \end{aligned}$$

Thus, the result is proved for 0-tensors and for exact 1-forms.

Now suppose that $A = B \otimes C$ for some smooth covariant tensor fields B and C, and assume that the proposition is true for B and C. (We include the possibility that B or C has rank 0, in which case the tensor product is just ordinary multiplication.) By the product rule for Lie derivatives (Proposition 12.32(c)), the right-hand side of (22.7) satisfies

$$\big(\mathcal{L}_{V_{t_0}}(B \otimes C)\big)_p = \big(\mathcal{L}_{V_{t_0}} B\big)_p \otimes C_p + B_p \otimes \big(\mathcal{L}_{V_{t_0}} C\big)_p.$$

On the other hand, by an argument entirely analogous to that in the proof of Proposition 12.32, the left-hand side satisfies a similar product rule:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_0} \big(\psi^*_{t,t_0}(B \otimes C)\big)_p = &\left(\frac{d}{dt}\bigg|_{t=t_0} \big(\psi^*_{t,t_0} B\big)_p \right) \otimes C_p \\ &+ B_p \otimes \left(\frac{d}{dt}\bigg|_{t=t_0} \big(\psi^*_{t,t_0} C\big)_p \right). \end{aligned}$$

This shows that (22.7) holds for $A = B \otimes C$, provided it holds for B and C. The case of arbitrary tensor fields now follows by induction, using the fact that any smooth covariant tensor field can be written locally as a sum of tensor fields of the form $A = f\, dx^{i_1} \otimes \dots \otimes dx^{i_k}$.

To handle arbitrary t_1, we use Theorem 9.48(d), which shows that $\psi_{t,t_0} = \psi_{t,t_1} \circ \psi_{t_1,t_0}$ wherever the right-hand side is defined. Therefore, because the linear map $d\big(\psi_{t_1,t_0}\big)^*_p \colon T^k\big(T^*_{\psi_{t_1,t_0}(p)} M\big) \to T^k(T^*_p M)$ does not depend on t,

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=t_1} \big(\psi^*_{t,t_0} A\big)_p &= \frac{d}{dt}\bigg|_{t=t_1} d(\psi_{t_1,t_0})^*_p \circ d(\psi_{t,t_1})^*_{\psi_{t_1,t_0}(p)}\big(A_{\psi_{t,t_0}(p)}\big) \\ &= d(\psi_{t_1,t_0})^*_p \, \frac{d}{dt}\bigg|_{t=t_1} d(\psi_{t,t_1})^*_{\psi_{t_1,t_0}(p)}\big(A_{\psi_{t,t_1} \circ \psi_{t_1,t_0}(p)}\big) \\ &= \big(\psi^*_{t_1,t_0}\big(\mathcal{L}_{V_{t_1}} A\big)\big)_p. \end{aligned}$$

□

A ***smooth time-dependent tensor field*** on a smooth manifold M is a smooth map $A\colon J \times M \to T^kT^*M$, where $J \subseteq \mathbb{R}$ is an interval, satisfying $A(t,p) \in T^k(T_p^*M)$ for each $(t,p) \in J \times M$.

Proposition 22.15. *Let M be a smooth manifold and $J \subseteq \mathbb{R}$ be an open interval. Suppose $V\colon J \times M$ is a smooth time-dependent vector field on M, $\psi\colon \mathcal{E} \to M$ is its time-dependent flow, and $A\colon J \times M \to T^kT^*M$ is a smooth time-dependent tensor field on M. Then for any $(t_1, t_0, p) \in \mathcal{E}$,*

$$\left.\frac{d}{dt}\right|_{t=t_1} \left(\theta_{t,t_0}^* A_t\right)_p = \left(\theta_{t_1,t_0}^*\left(\mathcal{L}_{V_{t_1}} A_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1} A_t\right)\right)_p. \tag{22.8}$$

Proof. For sufficiently small $\varepsilon > 0$, consider the smooth map $F\colon (t_1 - \varepsilon, t_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \to T^k(T_p^*M)$ defined by

$$F(u,v) = \left(\theta_{u,t_0}^* A_v\right)_p = d\left(\theta_{u,t_0}\right)_p^* \left(A_v|_{\theta_{u,t_0}(p)}\right).$$

Since F takes its values in the finite-dimensional vector space $T^k(T_p^*M)$, we can apply the chain rule together with Proposition 22.14 to conclude that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_1} F(t,t) &= \frac{\partial F}{\partial u}(t_1,t_1) + \frac{\partial F}{\partial v}(t_1,t_1)\\ &= \left(\theta_{t_1,t_0}^*\left(\mathcal{L}_{V_{t_1}} A_{t_1}\right)\right)_p + \left.\frac{\partial}{\partial v}\right|_{v=t_1} d\left(\theta_{t_1,t_0}\right)_p^* \left(A_v|_{\theta_{t_1,t_0}(p)}\right).\end{aligned}$$

Just as in the proof of Proposition 22.14, the linear map $d\left(\theta_{t_1,t_0}\right)_p^*$ commutes past $\partial/\partial v$, yielding (22.8). □

Proof of the Darboux theorem. Let ω_0 denote the given symplectic form on M, and let $p_0 \in M$ be arbitrary. The theorem will be proved if we can find a smooth coordinate chart (U_0, φ) centered at p_0 such that $\varphi^*\omega_1 = \omega_0$, where $\omega_1 = \sum_{i=1}^n dx^i \wedge dy^i$ is the standard symplectic form on $\mathbb{R}^{2n}$. Since this is a local question, by choosing smooth coordinates $\left(x^1, \dots, x^n, y^1, \dots, y^n\right)$ in a neighborhood of p_0, we may replace M with an open ball $U \subseteq \mathbb{R}^{2n}$. Proposition 22.7 shows that we can arrange by a linear change of coordinates that $\omega_0|_{p_0} = \omega_1|_{p_0}$.

Let $\eta = \omega_1 - \omega_0$. Because η is closed, the Poincaré lemma (Theorem 17.14) shows that we can find a smooth 1-form α on U such that $d\alpha = -\eta$. By subtracting a constant-coefficient (and thus closed) 1-form from α, we may assume without loss of generality that $\alpha_{p_0} = 0$.

For each $t \in \mathbb{R}$, define a closed 2-form ω_t on U by

$$\omega_t = \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1.$$

Let J be a bounded open interval containing $[0,1]$. Because $\omega_t|_{p_0} = \omega_0|_{p_0}$ is nondegenerate for all t, a simple compactness argument shows that there is some neighborhood $U_1 \subseteq U$ of p_0 such that ω_t is nondegenerate on U_1 for all $t \in \overline{J}$. Because of this nondegeneracy, the smooth bundle homomorphism $\widehat{\omega}_t \colon TU_1 \to T^*U_1$ defined by $\widehat{\omega}_t(X) = X \lrcorner\, \omega_t$ is an isomorphism for each $t \in \bar{J}$.

Define a smooth time-dependent vector field $V \colon J \times U_1 \to TU_1$ by $V_t = \widehat{\omega}_t^{-1}\alpha$, or equivalently

$$V_t \lrcorner\, \omega_t = \alpha.$$

Our assumption that $\alpha_{p_0} = 0$ implies that $V_t|_{p_0} = 0$ for each t. If $\theta \colon \mathcal{E} \to U_1$ denotes the time-dependent flow of V, it follows that $\theta(t,0,p_0) = p_0$ for all $t \in J$, so $J \times \{0\} \times \{p_0\} \subseteq \mathcal{E}$. Because $\mathcal{E}$ is open in $J \times J \times M$ and $[0,1]$ is compact, there is some neighborhood U_0 of p_0 such that $[0,1] \times \{0\} \times U_0 \subseteq \mathcal{E}$.

For each $t_1 \in [0,1]$, it follows from Proposition 22.15 that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} (\theta_{t,0}^* \omega_t) &= \theta_{t_1,0}^* \left(\mathcal{L}_{V_{t_1}} \omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1} \omega_t \right) \\
&= \theta_{t_1,0}^* \big(V_{t_1} \lrcorner\, d\omega_{t_1} + d(V_{t_1} \lrcorner\, \omega_{t_1}) + \eta \big) \\
&= \theta_{t_1,0}^* (0 + d\alpha + \eta) = 0.
\end{aligned}$$

Therefore, $\theta_{t,0}^*\omega_t = \theta_{0,0}^*\omega_0 = \omega_0$ for all t. In particular, $\theta_{1,0}^*\omega_1 = \omega_0$. It follows from Theorem 9.48(c) that $\theta_{1,0}$ is a diffeomorphism onto its image, so it is a coordinate map. Because $\theta_{1,0}(p_0) = p_0 = 0$, these coordinate are centered at p_0. □

Hamiltonian Vector Fields

One of the most useful constructions on symplectic manifolds is a symplectic analogue of the gradient, defined as follows. Suppose (M,ω) is a symplectic manifold. For any smooth function $f \in C^\infty(M)$, we define the ***Hamiltonian vector field of f*** to be the smooth vector field X_f defined by

$$X_f = \widehat{\omega}^{-1}(df),$$

where $\widehat{\omega} \colon TM \to T^*M$ is the bundle isomorphism determined by ω. Equivalently,

$$X_f \lrcorner\, \omega = df,$$

or for any vector field Y,

$$\omega(X_f, Y) = df(Y) = Yf.$$

In any Darboux coordinates, X_f can be computed explicitly as follows. Writing

$$X_f = \sum_{i=1}^n \left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \right)$$

for some coefficient functions (a^i, b^i) to be determined, we compute

$$X_f \lrcorner\, \omega = \sum_{j=1}^n \left(a^j \frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial y^j} \right) \lrcorner \sum_{i=1}^n dx^i \wedge dy^i = \sum_{i=1}^n \left(a^i\, dy^i - b^i\, dx^i \right).$$

On the other hand,

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i \right).$$

Setting these two expressions equal to each other, we find that $a^i = \partial f / \partial y^i$ and $b^i = -\partial f / \partial x^i$, which yields the following formula for the Hamiltonian vector field of f in Darboux coordinates:

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right). \tag{22.9}$$

This formula holds, in particular, on $\mathbb{R}^{2n}$ with its standard symplectic form.

