Chapter 3
Tangent Vectors

The central idea of calculus is *linear approximation*. This arises repeatedly in the study of calculus in Euclidean spaces, where, for example, a function of one variable can be approximated by its tangent line, a parametrized curve in $\mathbb{R}^n$ by its velocity vector, a surface in $\mathbb{R}^3$ by its tangent plane, or a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ by its total derivative (see Appendix C).

In order to make sense of calculus on manifolds, we need to introduce the *tangent space to a manifold at a point*, which we can think of as a sort of "linear model" for the manifold near the point. Because of the abstractness of the definition of a smooth manifold, this takes some work, which we carry out in this chapter.

We begin by studying much more concrete objects: *geometric tangent vectors* in $\mathbb{R}^n$, which can be visualized as "arrows" attached to points. Because the definition of smooth manifolds is built around the idea of identifying which functions are smooth, the property of a geometric tangent vector that is amenable to generalization is its action on smooth functions as a "directional derivative." The key observation, which we prove in the first section of this chapter, is that the process of taking directional derivatives gives a natural one-to-one correspondence between geometric tangent vectors and linear maps from $C^\infty(\mathbb{R}^n)$ to $\mathbb{R}$ satisfying the product rule. (Such maps are called *derivations*.) With this as motivation, we then *define* a tangent vector on a smooth manifold as a derivation of $C^\infty(M)$ at a point.

In the second section of the chapter, we show how a smooth map between manifolds yields a linear map between tangent spaces, called the *differential* of the map, which generalizes the total derivative of a map between Euclidean spaces. This allows us to connect the abstract definition of tangent vectors to our concrete geometric picture by showing that any smooth coordinate chart (U, φ) gives a natural isomorphism from the space of tangent vectors to M at p to the space of tangent vectors to $\mathbb{R}^n$ at $\varphi(p)$, which in turn is isomorphic to the space of geometric tangent vectors at $\varphi(p)$. Thus, any smooth coordinate chart yields a basis for each tangent space. Using this isomorphism, we describe how to do concrete computations in such a basis. Based on these coordinate computations, we show how the union of all the tangent spaces at all points of a smooth manifold can be "glued together" to form a new manifold, called the *tangent bundle* of the original manifold.

Next we show how a smooth curve determines a tangent vector at each point, called its *velocity*, which can be regarded as the derivation of $C^\infty(M)$ that takes the derivative of each function along the curve.

In the final two sections we discuss and compare several other approaches to defining tangent spaces, and give a brief overview of the terminology of *category theory*, which puts the tangent space and differentials in a larger context.

Tangent Vectors

Imagine a manifold in Euclidean space—for example, the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. What do we mean by a "tangent vector" at a point of $\mathbb{S}^{n-1}$? Before we can answer this question, we have to come to terms with a dichotomy in the way we think about elements of $\mathbb{R}^n$. On the one hand, we usually think of them as *points* in space, whose only property is location, expressed by the coordinates $(x^1, \ldots, x^n)$. On the other hand, when doing calculus we sometimes think of them instead as *vectors*, which are objects that have magnitude and direction, but whose location is irrelevant. A vector $v = v^i e_i$ (where e_i denotes the ith standard basis vector) can be visualized as an arrow with its initial point anywhere in $\mathbb{R}^n$; what is relevant from the vector point of view is only which direction it points and how long it is.

What we really have in mind here is a separate copy of $\mathbb{R}^n$ at each point. When we talk about vectors tangent to the sphere at a point a, for example, we imagine them as living in a copy of $\mathbb{R}^n$ with its origin translated to a.

Geometric Tangent Vectors

Here is a preliminary definition of tangent vectors in Euclidean space. Given a point $a \in \mathbb{R}^n$, let us define the ***geometric tangent space to $\mathbb{R}^n$ at a***, denoted by $\mathbb{R}_a^n$, to be the set $\{a\} \times \mathbb{R}^n = \{(a, v) : v \in \mathbb{R}^n\}$. A ***geometric tangent vector*** in $\mathbb{R}^n$ is an element of $\mathbb{R}_a^n$ for some $a \in \mathbb{R}^n$. As a matter of notation, we abbreviate (a, v) as v_a (or sometimes $v|_a$ if it is clearer, for example if v itself has a subscript). We think of v_a as the vector v with its initial point at a (Fig. 3.1).The set $\mathbb{R}_a^n$ is a real vector space under the natural operations

$$v_a + w_a = (v + w)_a, \qquad c(v_a) = (cv)_a.$$

The vectors $e_i|_a$, $i = 1, \ldots, n$, are a basis for $\mathbb{R}_a^n$. In fact, as a vector space, $\mathbb{R}_a^n$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}_a^n$ and $\mathbb{R}_b^n$ at distinct points a and b will be disjoint sets.

With this definition we could think of the tangent space to $\mathbb{S}^{n-1}$ at a point $a \in \mathbb{S}^{n-1}$ as a certain subspace of $\mathbb{R}_a^n$ (Fig. 3.2), namely the space of vectors that are orthogonal to the radial unit vector through a, using the inner product that $\mathbb{R}_a^n$ inherits from $\mathbb{R}^n$ via the natural isomorphism $\mathbb{R}^n \cong \mathbb{R}_a^n$. The problem with this definition, however, is that it gives us no clue as to how we might define tangent vectors on an arbitrary smooth manifold, where there is no ambient Euclidean space. So we
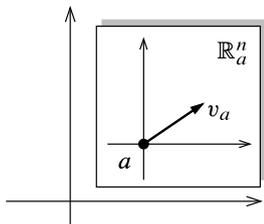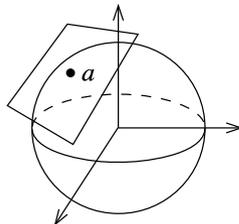
Fig. 3.1 Geometric tangent space

Fig. 3.2 Tangent space to $\mathbb{S}^{n-1}$

need to look for another characterization of tangent vectors that might make sense on a manifold.

The only things we have to work with on smooth manifolds so far are smooth functions, smooth maps, and smooth coordinate charts. One thing that a geometric tangent vector provides is a means of taking directional derivatives of functions. For example, any geometric tangent vector $v_a \in \mathbb{R}^n_a$ yields a map $D_v|_a \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$, which takes the directional derivative in the direction v at a:

$$D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a + tv). \tag{3.1}$$

This operation is linear over $\mathbb{R}$ and satisfies the product rule:

$$D_v|_a(fg) = f(a)D_v|_a g + g(a)D_v|_a f. \tag{3.2}$$

If $v_a = v^i e_i|_a$ in terms of the standard basis, then by the chain rule $D_v|_a f$ can be written more concretely as

$$D_v|_a f = v^i \frac{\partial f}{\partial x^i}(a).$$

(Here we are using the summation convention as usual, so the expression on the right-hand side is understood to be summed over $i = 1, \dots, n$. This sum is consistent with our index convention if we stipulate that an upper index "in the denominator" is to be regarded as a lower index.) For example, if $v_a = e_j|_a$, then

$$D_v|_a f = \frac{\partial f}{\partial x^j}(a).$$

With this construction in mind, we make the following definition. If a is a point of $\mathbb{R}^n$, a map $w \colon C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a ***derivation at a*** if it is linear over $\mathbb{R}$ and satisfies the following product rule:

$$w(fg) = f(a)wg + g(a)wf. \tag{3.3}$$

Let $T_a\mathbb{R}^n$ denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is a vector space under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f, \qquad (cw)f = c(wf).$$

The most important (and perhaps somewhat surprising) fact about $T_a \mathbb{R}^n$ is that it is finite-dimensional, and in fact is naturally isomorphic to the geometric tangent space $\mathbb{R}_a^n$ that we defined above. The proof will be based on the following lemma.

Lemma 3.1 (Properties of Derivations). *Suppose $a \in \mathbb{R}^n$, $w \in T_a \mathbb{R}^n$, and $f, g \in C^\infty (\mathbb{R}^n)$.*

(a) *If f is a constant function, then $wf = 0$.*
(b) *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Proof. It suffices to prove (a) for the constant function $f_1(x) \equiv 1$, for then $f(x) \equiv c$ implies $wf = w(cf_1) = cwf_1 = 0$ by linearity. For f_1, the product rule gives

$$wf_1 = w(f_1 f_1) = f_1(a)wf_1 + f_1(a)wf_1 = 2wf_1,$$

which implies that $wf_1 = 0$. Similarly, (b) also follows from the product rule:

$$w(fg) = f(a)wg + g(a)wf = 0 + 0 = 0. \qquad \square$$

The next proposition shows that derivations at a are in one-to-one correspondence with geometric tangent vectors.

