Chapter 11
The Cotangent Bundle

In this chapter we introduce a construction that is not typically seen in elementary calculus: *tangent covectors*, which are linear functionals on the tangent space at a point $p \in M$. The space of all covectors at p is a vector space called the *cotangent space* at p; in linear-algebraic terms, it is the dual space to $T_p M$. The union of all cotangent spaces at all points of M is a vector bundle called the *cotangent bundle*.

Whereas tangent vectors give us a coordinate-free interpretation of derivatives of curves, it turns out that derivatives of real-valued functions on a manifold are most naturally interpreted as tangent covectors. Thus we define the differential of a real-valued function as a covector field (a smooth section of the cotangent bundle); it is a coordinate-independent analogue of the gradient. We then explore the behavior of covector fields under smooth maps, and show that covector fields on the codomain of a smooth map always pull back to covector fields on the domain.

In the second half of the chapter we introduce line integrals of covector fields, which are the natural generalization of the line integrals of elementary calculus. Then we explore the relationships among three closely related types of covector fields: exact (those that are the differentials of functions), conservative (those whose line integrals around closed curves are zero), and closed (those that satisfy a certain differential equation in coordinates). This leads to a far-reaching generalization of the fundamental theorem of calculus to line integrals on manifolds.

Covectors

Let V be a finite-dimensional vector space. (As usual, all of our vector spaces are assumed to be real.) We define a ***covector on V*** to be a real-valued linear functional on V, that is, a linear map $\omega\colon V \to \mathbb{R}$. The space of all covectors on V is itself a real vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and called the ***dual space of V***.

The next proposition expresses the most important fact about V^* in the finite-dimensional case. Recall from Exercise B.13 that a linear map is uniquely determined by specifying its values on the elements of any basis.

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Proposition 11.1. *Let V be a finite-dimensional vector space. Given any basis $(E_1, \dots, E_n)$ for V, let $\varepsilon^1, \dots, \varepsilon^n \in V^*$ be the covectors defined by*

$$\varepsilon^i(E_j) = \delta^i_j,$$

where δ^i_j is the Kronecker delta symbol defined by (4.4). *Then $\left(\varepsilon^1, \dots, \varepsilon^n\right)$ is a basis for V^*, called the* ***dual basis to (E_j)***. *Therefore*, $\dim V^* = \dim V$.

► **Exercise 11.2.** Prove Proposition 11.1.

For example, we can apply this to the standard basis $(e_1, \dots, e_n)$ for $\mathbb{R}^n$. The dual basis is denoted by $\left(e^1, \dots, e^n\right)$ (note the upper indices), and is called the ***standard dual basis***. These basis covectors are the linear functionals on $\mathbb{R}^n$ given by

$$e^i(v) = e^i\left(v^1, \dots, v^n\right) = v^i.$$

In other words, e^i is the linear functional that picks out the ith component of a vector. In matrix notation, a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by a $1 \times n$ matrix, called a ***row matrix***. The basis covectors can therefore also be thought of as the linear functionals represented by the row matrices

$$e^1 = (1\ 0\ \dots\ 0), \quad e^2 = (0\ 1\ 0\ \dots\ 0), \quad \dots, \quad e^n = (0\ \dots\ 0\ 1).$$

In general, if (E_j) is a basis for V and (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$, we have (using the summation convention)

$$\varepsilon^i(v) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

Thus, just as in the case of $\mathbb{R}^n$, the ith basis covector ε^i picks out the ith component of a vector with respect to the basis (E_j). More generally, Proposition 11.1 shows that we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \varepsilon^i, \tag{11.1}$$

where the components are determined by $\omega_i = \omega(E_i)$. The action of ω on a vector $v = v^j E_j$ is

$$\omega(v) = \omega_i v^i. \tag{11.2}$$

We always write basis covectors with upper indices, and components of a covector with lower indices, because this helps to ensure that mathematically meaningful summations such as (11.1) and (11.2) always follow our index conventions.

Suppose V and W are vector spaces and $A\colon V \to W$ is a linear map. We define a linear map $A^*\colon W^* \to V^*$, called the ***dual map*** or ***transpose of A***, by

$$(A^*\omega)(v) = \omega(Av) \quad \text{for } \omega \in W^*,\ v \in V.$$

► **Exercise 11.3.** Show that $A^*\omega$ is actually a linear functional on V, and that A^* is a linear map.

Proposition 11.4. *The dual map satisfies the following properties*:

(a) $(A \circ B)^* = B^* \circ A^*$.
(b) $(\mathrm{Id}_V)^*\colon V^* \to V^*$ *is the identity map of* V^*.

► **Exercise 11.5.** Prove the preceding proposition.

Corollary 11.6. *The assignment that sends a vector space to its dual space and a linear map to its dual map is a contravariant functor from the category of real vector spaces to itself.* □

Apart from the fact that the dimension of V^* is the same as that of V, the second most important fact about dual spaces is the following characterization of the ***second dual space*** $V^{**} = (V^*)^*$. For each vector space V there is a natural, basis-independent map $\xi\colon V \to V^{**}$, defined as follows. For each vector $v \in V$, define a linear functional $\xi(v)\colon V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v) \quad \text{for } \omega \in V^*. \tag{11.3}$$

► **Exercise 11.7.** Let V be a vector space.

(a) For any $v \in V$, show that $\xi(v)(\omega)$ depends linearly on ω, so $\xi(v) \in V^{**}$.
(b) Show that the map $\xi\colon V \to V^{**}$ is linear.

Proposition 11.8. *For any finite-dimensional vector space* V*, the map* $\xi\colon V \to V^{**}$ *is an isomorphism.*

Proof. Because $\dim V = \dim V^{**}$, it suffices to verify that ξ is injective (see Exercise B.22(c)). Suppose $v \in V$ is not zero. Extend v to a basis $(v = E_1, \dots, E_n)$ for V, and let $\left(\varepsilon^1, \dots, \varepsilon^n\right)$ denote the dual basis for V^*. Then $\xi(v) \neq 0$ because

$$\xi(v)\left(\varepsilon^1\right) = \varepsilon^1(v) = \varepsilon^1(E_1) = 1. \qquad \square$$

The preceding proposition shows that when V is finite-dimensional, we can unambiguously identify V^{**} with V itself, because the map ξ is canonically defined, without reference to any basis. It is important to observe that although V^* is also isomorphic to V (for the simple reason that any two finite-dimensional vector spaces of the same dimension are isomorphic), there is no *canonical* isomorphism $V \cong V^*$. One way to make this statement precise is indicated in Problem 11-1. Note also that the conclusion of Proposition 11.8 is always false when V is infinite-dimensional (see Problem 11-2).