Although the definition of the Hamiltonian vector field is formally analogous to that of the gradient on a Riemannian manifold, Hamiltonian vector fields differ from gradients in some very significant ways, as the next lemma shows.

Proposition 22.16 (Properties of Hamiltonian Vector Fields). *Let (M, ω) be a symplectic manifold and let $f \in C^\infty(M)$.*

(a) *f is constant along each integral curve of X_f.*
(b) *At each regular point of f, the Hamiltonian vector field X_f is tangent to the level set of f.*

Proof. Both assertions follow from the fact that

$$X_f f = df(X_f) = \omega(X_f, X_f) = 0$$

because ω is alternating. □

A smooth vector field X on M is said to be ***symplectic*** if ω is invariant under the flow of X. It is said to be ***Hamiltonian*** (or ***globally Hamiltonian***) if there exists a function $f \in C^\infty(M)$ such that $X = X_f$, and ***locally Hamiltonian*** if each point p has a neighborhood on which X is Hamiltonian. Clearly, every globally Hamiltonian vector field is locally Hamiltonian.

Proposition 22.17 (Hamiltonian and Symplectic Vector Fields). *Let (M, ω) be a symplectic manifold. A smooth vector field on M is symplectic if and only if it is locally Hamiltonian. Every locally Hamiltonian vector field on M is globally Hamiltonian if and only if $H^1_{\mathrm{dR}}(M) = 0$.*

Proof. By Theorem 12.37, a smooth vector field X is symplectic if and only if $\mathcal{L}_X \omega = 0$. Using Cartan's magic formula, we compute

$$\mathcal{L}_X \omega = d(X \lrcorner\, \omega) + X \lrcorner\, (d\omega) = d(X \lrcorner\, \omega). \tag{22.10}$$

Therefore, X is symplectic if and only if the 1-form $X \lrcorner\, \omega$ is closed. On the one hand, if X is locally Hamiltonian, then in a neighborhood of each point there is a real-valued function f such that $X = X_f$, so $X \lrcorner\, \omega = X_f \lrcorner\, \omega = df$, which is certainly closed. Conversely, if X is symplectic, then by the Poincaré lemma each point $p \in M$ has a neighborhood U on which the closed 1-form $X \lrcorner \omega$ is exact. This means there is a smooth real-valued function f defined on U such that $X \lrcorner \omega = df$; because ω is nondegenerate, this implies that $X = X_f$ on U.

Now suppose M is a smooth manifold with $H^1_{dR}(M) = 0$. If X is a locally Hamiltonian vector field, then it is symplectic, so (22.10) shows that $X \lrcorner \omega$ is closed. The hypothesis then implies that there is a function $f \in C^\infty(M)$ such that $X \lrcorner \omega = df$. This means that $X = X_f$, so X is globally Hamiltonian. Conversely, suppose every locally Hamiltonian vector field is globally Hamiltonian. Let η be a closed 1-form, and let X be the vector field $X = \widehat{\omega}^{-1}\eta$. Then (22.10) shows that $\mathcal{L}_X \omega = d\eta = 0$, so X is symplectic and therefore locally Hamiltonian. By hypothesis, there is a global smooth real-valued function f such that $X = X_f$, and then unwinding the definitions, we find that $\eta = df$. □

A symplectic manifold (M, ω) together with a smooth function $H \in C^\infty(M)$ is called a ***Hamiltonian system***. The function H is called the ***Hamiltonian*** of the system; the flow of the Hamiltonian vector field X_H is called its ***Hamiltonian flow***, and the integral curves of X_H are called the ***trajectories*** or ***orbits*** of the system. In Darboux coordinates, formula (22.9) implies that the orbits are those curves $\gamma(t) = \big(x^i(t), y^i(t)\big)$ that satisfy

$$\begin{aligned} \dot{x}^i(t) &= \frac{\partial H}{\partial y^i}\big(x(t), y(t)\big), \\ \dot{y}^i(t) &= -\frac{\partial H}{\partial x^i}\big(x(t), y(t)\big) \end{aligned} \tag{22.11}$$

(with dots denoting ordinary derivatives of component functions with respect to t). These are called ***Hamilton's equations***.

Hamiltonian systems play a central role in classical mechanics. We illustrate how they arise with a simple example.

Example 22.18 (The *n*-Body Problem). Consider n physical particles moving in space, and suppose their masses are $m_1, \dots, m_n$. For many purposes, an effective model of such a system is obtained by idealizing the particles as points in $\mathbb{R}^3$, which we denote by $\mathbf{q}_1, \dots, \mathbf{q}_n$. Writing the coordinates of $\mathbf{q}_k$ at time t as $\big(q_k^1(t), q_k^2(t), q_k^3(t)\big)$, we can represent the evolution of the system over time by a curve in $\mathbb{R}^{3n}$:

$$q(t) = \big(q_1^1(t), q_1^2(t), q_1^3(t), \dots, q_n^1(t), q_n^2(t), q_n^3(t)\big).$$

The ***collision set*** is the subset $\mathcal{C} \subseteq \mathbb{R}^{3n}$ where two or more particles occupy the same position in space:

$$\mathcal{C} = \left\{q \in \mathbb{R}^{3n} : \mathbf{q}_k = \mathbf{q}_l \text{ for some } k \neq l\right\}.$$

We consider only motions with no collisions, so we are interested in curves in the open subset $Q = \mathbb{R}^{3n} \smallsetminus \mathcal{C}$.

Suppose the particles are acted upon by forces that depend only on the positions of all the particles in the system. (A typical example is gravitational forces.)

If we denote the components of the net force on the kth particle by $\mathbf{F}_k(q) = \big(F_k^1(q), F_k^2(q), F_k^3(q)\big)$, then Newton's second law of motion asserts that the particles' motion satisfies $m_k\ddot{\mathbf{q}}_k(t) = \mathbf{F}_k\big(\mathbf{q}(t)\big)$ for each k, which translates into the $3n \times 3n$ system of second-order ODEs

$$\begin{aligned}
m_k\ddot{q}_k^1(t) &= F_k^1\big(q(t)\big),\\
m_k\ddot{q}_k^2(t) &= F_k^2\big(q(t)\big),\\
m_k\ddot{q}_k^3(t) &= F_k^3\big(q(t)\big), \quad k = 1,\dots,n.
\end{aligned}$$

(There is no implied summation in these equations.)

This can be written in a more compact form if we relabel the $3n$ position coordinates as $q(t) = \big(q^1(t),\dots,q^{3n}(t)\big)$ and the $3n$ components of the forces as $F(q) = \big(F_1(q),\dots,F_{3n}(q)\big)$, and let $M = (M_{ij})$ denote the $3n \times 3n$ diagonal matrix whose diagonal entries are $(m_1, m_1, m_1, m_2, m_2, m_2, \dots, m_n, m_n, m_n)$. Then Newton's second law can be written

$$M_{ij}\ddot{q}^j(t) = F_i\big(q(t)\big). \tag{22.12}$$

(Here the summation convention is in force.) We assume that the forces depend smoothly on q, so we can interpret $F(q) = (F_1(q),\dots,F_{3n}(q))$ as the components of a smooth covector field on Q. We assume further that the forces are conservative, which by the results of Chapter 11 is equivalent to the existence of a smooth function $V \in C^\infty(Q)$ (called the ***potential energy*** of the system) such that $F = -dV$.

Under the physically reasonable assumption that all of the masses are positive, the matrix M is positive definite, and thus can be interpreted as a (constant-coefficient) Riemannian metric on Q. It therefore defines a smooth bundle isomorphism $\widehat{M}\colon TQ \to T^*Q$. If we denote the natural coordinates on TQ by $\big(q^i, v^i\big)$ and those on T^*Q by $\big(q^i, p_i\big)$, then $M(v,w) = M_{ij}v^i w^j$, and $\widehat{M}$ has the coordinate representation

$$\big(q^i, p_i\big) = \widehat{M}\left(q^i, v^i\right) = \big(q^i, M_{ij}v^j\big).$$

If $q'(t) = \big(\dot{q}^1(t),\dots,\dot{q}^{3n}(t)\big)$ is the velocity vector of the system of particles at time t, then the covector $p(t) = \widehat{M}\big(q'(t)\big)$ is given by the formula

$$p_i(t) = M_{ij}\dot{q}^j(t). \tag{22.13}$$

To give this equation a physical interpretation, we can revert to our original labeling of the coordinates and write

$$p(t) = \big(p_1^1, p_1^2, p_1^3, \dots, p_n^1, p_n^2, p_n^3\big),$$

and then $\mathbf{p}_k(t) = \big(p_k^1(t), p_k^2(t), p_k^3(t)\big) = m_k\dot{\mathbf{q}}_k(t)$ is interpreted as the ***momentum*** of the kth particle at time t.