Proposition 3.2. *Let $a \in \mathbb{R}^n$.*

(a) *For each geometric tangent vector $v_a \in \mathbb{R}_a^n$, the map $D_v|_a \colon C^\infty (\mathbb{R}^n) \to \mathbb{R}$ defined by (3.1) is a derivation at a.*
(b) *The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}_a^n$ onto $T_a \mathbb{R}^n$.*

Proof. The fact that $D_v|_a$ is a derivation at a is an immediate consequence of the product rule (3.2).

To prove that the map $v_a \mapsto D_v|_a$ is an isomorphism, we note first that it is linear, as is easily checked. To see that it is injective, suppose $v_a \in \mathbb{R}_a^n$ has the property that $D_v|_a$ is the zero derivation. Writing $v_a = v^i e_i|_a$ in terms of the standard basis, and taking f to be the jth coordinate function $x^j \colon \mathbb{R}^n \to \mathbb{R}$, thought of as a smooth function on $\mathbb{R}^n$, we obtain

$$0 = D_v|_a \left(x^j \right) = v^i \frac{\partial}{\partial x^i} \left(x^j \right) \Big|_{x=a} = v^j,$$

where the last equality follows because $\partial x^j / \partial x^i = 0$ except when $i = j$, in which case it is equal to 1. Since this is true for each j, it follows that v_a is the zero vector.

To prove surjectivity, let $w \in T_a \mathbb{R}^n$ be arbitrary. Motivated by the computation in the preceding paragraph, we define $v = v^i e_i$, where the real numbers $v^1, \dots, v^n$ are given by $v^i = w(x^i)$. We will show that $w = D_v|_a$.

To see this, let f be any smooth real-valued function on $\mathbb{R}^n$. By Taylor's theorem (Theorem C.15), we can write

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) \left(x^i - a^i \right)$$

$$+ \sum_{i,j=1}^{n} \left(x^i - a^i \right) \left(x^j - a^j \right) \int_0^1 (1 - t) \frac{\partial^2 f}{\partial x^i \partial x^j} \left(a + t(x - a) \right) dt.$$

Note that each term in the last sum above is a product of two smooth functions of x that vanish at $x = a$: one is $(x^i - a^i)$, and the other is $(x^j - a^j)$ times the integral. The derivation w annihilates this entire sum by Lemma 3.1(b). Thus

$$wf = w(f(a)) + \sum_{i=1}^{n} w\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right)$$

$$= 0 + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)\left(w(x^i) - w(a^i)\right)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)v^i = D_v|_a f. \qquad \square$$

Corollary 3.3. *For any $a \in \mathbb{R}^n$, the n derivations*

$$\frac{\partial}{\partial x^1}\bigg|_a, \ \ldots, \ \frac{\partial}{\partial x^n}\bigg|_a \quad \text{defined by} \quad \frac{\partial}{\partial x^i}\bigg|_a f = \frac{\partial f}{\partial x^i}(a)$$

form a basis for $T_a\mathbb{R}^n$, which therefore has dimension n.

Proof. Apply the previous proposition and note that $\partial/\partial x^i|_a = D_{e_i}|_a$. $\qquad \square$

Tangent Vectors on Manifolds

Now we are in a position to define tangent vectors on manifolds and manifolds with boundary. The definition is the same in both cases. Let M be a smooth manifold with or without boundary, and let p be a point of M. A linear map $v \colon C^\infty(M) \to \mathbb{R}$ is called a ***derivation at p*** if it satisfies

$$v(fg) = f(p)vg + g(p)vf \quad \text{for all } f, g \in C^\infty(M). \tag{3.4}$$

The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the ***tangent space to M at p***. An element of T_pM is called a ***tangent vector at p***.

The following lemma is the analogue of Lemma 3.1 for manifolds.

Lemma 3.4 (Properties of Tangent Vectors on Manifolds). *Suppose M is a smooth manifold with or without boundary, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$.*

(a) *If f is a constant function, then $vf = 0$.*
(b) *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

▶ **Exercise 3.5.** Prove Lemma 3.4.

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, you should visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given point. Proofs of theorems about tangent vectors must, of course, be based on the abstract definition in terms of derivations, but your intuition should be guided as much as possible by the geometric picture.
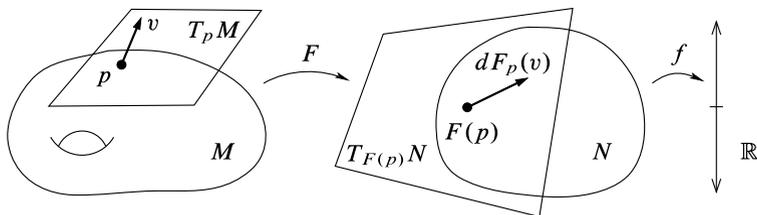
Fig. 3.3 The differential

The Differential of a Smooth Map

To relate the abstract tangent spaces we have defined on manifolds to geometric tangent spaces in $\mathbb{R}^n$, we have to explore the way smooth maps affect tangent vectors. In the case of a smooth map between Euclidean spaces, the total derivative of the map at a point (represented by its Jacobian matrix) is a linear map that represents the "best linear approximation" to the map near the given point. In the manifold case there is a similar linear map, but it makes no sense to talk about a linear map between manifolds. Instead, it will be a linear map between tangent spaces.

If M and N are smooth manifolds with or without boundary and $F\colon M \to N$ is a smooth map, for each $p \in M$ we define a map

$$dF_p\colon T_pM \to T_{F(p)}N,$$

called the ***differential of F at p*** (Fig. 3.3), as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F).$$

Note that if $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$, so $v(f \circ F)$ makes sense. The operator $dF_p(v)\colon C^\infty(N) \to \mathbb{R}$ is linear because v is, and is a derivation at $F(p)$ because for any $f, g \in C^\infty(N)$ we have

$$dF_p(v)(fg) = v\big((fg) \circ F\big) = v\big((f \circ F)(g \circ F)\big)$$
$$= f \circ F(p)v(g \circ F) + g \circ F(p)v(f \circ F)$$
$$= f\big(F(p)\big)dF_p(v)(g) + g\big(F(p)\big)dF_p(v)(f).$$

Proposition 3.6 (Properties of Differentials). *Let M, N, and P be smooth manifolds with or without boundary, let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$.*

(a) $dF_p\colon T_pM \to T_{F(p)}N$ *is linear.*
(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{G \circ F(p)}P.$
(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM.$
(d) *If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and* $(dF_p)^{-1} = d\big(F^{-1}\big)_{F(p)}.$

▶ **Exercise 3.7.** Prove Proposition 3.6.

Our first important application of the differential will be to use coordinate charts to relate the tangent space to a point on a manifold with the Euclidean tangent space. But there is an important technical issue that we must address first: while the tangent space is defined in terms of smooth functions on the whole manifold, coordinate charts are in general defined only on open subsets. The key point, expressed in the next proposition, is that tangent vectors act locally.

Proposition 3.8. *Let M be a smooth manifold with or without boundary, $p \in M$, and $v \in T_p M$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.*

Proof. Let $h = f - g$, so that h is a smooth function that vanishes in a neighborhood of p. Let $\psi \in C^\infty(M)$ be a smooth bump function that is identically equal to 1 on the support of h and is supported in $M \smallsetminus \{p\}$. Because $\psi \equiv 1$ where h is nonzero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, Lemma 3.4 implies that $vh = v(\psi h) = 0$. By linearity, this implies $vf = vg$. ∎

Using this proposition, we can identify the tangent space to an open submanifold with the tangent space to the whole manifold.

Proposition 3.9 (The Tangent Space to an Open Submanifold). *Let M be a smooth manifold with or without boundary, let $U \subseteq M$ be an open subset, and let $\iota \colon U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p \colon T_p U \to T_p M$ is an isomorphism.*

Proof. To prove injectivity, suppose $v \in T_p U$ and $d\iota_p(v) = 0 \in T_p M$. Let B be a neighborhood of p such that $\overline{B} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, the extension lemma for smooth functions guarantees that there exists $\widetilde{f} \in C^\infty(M)$ such that $\widetilde{f} \equiv f$ on $\overline{B}$. Then since f and $\widetilde{f}|_U$ are smooth functions on U that agree in a neighborhood of p, Proposition 3.8 implies

$$vf = v\big(\widetilde{f}|_U\big) = v\big(\widetilde{f} \circ \iota\big) = d\iota(v)_p \widetilde{f} = 0.$$

Since this holds for every $f \in C^\infty(U)$, it follows that $v = 0$, so $d\iota_p$ is injective.