Because of Proposition 11.8, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more symmetric-looking notations $\langle \omega, v\rangle$ and $\langle v, \omega\rangle$; both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $v \in V$, or as the action of the linear functional $\xi(v) \in V^{**}$ on the element $\omega \in V^*$. There should be no cause for confusion with the use of the same angle bracket notation for inner products: whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, v\rangle$ is always to be interpreted as the natural pairing between vectors and covectors, not as an inner product. We typically omit any mention of the map ξ, and think of $v \in V$ either as a vector or as a linear functional on V^*, depending on the context.

There is also a symmetry between bases and dual bases for a finite-dimensional vector space V: any basis for V determines a dual basis for V^*, and conversely, any basis for V^* determines a dual basis for $V^{**} = V$. If $\left(\varepsilon^i\right)$ is the basis for V^* dual to a basis (E_j) for V, then (E_j) is the basis dual to $\left(\varepsilon^i\right)$, because both statements are equivalent to the relation $\langle \varepsilon^i, E_j \rangle = \delta^i_j$.

Tangent Covectors on Manifolds

Now let M be a smooth manifold with or without boundary. For each $p \in M$, we define the ***cotangent space at p***, denoted by T_p^*M, to be the dual space to T_pM:

$$T_p^*M = \left(T_pM\right)^*.$$

Elements of T_p^*M are called ***tangent covectors at p***, or just ***covectors at p***.

Given smooth local coordinates $\left(x^i\right)$ on an open subset $U \subseteq M$, for each $p \in U$ the coordinate basis $\left(\partial/\partial x^i|_p\right)$ gives rise to a dual basis for T_p^*M, which we denote for the moment by $\left(\lambda^i|_p\right)$. (In a short while, we will come up with a better notation.) Any covector $\omega \in T_p^*M$ can thus be written uniquely as $\omega = \omega_i \lambda^i|_p$, where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

Suppose now that $\left(\widetilde{x}^j\right)$ is another set of smooth coordinates whose domain contains p, and let $\left(\widetilde{\lambda}^j|_p\right)$ denote the basis for T_p^*M dual to $\left(\partial/\partial \widetilde{x}^j|_p\right)$. We can compute the components of the same covector ω with respect to the new coordinate system as follows. First observe that the computations in Chapter 3 show that the coordinate vector fields transform as follows:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p) \left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p. \tag{11.4}$$

(Here we use the same notation p to denote either a point in M or its coordinate representation as appropriate.) Writing ω in both systems as $\omega = \omega_i \lambda^i|_p = \widetilde{\omega}_j \widetilde{\lambda}^j|_p$, we can use (11.4) to compute the components ω_i in terms of $\widetilde{\omega}_j$:

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p) \left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\widetilde{\omega}_j. \tag{11.5}$$

As we mentioned in Chapter 3, in the early days of smooth manifold theory, before most of the abstract coordinate-free definitions we are using were developed, mathematicians tended to think of a tangent vector at a point p as an assignment of an n-tuple of real numbers to each smooth coordinate system, with the property that the n-tuples $\left(v^1, \dots, v^n\right)$ and $\left(\widetilde{v}^1, \dots, \widetilde{v}^n\right)$ assigned to two different coordinate systems $\left(x^i\right)$ and $\left(\widetilde{x}^j\right)$ were related by the transformation law that we derived in

Chapter 3:

$$\widetilde{v}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p) v^i. \tag{11.6}$$

Similarly, a tangent covector was thought of as an n-tuple $(\omega_1, \dots, \omega_n)$ that transforms, by virtue of (11.5), according to the following slightly different rule:

$$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p) \widetilde{\omega}_j. \tag{11.7}$$

Since the transformation law (11.4) for the coordinate partial derivatives follows directly from the chain rule, it can be thought of as fundamental. Thus it became customary to call tangent covectors ***covariant vectors*** because their components transform in the same way as ("vary with") the coordinate partial derivatives, with the Jacobian matrix $\left(\partial \widetilde{x}^j / \partial x^i\right)$ multiplying the objects associated with the "new" coordinates $\left(\widetilde{x}^j\right)$ to obtain those associated with the "old" coordinates $\left(x^i\right)$. Analogously, tangent vectors were called ***contravariant vectors***, because their components transform in the opposite way. (Remember, it was the component n-tuples that were thought of as the objects of interest.) Admittedly, these terms do not make a lot of sense, but by now they are well entrenched, and we will see them again in Chapter 12. Note that this use of the terms covariant and contravariant has nothing to do with the covariant and contravariant functors of category theory!

Covector Fields

For any smooth manifold M with or without boundary, the disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M$$

is called the ***cotangent bundle of M***. It has a natural projection map $\pi\colon T^*M \to M$ sending $\omega \in T_p^*M$ to $p \in M$. As above, given any smooth local coordinates $\left(x^i\right)$ on an open subset $U \subseteq M$, for each $p \in U$ we denote the basis for T_p^*M dual to $\left(\partial/\partial x^i|_p\right)$ by $\left(\lambda^i|_p\right)$. This defines n maps $\lambda^1, \dots, \lambda^n\colon U \to T^*M$, called ***coordinate covector fields***.

Proposition 11.9 (The Cotangent Bundle as a Vector Bundle). *Let M be a smooth n-manifold with or without boundary. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth rank-n vector bundle over M for which all coordinate covector fields are smooth local sections.*

Proof. The proof is just like that of Theorem 10.4. Given a smooth chart (U, φ) on M, with coordinate functions $\left(x^i\right)$, define $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(\xi_i \lambda^i\big|_p\right) = \left(p, (\xi_1, \dots, \xi_n)\right),$$

where λ^i is the ith coordinate covector field associated with (x^i). Suppose $(\widetilde{U},\widetilde{\varphi})$ is another smooth chart with coordinate functions $(\widetilde{x}^j)$, and let $\widetilde{\Phi}\colon \pi^{-1}(\widetilde{U}) \to \widetilde{U} \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \widetilde{U})$, it follows from (11.5) that

$$\Phi \circ \widetilde{\Phi}^{-1}\big(p,(\widetilde{\xi}_1,\dots,\widetilde{\xi}_n)\big) = \left(p, \left(\frac{\partial \widetilde{x}^j}{\partial x^1}(p)\widetilde{\xi}_j,\dots,\frac{\partial \widetilde{x}^j}{\partial x^n}(p)\widetilde{\xi}_j\right)\right).$$

The $\mathrm{GL}(n,\mathbb{R})$-valued function $(\partial\widetilde{x}^j/\partial x^i)$ is smooth, so it follows from the vector bundle chart lemma that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are smooth local trivializations. Uniqueness follows as in the proof of Proposition 10.24. □

► **Exercise 11.10.** Suppose M is a smooth manifold and $E \to M$ is a smooth vector bundle over M. Define the ***dual bundle to E*** to be the bundle $E^* \to M$ whose total space is the disjoint union $E^* = \coprod_{p\in M} E_p^*$, where E_p^* is the dual space to E_p, with the obvious projection. Show that $E^* \to M$ is a smooth vector bundle, whose transition functions are given by $\tau^*(p) = \big(\tau(p)^{-1}\big)^T$ for any transition function $\tau\colon U \to \mathrm{GL}(k,\mathbb{R})$ of E.