Using (22.13), we see that a curve $q(t)$ in Q satisfies Newton's second law (22.12) if and only if the curve $\gamma(t) = \big(q(t), p(t)\big)$ in T^*Q satisfies the first-order system of ODEs

$$\dot{q}^i(t) = M^{ij} p_j(t),$$
$$\dot{p}_i(t) = -\frac{\partial V}{\partial q^i}\big(q(t)\big), \tag{22.14}$$

where $\left(M^{ij}\right)$ is the inverse of the matrix of (M_{ij}). Define a function $E \in C^\infty(T^*Q)$, called the ***total energy*** of the system, by

$$E(q,p) = V(q) + K(p),$$

where V is the potential energy introduced above, and K is the ***kinetic energy***, defined by

$$K(p) = \tfrac{1}{2} M^{ij} p_i p_j.$$

Since $\left(q^i, p_i\right)$ are Darboux coordinates on T^*Q, a comparison of (22.14) with (22.11) shows that (22.14) is precisely Hamilton's equations for the Hamiltonian flow of E. The fact that E is constant along the trajectories of its own Hamiltonian flow is known as the ***law of conservation of energy***. //

An elaboration of the same technique can be applied to virtually any classical dynamical system in which the forces are conservative. For example, if the positions of a system of particles are subject to constraints, as are the constituent particles of a rigid body, for example, then the configuration space is typically a submanifold of $\mathbb{R}^{3n}$ rather than an open subset. Under very general hypotheses, the equations of motion of such a system can be formulated as a Hamiltonian system on the cotangent bundle of the constraint manifold. For much more on Hamiltonian systems, see [AM78].

Poisson Brackets

Hamiltonian vector fields allow us to define an operation on real-valued functions on a symplectic manifold M similar to the Lie bracket of vector fields. Given $f, g \in C^\infty(M)$, we define their ***Poisson bracket*** $\{f, g\} \in C^\infty(M)$ by any of the following equivalent formulas:

$$\{f, g\} = \omega\left(X_f, X_g\right) = df\left(X_g\right) = X_g f. \tag{22.15}$$

Two functions are said to ***Poisson commute*** if their Poisson bracket is zero.

The geometric interpretation of the Poisson bracket is evident from the characterization $\{f, g\} = X_g f$: it is a measure of the rate of change of f along the Hamiltonian flow of g. In particular, f and g Poisson commute if and only if f is constant along the Hamiltonian flow of g.

Using (22.9), we can readily compute the Poisson bracket of two functions f, g in Darboux coordinates:

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}. \tag{22.16}$$

Proposition 22.19 (Properties of the Poisson Bracket). *Suppose (M,ω) is a symplectic manifold, and $f,g,h \in C^\infty(M)$.*

(a) BILINEARITY: *$\{f,g\}$ is linear over $\mathbb{R}$ in f and in g.*
(b) ANTISYMMETRY: *$\{f,g\} = -\{g,f\}$.*
(c) JACOBI IDENTITY: *$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$.*
(d) *$X_{\{f,g\}} = -[X_f, X_g]$.*

Proof. Parts (a) and (b) are obvious from the characterization $\{f,g\} = \omega\left(X_f, X_g\right)$ together with the fact that $X_f = \widehat{\omega}^{-1}(df)$ depends linearly on f. Because of the nondegeneracy of ω, to prove (d), it suffices to show that the following holds for every vector field Y:

$$\omega\left(X_{\{f,g\}}, Y\right) + \omega\left([X_f, X_g], Y\right) = 0. \tag{22.17}$$

On the one hand, note that $\omega\left(X_{\{f,g\}}, Y\right) = d(\{f,g\})(Y) = Y\{f,g\} = YX_g f$. On the other hand, because Hamiltonian vector fields are symplectic, the Lie derivative formula of Corollary 12.33 yields

$$\begin{aligned} 0 &= \left(\mathcal{L}_{X_g}\omega\right)(X_f, Y) \\ &= X_g\left(\omega(X_f, Y)\right) - \omega\left([X_g, X_f], Y\right) - \omega\left(X_f, [X_g, Y]\right). \end{aligned} \tag{22.18}$$

The first and third terms on the right-hand side can be simplified as follows:

$$\begin{aligned} X_g\left(\omega(X_f, Y)\right) &= X_g\left(df(Y)\right) = X_g Y f. \\ \omega\left(X_f, [X_g, Y]\right) &= df\left([X_g, Y]\right) = [X_g, Y]f = X_g Y f - Y X_g f \\ &= X_g Y f - \omega\left(X_{\{f,g\}}, Y\right). \end{aligned}$$

Inserting these into (22.18), we obtain (22.17).

Finally, (c) follows from (d), (b), and (22.15):

$$\begin{aligned} \{f,\{g,h\}\} &= X_{\{g,h\}} f = -[X_g, X_h] f = -X_g X_h f + X_h X_g f \\ &= -X_g\{f,h\} + X_h\{f,g\} = -\{\{f,h\},g\} + \{\{f,g\},h\} \\ &= -\{g,\{h,f\}\} - \{h,\{f,g\}\}. \end{aligned}$$

□

The following corollary is immediate.

Corollary 22.20. *If (M,ω) is a symplectic manifold, the vector space $C^\infty(M)$ is a Lie algebra under the Poisson bracket.* □

If (M,ω,H) is a Hamiltonian system, any function $f \in C^\infty(M)$ that is constant on every integral curve of X_H is called a ***conserved quantity*** of the system. Conserved quantities turn out to be deeply related to symmetries, as we now show.

A smooth vector field V on M is called an ***infinitesimal symmetry*** of (M,ω,H) if both ω and H are invariant under the flow of V.

Proposition 22.21. *Let (M, ω, H) be a Hamiltonian system.*

(a) *A function $f \in C^\infty(M)$ is a conserved quantity if and only if $\{f, H\} = 0$.*
(b) *The infinitesimal symmetries of (M, ω, H) are precisely the symplectic vector fields V that satisfy $VH = 0$.*
(c) *If θ is the flow of an infinitesimal symmetry and γ is a trajectory of the system, then for each $s \in \mathbb{R}$, $\theta_s \circ \gamma$ is also a trajectory on its domain of definition.*

Proof. Problem 22-18. □

The following theorem, first proved (in a somewhat different form) by Emmy Noether in 1918 [Noe71], has had a profound influence on both physics and mathematics. It shows that for many Hamiltonian systems, there is a one-to-one correspondence between conserved quantities (modulo constants) and infinitesimal symmetries.

Theorem 22.22 (Noether's Theorem). *Let (M, ω, H) be a Hamiltonian system. If f is any conserved quantity, then its Hamiltonian vector field is an infinitesimal symmetry. Conversely, if $H^1_{dR}(M) = 0$, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to addition of a function that is constant on each component of M.*

Proof. Suppose f is a conserved quantity. Proposition 22.21 shows that $\{f, H\} = 0$. This in turn implies that $X_f H = \{H, f\} = 0$, so H is constant along the flow of X_f. Since ω is invariant along the flow of any Hamiltonian vector field by Proposition 22.17, this shows that X_f is an infinitesimal symmetry.

Now suppose that M is a smooth manifold with $H^1_{dR}(M) = 0$. Let V be an infinitesimal symmetry of (M, ω, H). Then V is symplectic by definition, and globally Hamiltonian by Proposition 22.17. Writing $V = X_f$, the fact that H is constant along the flow of V implies that $\{H, f\} = X_f H = VH = 0$, so f is a conserved quantity. If $\widetilde{f}$ is any other function that satisfies $X_{\widetilde{f}} = V = X_f$, then $d(\widetilde{f} - f) = (X_{\widetilde{f}} - X_f) \lrcorner\, \omega = 0$, so $\widetilde{f} - f$ must be constant on each component of M. □

There is one conserved quantity that every Hamiltonian system possesses: the Hamiltonian H itself. The infinitesimal symmetry corresponding to it, of course, generates the Hamiltonian flow of the system, which describes how the system evolves over time. Since H is typically interpreted as the total energy of the system (as in Example 22.18), one usually says that the symmetry corresponding to conservation of energy is "translation in the time variable."

Hamiltonian Flowouts

Hamiltonian vector fields are powerful tools for constructing isotropic and Lagrangian submanifolds. Because Lagrangian submanifolds of T^*M correspond to closed 1-forms (Proposition 22.12), which in turn correspond locally to differentials of functions, such constructions have numerous applications in PDE theory. We will see one such application later in this chapter.

Theorem 22.23 (Hamiltonian Flowout Theorem). *Suppose (M, ω) is a symplectic manifold, $H \in C^\infty(M)$, Γ is an embedded isotropic submanifold of M that is contained in a single level set of H, and the Hamiltonian vector field X_H is nowhere tangent to Γ. If $\mathcal{S}$ is a flowout from Γ along X_H, then $\mathcal{S}$ is also isotropic and contained in the same level set of H.*

Proof. Let θ be the flow of X_H. Recall from Theorem 9.20 that the flowout is parametrized by the restriction of θ to a neighborhood $\mathcal{O}_\delta$ of $\{0\} \times \Gamma$ in $\mathbb{R} \times \Gamma$. First consider a point $p \in \Gamma \subseteq \mathcal{S}$. If we choose a basis $E_1, \dots, E_k$ for $T_p\Gamma$, then $T_p\mathcal{S}$ is spanned by $\left(X_H|_p, E_1, \dots, E_k\right)$. The assumption that Γ is isotropic implies that $\omega_p\left(E_i, E_j\right) = 0$ for all i and j. On the other hand, by definition of the Hamiltonian vector field,

$$\omega_p\left(X_H|_p, E_j\right) = dH_p\left(E_j\right) = 0,$$

because E_j is tangent to Γ, which is contained in a level set of H. This shows that the restriction of ω to $T_p\mathcal{S}$ is zero when $p \in \Gamma$.