On the other hand, to prove surjectivity, suppose $w \in T_p M$ is arbitrary. Define an operator $v \colon C^\infty(U) \to \mathbb{R}$ by setting $vf = w\widetilde{f}$, where $\widetilde{f}$ is any smooth function on all of M that agrees with f on $\overline{B}$. By Proposition 3.8, vf is independent of the choice of $\widetilde{f}$, so v is well defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$,

$$d\iota_p(v)g = v(g \circ \iota) = w\big(\widetilde{g \circ \iota}\big) = wg,$$

where the last two equalities follow from the facts that $g \circ \iota$, $\widetilde{g \circ \iota}$, and g all agree on B. Therefore, $d\iota_p$ is also surjective. ∎

Given an open subset $U \subseteq M$, the isomorphism $d\iota_p$ between $T_p U$ and $T_p M$ is canonically defined, independently of any choices. From now on we *identify* $T_p U$ with $T_p M$ for any point $p \in U$. This identification just amounts to the observation
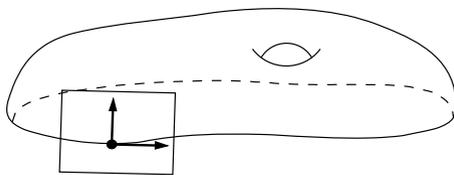
Fig. 3.4 The tangent space to a manifold with boundary

that $d\iota_p(v)$ is the *same derivation as* v, thought of as acting on functions on the bigger manifold M instead of functions on U. Since the action of a derivation on a function depends only on the values of the function in an arbitrarily small neighborhood, this is a harmless identification. In particular, this means that any tangent vector $v \in T_p M$ can be unambiguously applied to functions defined only in a neighborhood of p, not necessarily on all of M.

Proposition 3.10 (Dimension of the Tangent Space). *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space $T_p M$ is an n-dimensional vector space.*

Proof. Given $p \in M$, let (U, φ) be a smooth coordinate chart containing p. Because φ is a diffeomorphism from U onto an open subset $\widehat{U} \subseteq \mathbb{R}^n$, it follows from Proposition 3.6(d) that $d\varphi_p$ is an isomorphism from $T_p U$ to $T_{\varphi(p)}\widehat{U}$. Since Proposition 3.9 guarantees that $T_p M \cong T_p U$ and $T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{R}^n$, it follows that $\dim T_p M = \dim T_{\varphi(p)}\mathbb{R}^n = n$. $\square$

Next we need to prove an analogous result for manifolds with boundary. In fact, if M is an n-manifold with boundary, it might not be immediately clear what one should expect the tangent space at a boundary point of M to look like. Should it be an n-dimensional vector space, like the tangent space at an interior point? Or should it be $(n-1)$-dimensional, like the boundary? Or should it be an n-dimensional half-space, like the space $\mathbb{H}^n$ on which M is modeled locally?

As we will show below, our definition implies that the tangent space at a boundary point is an n-dimensional vector space (Fig. 3.4), just like the tangent spaces at interior points. This may or may not seem like the most geometrically intuitive choice, but it has the advantage of making most of the definitions of geometric objects on a manifold with boundary look exactly the same as those on a manifold.

First, we need to relate the tangent spaces $T_a \mathbb{H}^n$ and $T_a \mathbb{R}^n$ for points $a \in \partial \mathbb{H}^n$. Since $\mathbb{H}^n$ is not an open subset of $\mathbb{R}^n$, Proposition 3.9 does not apply. As a substitute, we have the following lemma.

Lemma 3.11. *Let $\iota: \mathbb{H}^n \hookrightarrow \mathbb{R}^n$ denote the inclusion map. For any $a \in \partial \mathbb{H}^n$, the differential $d\iota_a: T_a \mathbb{H}^n \to T_a \mathbb{R}^n$ is an isomorphism.*

Proof. Suppose $a \in \partial \mathbb{H}^n$. To show that $d\iota_a$ is injective, assume $d\iota_a(v) = 0$. Suppose $f: \mathbb{H}^n \to \mathbb{R}$ is smooth, and let $\widetilde{f}$ be any extension of f to a smooth function defined on all of $\mathbb{R}^n$. (Such an extension exists by the extension lemma for smooth

functions, Lemma 2.26.) Then $\widetilde{f} \circ \iota = f$, so

$$vf = v\big(\widetilde{f} \circ \iota\big) = d\iota_a(v)\widetilde{f} = 0,$$

which implies that $d\iota_a$ is injective.

To show surjectivity, let $w \in T_a\mathbb{R}^n$ be arbitrary. Define $v \in T_a\mathbb{H}^n$ by

$$vf = w\widetilde{f},$$

where $\widetilde{f}$ is any smooth extension of f. Writing $w = w^i \partial/\partial x^i|_a$ in terms of the standard basis for $T_a\mathbb{R}^n$, this means that

$$vf = w^i \frac{\partial \widetilde{f}}{\partial x^i}(a).$$

This is independent of the choice of $\widetilde{f}$, because by continuity the derivatives of $\widetilde{f}$ at a are determined by those of f in $\mathbb{H}^n$. It is easy to check that v is a derivation at a and that $w = d\iota_a(v)$, so $d\iota_a$ is surjective. $\qquad\square$

Just as we use Proposition 3.9 to identify T_pU with T_pM when U is an open subset of M, we use this lemma to identify $T_a\mathbb{H}^n$ with $T_a\mathbb{R}^n$ when $a \in \partial\mathbb{H}^n$, and we do not distinguish notationally between an element of $T_a\mathbb{H}^n$ and its image in $T_a\mathbb{R}^n$.

Proposition 3.12 (Dimension of Tangent Spaces on a Manifold with Boundary).
Suppose M is an n-dimensional smooth manifold with boundary. For each $p \in M$, T_pM is an n-dimensional vector space.

Proof. Let $p \in M$ be arbitrary. If p is an interior point, then because $\mathrm{Int}\,M$ is an open submanifold of M, Proposition 3.9 implies that $T_p(\mathrm{Int}\,M) \cong T_pM$. Since $\mathrm{Int}\,M$ is a smooth n-manifold without boundary, its tangent spaces all have dimension n.

On the other hand, if $p \in \partial M$, let (U, φ) be a smooth boundary chart containing p, and let $\widehat{U} = \varphi(U) \subseteq \mathbb{H}^n$. There are isomorphisms $T_pM \cong T_pU$ (by Proposition 3.9); $T_pU \cong T_{\varphi(p)}\widehat{U}$ (by Proposition 3.6(d), because φ is a diffeomorphism); $T_{\varphi(p)}\widehat{U} \cong T_{\varphi(p)}\mathbb{H}^n$ (by Proposition 3.9 again); and $T_{\varphi(p)}\mathbb{H}^n \cong T_{\varphi(p)}\mathbb{R}^n$ (by Lemma 3.11). The result follows. $\qquad\square$

Recall from Example 1.24 that every finite-dimensional vector space has a natural smooth manifold structure that is independent of any choice of basis or norm. The following proposition shows that the tangent space to a vector space can be naturally identified with the vector space itself.

Suppose V is a finite-dimensional vector space and $a \in V$. Just as we did earlier in the case of $\mathbb{R}^n$, for any vector $v \in V$, we define a map $D_v|_a \colon C^\infty(V) \to \mathbb{R}$ by

$$D_v|_a f = \frac{d}{dt}\bigg|_{t=0} f(a + tv). \tag{3.5}$$

Proposition 3.13 (The Tangent Space to a Vector Space). *Suppose V is a finite-dimensional vector space with its standard smooth manifold structure. For each point $a \in V$, the map $v \mapsto D_v|_a$ defined by (3.5) is a canonical isomorphism from V to T_aV, such that for any linear map $L\colon V \to W$, the following diagram commutes:*

$$
\begin{array}{ccc}
V & \xrightarrow{\cong} & T_aV \\
\Big\downarrow{\scriptstyle L} & & \Big\downarrow{\scriptstyle dL_a} \\
W & \xrightarrow[\cong]{} & T_{La}W.
\end{array}
\tag{3.6}
$$

Proof. Once we choose a basis for V, we can use the same argument as in the proof of Proposition 3.2 to show that $D_v|_a$ is indeed a derivation at a, and that the map $v \mapsto D_v|_a$ is an isomorphism.

Now suppose $L\colon V \to W$ is a linear map. Because its components with respect to any choices of bases for V and W are linear functions of the coordinates, L is smooth. Unwinding the definitions and using the linearity of L, we compute

$$
dL_a\big(D_v|_a\big)f = D_v|_a(f \circ L)
$$

$$
= \frac{d}{dt}\bigg|_{t=0} f\big(L(a + tv)\big) = \frac{d}{dt}\bigg|_{t=0} f(La + tLv)
$$

$$
= D_{Lv}|_{La} f. \qquad \square
$$

It is important to understand that each isomorphism $V \cong T_aV$ is canonically defined, independently of any choice of basis (notwithstanding the fact that we used a choice of basis to prove that it is an isomorphism). Because of this result, we can routinely *identify* tangent vectors to a finite-dimensional vector space with elements of the space itself. More generally, if M is an open submanifold of a vector space V, we can combine our identifications $T_pM \leftrightarrow T_pV \leftrightarrow V$ to obtain a canonical identification of each tangent space to M with V. For example, since $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of the vector space $\mathrm{M}(n, \mathbb{R})$, we can identify its tangent space at each point $X \in \mathrm{GL}(n, \mathbb{R})$ with the full space of matrices $\mathrm{M}(n, \mathbb{R})$.