As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subseteq M$, Corollary 10.21 shows that the map from $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i\lambda^i\big|_p \mapsto \big(x^1(p),\dots,x^n(p),\xi_1,\dots,\xi_n\big)$$

is a smooth coordinate chart for T^*M. We call (x^i,ξ_i) the ***natural coordinates for T^*M*** associated with (x^i). (In this situation, we must forgo our insistence that coordinate functions have upper indices, because the fiber coordinates ξ_i are already required by our index conventions to have lower indices. Nonetheless, the convention still holds that each index to be summed over in a given term appears once as a superscript and once as a subscript.)

A (local or global) section of T^*M is called a ***covector field*** or a ***(differential) 1-form***. (The reason for the latter terminology will become clear in Chapter 14, when we define differential k-forms for $k > 1$.) Like sections of other bundles, covector fields without further qualification are assumed to be merely continuous; when we make different assumptions, we use the terms ***rough covector field*** and ***smooth covector field*** with the obvious meanings. As we did with vector fields, we write the value of a covector field ω at a point $p \in M$ as ω_p instead of $\omega(p)$, to avoid conflict with the notation for the action of a covector on a vector. If ω itself has subscripts or superscripts, we usually use the notation $\omega|_p$ instead. In any smooth local coordinates on an open subset $U \subseteq M$, a (rough) covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i\lambda^i$ for n functions $\omega_i\colon U \to \mathbb{R}$ called the ***component functions of ω***. They are characterized by

$$\omega_i(p) = \omega_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

If ω is a (rough) covector field and X is a vector field on M, then we can form a function $\omega(X)\colon M \to \mathbb{R}$ by

$$\omega(X)(p) = \omega_p(X_p), \qquad p \in M.$$

If we write $\omega = \omega_i \lambda^i$ and $X = X^j \partial/\partial x^j$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation $\omega(X) = \omega_i X^i$.

Just as in the case of vector fields, there are several ways to check for smoothness of a covector field.

Proposition 11.11 (Smoothness Criteria for Covector Fields). *Let M be a smooth manifold with or without boundary, and let $\omega\colon M \to T^*M$ be a rough covector field. The following are equivalent*:

(a) *ω is smooth.*
(b) *In every smooth coordinate chart, the component functions of ω are smooth.*
(c) *Each point of M is contained in some coordinate chart in which ω has smooth component functions.*
(d) *For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.*
(e) *For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on U.*

► **Exercise 11.12.** Prove this proposition. [Suggestion: try proving (a) ⇒ (b) ⇒ (c) ⇒ (a), and (c) ⇒ (d) ⇒ (e) ⇒ (b). The only tricky part is (d) ⇒ (e); look at the proof of Proposition 8.14 for ideas.]

Of course, since any open subset of a smooth manifold (with boundary) is again a smooth manifold (with boundary), the preceding proposition applies equally well to covector fields defined only on some open subset of M.

Coframes

Let M be a smooth manifold with or without boundary, and let $U \subseteq M$ be an open subset. A ***local coframe for M over U*** is an ordered n-tuple of covector fields $(\varepsilon^1, \dots, \varepsilon^n)$ defined on U such that $(\varepsilon^i|_p)$ forms a basis for T_p^*M at each point $p \in U$. If $U = M$, it is called a ***global coframe***. (A local coframe for M is just a local frame for the vector bundle T^*M, in the terminology of Chapter 10.)

Example 11.13 (Coordinate Coframes). For any smooth chart $(U, (x^i))$, the coordinate covector fields (λ^i) defined above constitute a local coframe over U, called a ***coordinate coframe***. By Proposition 11.11(c), every coordinate frame is smooth, because its component functions in the given chart are constants. //

Given a local frame $(E_1, \dots, E_n)$ for TM over an open subset U, there is a uniquely determined (rough) local coframe $(\varepsilon^1, \dots, \varepsilon^n)$ over U such that $(\varepsilon^i|_p)$ is the dual basis to $(E_i|_p)$ for each $p \in U$, or equivalently $\varepsilon^i(E_j) = \delta^i_j$. This coframe is called the ***coframe dual to (E_i)***. Conversely, if we start with a local coframe (ε^i)

over an open subset $U \subseteq M$, there is a uniquely determined local frame (E_i), called the ***frame dual to*** $(\boldsymbol{\varepsilon^i})$, determined by $\varepsilon^i(E_j) = \delta^i_j$. For example, in a smooth chart, the coordinate frame $(\partial/\partial x^i)$ and the coordinate coframe $\left(\lambda^i\right)$ are dual to each other.

Lemma 11.14. *Let M be a smooth manifold with or without boundary. If (E_i) is a rough local frame over an open subset $U \subseteq M$ and $\left(\varepsilon^i\right)$ is its dual coframe, then (E_i) is smooth if and only if $\left(\varepsilon^i\right)$ is smooth.*

Proof. It suffices to show that for each $p \in U$, the frame (E_i) is smooth in a neighborhood of p if and only if $\left(\varepsilon^i\right)$ is. Given $p \in U$, let $\left(V, \left(x^i\right)\right)$ be a smooth coordinate chart such that $p \in V \subseteq U$. In V, we can write

$$E_i = a_i^k \frac{\partial}{\partial x^k}, \qquad \varepsilon^j = b_l^j \lambda^l,$$

for some matrices of real-valued functions $\left(a_i^k\right)$ and $\left(b_l^j\right)$ defined on V. By virtue of Propositions 8.1 and 11.11, the vector fields E_i are smooth on V if and only if the functions a_i^k are smooth, and the covector fields ε^j are smooth on V if and only if the functions b_l^j are smooth. The fact that $\varepsilon^j(E_i) = \delta_i^j$ implies that the matrices $\left(a_i^k\right)$ and $\left(b_l^j\right)$ are inverses of each other. Because matrix inversion is a smooth map from $\mathrm{GL}(n, \mathbb{R})$ to itself, either one of these matrix-valued functions is smooth if and only if the other one is smooth. $\square$

Given a local coframe $\left(\varepsilon^i\right)$ over an open subset $U \subseteq M$, every (rough) covector field ω on U can be expressed in terms of the coframe as $\omega = \omega_i \varepsilon^i$ for some functions $\omega_1, \dots, \omega_n \colon U \to \mathbb{R}$, called the ***component functions of*** $\boldsymbol{\omega}$ ***with respect to the given coframe***. The component functions are determined by $\omega_i = \omega(E_i)$, where (E_i) is the frame dual to $\left(\varepsilon^i\right)$. This leads to another way of characterizing smoothness of covector fields.