Any other point $p' \in \mathcal{S}$ is of the form $p' = \theta_t(p)$ for some $(t, p) \in \mathcal{O}_\delta \subseteq \mathbb{R} \times \Gamma$. Because θ_t is a local diffeomorphism that maps a neighborhood of p in $\mathcal{S}$ to a neighborhood of p' in $\mathcal{S}$, its differential takes $T_p\mathcal{S}$ isomorphically onto $T_{p'}\mathcal{S}$. Thus, for any vectors $v, w \in T_{p'}\mathcal{S}$, there are vectors $\widehat{v}, \widehat{w} \in T_p\mathcal{S}$ such that $d(\theta_t)_p\left(\widehat{v}\right) = v$ and $d(\theta_t)_p\left(\widehat{w}\right) = w$. Moreover, because X_H is a symplectic vector field, its flow preserves ω. Therefore,

$$\omega_{p'}(v, w) = \omega_{p'}\left(d(\theta_t)_p\left(\widehat{v}\right), d(\theta_t)_p\left(\widehat{w}\right)\right) = \left(\theta_t^*\omega\right)_p\left(\widehat{v}, \widehat{w}\right) = \omega_p\left(\widehat{v}, \widehat{w}\right) = 0.$$

It follows that $\mathcal{S}$ is isotropic. By Proposition 22.16, H is constant along each integral curve of X_H, so $\mathcal{S}$ is contained in the same level set of H as Γ. □

Contact Structures

As we have seen, symplectic manifolds must be even-dimensional; but there is a closely related structure called a *contact structure* that one can define on odd-dimensional manifolds. It also has important applications in geometry and analysis. In this section, we introduce the main elements of contact geometry.

Suppose M is a smooth manifold of odd dimension $2n + 1$. A ***contact form*** on M is a nonvanishing smooth 1-form θ with the property that for each $p \in M$, the restriction of $d\theta_p$ to the subspace $\operatorname{Ker} \theta_p \subseteq T_pM$ is nondegenerate, which is to say it is a symplectic tensor. A ***contact structure on*** $\boldsymbol{M}$ is a smooth distribution $H \subseteq TM$ of rank $2n$ with the property that any smooth local defining form θ for H is a contact form. A ***contact manifold*** is a smooth manifold M together with a contact structure on M. If (M, H) is a contact manifold, any (local or global) defining form for H is called a ***contact form for*** $\boldsymbol{H}$. It was proved in 1971 by Jean Martinet [Mar71] that every oriented compact smooth 3-manifold admits a contact structure; but the question of which higher-dimensional manifolds admit contact structures is still unresolved.

Proposition 22.24. *A smooth* 1*-form* θ *on a* $(2n+1)$*-manifold is a contact form if and only if* $\theta \wedge d\theta^n$ *is nonzero everywhere on* M, *where* $d\theta^n$ *represents the* n*-fold wedge product* $d\theta \wedge \cdots \wedge d\theta$.

► **Exercise 22.25.** Prove the preceding proposition.

► **Exercise 22.26.** Suppose H is a contact structure on a smooth manifold M. Show that if θ_1 and θ_2 are any two local contact forms for H, then on their common domain there is a smooth nonvanishing function f such that $\theta_2 = f\theta_1$.

It follows from the result of Problem 19-2 that a codimension-1 distribution $H \subseteq TM$ is integrable if and only if any local defining form θ satisfies $\theta \wedge d\theta \equiv 0$. If H is a contact structure, by contrast, not only is $\theta \wedge d\theta$ nonzero everywhere on M, but it remains nonzero after taking $n-1$ more wedge products with $d\theta$. Thus, a contact structure is, in a sense, a "maximally nonintegrable distribution."

Example 22.27 (Contact Forms).

(a) On $\mathbb{R}^{2n+1}$ with coordinates $(x^1,\dots,x^n,y^1,\dots,y^n,z)$, define a 1-form θ by

$$\theta = dz - \sum_{i=1}^{n} y^i\,dx^i, \tag{22.19}$$

and let $H \subseteq T\mathbb{R}^{2n+1}$ be the rank-$2n$ distribution annihilated by θ. Then $d\theta = \sum_{i=1}^n dx^i \wedge dy^i$. If we define vector fields $\{X_i, Y_i : i = 1,\dots,n\}$ by

$$X_i = \frac{\partial}{\partial x^i} + y^i\frac{\partial}{\partial z}, \qquad Y_i = \frac{\partial}{\partial y^i},$$

then (X_i, Y_i) is a smooth frame for H, and it is straightforward to check that it satisfies $d\theta(X_i,X_j) = d\theta(Y_i,Y_j) = 0$ and $d\theta(X_i,Y_j) = \delta_{ij}$. It follows just as in Example 22.2 that $d\theta|_H$ is nondegenerate, so θ is a contact form. Theorem 22.31 below shows that every contact form can be put into this form locally by a change of coordinates.

(b) Let M be a smooth n-manifold, and define a smooth 1-form θ on the $(2n+1)$-manifold $\mathbb{R} \times T^*M$ by $\theta = dz - \tau$, where z is the standard coordinate on $\mathbb{R}$ and τ is the tautological 1-form on T^*M. In terms of natural coordinates (x^i, ξ_i) for T^*M, the form θ has the coordinate representation

$$\theta = dz - \sum_{i=1}^{n} \xi_i\,dx^i,$$

so it is a contact form by the same argument as in part (a).

(c) On $\mathbb{R}^{2n+2}$ with coordinates $(x^1,\dots,x^{n+1},y^1,\dots,y^{n+1})$, consider the 1-form

$$\Theta = \sum_{i=1}^{n+1} \left(x^i\,dy^i - y^i\,dx^i\right).$$

The ***standard contact form on*** $\mathbb{S}^{2n+1}$ is the smooth 1-form $\theta = \iota^*\Theta$, where $\iota\colon \mathbb{S}^{2n+1} \hookrightarrow \mathbb{R}^{2n+2}$ is inclusion. To see that θ is indeed a contact form, note first that $d\Theta = 2\sum_{i=1}^{n+1} dx^i \wedge dy^i$ is a symplectic form on $\mathbb{R}^{2n+2}$. Consider the following vector fields on $\mathbb{R}^{2n+1} \smallsetminus \{0\}$:

$$N = x^j \frac{\partial}{\partial x^j} + y^j \frac{\partial}{\partial y^j}, \qquad T = x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j}.$$

A computation shows that N is normal to $\mathbb{S}^{2n+1}$ (with respect to the Euclidean metric) and T is tangent to it. Let $S \subseteq T\left(\mathbb{R}^{2n+2} \smallsetminus \{0\}\right)$ denote the subbundle spanned by N and T, and let $S^\perp$ denote its symplectic complement with respect to $d\Theta$. For each $p \in \mathbb{S}^{2n+1}$, $S_p^\perp$ is the set of vectors $X \in T_p\mathbb{R}^{2n+2}$ such that $d\Theta_p\left(N_p, X_p\right) = d\Theta_p\left(T_p, X_p\right) = 0$. We compute

$$N \lrcorner\, d\Theta = 2\sum_{i=1}^{n+1}\left(x^i\, dy^i - y^i\, dx^i\right) = 2\Theta,$$

$$T \lrcorner\, d\Theta = 2\sum_{i=1}^{n+1}\left(x^i\, dx^i + y^i\, dy^i\right) = d\left(|x|^2 + |y|^2\right).$$

It follows that $S_p^\perp$ is the common kernel of Θ_p and $d\left(|x|^2 + |y|^2\right)_p$, which is $\operatorname{Ker}\Theta_p \cap T_p\mathbb{S}^{2n+1} = \operatorname{Ker}\theta_p$. Because $d\Theta(N,T) = |x|^2 + |y|^2 \neq 0$ on $\mathbb{R}^{2n+1} \smallsetminus \{0\}$, S_p is a symplectic subspace of $T_p\mathbb{R}^{2n+2}$, and thus $\operatorname{Ker}\theta_p = S_p^\perp$ is also a symplectic subspace by Proposition 22.5(a). Because the restriction of $d\theta_p$ to $\operatorname{Ker}\theta_p$ is the same as the restriction of $d\Theta_p$, it is nondegenerate, so θ is a contact form. //

Theorem 22.28 (The Reeb Field). *Let (M,H) be a contact manifold, and suppose θ is a contact form for H. There is a unique vector field $T \in \mathfrak{X}(M)$, called the **Reeb field of θ**, that satisfies the following two conditions*:

$$T \lrcorner\, d\theta = 0, \qquad \theta(T) = 1. \tag{22.20}$$

Proof. Define a smooth bundle homomorphism $\Phi\colon TM \to T^*M$ by $\Phi(X) = X \lrcorner\, d\theta$, and for each $p \in M$, let Φ_p denote the linear map $\Phi|_{T_pM}\colon T_pM \to T_p^*M$. The fact that $d\theta_p$ restricts to a nondegenerate 2-tensor on H_p implies that $\Phi_p|_{H_p}$ is injective, so Φ_p has rank at least $2n$ (where $2n+1$ is the dimension of M). On the other hand, Φ_p cannot have rank $2n+1$, because then $d\theta_p$ would be nondegenerate, which is impossible because T_pM is odd-dimensional. Thus Φ_p has rank exactly $2n$, so $\dim \operatorname{Ker}\Phi_p = 1$. Since $\operatorname{Ker}\Phi_p$ is not contained in $H_p = \operatorname{Ker}\theta_p$, there is a unique vector $T_p \in \operatorname{Ker}\Phi_p$ satisfying $\theta_p(T_p) = 1$. This shows that there is a unique rough vector field T satisfying (22.20).