There is another natural identification for tangent spaces to a product manifold.

Proposition 3.14 (The Tangent Space to a Product Manifold). *Let $M_1, \dots, M_k$ be smooth manifolds, and for each j, let $\pi_j\colon M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$, the map*

$$
\alpha\colon T_p(M_1 \times \cdots \times M_k) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k
$$

defined by

$$
\alpha(v) = \big(d(\pi_1)_p(v), \dots, d(\pi_k)_p(v)\big)
\tag{3.7}
$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

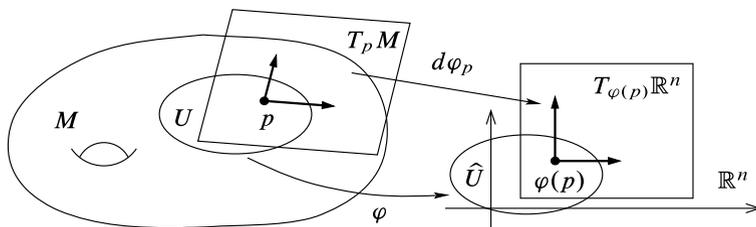*Proof.* See Problem 3-2. $\qquad \square$

Fig. 3.5 Tangent vectors in coordinates

Once again, because the isomorphism (3.7) is canonically defined, independently of any choice of coordinates, we can consider it as a canonical identification, and we will always do so. Thus, for example, we identify $T_{(p,q)}(M \times N)$ with $T_p M \oplus T_q N$, and treat $T_p M$ and $T_q N$ as subspaces of $T_{(p,q)}(M \times N)$.

Computations in Coordinates

Our treatment of the tangent space to a manifold so far might seem hopelessly abstract. To bring it down to earth, we will show how to do computations with tangent vectors and differentials in local coordinates.

First, suppose M is a smooth manifold (without boundary), and let (U, φ) be a smooth coordinate chart on M. Then φ is, in particular, a diffeomorphism from U to an open subset $\widehat{U} \subseteq \mathbb{R}^n$. Combining Propositions 3.9 and 3.6(d), we see that $d\varphi_p \colon T_p M \to T_{\varphi(p)} \mathbb{R}^n$ is an isomorphism.

By Corollary 3.3, the derivations $\partial/\partial x^1|_{\varphi(p)}, \ldots, \partial/\partial x^n|_{\varphi(p)}$ form a basis for $T_{\varphi(p)} \mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$ form a basis for $T_p M$ (Fig. 3.5). In keeping with our standard practice of treating coordinate maps as identifications whenever possible, we use the notation $\partial/\partial x^i|_p$ for these vectors, characterized by either of the following expressions:

$$\frac{\partial}{\partial x^i}\bigg|_p = (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) = d\left(\varphi^{-1}\right)_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right). \tag{3.8}$$

Unwinding the definitions, we see that $\partial/\partial x^i|_p$ acts on a function $f \in C^\infty(U)$ by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\left(f \circ \varphi^{-1}\right) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}),$$

where $\widehat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f, and $\widehat{p} = (p^1, \ldots, p^n) = \varphi(p)$ is the coordinate representation of p. In other words, $\partial/\partial x^i|_p$ is just the derivation that takes the ith partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors $\partial/\partial x^i|_p$ are called the ***coordinate vectors at p*** associated with the given coordinate system. In the special case of standard coordinates on $\mathbb{R}^n$, the vectors $\partial/\partial x^i|_p$ are literally the partial derivative operators.

When M is a smooth manifold with boundary and p is an interior point, the discussion above applies verbatim. For $p \in \partial M$, the only change that needs to be made is to substitute $\mathbb{H}^n$ for $\mathbb{R}^n$, with the understanding that the notation $\partial/\partial x^i|_{\varphi(p)}$ can be used interchangeably to denote either an element of $T_{\varphi(p)}\mathbb{R}^n$ or an element of $T_{\varphi(p)}\mathbb{H}^n$, in keeping with our convention of considering the isomorphism $d\iota_{\varphi(p)}\colon T_{\varphi(p)}\mathbb{H}^n \to T_{\varphi(p)}\mathbb{R}^n$ as an identification. The nth coordinate vector $\partial/\partial x^n|_p$ should be interpreted as a one-sided derivative in this case.

The following proposition summarizes the discussion so far.

Proposition 3.15. *Let M be a smooth n-manifold with or without boundary, and let $p \in M$. Then $T_p M$ is an n-dimensional vector space, and for any smooth chart $\big(U, (x^i)\big)$ containing p, the coordinate vectors $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ form a basis for $T_p M$.* $\qquad\square$

Thus, a tangent vector $v \in T_p M$ can be written uniquely as a linear combination

$$v = v^i \frac{\partial}{\partial x^i}\bigg|_p,$$

where we use the summation convention as usual, with an upper index in the denominator being considered as a lower index, as explained on p. . The ordered basis $\big(\partial/\partial x^i|_p\big)$ is called a *coordinate basis for $T_p M$*, and the numbers $\big(v^1, \ldots, v^n\big)$ are called the *components of v* with respect to the coordinate basis. If v is known, its components can be computed easily from its action on the coordinate functions. For each j, the components of v are given by $v^j = v\big(x^j\big)$ (where we think of x^j as a smooth real-valued function on U), because

$$v\big(x^j\big) = \left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right)\big(x^j\big) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j.$$

The Differential in Coordinates

Next we explore how differentials look in coordinates. We begin by considering the special case of a smooth map $F\colon U \to V$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets of Euclidean spaces. For any $p \in U$, we will determine the matrix of $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Using $\big(x^1, \ldots, x^n\big)$ to denote the coordinates in the domain and $\big(y^1, \ldots, y^m\big)$ to denote those in the codomain, we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$dF_p\left(\frac{\partial}{\partial x^i}\bigg|_p \right)f = \frac{\partial}{\partial x^i}\bigg|_p (f \circ F) = \frac{\partial f}{\partial y^j}\big(F(p)\big)\frac{\partial F^j}{\partial x^i}(p)$$

$$= \left(\frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)} \right)f.$$
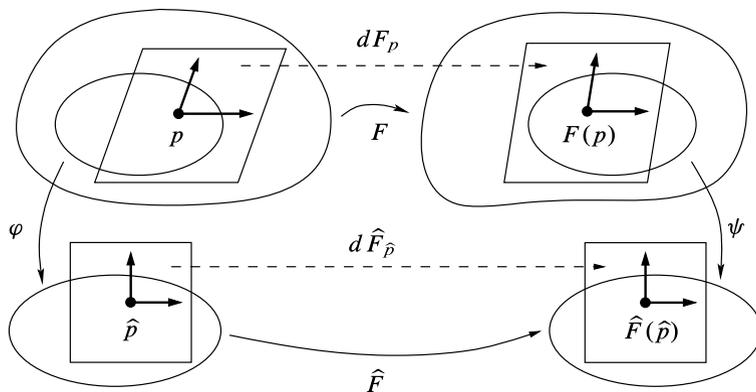
Fig. 3.6 The differential in coordinates

Thus

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \tag{3.9}$$

In other words, the matrix of dF_p in terms of the coordinate bases is

$$\begin{pmatrix} \dfrac{\partial F^1}{\partial x^1}(p) & \cdots & \dfrac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^m}{\partial x^1}(p) & \cdots & \dfrac{\partial F^m}{\partial x^n}(p) \end{pmatrix}.$$

(Recall that the columns of the matrix are the components of the images of the basis vectors.) This matrix is none other than the Jacobian matrix of F at p, which is the matrix representation of the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $dF_p\colon T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $DF(p)\colon \mathbb{R}^n \to \mathbb{R}^m$, under our usual identification of Euclidean spaces with their tangent spaces. The same calculation applies if U is an open subset of $\mathbb{H}^n$ and V is an open subset of $\mathbb{H}^m$.

Now consider the more general case of a smooth map $F\colon M \to N$ between smooth manifolds with or without boundary. Choosing smooth coordinate charts (U,φ) for M containing p and (V,ψ) for N containing $F(p)$, we obtain the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}\colon \varphi(U \cap F^{-1}(V)) \to \psi(V)$ (Fig. 3.6). Let $\widehat{p} = \varphi(p)$ denote the coordinate representation of p. By the computation above, $d\widehat{F}_{\widehat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\widehat{F}$ at $\widehat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = dF_p\left(d\left(\varphi^{-1}\right)_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) = d\left(\psi^{-1}\right)_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right)$$

$$= d\left(\psi^{-1}\right)_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^{j}}{\partial x^{i}}(\hat{p}) \left.\frac{\partial}{\partial y^{j}}\right|_{\hat{F}(\hat{p})}\right)$$

$$= \frac{\partial \hat{F}^{j}}{\partial x^{i}}(\hat{p}) \left.\frac{\partial}{\partial y^{j}}\right|_{F(p)}. \tag{3.10}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. In fact, the definition of the differential was cooked up precisely to give a coordinate-independent meaning to the Jacobian matrix.