Proposition 11.15 (Coframe Criterion for Smoothness of Covector Fields). *Let M be a smooth manifold with or without boundary, and let ω be a rough covector field on M. If $\left(\varepsilon^i\right)$ is a smooth coframe on an open subset $U \subseteq M$, then ω is smooth on U if and only if its component functions with respect to $\left(\varepsilon^i\right)$ are smooth.*

► **Exercise 11.16.** Prove the preceding proposition.

We denote the real vector space of all smooth covector fields on M by $\mathfrak{X}^*(M)$. As smooth sections of a vector bundle, elements of $\mathfrak{X}^*(M)$ can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $\omega \in \mathfrak{X}^*(M)$, the covector field $f\omega$ is defined by

$$(f\omega)_p = f(p)\omega_p. \tag{11.8}$$

Because it is the space of smooth sections of a vector bundle, $\mathfrak{X}^*(M)$ is a module over $C^\infty(M)$.

Geometrically, we think of a vector field on M as an arrow attached to each point of M. What kind of geometric picture can we form of a covector field? The key idea is that a nonzero linear functional $\omega_p \in T_p^* M$ is completely determined by

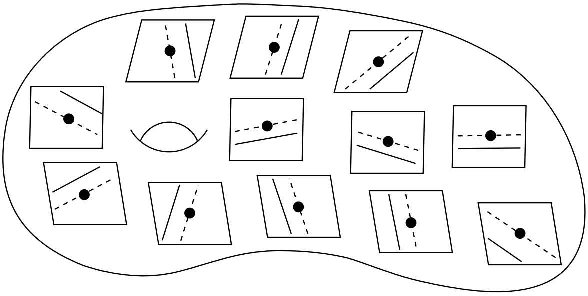

Fig. 11.1 A covector field

two pieces of data: its kernel, which is a linear hyperplane in T_pM (a codimension-1 linear subspace); and the set of vectors v for which $\omega_p(v) = 1$, which is an affine hyperplane parallel to the kernel (Fig. 11.1). (Actually, the set where $\omega_p(v) = 1$ alone suffices, but it is useful to visualize the two parallel hyperplanes.) The value of $\omega_p(v)$ for any other vector v is then obtained by linear interpolation or extrapolation.

Thus, you can visualize a covector field as defining a pair of hyperplanes in each tangent space, one through the origin and another parallel to it, and varying continuously from point to point. Where the covector field is small, one of the hyperplanes becomes very far from the kernel, eventually disappearing altogether at points where the covector field takes the value zero.

The Differential of a Function

In elementary calculus, the gradient of a smooth real-valued function f on $\mathbb{R}^n$ is defined as the vector field whose components are the partial derivatives of f. In our notation, this would read

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \tag{11.9}$$

Unfortunately, in this form, the gradient does not make sense independently of coordinates. (The fact that it violates our index conventions is a strong clue.)

► **Exercise 11.17.** Let $f(x, y) = x^2$ on $\mathbb{R}^2$, and let X be the vector field

$$X = \operatorname{grad} f = 2x \frac{\partial}{\partial x}.$$

Compute the coordinate expression for X in polar coordinates (on some open subset on which they are defined) using (11.4) and show that it is *not* equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

Although the partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field, it turns out that they *can* be interpreted as the components of a covector field. This is the most important application of covector fields.

Let f be a smooth real-valued function on a smooth manifold M with or without boundary. (As usual, all of this discussion applies to functions defined on an open subset $U \subseteq M$, simply by replacing M with U throughout.) We define a covector field df, called the ***differential of*** $\boldsymbol{f}$, by

$$df_p(v) = vf \quad \text{for } v \in T_pM.$$

(We will discuss the relationship between this differential and the differential of a smooth map below; see the paragraph just after Exercise 11.24.)

Proposition 11.18. *The differential of a smooth function is a smooth covector field.*

Proof. It is straightforward to verify that at each point $p \in M$, $df_p(v)$ depends linearly on v, so that df_p is indeed a covector at p. To see that df is smooth, we use Proposition 11.11(d): for any smooth vector field X on M, the function $df(X)$ is smooth because it is equal to Xf. □

To see what df looks like more concretely, we need to compute its coordinate representation. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$, and let (λ^i) be the corresponding coordinate coframe on U. Write df in coordinates as $df_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$; then the definition of df implies

$$A_i(p) = df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p).$$

This yields the following formula for the coordinate representation of df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i\big|_p. \tag{11.10}$$

Thus, the component functions of df in any smooth coordinate chart are the partial derivatives of f with respect to those coordinates. Because of this, we can think of df as an analogue of the classical gradient, reinterpreted in a way that makes coordinate-independent sense on a manifold.

If we apply (11.10) to the special case in which f is one of the coordinate functions $x^j : U \to \mathbb{R}$, we obtain

$$dx^j\big|_p = \frac{\partial x^j}{\partial x^i}(p)\lambda^i\big|_p = \delta^j_i\lambda^i\big|_p = \lambda^j\big|_p.$$

In other words, *the coordinate covector field λ^j is none other than the differential dx^j*! Therefore, the formula (11.10) for df_p can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i\big|_p,$$

or as an equation between covector *fields* instead of covectors:

$$df = \frac{\partial f}{\partial x^i}dx^i. \tag{11.11}$$

In particular, in the 1-dimensional case, this reduces to

$$df = \frac{df}{dx}dx.$$

Thus, we have recovered the familiar classical expression for the differential of a function f in coordinates. Henceforth, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

Example 11.19. If $f(x, y) = x^2 y \cos x$ on $\mathbb{R}^2$, then df is given by the formula

$$\begin{aligned} df &= \frac{\partial\left(x^2 y \cos x\right)}{\partial x} dx + \frac{\partial\left(x^2 y \cos x\right)}{\partial y} dy \\ &= \left(2xy \cos x - x^2 y \sin x\right) dx + x^2 \cos x \, dy. \end{aligned}$$ //

Proposition 11.20 (Properties of the Differential). *Let M be a smooth manifold with or without boundary, and let $f, g \in C^\infty(M)$.*

(a) *If a and b are constants, then $d(af + bg) = a\,df + b\,dg$.*
(b) *$d(fg) = f\,dg + g\,df$.*
(c) *$d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.*
(d) *If $J \subseteq \mathbb{R}$ is an interval containing the image of f, and $h\colon J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\,df$.*
(e) *If f is constant, then $df = 0$.*

► **Exercise 11.21.** Prove Proposition 11.20.