To see that T is smooth, note that $\operatorname{Ker}\Phi$ is a smooth rank-1 subbundle of TM by Theorem 10.34. Given $p \in M$, let X be any smooth nonvanishing section of $\operatorname{Ker}\Phi$ on a neighborhood of p. Because $\theta(X) \neq 0$, we can write the Reeb field locally as $T = \theta(X)^{-1}X$, which is also smooth. □

► **Exercise 22.29.** Show that the Reeb fields of the three contact forms described in Example 22.27 are as follows:

$$\text{(a)}\quad T = \frac{\partial}{\partial z}; \qquad \text{(b)}\quad T = \frac{\partial}{\partial t}; \qquad \text{(c)}\quad T = \left(x^j \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial x^j} \right)\bigg|_{\mathbb{S}^{2n+1}}.$$

► **Exercise 22.30.** Let θ be a contact form and T be its Reeb field. Show that $\mathcal{L}_T \theta = 0$.

Many of the constructs that we described for symplectic manifolds have analogues in contact geometry. We begin with an analogue of the Darboux theorem.

Theorem 22.31 (Contact Darboux Theorem). *Suppose θ is a contact form on a $(2n+1)$-dimensional manifold M. For each $p \in M$, there are smooth coordinates $\left(x^1, \dots, x^n, y^1, \dots, y^n, z\right)$ centered at p in which θ has the form (22.19).*

Proof. Let $p \in M$ be arbitrary. Let $\left(U, \left(u^i\right)\right)$ be a smooth coordinate cube centered at p in which the Reeb field of θ has the form $T = \partial/\partial u^1$, and let $Y \subseteq U$ be the slice defined by $u^1 = 0$. Because T is nowhere tangent to Y, it follows that the pullback of $d\theta$ to Y is a symplectic form. After shrinking U and Y if necessary, we can find Darboux coordinates $\left(x^1, \dots, x^n, y^1, \dots, y^n\right)$ for Y centered at p, and extend them to U by requiring them to be independent of u^1 (or equivalently, to be constant on the integral curves of T). Let α be the 1-form $\sum_i y^i\, dx^i$ on U, so the pullbacks of $d\theta$ and $-d\alpha$ to Y agree. Because $T \lrcorner\, d\theta = T \lrcorner\, d\alpha = 0$, it follows that $d\theta + d\alpha = 0$ at points of Y. Then $\mathcal{L}_T \theta = \mathcal{L}_T \alpha = 0$ implies that $d(\theta + \alpha)$ is invariant under the flow of T, so in fact $d(\theta + \alpha) = 0$ on all of U. By the Poincaré lemma, there is a smooth function z on U such that $dz = \theta + \alpha$; by subtracting a constant, we may arrange that $z(p) = 0$. Because $dz_p(T_p) = \theta_p(T_p) = 1$, it follows that $\left\{dx^i|_p, dy^i|_p, dz|_p\right\}$ are linearly independent, so Problem 11-6 shows that there is a neighborhood of p on which $\left(x^1, \dots, x^n, y^1, \dots, y^n, z\right)$ are the coordinates we seek. □

The next proposition describes a contact analogue of Hamiltonian vector fields.

Proposition 22.32. *Suppose (M, H) is a contact manifold and θ is a contact form for H. For any function $f \in C^\infty(M)$, there is a unique vector field X_f, called the* ***contact Hamiltonian vector field of f****, that satisfies $\theta(X_f) = f$ and $(X_f \lrcorner\, d\theta)|_H = -df|_H$.*

Proof. Suppose $f \in C^\infty(M)$. Because the restriction of $d\theta$ to H is nondegenerate, there is a unique smooth vector field $B \in \Gamma(H)$ such that $B \lrcorner\, d\theta|_H = df|_H$. If we set $X_f = fT - B$, where T is the Reeb field for θ, then it is easy to check that the required conditions are satisfied. □

Suppose (M, H) is a contact manifold. A smooth vector field $X \in \mathfrak{X}(M)$ is called a ***contact vector field*** if its flow θ preserves the contact structure, in the sense that $d(\theta_t)_p(H_p) = H_{\theta_t(p)}$ for all (t, p) in the domain of θ.

Theorem 22.33 (Characterization of Contact Vector Fields). *If (M, H) is a contact manifold and θ is a contact form for H, then a smooth vector field on M is a contact vector field if and only if it is a contact Hamiltonian vector field.*

Proof. Problem 22-21. □

If (M, H) is a contact manifold, a smooth submanifold $S \subseteq M$ is said to be ***isotropic*** if $TS \subseteq H$, or equivalently if $\iota^*\theta = 0$ for any contact form θ, where $\iota\colon S \hookrightarrow M$ is inclusion. If $S \subseteq M$ is isotropic, then $\iota^* d\theta = d(\iota^*\theta) = 0$. This implies that for each $p \in S$, the tangent space $T_p S$ is an isotropic subspace of the symplectic vector space H_p, and thus its dimension cannot be any larger than n, where $2n + 1$ is the dimension of M. An isotropic submanifold of the maximum possible dimension n is called a ***Legendrian submanifold***.

The next theorem is a contact analogue of the Hamiltonian flowout theorem, and is proved in much the same way. It is the main tool for constructing solutions of fully nonlinear PDEs (see Theorem 22.39 below).

Theorem 22.34 (Contact Flowout Theorem). *Suppose (M, H) is a contact manifold, $F \in C^\infty(M)$, Γ is an embedded isotropic submanifold of M that is contained in the zero set of F, and the contact Hamiltonian vector field X_F is nowhere tangent to Γ. If $\mathcal{S}$ is a flowout from Γ along X_F, then $\mathcal{S}$ is also isotropic and contained in the zero set of H.*

Proof. Problem 22-23. □

Nonlinear First-Order PDEs

In Chapter 9, we discussed first-order partial differential equations, and showed how to use the theory of flows to solve them in the linear and quasilinear cases. In this section, we show how to use symplectic and contact geometry to solve fully nonlinear first-order equations (i.e., equations that are not quasilinear).

We begin with a somewhat special case. A first-order partial differential equation that involves only the first derivatives of the unknown function but not the values of the function itself is called a ***Hamilton–Jacobi equation***. Such an equation for an unknown function $u(x^1, \dots, x^n)$ can be written in the form

$$F\left(x^1, \dots, x^n, \frac{\partial u}{\partial x^1}(x), \dots, \frac{\partial u}{\partial x^n}(x)\right) = 0, \tag{22.21}$$

where F is a smooth function defined on an open subset of $\mathbb{R}^{2n}$. (The terminology regarding Hamilton–Jacobi equations is not universally agreed upon. Some authors reserve the term *Hamilton–Jacobi equation* for the special case of an equation of the form

$$\frac{\partial u}{\partial x^1} + H\left(x^1, \dots, x^n, \frac{\partial u}{\partial x^2}(x), \dots, \frac{\partial u}{\partial x^n}(x)\right) = 0. \tag{22.22}$$

The implicit function theorem shows that an equation of the general form (22.21) can be locally rewritten in this special form if and only if the partial derivative of F with respect to its $(n + 1)$st variable is nonzero. On the other hand, other authors use the term *eikonal equation* to refer to any equation of the form (22.21).

We reserve that term for another special case to which it was originally applied; see Problem 22-24.)

More generally, if M is a smooth manifold, a Hamilton–Jacobi equation on M is given by a smooth real-valued function F defined on an open subset $W \subseteq T^*M$, and a solution to the equation is a smooth real-valued function u defined on an open subset $U \subseteq M$ such that the image of du lies in the zero set of F:

$$F\big(x, du(x)\big) = 0 \quad \text{for all } x \in U. \tag{22.23}$$

(We write the covector $du_x \in T_x^*M$ as $\big(x, du(x)\big)$, in order to be more consistent with the coordinate representation (22.21) of the equation.) We are interested in solving a Cauchy problem for (22.23): given an embedded hypersurface $S \subseteq M$ and a smooth function $\varphi\colon S \to \mathbb{R}$, we wish to find a smooth function u defined on a neighborhood of S in M and satisfying (22.23) together with the initial condition

$$u\big|_S = \varphi. \tag{22.24}$$

Just as in Chapter 9, in order to obtain solutions we need to assume that the problem is of a type called *noncharacteristic*; we will describe what this means below.

Because Equation (22.23) involves only du, not u itself, we look first for a closed 1-form α satisfying $F\big(x, \alpha(x)\big) \equiv 0$; then the Poincaré lemma guarantees that locally $\alpha = du$ for some function u, which then satisfies (22.23). By Proposition 22.12, it suffices to construct a Lagrangian submanifold of T^*M that is the image of a 1-form and is contained in $F^{-1}(0)$. The key to finding such a submanifold is the Hamiltonian flowout theorem (Theorem 22.23): after identifying an appropriate isotropic embedded initial submanifold $\Gamma \subseteq T^*M$, we will construct the image of α as the flowout from Γ along the Hamiltonian field of F.