In the differential geometry literature, the differential is sometimes called the *tangent map*, the *total derivative*, or simply the *derivative of F*. Because it "pushes" tangent vectors forward from the domain manifold to the codomain, it is also called the (*pointwise*) *pushforward*. Different authors denote it by symbols such as

$$F'(p), \quad DF, \quad DF(p), \quad F_{*}, \quad TF, \quad T_{p}F.$$

We will stick with the notation dF_p for the differential of a smooth map between manifolds, and reserve $DF(p)$ for the total derivative of a map between finite-dimensional vector spaces, which in the case of Euclidean spaces we identify with the Jacobian matrix of F.

Change of Coordinates

Suppose (U, φ) and (V, ψ) are two smooth charts on M, and $p \in U \cap V$. Let us denote the coordinate functions of φ by (x^i) and those of ψ by $(\tilde{x}^i)$. Any tangent vector at p can be represented with respect to either basis $(\partial/\partial x^i|_p)$ or $(\partial/\partial \tilde{x}^i|_p)$. How are the two representations related?

In this situation, it is customary to write the transition map $\psi \circ \varphi^{-1} \colon \varphi(U \cap V) \to \psi(U \cap V)$ in the following shorthand notation:

$$\psi \circ \varphi^{-1}(x) = \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x)\right).$$

Here we are indulging in a typical abuse of notation: in the expression $\tilde{x}^i(x)$, we think of $\tilde{x}^i$ as a coordinate *function* (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$); but we think of x as representing a *point* (in this case, in $\varphi(U \cap V)$). By (3.9), the differential $d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)}$ can be written

$$d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^{i}}\right|_{\varphi(p)}\right) = \frac{\partial \tilde{x}^{j}}{\partial x^{i}}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^{j}}\right|_{\psi(p)}.$$
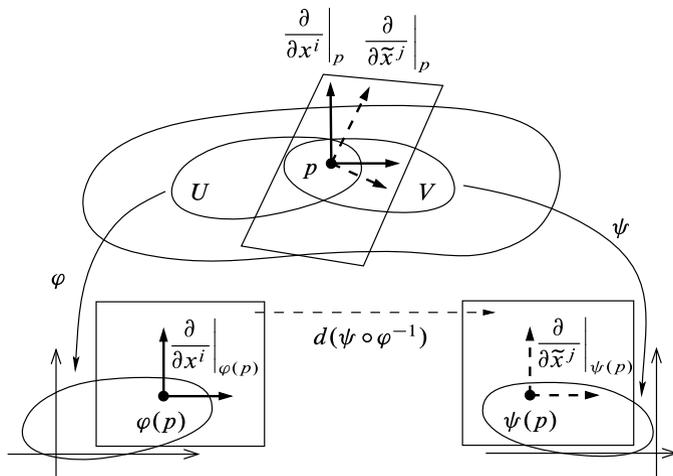
Fig. 3.7 Change of coordinates

(See Fig. 3.7.) Using the definition of coordinate vectors, we obtain

$$\frac{\partial}{\partial x^i}\bigg|_p = d\left(\varphi^{-1}\right)_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right)$$

$$= d\left(\psi^{-1}\right)_{\psi(p)} \circ d\left(\psi \circ \varphi^{-1}\right)_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right)$$

$$= d\left(\psi^{-1}\right)_{\psi(p)}\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p))\frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\psi(p)}\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(\hat{p})\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p, \qquad (3.11)$$

where again we have written $\hat{p} = \varphi(p)$. (This formula is easy to remember, because it looks exactly the same as the chain rule for partial derivatives in $\mathbb{R}^n$.) Applying this to the components of a vector $v = v^i \partial/\partial x^i|_p = \widetilde{v}^j \partial/\partial \widetilde{x}^j|_p$, we find that the components of v transform by the rule

$$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(\hat{p})v^i. \qquad (3.12)$$

Example 3.16. The transition map between polar coordinates and standard coordinates in suitable open subsets of the plane is given by $(x, y) = (r\cos\theta, r\sin\theta)$. Let p be the point in $\mathbb{R}^2$ whose polar coordinate representation is $(r, \theta) = (2, \pi/2)$, and let $v \in T_p\mathbb{R}^2$ be the tangent vector whose polar coordinate representation is

$$v = 3\frac{\partial}{\partial r}\bigg|_p - \frac{\partial}{\partial \theta}\bigg|_p.$$

Applying (3.11) to the coordinate vectors, we find

$$
\left.\frac{\partial}{\partial r}\right|_p = \cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + \sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = \left.\frac{\partial}{\partial y}\right|_p,
$$

$$
\left.\frac{\partial}{\partial \theta}\right|_p = -2\sin\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial x}\right|_p + 2\cos\left(\frac{\pi}{2}\right)\left.\frac{\partial}{\partial y}\right|_p = -2\left.\frac{\partial}{\partial x}\right|_p,
$$

and thus v has the following coordinate representation in standard coordinates:

$$
v = 3\left.\frac{\partial}{\partial y}\right|_p + 2\left.\frac{\partial}{\partial x}\right|_p. \qquad\qquad \text{//}
$$

One important fact to bear in mind is that each coordinate vector $\partial/\partial x^i|_p$ depends on the entire *coordinate system*, not just on the single coordinate function x^i. Geometrically, this reflects the fact that $\partial/\partial x^i|_p$ is the derivation obtained by differentiating with respect to x^i while *all the other coordinates are held constant*. If the coordinate functions other than x^i are changed, then the direction of this coordinate derivative can change. The next exercise illustrates how this can happen.

▶ **Exercise 3.17.** Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$
\tilde{x} = x, \qquad \tilde{y} = y + x^3.
$$

Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that

$$
\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p,
$$

even though the coordinate functions x and $\tilde{x}$ are identically equal.

The Tangent Bundle

Often it is useful to consider the set of all tangent vectors at all points of a manifold. Given a smooth manifold M with or without boundary, we define the **tangent bundle of M**, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$
TM = \coprod_{p \in M} T_p M.
$$

We usually write an element of this disjoint union as an ordered pair (p, v), with $p \in M$ and $v \in T_p M$ (instead of putting the point p in the second position, as elements of a disjoint union are more commonly written). The tangent bundle comes equipped with a natural **projection map** $\pi\colon TM \to M$, which sends each vector in $T_p M$ to the point p at which it is tangent: $\pi(p, v) = p$. We will often commit the usual mild sin of identifying $T_p M$ with its image under the canonical injection $v \mapsto (p, v)$, and will use any of the notations (p, v), v_p, and v for a tangent vector in $T_p M$, depending on how much emphasis we wish to give to the point p.
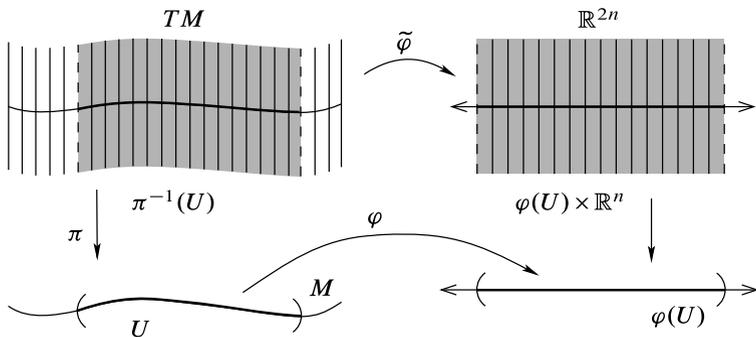
Fig. 3.8 Coordinates for the tangent bundle

For example, in the special case $M = \mathbb{R}^n$, using Proposition 3.2, we see that the tangent bundle of $\mathbb{R}^n$ can be canonically identified with the union of its geometric tangent spaces, which in turn is just the Cartesian product of $\mathbb{R}^n$ with itself:

$$T\mathbb{R}^n = \coprod_{a \in \mathbb{R}^n} T_a\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \mathbb{R}^n_a = \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

An element (a, v) of this Cartesian product can be thought of as representing either the geometric tangent vector v_a or the derivation $D_v|_a$ defined by (3.1). Be warned, however, that in general the tangent bundle of a smooth manifold cannot be identified in any natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at different points with each other. We will have more to say about this below.

If M is a smooth manifold, the tangent bundle TM can be thought of simply as a disjoint union of vector spaces; but it is much more than that. The next proposition shows that TM can be considered as a smooth manifold in its own right.