One very important property of the differential is the following characterization of smooth functions with vanishing differentials.

Proposition 11.22 (Functions with Vanishing Differentials). *If f is a smooth real-valued function on a smooth manifold M with or without boundary, then $df = 0$ if and only if f is constant on each component of M.*

Proof. It suffices to assume that M is connected and show that $df = 0$ if and only if f is constant. One direction is immediate: if f is constant, then $df = 0$ by Proposition 11.20(e). Conversely, suppose $df = 0$, let $p \in M$, and let $\mathcal{C} = \{q \in M : f(q) = f(p)\}$. If q is any point in $\mathcal{C}$, let U be a smooth coordinate ball (or half-ball, in case $q \in \partial M$) centered at q. From (11.11) we see that $\partial f / \partial x^i \equiv 0$ in U for each i, so by elementary calculus f is constant on U. This shows that $\mathcal{C}$ is open, and since it is closed by continuity, it must be all of M. Thus, f is everywhere equal to the constant $f(p)$. □

In elementary calculus, one thinks of df as an approximation for the small change in the value of f caused by small changes in the independent variables x^i. In our present context, df has the same meaning, provided we interpret everything appropriately. Suppose M is a smooth manifold and $f \in C^\infty(M)$, and let p be a point in M. By choosing smooth coordinates on a neighborhood of p, we can think of f as a function on an open subset $U \subseteq \mathbb{R}^n$. Recall that $dx^i|_p$ is the linear functional that picks out the ith component of a tangent vector at p. Writing

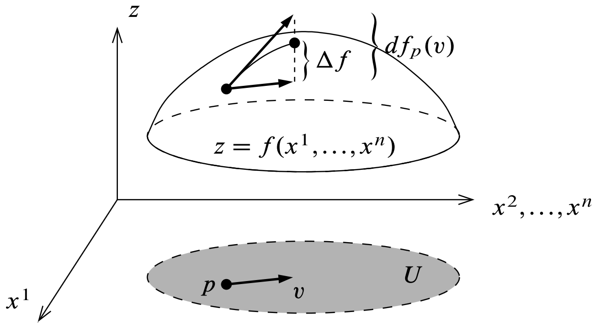

Fig. 11.2 The differential as an approximation to Δf

$\Delta f = f(p+v) - f(p)$ for $v \in \mathbb{R}^n$, Taylor's theorem shows that Δf is well approximated when v is small by

$$\Delta f \approx \frac{\partial f}{\partial x^i}(p)v^i = \frac{\partial f}{\partial x^i}(p)dx^i\big|_p(v) = df_p(v)$$

(where now we are considering v as an element of $T_p\mathbb{R}^n$ via our usual identification $T_p\mathbb{R}^n \leftrightarrow \mathbb{R}^n$). In other words, df_p is the linear functional that best approximates Δf near p (Fig. 11.2). The great power of the concept of the differential comes from the fact that we can define df invariantly on any manifold, without resorting to vague arguments involving infinitesimals.

The next result is an analogue of Proposition 3.24 for the differential.

Proposition 11.23 (Derivative of a Function Along a Curve). *Suppose M is a smooth manifold with or without boundary, $\gamma\colon J \to M$ is a smooth curve, and $f\colon M \to \mathbb{R}$ is a smooth function. Then the derivative of the real-valued function $f \circ \gamma\colon J \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\big(\gamma'(t)\big). \tag{11.12}$$

Proof. See Fig. 11.3. Directly from the definitions, for any $t_0 \in J$,

$$\begin{aligned}
df_{\gamma(t_0)}\big(\gamma'(t_0)\big) &= \gamma'(t_0)f && \text{(definition of } df) \\
&= d\gamma_{t_0}\left(\frac{d}{dt}\bigg|_{t_0}\right)f && \text{(definition of } \gamma'(t)) \\
&= \frac{d}{dt}\bigg|_{t_0}(f \circ \gamma) && \text{(definition of } d\gamma) \\
&= (f \circ \gamma)'(t_0) && \text{(definition of } d/dt|_{t_0}).
\end{aligned}$$

□

► **Exercise 11.24.** For a smooth real-valued function $f\colon M \to \mathbb{R}$, show that $p \in M$ is a critical point of f if and only if $df_p = 0$.

You may have noticed that for a smooth real-valued function $f\colon M \to \mathbb{R}$, we now have two different definitions for the differential of f at a point $p \in M$. In

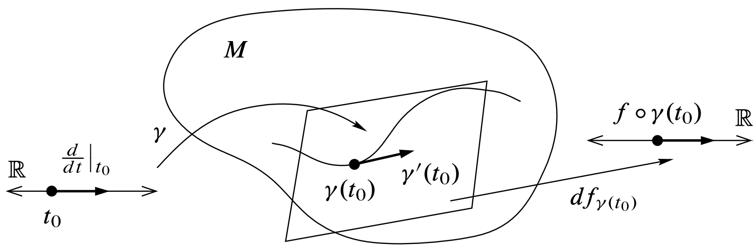

Fig. 11.3 Derivative of a function along a curve

Chapter 3, we defined df_p as a linear map from T_pM to $T_{f(p)}\mathbb{R}$. In this chapter, we defined df_p as a covector at p, which is to say a linear map from T_pM to $\mathbb{R}$. These are really the same object, once we take into account the canonical identification between $\mathbb{R}$ and $T_{f(p)}\mathbb{R}$; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f.

Similarly, if γ is a smooth curve in M, we have two different meanings for the expression $(f \circ \gamma)'(t)$. On the one hand, $f \circ \gamma$ can be interpreted as a smooth curve in $\mathbb{R}$, and thus $(f \circ \gamma)'(t)$ is its velocity at the point $f \circ \gamma(t)$, which is an element of the tangent space $T_{f\circ\gamma(t)}\mathbb{R}$. Proposition 3.24 shows that this tangent vector is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as an element of $T_{f\circ\gamma(t)}\mathbb{R}$. On the other hand, $f \circ \gamma$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ordinary derivative. Proposition 11.23 shows that this derivative is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as a real number. Which of these interpretations we choose depends on the purpose we have in mind.