The first challenge is to construct an appropriate initial submanifold $\Gamma \subseteq T^*M$. The image of $d\varphi$ will not do, because it lies in T^*S, not T^*M (and there is no canonical way to identify T^*S as a subset of T^*M). Thus, we must first look for an appropriate section of the restricted bundle $T^*M|_S$, that is, a smooth map $\sigma\colon S \to T^*M$ such that $\sigma(x) \in T_x^*M$ for each $x \in S$. This will be the value of du along S for our eventual solution u. Thus, we should expect that it matches $d\varphi$ when restricted to vectors tangent to S, and that it satisfies the PDE at points of S. In summary, we require σ to satisfy the following conditions:

$$\sigma(x)\big|_{T_xS} = d\varphi(x) \quad \text{for all } x \in S, \tag{22.25}$$

$$F\big(x, \sigma(x)\big) = 0 \qquad \text{for all } x \in S. \tag{22.26}$$

To find such a σ, at least locally, begin by extending φ to a smooth function $\widetilde{\varphi}$ in a neighborhood of S and choosing a smooth local defining function ψ for S. Since σ must agree with $d\varphi$ when restricted to TS, and the annihilator of TS at each point is spanned by $d\psi$, the only possibility for σ is a section of the form $\sigma = d\widetilde{\varphi} + f\, d\psi$ for some unknown real-valued function f defined in a neighborhood of S. You can then insert this into the equation $F\big(x, \sigma(x)\big) = 0$, and attempt to solve for the values of f along S.

The Cauchy problem (22.23)–(22.24) is said to be ***noncharacteristic*** if there exists a smooth section $\sigma \in \Gamma(T^*M|_S)$ satisfying (22.25)–(22.26), with the additional property that if (x^i) are any local coordinates on M and $(x^1,\dots,x^n,\xi_1,\dots,\xi_n)$ are the corresponding natural coordinates on T^*M, the following vector field along S is nowhere tangent to S:

$$A^\sigma\big|_x = \frac{\partial F}{\partial \xi_1}\big(x,\sigma(s)\big)\frac{\partial}{\partial x^1} + \cdots + \frac{\partial F}{\partial \xi_n}\big(x,\sigma(s)\big)\frac{\partial}{\partial x^n}. \tag{22.27}$$

(As we will see in the proof of the next theorem, A^σ is actually globally defined as a vector field along S, and does not depend on the choice of coordinates.) When this condition is satisfied, we can solve the Cauchy problem.

Theorem 22.35 (The Cauchy Problem for a Hamilton–Jacobi Equation). *Suppose M is a smooth manifold, $W \subseteq T^*M$ is an open subset, $F\colon W \to \mathbb{R}$ is a smooth function, $S \subseteq M$ is an embedded hypersurface, and $\varphi\colon S \to \mathbb{R}$ is a smooth function. If the Cauchy problem (22.23)–(22.24) is noncharacteristic, then for each $p \in S$ there is a smooth solution defined on some neighborhood of p in M.*

Proof. Given $\sigma\colon S \to T^*M|_S$ satisfying (22.25)–(22.26), let $\Gamma \subseteq W$ be the image of σ. Then Γ is an embedded submanifold of dimension $n-1$, where $n = \dim M$. In order to apply the Hamiltonian flowout theorem, we need to check first that Γ is isotropic with respect to the canonical symplectic structure ω on T^*M. Since $\sigma\colon S \to T^*M$ is a smooth embedding whose image is Γ, this is equivalent to showing that $\sigma^*\omega = 0$. Let $\pi\colon T^*M \to M$ be the projection; then $\pi\circ\sigma$ is equal to the inclusion $\iota\colon S \hookrightarrow M$. If τ denotes the tautological 1-form on T^*M, the defining equation (22.3) for τ implies

$$(\sigma^*\tau)(p) = d\sigma_p^*\big(d\pi^*_{\sigma(p)}\sigma(p)\big) = d(\pi\circ\sigma)^*_p\sigma(p) = d\iota_p^*\sigma(p) = d\varphi(p).$$

Thus $\sigma^*\tau = d\varphi$, and it follows that $\sigma^*\omega = \sigma^*(-d\tau) = -d(\sigma^*\tau) = -d(d\varphi) = 0$. Thus Γ is isotropic.

Next we need to check that the Hamiltonian vector field X_F is nowhere tangent to Γ. This follows from the noncharacteristic condition just as in the proof of Theorem 9.53: because $\pi\colon T^*M \to M$ restricts to a diffeomorphism from Γ to S, if X_F were tangent to Γ at some point $\big(p,\sigma(p)\big) \in \Gamma$, then $d\pi\big(X_F|_{(p,\sigma(p))}\big)$ would be tangent to S at p. Using (22.9) in natural coordinates $\big(x^i,\xi_i\big)$ on T^*M (which are Darboux coordinates for the canonical symplectic form), we find that

$$X_F = \frac{\partial F}{\partial \xi^1}\frac{\partial}{\partial x^1} + \cdots + \frac{\partial F}{\partial \xi^n}\frac{\partial}{\partial x^n} - \frac{\partial F}{\partial x^1}\frac{\partial}{\partial \xi_1} - \cdots - \frac{\partial F}{\partial x^n}\frac{\partial}{\partial \xi_n}.$$

Thus $d\pi\big(X_F|_{(p,\sigma(p))}\big) = A^\sigma|_p$, so the assumption that the Cauchy problem is noncharacteristic guarantees that X_F is nowhere tangent to Γ. (This calculation also shows that A^σ is well defined independently of coordinates, because it is the pushforward of X_F from points of Γ.)

Let $\mathcal{S}$ be a flowout from Γ along X_F. The Hamiltonian flowout theorem guarantees that $\mathcal{S}$ is an n-dimensional isotropic—and therefore Lagrangian—submanifold of T^*M contained in $F^{-1}(0)$. Using the result of Problem 22-11, we conclude that it will be the image of a closed 1-form on a neighborhood of p provided that it is transverse to the fiber of π at $(p,\sigma(p))$. Once again, we use the fact that $T_{(p,\sigma(p))}\mathcal{S}$ is spanned by $T_{(p,\sigma(p))}\Gamma$ and $X_F|_{(p,\sigma(p))}$. Because $d\pi$ maps $X_F|_{(p,\sigma(p))}$ to $A^\sigma|_p$ and maps $T_{(p,\sigma(p))}\Gamma$ isomorphically onto T_pS, the noncharacteristic assumption guarantees that $T_{(p,\sigma(p))}T^*M = T_{(p,\sigma(p))}\mathcal{S} \oplus \operatorname{Ker} d\pi_{(p,\sigma(p))}$, and thus $\mathcal{S}$ intersects the fiber transversely at $(p,\sigma(p))$. By Problem 22-11, there is a closed 1-form α defined on a neighborhood U of p whose graph is an open subset of $\mathcal{S}$. Because the image of σ is contained in $\mathcal{S}$, it follows that

$$\alpha(x) = \sigma(x) \quad \text{for } x \in S \cap U. \tag{22.28}$$

By the Poincaré lemma, after shrinking U further if necessary, we can find a smooth function $u\colon U \to \mathbb{R}$ such that $du = \alpha$. Because $\mathcal{S} \subseteq F^{-1}(0)$, we conclude that u satisfies (22.23). To ensure that the initial condition is also satisfied, shrink U further so that $S \cap U$ is connected. By adding a constant to u, we may arrange that $u(p) = \varphi(p)$. Then for any $x \in S$, it follows from (22.25) and (22.28) that

$$du(x)\big|_{T_xS} = \alpha(x)\big|_{T_xS} = \sigma(x)\big|_{T_xS} = d\varphi(x).$$

Because $S \cap U$ is connected, this means that $u - \varphi$ is constant on $S \cap U$. Since this difference vanishes at p, it vanishes identically, so (22.24) is satisfied on $S \cap U$. □

Note that we did not claim any uniqueness in this theorem. In Cauchy problems for fully nonlinear equations, even local uniqueness can fail. For example, consider the following Cauchy problem in the plane:

$$\left(\frac{\partial u}{\partial x}\right)^2 = 1, \qquad u(0,y) = 0.$$

This is noncharacteristic, as you can check. Both $u(x,y) = x$ and $u(x,y) = -x$ are solutions to this problem, but they are not equal in any open subset. The problem here is that there are two possible choices for the initial 1-form σ (namely, $\sigma = dx$ and $\sigma = -dx$), and they yield different initial manifolds Γ and therefore different solutions to the Cauchy problem. As Problem 22-25 shows, once σ is chosen, local uniqueness holds just as in the quasilinear case.

Example 22.36 (A Hamilton–Jacobi Equation). Consider the following Cauchy problem in the plane:

$$\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial y}\right)^2 = 0, \qquad u(0,y) = y^2.$$

The corresponding function on $T^*\mathbb{R}^2$ is $F(x,y,\xi,\eta) = \xi - \eta^2$, where we use (x,y,ξ,η) to denote natural coordinates on $T^*\mathbb{R}^2$ associated with (x,y).

To check that the problem is noncharacteristic, we need to find a suitable 1-form σ along the initial manifold $S = \{(x,y) : x = 0\}$. Since x is a defining function for S, we can write $\sigma = d(y^2) + f(y)\,dx = 2y\,dy + f(y)\,dx$ and solve the equation

$F(0, y, f(y), 2y) = f(y) - (2y)^2 = 0$ to obtain $f(y) = 4y^2$, and thus we can set $\sigma(y) = 2y\,dy + 4y^2\,dx$. The vector field A^σ is given by

$$A^\sigma\big|_{(x,y)} = \frac{\partial}{\partial x} - 4y\frac{\partial}{\partial y},$$

which is nowhere tangent to S.