Proposition 3.18. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a 2n-dimensional smooth manifold. With respect to this structure, the projection $\pi\colon TM \to M$ is smooth.*

Proof. We begin by defining the maps that will become our smooth charts. Given any smooth chart (U, φ) for M, note that $\pi^{-1}(U) \subseteq TM$ is the set of all tangent vectors to M at all points of U. Let $(x^1, \ldots, x^n)$ denote the coordinate functions of φ, and define a map $\widetilde{\varphi}\colon \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\Big|_p \right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n \right). \tag{3.13}$$

(See Fig. 3.8.) Its image set is $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be written explicitly as

$$\widetilde{\varphi}^{-1}\left(x^1, \ldots, x^n, v^1, \ldots, v^n \right) = v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)}.$$

Now suppose we are given two smooth charts (U, φ) and (V, ψ) for M, and let $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$, $\left(\pi^{-1}(V), \widetilde{\psi}\right)$ be the corresponding charts on TM. The sets

$$\widetilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \varphi(U \cap V) \times \mathbb{R}^n \quad \text{and}$$

$$\widetilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map $\widetilde{\psi} \circ \widetilde{\varphi}^{-1} \colon \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ can be written explicitly using (3.12) as

$$\widetilde{\psi} \circ \widetilde{\varphi}^{-1}\left(x^1, \dots, x^n, v^1, \dots, v^n\right)$$

$$= \left(\widetilde{x}^1(x), \dots, \widetilde{x}^n(x), \frac{\partial \widetilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \widetilde{x}^n}{\partial x^j}(x) v^j\right).$$

This is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\left\{\pi^{-1}(U_i)\right\}$ satisfying conditions (i)–(iv) of the smooth manifold chart lemma (Lemma 1.35). To check the Hausdorff condition (v), just note that any two points in the same fiber of π lie in one chart, while if (p, v) and (q, w) lie in different fibers, there exist disjoint smooth coordinate domains U, V for M such that $p \in U$ and $q \in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p, v) and (q, w), respectively.

To see that π is smooth, note that with respect to charts (U, φ) for M and $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$ for TM, its coordinate representation is $\pi(x, v) = x$. $\qquad\square$

The coordinates $\left(x^i, v^i\right)$ given by (3.13) are called ***natural coordinates on TM***.

▶ **Exercise 3.19.** Suppose M is a smooth manifold with boundary. Show that TM has a natural topology and smooth structure making it into a smooth manifold with boundary, such that if $\left(U, \left(x^i\right)\right)$ is any smooth boundary chart for M, then rearranging the coordinates in the natural chart $\left(\pi^{-1}(U), \left(x^i, v^i\right)\right)$ for TM yields a boundary chart $\left(\pi^{-1}(U), \left(v^i, x^i\right)\right)$.

Proposition 3.20. *If M is a smooth n-manifold with or without boundary, and M can be covered by a single smooth chart, then TM is diffeomorphic to $M \times \mathbb{R}^n$.*

Proof. If (U, φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U = M$ to an open subset $\widehat{U} \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$. The proof of the previous proposition showed that the natural coordinate chart $\widetilde{\varphi}$ is a bijection from TM to $\widehat{U} \times \mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\widetilde{\varphi}$ to be a diffeomorphism. $\qquad\square$

Although the picture of a product $U \times \mathbb{R}^n$ is a useful way to visualize the smooth structure on a tangent bundle locally as in Fig. 3.8, do not be misled into imagining that every tangent bundle is *globally* diffeomorphic (or even homeomorphic) to a product of the manifold with $\mathbb{R}^n$. This is not the case for most smooth manifolds. We will revisit this question in Chapters 8, 10, and 16.

By putting together the differentials of F at all points of M, we obtain a globally defined map between tangent bundles, called the ***global differential*** or ***global tangent map*** and denoted by $dF: TM \to TN$. This is just the map whose restriction to each tangent space $T_pM \subseteq TM$ is dF_p. When we apply the differential of F to a specific vector $v \in T_pM$, we can write either $dF_p(v)$ or $dF(v)$, depending on how much emphasis we wish to give to the point p. The former notation is more informative, while the second is more concise.

One important feature of the smooth structure we have defined on TM is that it makes the differential of a smooth map into a smooth map between tangent bundles.

Proposition 3.21. *If* $F: M \to N$ *is a smooth map*, *then its global differential* $dF: TM \to TN$ *is a smooth map.*

Proof. From the local expression (3.9) for dF_p in coordinates, it follows that dF has the following coordinate representation in terms of natural coordinates for TM and TN:

$$dF\left(x^1,\ldots,x^n,v^1,\ldots,v^n\right) = \left(F^1(x),\ldots,F^n(x), \frac{\partial F^1}{\partial x^i}(x)v^i,\ldots,\frac{\partial F^n}{\partial x^i}(x)v^i\right).$$

This is smooth because F is. □

The following properties of the global differential follow immediately from Proposition 3.6.

Corollary 3.22 (Properties of the Global Differential). *Suppose* $F: M \to N$ *and* $G: N \to P$ *are smooth maps.*

(a) $d(G \circ F) = dG \circ dF$.
(b) $d(\mathrm{Id}_M) = \mathrm{Id}_{TM}$.
(c) *If* F *is a diffeomorphism, then* $dF: TM \to TN$ *is also a diffeomorphism, and* $(dF)^{-1} = d\left(F^{-1}\right)$. □

Because of part (c) of this corollary, when F is a diffeomorphism we can use the notation dF^{-1} unambiguously to mean either $(dF)^{-1}$ or $d\left(F^{-1}\right)$.

Velocity Vectors of Curves

The *velocity* of a smooth parametrized curve in $\mathbb{R}^n$ is familiar from elementary calculus. It is just the vector whose components are the derivatives of the component functions of the curve. In this section we extend this notion to curves in manifolds.

If M is a manifold with or without boundary, we define a ***curve in M*** to be a continuous map $\gamma: J \to M$, where $J \subseteq \mathbb{R}$ is an interval. (Most of the time, we will be interested in curves whose domains are open intervals, but for some purposes it is useful to allow J to have one or two endpoints; the definitions all make sense with minor modifications in that case, either by considering J as a manifold with boundary or by interpreting derivatives as one-sided derivatives.) Note that in this
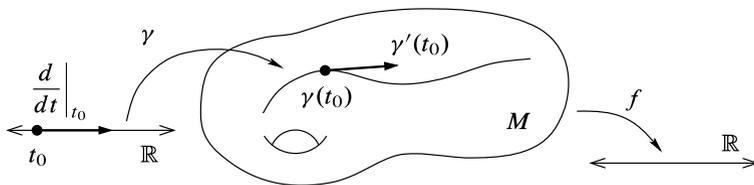
Fig. 3.9 The velocity of a curve

book the term *curve* always refers to a map from an interval into M (a parametrized curve), not just a set of points in M.

Now let M be a smooth manifold, still with or without boundary. Our definition of tangent spaces leads to a natural interpretation of velocity vectors: given a smooth curve $\gamma \colon J \to M$ and $t_0 \in J$, we define the **velocity of $\boldsymbol{\gamma}$ at $\boldsymbol{t_0}$** (Fig. 3.9), denoted by $\gamma'(t_0)$, to be the vector

$$\gamma'(t_0) = d\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) \in T_{\gamma(t_0)} M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$. (As in ordinary calculus, it is customary to use d/dt instead of $\partial/\partial t$ when the manifold is 1-dimensional.) Other common notations for the velocity are

$$\dot{\gamma}(t_0), \quad \frac{d\gamma}{dt}(t_0), \quad \text{and} \quad \left. \frac{d\gamma}{dt} \right|_{t=t_0}.$$

This tangent vector acts on functions by

$$\gamma'(t_0) f = d\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) f = \left. \frac{d}{dt} \right|_{t_0} (f \circ \gamma) = (f \circ \gamma)'(t_0).$$

In other words, $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. (If t_0 is an endpoint of J, this still holds, provided that we interpret the derivative with respect to t as a one-sided derivative, or equivalently as the derivative of any smooth extension of $f \circ \gamma$ to an open subset of $\mathbb{R}$.)

Now let (U, φ) be a smooth chart with coordinate functions (x^i). If $\gamma(t_0) \in U$, we can write the coordinate representation of γ as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, at least for t sufficiently close to t_0, and then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t_0)}.$$

This means that $\gamma'(t_0)$ is given by essentially the same formula as it would be in Euclidean space: it is the tangent vector whose components in a coordinate basis are the derivatives of the component functions of γ.

The next proposition shows that every tangent vector on a manifold is the velocity vector of some curve. This gives a different and somewhat more geometric way to

think about the tangent bundle: it is just the set of all velocity vectors of smooth curves in M.