Pullbacks of Covector Fields

As we have seen, a smooth map yields a linear map on tangent vectors called the differential. Dualizing this leads to a linear map on covectors going in the opposite direction.

Let $F: M \to N$ be a smooth map between smooth manifolds with or without boundary, and let $p \in M$ be arbitrary. The differential $dF_p: T_pM \to T_{F(p)}N$ yields a dual linear map

$$dF_p^*: T_{F(p)}^*N \to T_p^*M,$$

called the ***(pointwise) pullback by F at p***, or the ***cotangent map of F***. Unraveling the definitions, we see that dF_p^* is characterized by

$$dF_p^*(\omega)(v) = \omega\big(dF_p(v)\big), \quad \text{for } \omega \in T_{F(p)}^*N,\ v \in T_pM.$$

Observe that the assignments $(M, p) \mapsto T_p^*M$ and $F \mapsto dF_p^*$ yield a contravariant functor from the category of pointed smooth manifolds to the category of real vector spaces. Because of this, the convention of calling elements of T^*M “covariant vectors” is particularly unfortunate; but this terminology is so deeply entrenched that one has no choice but to go along with it.

When we discussed vector fields, we made a point of noting that pushforwards of vector fields under smooth maps are defined only in the special cases of diffeomorphisms or Lie group homomorphisms. The surprising thing about covectors is that covector fields *always* pull back to covector fields. Given a smooth map $F: M \to N$ and a covector field ω on N, define a rough covector field $F^*\omega$ on M, called the ***pullback of ω by F***, by

$$(F^*\omega)_p = dF_p^*\left(\omega_{F(p)}\right). \tag{11.13}$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}\left(dF_p(v)\right).$$

In contrast to the vector field case, there is no ambiguity here about what point to pull back from: the value of $F^*\omega$ at p is the pullback of ω at $F(p)$. We will prove in Proposition 11.26 below that $F^*\omega$ is continuous, and is smooth when ω is smooth. Before we do so, let us prove two important properties of the pullback.

Proposition 11.25. *Let $F: M \to N$ be a smooth map between smooth manifolds with or without boundary. Suppose u is a continuous real-valued function on N, and ω is a covector field on N. Then*

$$F^*(u\omega) = (u \circ F)F^*\omega. \tag{11.14}$$

If in addition u is smooth, then

$$F^*du = d(u \circ F). \tag{11.15}$$

Proof. To prove (11.14) we compute

$$\begin{aligned}
\left(F^*(u\omega)\right)_p &= dF_p^*\left((u\omega)_{F(p)}\right) && \text{(by (11.13))}\\
&= dF_p^*\left(u\big(F(p)\big)\omega_{F(p)}\right) && \text{(by (11.8))}\\
&= u\big(F(p)\big)dF_p^*\left(\omega_{F(p)}\right) && \text{(by linearity of } dF_p^*)\\
&= u\big(F(p)\big)(F^*\omega)_p && \text{(by (11.13))}\\
&= \left((u \circ F)F^*\omega\right)_p && \text{(by (11.8))}.
\end{aligned}$$

For (11.15), we let $v \in T_pM$ be arbitrary, and compute

$$\begin{aligned}
(F^*du)_p(v) &= \left(dF_p^*\left(du_{F(p)}\right)\right)(v) && \text{(by (11.13))}\\
&= du_{F(p)}\left(dF_p(v)\right) && \text{(by definition of } dF_p^*)\\
&= dF_p(v)u && \text{(by definition of } du)\\
&= v(u \circ F) && \text{(by definition of } dF_p)\\
&= d(u \circ F)_p(v) && \text{(by definition of } d(u \circ F)).
\end{aligned}$$
□

Proposition 11.26. *Suppose $F: M \to N$ is a smooth map between smooth manifolds with or without boundary, and let ω be a covector field on N. Then $F^*\omega$ is a (continuous) covector field on M. If ω is smooth, then so is $F^*\omega$.*

Proof. Let $p \in M$ be arbitrary, and choose smooth coordinates $\left(y^j\right)$ for N in a neighborhood V of $F(p)$. Let $U = F^{-1}(V)$, which is a neighborhood of p. Writing ω in coordinates as $\omega = \omega_j \, dy^j$ for continuous functions ω_j on V and using Proposition 11.25 twice (applied to $F|_U$), we have the following computation in U:

$$F^*\omega = F^*\left(\omega_j \, dy^j\right) = (\omega_j \circ F) F^* \, dy^j = (\omega_j \circ F) d\left(y^j \circ F\right). \tag{11.16}$$

This expression is continuous, and is smooth if ω is smooth. □

Formula (11.16) for the pullback of a covector field can also be written in the following way:

$$F^*\omega = \left(\omega_j \circ F\right) d\left(y^j \circ F\right) = \left(\omega_j \circ F\right) dF^j, \tag{11.17}$$

where F^j is the jth component function of F in these coordinates. Using either of these formulas, the computation of pullbacks in coordinates is exceedingly simple, as the next example shows.

Example 11.27. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by

$$(u, v) = F(x, y, z) = \left(x^2 y, y \sin z\right),$$

and let $\omega \in \mathfrak{X}^*\left(\mathbb{R}^2\right)$ be the covector field

$$\omega = u \, dv + v \, du.$$

According to (11.16), the pullback $F^*\omega$ is given by

$$\begin{aligned} F^*\omega &= (u \circ F) d(v \circ F) + (v \circ F) d(u \circ F) \\ &= \left(x^2 y\right) d(y \sin z) + (y \sin z) d\left(x^2 y\right) \\ &= x^2 y(\sin z \, dy + y \cos z \, dz) + y \sin z\left(2xy \, dx + x^2 \, dy\right) \\ &= 2xy^2 \sin z \, dx + 2x^2 y \sin z \, dy + x^2 y^2 \cos z \, dz. \end{aligned}$$ //

In other words, to compute $F^*\omega$, all you need to do is substitute the component functions of F for the coordinate functions of N everywhere they appear in ω!

This also yields an easy way to remember the transformation law for a covector field under a change of coordinates. Again, an example will convey the idea better than a general formula.