The initial curve S can be parametrized by $X(s) = (0, s)$, and therefore the initial curve $\Gamma \subseteq T^*\mathbb{R}^2$ (the image of σ) can be parametrized by $\widetilde{X}(s) = \big(0, s, 4s^2, 2s\big)$. The Hamiltonian field of F is

$$X_F\big|_{(x,y,\xi,\eta)} = \frac{\partial}{\partial x} - 2\eta\frac{\partial}{\partial y},$$

and it is an easy matter to solve the corresponding system of ODEs with initial conditions $(x, y, \xi, \eta) = \big(0, s, 4s^2, 2s\big)$ to obtain the following parametrization of $\mathcal{S}$:

$$\Psi(t, s) = \big(t, s - 4st, 4s^2, 2s\big).$$

Solving $(x, y) = (t, s - 4st)$ for (t, s) and inserting into the formulas for (ξ, η), we find that $\mathcal{S}$ is the image of the following 1-form:

$$\alpha = \frac{4y^2}{(1-4x)^2}\,dx + \frac{2y}{1-4x}\,dy.$$

This is indeed a closed 1-form, and using the procedure sketched at the end of Chapter 11 we find that $\alpha = du$ on the set $\{(x, y) : x < 1/4\}$, where

$$u(x, y) = \frac{y^2}{1-4x}.$$

In principle, we might have to add a constant to u to satisfy the initial condition, but in this case $u(0, y) = y^2$ already, so this is the solution to our Cauchy problem. //

General Nonlinear Equations

Finally, we show how the preceding method can be adapted to solve arbitrary first-order PDEs by using contact geometry in place of symplectic geometry. For this purpose, we introduce one last geometric construction. If M is a smooth manifold, the ***1-jet bundle of M*** is the smooth vector bundle $J^1M = \mathbb{R} \times T^*M \to M$, whose fiber at $x \in M$ is $\mathbb{R} \times T_x^*M$. (It is the Whitney sum of a trivial $\mathbb{R}$-bundle with T^*M.) If $u\colon M \to \mathbb{R}$ is a smooth function, the ***1-jet of u*** is the section $j^1u\colon M \to J^1M$ defined by $j^1u(x) = \big(u(x), du(x)\big)$. A point in the fiber of J^1M over $x \in M$ can be viewed as a first-order Taylor polynomial at x of a smooth function on M, represented invariantly as the values of the function and its differential at x. (One can also define higher-order jet bundles that give invariant representations of higher-order Taylor polynomials. They are useful for studying higher-order PDEs, but we do not pursue them here.)

The ***canonical contact form*** on J^1M is the 1-form $\theta = dz - \tau$ defined in Example 22.27(b). A smooth (local or global) section $\eta\colon M \to J^1M$ is said to be ***Legendrian*** if its image is a Legendrian submanifold of J^1M, or equivalently if $\eta^*\theta = 0$. The next proposition is a contact analogue of Proposition 22.12.

Proposition 22.37. *Let M be a smooth manifold. A smooth local section of J^1M is the 1-jet of a smooth function if and only if it is Legendrian.*

► **Exercise 22.38.** Prove the preceding proposition.

The 1-jet bundle provides the most general setting in which to consider first-order partial differential equations. If M is a smooth manifold, a first-order PDE for a function $u\colon M \to \mathbb{R}$ can be viewed as a real-valued function F on the 1-jet bundle of M, and a solution is a function whose 1-jet takes its values in the zero set of F.

Let M be a smooth manifold, and suppose we are given a function $F \in C^\infty(W)$ on some open subset $W \subseteq J^1M$, a smooth hypersurface $S \subseteq M$, and a smooth function $\varphi\colon S \to \mathbb{R}$. We wish to solve the following Cauchy problem for u:

$$F\big(x, u(x), du(x)\big) \equiv 0, \tag{22.29}$$

$$u\big|_S = \varphi. \tag{22.30}$$

This problem is said to be ***noncharacteristic*** if there exists a smooth section $\sigma \in \Gamma\big(T^*M|_S\big)$ taking its values in W and satisfying

$$\sigma(x)\big|_{T_xS} = d\varphi(x) \quad \text{for all } x \in S, \tag{22.31}$$

$$F\big(x, \varphi(x), \sigma(x)\big) = 0 \qquad \text{for all } x \in S, \tag{22.32}$$

and such that the following vector field along S is nowhere tangent to S:

$$A^{\varphi,\sigma}\big|_x = \frac{\partial F}{\partial \xi_1}\big(x, \varphi(x), \sigma(x)\big)\frac{\partial}{\partial x^1} + \cdots + \frac{\partial F}{\partial \xi_n}\big(x, \varphi(x), \sigma(x)\big)\frac{\partial}{\partial x^n}. \tag{22.33}$$

The proof of the next theorem is very similar to that of Theorem 22.35, but uses the contact flowout theorem instead of the Hamiltonian one.

Theorem 22.39 (The General First-Order Cauchy Problem). *Suppose M is a smooth manifold, $W \subseteq J^1M$ is an open subset, $F\colon W \to \mathbb{R}$ is a smooth function, $S \subseteq M$ is an embedded hypersurface, and $\varphi\colon S \to \mathbb{R}$ is a smooth function. If the Cauchy problem (22.29)–(22.30) is noncharacteristic, then for each $p \in S$ there is a smooth solution on some neighborhood of p in M.*

Proof. Problem 22-26. □

Problems

22-1. Prove Proposition 22.5 (properties of symplectic, isotropic, coisotropic, and Lagrangian subspaces).

22-2. Let (V,ω) be a symplectic vector space of dimension $2n$. Show that for each symplectic, isotropic, coisotropic, or Lagrangian subspace $S \subseteq V$, there exists a symplectic basis (A_i, B_i) for V with the following property:
(a) If S is symplectic, $S = \operatorname{span}(A_1, B_1, \dots, A_k, B_k)$ for some k.
(b) If S is isotropic, $S = \operatorname{span}(A_1, \dots, A_k)$ for some k.
(c) If S is coisotropic, $S = \operatorname{span}(A_1, \dots, A_n, B_1, \dots, B_k)$ for some k.
(d) If S is Lagrangian, $S = \operatorname{span}(A_1, \dots, A_n)$.

22-3. The ***real symplectic group*** is the subgroup $\mathrm{Sp}(2n,\mathbb{R}) \subseteq \mathrm{GL}(2n,\mathbb{R})$ consisting of all $2n \times 2n$ matrices that leave the standard symplectic tensor $\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$ invariant, that is, the set of invertible linear maps $Z\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\omega(Zx, Zy) = \omega(x,y)$ for all $x, y \in \mathbb{R}^{2n}$.
(a) Show that a matrix Z is in $\mathrm{Sp}(2n,\mathbb{R})$ if and only if it takes the standard basis to a symplectic basis.
(b) Show that $Z \in \mathrm{Sp}(2n,\mathbb{R})$ if and only if $Z^T J Z = J$, where J is the $2n \times 2n$ block diagonal matrix

$$J = \begin{pmatrix} j & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & j \end{pmatrix},$$

with copies of the 2×2 block $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ along the main diagonal, and zeros elsewhere.
(c) Show that $\mathrm{Sp}(2n,\mathbb{R})$ is an embedded Lie subgroup of $\mathrm{GL}(2n,\mathbb{R})$, and determine its dimension.
(d) Determine the Lie algebra of $\mathrm{Sp}(2n,\mathbb{R})$ as a subalgebra of $\mathfrak{gl}(2n,\mathbb{R})$.
(e) Is $\mathrm{Sp}(2n,\mathbb{R})$ compact?

22-4. Let (M,ω) be a symplectic manifold, and suppose $F\colon N \to M$ is a smooth map such that $F^*\omega$ is symplectic. Show that F is a smooth immersion.

22-5. Suppose (M,ω) is a $2n$-dimensional compact symplectic manifold.
(a) Show that ω^n (the n-fold wedge product of ω with itself) is not exact.
(b) Show that $H^{2p}_{\mathrm{dR}}(M) \neq 0$ for $p = 1, \dots, n$.
(c) Show that $\mathbb{S}^2$ is the only sphere that admits a symplectic structure.

22-6. Prove that $\mathbb{R}^{2n}$ (with its standard symplectic structure) does not have any compact symplectic submanifolds.

22-7. Let (M,ω) and $\big(\widetilde{M}, \widetilde{\omega}\big)$ be symplectic manifolds. Define a 2-form Ω on $M \times \widetilde{M}$ by $\Omega = \pi^*\omega - \widetilde{\pi}^*\widetilde{\omega}$, where $\pi\colon M \times \widetilde{M} \to M$ and $\widetilde{\pi}\colon M \times \widetilde{M} \to \widetilde{M}$ are the projections.
(a) Show that Ω is symplectic.
(b) Show that a diffeomorphism $F\colon M \to \widetilde{M}$ is a symplectomorphism if and only if its graph $\Gamma(F) = \big\{(x,y) \in M \times \widetilde{M} : y = F(x)\big\}$ is a Lagrangian submanifold of $\big(M \times \widetilde{M}, \Omega\big)$.

22-8. Suppose (M,ω) is a symplectic manifold and $S \subseteq M$ is a coisotropic submanifold. An immersed submanifold $N \subseteq S$ is said to be ***characteristic*** if

$T_pN = (T_pS)^{\perp}$ for each $p \in N$. Show that there is a foliation of S by connected characteristic submanifolds of S whose dimension is equal to the codimension of S in M.

22-9. Considering $\mathbb{R}^{2n}$ as a symplectic manifold with its standard symplectic structure $\omega = \sum_i dx^i \wedge dy^i$, let $\Lambda_n \subseteq G_n\left(\mathbb{R}^{2n}\right)$ denote the set of Lagrangian subspaces of $\mathbb{R}^{2n}$.
(a) Show that the real symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ acts transitively on Λ_n (see Problem 22-3).
(b) Show that Λ_n has a unique smooth manifold structure such that the action of $\mathrm{Sp}(2n,\mathbb{R})$ is smooth, and determine its dimension.
(c) Is Λ_n compact?