Proposition 3.23. *Suppose M is a smooth manifold with or without boundary and $p \in M$. Every $v \in T_p M$ is the velocity of some smooth curve in M.*

Proof. First suppose that $p \in \operatorname{Int} M$ (which includes the case $\partial M = \varnothing$). Let (U, φ) be a smooth coordinate chart centered at p, and write $v = v^i \partial/\partial x^i|_p$ in terms of the coordinate basis. For sufficiently small $\varepsilon > 0$, let $\gamma \colon (-\varepsilon, \varepsilon) \to U$ be the curve whose coordinate representation is

$$\gamma(t) = \left(t v^1, \dots, t v^n \right). \tag{3.14}$$

(Remember, this really means $\gamma(t) = \varphi^{-1} \left(t v^1, \dots, t v^n \right)$.) This is a smooth curve with $\gamma(0) = p$, and the computation above shows that $\gamma'(0) = v^i \partial/\partial x^i|_{\gamma(0)} = v$.

Now suppose $p \in \partial M$. Let (U, φ) be a smooth boundary chart centered at p, and write $v = v^i \partial/\partial x^i|_p$ as before. We wish to let γ be the curve whose coordinate representation is (3.14), but this formula represents a point of M only when $t v^n \geq 0$. We can accommodate this requirement by suitably restricting the domain of γ: if $v^n = 0$, we define $\gamma \colon (-\varepsilon, \varepsilon) \to U$ as before; if $v^n > 0$, we let the domain be $[0, \varepsilon)$; and if $v^n < 0$, we let it be $(-\varepsilon, 0]$. In each case, γ is a smooth curve in M with $\gamma(0) = p$ and $\gamma'(0) = v$. $\qquad\square$

The next proposition shows that velocity vectors behave well under composition with smooth maps.

Proposition 3.24 (The Velocity of a Composite Curve). *Let $F \colon M \to N$ be a smooth map, and let $\gamma \colon J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma \colon J \to N$ is given by*

$$(F \circ \gamma)'(t_0) = dF \left(\gamma'(t_0) \right).$$

Proof. Just go back to the definition of the velocity of a curve:

$$(F \circ \gamma)'(t_0) = d(F \circ \gamma)\left(\left. \frac{d}{dt} \right|_{t_0} \right) = dF \circ d\gamma \left(\left. \frac{d}{dt} \right|_{t_0} \right) = dF \left(\gamma'(t_0) \right). \qquad\square$$

On the face of it, the preceding proposition tells us how to compute the velocity of a composite curve in terms of the differential. However, it is often much more useful to turn it around the other way, and use it as a streamlined way to compute differentials. Suppose $F \colon M \to N$ is a smooth map, and we need to compute the differential dF_p at some point $p \in M$. We can compute $dF_p(v)$ for any $v \in T_p M$ by choosing a smooth curve γ whose initial tangent vector is v, and then applying Proposition 3.23 to the composite curve $F \circ \gamma$. The next corollary summarizes the result.

Corollary 3.25 (Computing the Differential Using a Velocity Vector). *Suppose $F \colon M \to N$ is a smooth map, $p \in M$, and $v \in T_p M$. Then*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma \colon J \to M$ such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$. $\qquad\square$

This corollary frequently yields a much more succinct computation of dF, especially if F is presented in some form other than an explicit coordinate representation. We will see many examples of this technique in later chapters.

Alternative Definitions of the Tangent Space

In the literature you will find tangent vectors to a smooth manifold defined in several different ways. Here we describe the most common ones. (Yet another definition is suggested in the remark following Problem 11-4.) It is good to be conversant with all of them. Throughout this section, M represents an arbitrary smooth manifold with or without boundary.

Tangent Vectors as Derivations of the Space of Germs

The most common alternative definition is based on the notion of "germs" of smooth functions, which we now define.

A ***smooth function element*** on M is an ordered pair (f, U), where U is an open subset of M and $f: U \to \mathbb{R}$ is a smooth function. Given a point $p \in M$, let us define an equivalence relation on the set of all smooth function elements whose domains contain p by setting $(f, U) \sim (g, V)$ if $f \equiv g$ on some neighborhood of p. The equivalence class of a function element (f, U) is called the ***germ of f at p***. The set of all germs of smooth functions at p is denoted by $C_p^\infty(M)$. It is a real vector space and an associative algebra under the operations

$$c\big[(f, U)\big] = \big[(cf, U)\big],$$
$$\big[(f, U)\big] + \big[(g, V)\big] = \big[(f + g, U \cap V)\big],$$
$$\big[(f, U)\big]\big[(g, V)\big] = \big[(fg, U \cap V)\big].$$

(The zero element of this algebra is the equivalence class of the zero function on M.) Let us denote the germ at p of the function element (f, U) simply by $[f]_p$; there is no need to include the domain U in the notation, because the same germ is represented by the restriction of f to any neighborhood of p. To say that two germs $[f]_p$ and $[g]_p$ are equal is simply to say that $f \equiv g$ on some neighborhood of p, however small.

A ***derivation of $C_p^\infty(M)$*** is a linear map $v: C_p^\infty(M) \to \mathbb{R}$ satisfying the following product rule analogous to (3.4):

$$v[fg]_p = f(p)v[g]_p + g(p)v[f]_p.$$

It is common to define the tangent space to M at p as the vector space $\mathcal{D}_p M$ of derivations of $C_p^\infty(M)$. Thanks to Proposition 3.8, it is a simple matter to prove that $\mathcal{D}_p M$ is naturally isomorphic to the tangent space as we have defined it (see Problem 3-7).

The germ definition has a number of advantages. One of the most significant is that it makes the local nature of the tangent space clearer, without requiring the use of bump functions. Because there do not exist analytic bump functions, the germ definition of tangent vectors is the only one available on real-analytic or complex-analytic manifolds. The chief disadvantage of the germ approach is simply that it adds an additional level of complication to an already highly abstract definition.

Tangent Vectors as Equivalence Classes of Curves

Another common approach to tangent vectors is to define an intrinsic equivalence relation on the set of smooth curves with the same starting point, which captures the idea of "having the same velocity," and to define a tangent vector as an equivalence class of curves. Here we describe one such equivalence relation.

Suppose p is a point of M. We wish to define an equivalence relation on the set of all smooth curves of the form $\gamma \colon J \to M$, where J is an interval containing 0 and $\gamma(0) = p$. Given two such curves $\gamma_1 \colon J_1 \to M$ and $\gamma_2 \colon J_2 \to M$, let us say that $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Let $\mathcal{V}_p M$ denote the set of equivalence classes. The tangent space to M at p is often defined to be the set $\mathcal{V}_p M$.

Using this definition, it is very easy to define the differential of a smooth map $F \colon M \to N$ as the map that sends $[\gamma] \in \mathcal{V}_p M$ to $[F \circ \gamma] \in \mathcal{V}_{F(p)} N$. Velocity vectors of smooth curves are almost as easy to define. Suppose $\gamma \colon J \to M$ is any smooth curve. If $0 \in J$, then the velocity of γ at 0 is just the equivalence class of γ in $\mathcal{V}_{\gamma(0)} M$. The velocity at any other point $t_0 \in J$ can be defined as the equivalence class in $\mathcal{V}_{\gamma(t_0)} M$ of the curve γ_{t_0} defined by $\gamma_{t_0}(t) = \gamma(t_0 + t)$.

Problem 3-8 shows that there is a natural one-to-one correspondence between $\mathcal{V}_p M$ and $T_p M$. This definition has the advantage of being geometrically more intuitive, but it has the serious drawback that the existence of a vector space structure on $\mathcal{V}_p M$ is not at all obvious.

Tangent Vectors as Equivalence Classes of n-Tuples

Yet another approach to defining the tangent space is based on the transformation rule (3.12) for the components of tangent vectors in coordinates. One defines a tangent vector at a point $p \in M$ to be a rule that assigns an ordered n-tuple $(v^1, \dots, v^n) \in \mathbb{R}^n$ to each smooth coordinate chart containing p, with the property that the n-tuples assigned to overlapping charts transform according to (3.12). (This is, in fact, the oldest definition of all, and many physicists are still apt to think of tangent vectors this way.)

In this approach, the velocity of a curve is defined by the usual Euclidean formula in coordinates, and the differential of $F \colon M \to N$ is defined as the linear map determined by the Jacobian matrix of F in coordinates. One then has to show, by means of tedious computations involving the chain rule, that these operations are well defined, independently of the choices of coordinates.

It is a matter of individual taste which of the various characterizations of $T_p M$ one chooses to take as the definition. The definition we have chosen, however abstract it may seem at first, has several advantages: it is relatively concrete (tangent vectors are actual derivations of $C^\infty(M)$, with no equivalence classes involved); it makes the vector space structure on $T_p M$ obvious; and it leads to straightforward coordinate-independent definitions of differentials, velocities, and many of the other geometric objects we will be studying.

Categories and Functors

Another useful perspective on tangent spaces and differentials is provided by the theory of categories. In this section we summarize the basic definitions of category theory. We do not do much with the theory in this book, but we mention it because it provides a convenient and powerful language for talking about many of the mathematical structures we will meet.