Example 11.28. Let (r, θ) be polar coordinates on, say, the right half-plane $H = \{(x, y) : x > 0\}$. We can think of the change of coordinates $(x, y) = (r \cos\theta, r \sin\theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) for the codomain. Then the pullback formula (11.17) tells us that we can compute the polar coordinate expression for a covector field simply by substituting $x = r\cos\theta$, $y = r \sin\theta$ and expanding. For example,

$$\begin{aligned} x \, dy - y \, dx &= \mathrm{Id}^*(x \, dy - y \, dx) \\ &= (r\cos\theta) d(r \sin\theta) - (r \sin\theta) d(r \cos\theta) \end{aligned}$$

$$= (r\cos\theta)(\sin\theta\,dr + r\cos\theta\,d\theta) - (r\sin\theta)(\cos\theta\,dr - r\sin\theta\,d\theta)$$
$$= (r\cos\theta\sin\theta - r\sin\theta\cos\theta)\,dr + \left(r^2\cos^2\theta + r^2\sin^2\theta\right)d\theta$$
$$= r^2\,d\theta. \qquad //$$

Restricting Covector Fields to Submanifolds

In Chapter 8, we considered the conditions under which a vector field restricts to a submanifold. The restriction of covector fields to submanifolds is much simpler.

Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed submanifold with or without boundary, and $\iota\colon S \hookrightarrow M$ is the inclusion map. If ω is any smooth covector field on M, the pullback by ι yields a smooth covector field $\iota^*\omega$ on S. To see what this means, let $v \in T_pS$ be arbitrary, and compute

$$(\iota^*\omega)_p(v) = \omega_p\left(d\iota_p(v)\right) = \omega_p(v),$$

since $d\iota_p\colon T_pS \to T_pM$ is just the inclusion map, under our usual identification of T_pS with a subspace of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called the ***restriction of ω to S***. Be warned, however, that $\iota^*\omega$ might equal zero at a given point of S, even though *considered as a covector field on M*, ω might not vanish there. An example will help to clarify this distinction.

Example 11.29. Let $\omega = dy$ on $\mathbb{R}^2$, and let S be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is nonzero everywhere, because one of its component functions is always 1. However, the restriction $\iota^*\omega$ is identically zero, because y vanishes identically on S:

$$\iota^*\omega = \iota^*\,dy = d(y \circ \iota) = 0. \qquad //$$

To distinguish the two ways in which we might interpret the statement "ω vanishes on S," one usually says that ***ω vanishes along S*** or ***vanishes at points of S*** if $\omega_p = 0$ for every point $p \in S$. The weaker condition that $\iota^*\omega = 0$ is expressed by saying that ***the restriction of ω to S vanishes***, or ***the pullback of ω to S vanishes***.

► **Exercise 11.30.** Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an immersed submanifold with or without boundary. For any $f \in C^\infty(M)$, show that $d\left(f|_S\right) = \iota^*(df)$. Conclude that the pullback of df to S is zero if and only if f is constant on each component of S.

Line Integrals

Another important application of covector fields is to make coordinate-independent sense of the notion of a line integral.

We begin with the simplest case: an interval in the real line. Suppose $[a,b] \subseteq \mathbb{R}$ is a compact interval, and ω is a smooth covector field on $[a,b]$. (This means that the component function of ω admits a smooth extension to some neighborhood of $[a,b]$.) If we let t denote the standard coordinate on $\mathbb{R}$, then ω can be written $\omega_t = f(t)\,dt$ for some smooth function $f\colon [a,b] \to \mathbb{R}$. The similarity between this and the usual notation $\int f(t)\,dt$ for an integral suggests that there might be a connection between covector fields and integrals, and indeed there is. We define the ***integral of ω over $[a,b]$*** to be

$$\int_{[a,b]} \omega = \int_a^b f(t)\,dt.$$

The next proposition should convince you that this is more than just a trick of notation.

Proposition 11.31 (Diffeomorphism Invariance of the Integral). *Let ω be a smooth covector field on the compact interval $[a,b] \subseteq \mathbb{R}$. If $\varphi\colon [c,d] \to [a,b]$ is an increasing diffeomorphism (meaning that $t_1 < t_2$ implies $\varphi(t_1) < \varphi(t_2)$), then*

$$\int_{[c,d]} \varphi^*\omega = \int_{[a,b]} \omega.$$

Proof. If we let s denote the standard coordinate on $[c,d]$ and t that on $[a,b]$, then (11.17) shows that the pullback $\varphi^*\omega$ has the coordinate expression $(\varphi^*\omega)_s = f\big(\varphi(s)\big)\varphi'(s)\,ds$. Inserting this into the definition of the line integral and using the change of variables formula for ordinary integrals, we obtain

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f\big(\varphi(s)\big)\varphi'(s)\,ds = \int_a^b f(t)\,dt = \int_{[a,b]} \omega. \qquad \square$$

► **Exercise 11.32.** Prove that if $\varphi\colon [c,d] \to [a,b]$ is a decreasing diffeomorphism, then

$$\int_{[c,d]} \varphi^*\omega = -\int_{[a,b]} \omega.$$

Now let M be a smooth manifold with or without boundary. By a ***curve segment in M*** we mean a continuous curve $\gamma\colon [a,b] \to M$ whose domain is a compact interval. It is a ***smooth curve segment*** if it is smooth when $[a,b]$ is considered as a manifold with boundary (or, equivalently, if γ has an extension to a smooth curve defined in a neighborhood of each endpoint). It is a ***piecewise smooth curve segment*** if there exists a finite partition $a = a_0 < a_1 < \cdots < a_k = b$ of $[a,b]$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for each i (Fig. 11.4). Continuity of γ means that $\gamma(t)$ approaches the same value as t approaches any of the points a_i (other than a_0 or a_k) from the left or the right. Smoothness of γ on each subinterval means that γ has one-sided velocity vectors at each such a_i when approaching from the left or the right, but these one-sided velocities need not be equal.

Proposition 11.33. *If M is a connected smooth manifold with or without boundary, any two points of M can be joined by a piecewise smooth curve segment.*

Proof. Let p be an arbitrary point of M, and define a subset $\mathcal{C} \subseteq M$ by

$$\mathcal{C} = \{q \in M : \text{there is a piecewise smooth curve segment in } M \text{ from } p \text{ to } q\}.$$

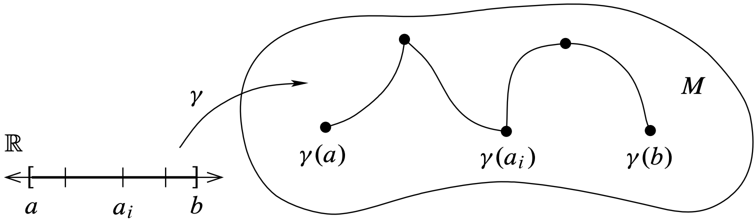

Fig. 11.4 A piecewise smooth curve segment

Clearly, $p \in \mathcal{C}$, so $\mathcal{C}$ is nonempty. To show that $\mathcal{C} = M$, we need to show that it is open and closed in M.