22-10. Show that the canonical symplectic form on the cotangent bundle is invariant under diffeomorphisms, in the following sense: suppose Q and $\widetilde{Q}$ are smooth manifolds and $F\colon Q \to \widetilde{Q}$ is a diffeomorphism. Let $dF^*\colon T^*\widetilde{Q} \to T^*Q$ be the smooth bundle homomorphism described in Problem 11-8. Show that dF^* is a symplectomorphism when both T^*Q and $T^*\widetilde{Q}$ are endowed with their canonical symplectic forms.

22-11. Let Q be a smooth manifold, and let S be an embedded Lagrangian submanifold of the total space of T^*Q. Prove the following statements.
(a) If S is transverse to the fiber of T^*Q at a point $q \in T^*Q$, then there exist a neighborhood V of q in S and a neighborhood U of $\pi(q)$ in Q such that V is the image of a smooth closed 1-form defined on U.
(b) S is the image of a globally defined smooth closed 1-form on Q if and only if S intersects each fiber transversely in exactly one point.
(Cf. Theorem 6.32 and Corollary 6.33.) (*Used on p. 588.*)

22-12. Let M be a smooth manifold of dimension at least 1. Show that there is no 1-form σ on M such that the tautological form $\tau \in \Omega^1(T^*M)$ is equal to the pullback $\pi^*\sigma$.

22-13. Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Define the ***conormal bundle of S*** to be the subset $N^*S \subseteq T^*M$ defined by

$$N^*S = \left\{(q,\eta) \in T^*M : q \in S,\ \eta\big|_{T_qS} \equiv 0\right\}.$$

Show that N^*S is a smooth subbundle of $T^*M|_S$, and an embedded Lagrangian submanifold of T^*M (with respect to the canonical symplectic structure on T^*M).

22-14. Prove the following global version of the Darboux theorem, due to Moser [Mos65]: Let M be a compact smooth manifold, and let ω_0 be a symplectic form on M. Suppose there is a smooth time-dependent 1-form $\alpha\colon [0,1] \times M \to T^*M$ such that $\omega_t = \omega_0 + d\alpha_t$ is symplectic for each $t \in [0,1]$. Show that there is a diffeomorphism $F\colon M \to M$ such that $F^*\omega_1 = \omega_0$.

22-15. Using the same technique as in the proof of Theorem 22.13, prove the following theorem of Moser [Mos65]: If M is an oriented compact smooth

n-manifold, $n \geq 1$, and ω_0, ω_1 are smooth orientation forms on M such that $\int_M \omega_0 = \int_M \omega_1$, then there exists a diffeomorphism $F\colon M \to M$ such that $F^*\omega_1 = \omega_0$. [Hint: for any orientation form ω, show that the map $\beta\colon \mathfrak{X}(M) \to \Omega^{n-1}(M)$ defined by $\beta(X) = X \lrcorner\, \omega$ as in (16.11) is a bundle isomorphism.]

22-16. Let (M, ω) be a symplectic manifold. Let $\mathcal{S}(M) \subseteq \mathfrak{X}(M)$ denote the set of symplectic vector fields on M, and $\mathcal{H}(M) \subseteq \mathfrak{X}(M)$ the set of Hamiltonian vector fields.

(a) Show that $\mathcal{S}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, and $\mathcal{H}(M)$ is a Lie subalgebra of $\mathcal{S}(M)$.

(b) Show that the map from $\mathcal{S}(M)$ to $\Omega^1(M)$ given by $X \mapsto X \lrcorner\, \omega$ descends to a vector space isomorphism between $\mathcal{S}(M)/\mathcal{H}(M)$ and $H^1_{\mathrm{dR}}(M)$.

22-17. Consider the 2-*body problem*, that is, the Hamiltonian system (T^*Q, ω, E) described in Example 22.18 in the special case $n = 2$. Suppose that the potential energy V depends only on the distance between the particles. More precisely, suppose that $V(q) = v\big(r(q)\big)$ for some smooth function $v\colon (0, \infty) \to \mathbb{R}$, where

$$r(q) = |\mathbf{q}_1 - \mathbf{q}_2| = \sqrt{\big(q_1^1 - q_2^1\big)^2 + \big(q_1^2 - q_2^2\big)^2 + \big(q_1^3 - q_2^3\big)^2}.$$

(a) Let $\mathbf{u} = \big(u^1, u^2, u^3\big)$ be a unit vector in $\mathbb{R}^3$, and show that the function $P\colon T^*Q \to \mathbb{R}$ defined by

$$\begin{aligned} P(q, p) &= \mathbf{u} \cdot \mathbf{p}_1 + \mathbf{u} \cdot \mathbf{p}_2 \\ &= u^1 p_1^1 + u^2 p_1^2 + u^3 p_1^3 + u^1 p_2^1 + u^2 p_2^2 + u^3 p_2^3 \end{aligned}$$

is a conserved quantity (called the ***total linear momentum in the u-direction***), and that the corresponding infinitesimal symmetry generates translations in the $\mathbf{u}$-direction:

$$\begin{aligned} \theta_t(q, p) &= (\mathbf{q}_1 + t\mathbf{u}, \mathbf{q}_2 + t\mathbf{u}, \mathbf{p}_1, \mathbf{p}_2) \\ &= \big(q_1^1 + tu^1, q_1^2 + tu^2, q_1^3 + tu^3, q_2^1 + tu^1, q_2^2 + tu^2, \\ &\qquad q_2^3 + tu^3, p_1^1, p_1^2, p_1^3, p_2^1, p_2^2, p_2^3\big). \end{aligned}$$

(b) Show that the function $L\colon T^*Q \to \mathbb{R}$ defined by

$$L(q, p) = q_1^1 p_1^2 - q_1^2 p_1^1 + q_2^1 p_2^2 - q_2^2 p_2^1$$

is a conserved quantity (called the ***total angular momentum about the z-axis***), and find the flow of the corresponding infinitesimal symmetry. Explain what this has to do with rotational symmetry.

22-18. Prove Proposition 22.21 (properties of conserved quantities and infinitesimal symmetries).

22-19. This problem outlines a different proof of the Darboux theorem. Let (M, ω) be a $2n$-dimensional symplectic manifold and $p \in M$.

(a) Show that smooth coordinates $\left(x^1, \dots, x^n, y^1, \dots, y^n\right)$ on an open subset $U \subseteq M$ are Darboux coordinates if and only if their Poisson brackets satisfy

$$\left\{x^i, y^j\right\} = \delta^{ij}; \qquad \left\{x^i, x^j\right\} = \left\{y^i, y^j\right\} = 0. \tag{22.34}$$

(b) Prove the following statement by induction on k: for each $k = 0, \dots, n$, there are smooth functions $\left(x^1, \dots, x^k, y^1, \dots, y^k\right)$ vanishing at p and satisfying (22.34) in a neighborhood of p such that the $2k$-tuple of 1-forms $\left(dx^1, \dots, dx^k, dy^1, \dots, dy^k\right)$ is linearly independent at p. When $k = n$, this proves the theorem. [Hint: for the inductive step, assuming that $\left(x^1, \dots, x^k, y^1, \dots, y^k\right)$ have been found, find smooth coordinates $\left(u^1, \dots, u^{2n}\right)$ such that

$$\frac{\partial}{\partial u^i} = X_{x^i}, \qquad \frac{\partial}{\partial u^{i+k}} = X_{y^i}, \quad i = 1, \dots, k,$$

and let $y^{k+1} = u^{2k+1}$. Then find new coordinates $\left(v^1, \dots, v^{2n}\right)$ with

$$\frac{\partial}{\partial v^i} = X_{x^i}, \quad i = 1, \dots, k,$$
$$\frac{\partial}{\partial v^{i+k}} = X_{y^i}, \quad i = 1, \dots, k+1,$$

and let $x^{k+1} = v^{2k+1}$.]

22-20. Suppose (M, H) is a contact manifold of dimension $2n + 1$. Show that if n is odd, then M is orientable, while if n is even, then M is orientable if and only if there exists a global contact form for H.

22-21. Prove Theorem 22.33 (characterization of contact vector fields).

22-22. Suppose (M, H) is a contact manifold and X is a smooth vector field on M. Prove that X is the Reeb field of some contact form for H if and only if it is a contact vector field that takes no values in H.

22-23. Prove Theorem 22.34 (the contact flowout theorem).

22-24. The classical ***eikonal equation*** for a real-valued function u on an open subset $U \subseteq \mathbb{R}^n$ is

$$\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x^i}\right)^2 = f(x), \tag{22.35}$$

where f is a given smooth real-valued function on u. It plays an important role in the theory of optics. (The word "eikonal" stems from the Greek word for "image," the same root from which our word "icon" is derived.) In the special case $f(x) \equiv 1$, find an explicit solution u to (22.35) on an open subset of $\mathbb{R}^n$ with $u = 0$ on the unit sphere.

22-25. Suppose (22.24) is a noncharacteristic initial condition for a Hamilton–Jacobi equation (22.23). For any choice of $\sigma\colon S \to T^*\mathbb{R}^n$ satisfying (22.25), (22.26), and (22.27), and any $p \in S$, show that there is a neighborhood U of p on which there is a *unique* solution to the Cauchy problem (22.23)–(22.24) satisfying $du(x) = \sigma(x)$ for $x \in S$.

22-26. Prove Theorem 22.39 (solution to the general first-order Cauchy problem).