A *category* C consists of the following things:

- a class $\mathrm{Ob}(C)$, whose elements are called *objects of* C,
- a class $\mathrm{Hom}(C)$, whose elements are called *morphisms of* C,
- for each morphism $f \in \mathrm{Hom}(C)$, two objects $X, Y \in \mathrm{Ob}(C)$ called the *source* and *target of* f, respectively,
- for each triple $X, Y, Z \in \mathrm{Ob}(C)$, a mapping called *composition*:

$$\mathrm{Hom}_C(X, Y) \times \mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z),$$

written $(f, g) \mapsto g \circ f$, where $\mathrm{Hom}_C(X, Y)$ denotes the class of all morphisms with source X and target Y.

The morphisms are required to satisfy the following axioms:

(i) ASSOCIATIVITY: $(f \circ g) \circ h = f \circ (g \circ h)$.
(ii) EXISTENCE OF IDENTITIES: For each object $X \in \mathrm{Ob}(C)$, there exists an *identity morphism* $\mathrm{Id}_X \in \mathrm{Hom}_C(X, X)$, such that $\mathrm{Id}_Y \circ f = f = f \circ \mathrm{Id}_X$ for all $f \in \mathrm{Hom}_C(X, Y)$.

A morphism $f \in \mathrm{Hom}_C(X, Y)$ is called an *isomorphism in* C if there exists a morphism $g \in \mathrm{Hom}_C(Y, X)$ such that $f \circ g = \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.

Example 3.26 (Categories). In most of the categories that one meets "in nature," the objects are sets with some extra structure, the morphisms are maps that preserve that structure, and the composition laws and identity morphisms are the obvious ones. Some of the categories of this type that appear in this book (implicitly or explicitly) are listed below. In each case, we describe the category by giving its objects and its morphisms.

- Set: sets and maps
- Top: topological spaces and continuous maps
- Man: topological manifolds and continuous maps

- $\mathsf{Man_b}$: topological manifolds with boundary and continuous maps
- Diff: smooth manifolds and smooth maps
- $\mathsf{Diff_b}$: smooth manifolds with boundary and smooth maps
- $\mathsf{Vec_\mathbb{R}}$: real vector spaces and real-linear maps
- $\mathsf{Vec_\mathbb{C}}$: complex vector spaces and complex-linear maps
- Grp: groups and group homomorphisms
- Ab: abelian groups and group homomorphisms
- Rng: rings and ring homomorphisms
- CRng: commutative rings and ring homomorphisms

There are also important categories whose objects are sets with distinguished base points, in addition to (possibly) other structures. A ***pointed set*** is an ordered pair (X, p), where X is a set and p is an element of X. Other pointed objects such as ***pointed topological spaces*** or ***pointed smooth manifolds*** are defined similarly. If (X, p) and (X', p') are pointed sets (or topological spaces, etc.), a map $F\colon X \to X'$ is said to be a ***pointed map*** if $F(p) = p'$; in this case, we write $F\colon (X, p) \to (X', p')$. Here are some important examples of categories of pointed objects.

- $\mathsf{Set_*}$: pointed sets and pointed maps
- $\mathsf{Top_*}$: pointed topological spaces and pointed continuous maps
- $\mathsf{Man_*}$: pointed topological manifolds and pointed continuous maps
- $\mathsf{Diff_*}$: pointed smooth manifolds and pointed smooth maps   //

We use the word *class* instead of *set* for the collections of objects and morphisms in a category because in some categories they are "too large" to be considered sets. For example, in the category Set, Ob(Set) is the class of all sets; any attempt to treat it as a set in its own right leads to the well-known Russell paradox of set theory. (See [LeeTM, Appendix A] or almost any book on set theory for more.) Even though the classes of objects and morphisms might not constitute sets, we still use notations such as $X \in \mathrm{Ob(C)}$ and $f \in \mathrm{Hom(C)}$ to indicate that X is an object and f is a morphism in C. A category in which both Ob(C) and Hom(C) are sets is called a ***small category***, and one in which each class of morphisms $\mathrm{Hom_C}(X, Y)$ is a set is called ***locally small***. All the categories listed above are locally small but not small.

If C and D are categories, a ***covariant functor from*** **C** ***to*** **D** is a rule $\mathscr{F}$ that assigns to each object $X \in \mathrm{Ob(C)}$ an object $\mathscr{F}(X) \in \mathrm{Ob(D)}$, and to each morphism $f \in \mathrm{Hom_C}(X, Y)$ a morphism $\mathscr{F}(f) \in \mathrm{Hom_D}(\mathscr{F}(X), \mathscr{F}(Y))$, so that identities and composition are preserved:

$$\mathscr{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{F}(X)}; \qquad \mathscr{F}(g \circ h) = \mathscr{F}(g) \circ \mathscr{F}(h).$$

We also need to consider functors that reverse morphisms: a ***contravariant functor from*** **C** ***to*** **D** is a rule $\mathscr{F}$ that assigns to each object $X \in \mathrm{Ob(C)}$ an object $\mathscr{F}(X) \in \mathrm{Ob(D)}$, and to each morphism $f \in \mathrm{Hom_C}(X, Y)$ a morphism $\mathscr{F}(f) \in \mathrm{Hom_D}(\mathscr{F}(Y), \mathscr{F}(X))$, such that

$$\mathscr{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathscr{F}(X)}; \qquad \mathscr{F}(g \circ h) = \mathscr{F}(h) \circ \mathscr{F}(g).$$

▶ **Exercise 3.27.** Show that any (covariant or contravariant) functor from C to D takes isomorphisms in C to isomorphisms in D.

One trivial example of a covariant functor is the ***identity functor*** from any category to itself: it takes each object and each morphism to itself. Another example is the ***forgetful functor***: if C is a category whose objects are sets with some additional structure and whose morphisms are maps preserving that structure (as are all the categories listed in the first part of Example 3.26 except Set itself), the forgetful functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{Set}$ assigns to each object its underlying set, and to each morphism the same map thought of as a map between sets.

More interesting functors arise when we associate "invariants" to classes of mathematical objects. For example, the fundamental group is a covariant functor from Top_* to Grp. The results of Problem 2-10 show that there is a contravariant functor from Diff to $\mathsf{Vec}_\mathbb{R}$ defined by assigning to each smooth manifold M the vector space $C^\infty(M)$, and to each smooth map $F\colon M \to N$ the linear map $F^*\colon C^\infty(N) \to C^\infty(M)$ defined by $F^*(f) = f \circ F$.

The discussion in this chapter has given us some other important examples of functors. First, the ***tangent space functor*** is a covariant functor from the category Diff_* of pointed smooth manifolds to the category $\mathsf{Vec}_\mathbb{R}$ of real vector spaces. To each pointed smooth manifold (M, p) it assigns the tangent space $T_p M$, and to each pointed smooth map $F\colon (M, p) \to (N, F(p))$ it assigns the differential dF_p. The fact that this is a functor is the content of parts (b) and (c) of Proposition 3.6.

Similarly, we can think of the assignments $M \mapsto TM$ and $F \mapsto dF$ (sending each smooth manifold to its tangent bundle and each smooth map to its global differential) as a covariant functor from Diff to itself, called the ***tangent functor***.

Problems

3-1. Suppose M and N are smooth manifolds with or without boundary, and $F\colon M \to N$ is a smooth map. Show that $dF_p\colon T_p M \to T_{F(p)} N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

3-2. Prove Proposition 3.14 (the tangent space to a product manifold).

3-3. Prove that if M and N are smooth manifolds, then $T(M \times N)$ is diffeomorphic to $TM \times TN$.

3-4. Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

3-5. Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$, but there is no *diffeomorphism* with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF(\gamma'(t))$ on the coordinate functions x and y.] (*Used on p. 123.*)

3-6. Consider $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$ under the usual identification $\mathbb{C}^2 \leftrightarrow \mathbb{R}^4$. For each $z = (z^1, z^2) \in \mathbb{S}^3$, define a curve $\gamma_z\colon \mathbb{R} \to \mathbb{S}^3$ by $\gamma_z(t) = (e^{it}z^1, e^{it}z^2)$. Show that γ_z is a smooth curve whose velocity is never zero.

3-7. Let M be a smooth manifold with or without boundary and p be a point of
      M. Let $C_p^\infty(M)$ denote the algebra of germs of smooth real-valued func-
      tions at p, and let $\mathcal{D}_p M$ denote the vector space of derivations of $C_p^\infty(M)$.
      Define a map $\Phi\colon \mathcal{D}_p M \to T_p M$ by $(\Phi v)f = v([f]_p)$. Show that Φ is an
      isomorphism. (*Used on p. .*)

3-8. Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_p M$
      denote the set of equivalence classes of smooth curves starting at p under the
      relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued
      function f defined in a neighborhood of p. Show that the map $\Psi\colon \mathcal{V}_p M \to$
      $T_p M$ defined by $\Psi[\gamma] = \gamma'(0)$ is well defined and bijective. (*Used on p. .*)