Let $q \in \mathcal{C}$ be arbitrary, which means that there is a piecewise smooth curve segment γ going from p to q. Let U be a smooth coordinate ball (or half-ball) centered at q. If q' is any point in U, then it is easy to construct a piecewise smooth curve segment from p to q' by first following γ from p to q, and then following a straight-line path in coordinates from q to q'. Thus $U \subseteq \mathcal{C}$, which shows that $\mathcal{C}$ is open in M. On the other hand, if $q \in \partial\mathcal{C}$, let U be a smooth coordinate ball or half-ball around q as above. The fact that q is a boundary point of $\mathcal{C}$ means that there is some point $q' \in \mathcal{C} \cap U$. In this case, we can construct a piecewise smooth curve from p to q by first following one from p to q' and then following a straight-line path in coordinates from q' to q. This shows that $q \in \mathcal{C}$, so $\mathcal{C}$ is also closed. □

If $\gamma\colon [a,b] \to M$ is a smooth curve segment and ω is a smooth covector field on M, we define the ***line integral of ω over γ*** to be the real number

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega.$$

Because $\gamma^*\omega$ is a smooth covector field on $[a,b]$, this definition makes sense. More generally, if γ is piecewise smooth, we define

$$\int_\gamma \omega = \sum_{i=1}^{k} \int_{[a_{i-1},a_i]} \gamma^*\omega,$$

where $[a_{i-1},a_i]$, $i = 1,\dots,k$, are subintervals on which γ is smooth.

Proposition 11.34 (Properties of Line Integrals). *Let M be a smooth manifold with or without boundary. Suppose $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, and $\omega, \omega_1, \omega_2 \in \mathfrak{X}^*(M)$.*

(a) *For any $c_1, c_2 \in \mathbb{R}$,*

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) *If γ is a constant map, then $\int_\gamma \omega = 0$.*

(c) *If* $\gamma_1 = \gamma|_{[a,c]}$ *and* $\gamma_2 = \gamma|_{[c,b]}$ *with* $a < c < b$, *then*

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.$$

(d) *If* $F\colon M \to N$ *is any smooth map and* $\eta \in \mathfrak{X}^*(N)$, *then*

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

► **Exercise 11.35.** Prove Proposition 11.34.

Example 11.36. Let $M = \mathbb{R}^2 \smallsetminus \{0\}$, let ω be the covector field on M given by

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2},$$

and let $\gamma\colon [0, 2\pi] \to M$ be the curve segment defined by $\gamma(t) = (\cos t, \sin t)$. Since $\gamma^*\omega$ can be computed by substituting $x = \cos t$ and $y = \sin t$ everywhere in the formula for ω, we find that

$$\int_\gamma \omega = \int_{[0,2\pi]} \frac{\cos t(\cos t\,dt) - \sin t(-\sin t\,dt)}{\sin^2 t + \cos^2 t} = \int_0^{2\pi} dt = 2\pi. \qquad //$$

One of the most significant features of line integrals is that they are independent of parametrization, in a sense we now make precise. If $\gamma\colon [a,b] \to M$ and $\widetilde{\gamma}\colon [c,d] \to M$ are piecewise smooth curve segments, we say that $\widetilde{\gamma}$ is a ***reparametrization of γ*** if $\widetilde{\gamma} = \gamma \circ \varphi$ for some diffeomorphism $\varphi\colon [c,d] \to [a,b]$. If φ is an increasing function, we say that $\widetilde{\gamma}$ is a ***forward reparametrization***, and if φ is decreasing, it is a ***backward reparametrization***. (More generally, with obvious modifications one can allow φ to be piecewise smooth.)

Proposition 11.37 (Parameter Independence of Line Integrals). *Suppose* M *is a smooth manifold with or without boundary,* $\omega \in \mathfrak{X}^*(M)$, *and* γ *is a piecewise smooth curve segment in* M. *For any reparametrization* $\widetilde{\gamma}$ *of* γ, *we have*

$$\int_{\widetilde{\gamma}} \omega = \begin{cases} \displaystyle\int_\gamma \omega & \text{if } \widetilde{\gamma} \text{ is a forward reparametrization,} \\ \displaystyle -\int_\gamma \omega & \text{if } \widetilde{\gamma} \text{ is a backward reparametrization.} \end{cases}$$

Proof. First assume that $\gamma\colon [a,b] \to M$ is smooth, and suppose $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi\colon [c,d] \to [a,b]$ is an increasing diffeomorphism. Then Proposition 11.31 implies

$$\int_{\widetilde{\gamma}} \omega = \int_{[c,d]} (\gamma\circ\varphi)^*\omega = \int_{[c,d]} \varphi^*\gamma^*\omega = \int_{[a,b]} \gamma^*\omega = \int_\gamma \omega.$$

When φ is decreasing, the analogous result follows from Exercise 11.32. If γ is only piecewise smooth, the result follows simply by applying the preceding argument on each subinterval where γ is smooth. □

The next proposition gives a useful alternative expression for a line integral.

Proposition 11.38. *If $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment, the line integral of ω over γ can also be expressed as the ordinary integral*

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\,dt. \tag{11.18}$$

Proof. First suppose that γ is smooth and that its image is contained in the domain of a single smooth chart. Writing the coordinate representations of γ and ω as $\big(\gamma^1(t),\dots,\gamma^n(t)\big)$ and $\omega_i\,dx^i$, respectively, we have

$$\omega_{\gamma(t)}\left(\gamma'(t)\right) = \omega_i\left(\gamma(t)\right)dx^i\left(\gamma'(t)\right) = \omega_i\left(\gamma(t)\right)\dot\gamma^i(t).$$

Combining this with the coordinate formula (11.17) for the pullback, we obtain

$$(\gamma^*\omega)_t = (\omega_i \circ \gamma)(t)\,d\left(\gamma^i\right)_t = \omega_i\big(\gamma(t)\big)\,\dot\gamma^i(t)\,dt = \omega_{\gamma(t)}\big(\gamma'(t)\big)\,dt.$$

Therefore, by definition of the line integral,

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = \int_a^b \omega_{\gamma(t)}\big(\gamma'(t)\big)\,dt.$$

If γ is an arbitrary smooth curve segment, by compactness there exists a finite partition $a = a_0 < a_1 < \dots < a_k = b$ of $[a,b]$ such that $\gamma\big([a_{i-1},a_i]\big)$ is contained in the domain of a single smooth chart for each $i = 1,\dots,k$, so we can apply the computation above on each such subinterval. Finally, if γ is only piecewise smooth, we simply apply the same argument on each subinterval on which γ is smooth. $\square$

There is one special case in which a line integral is trivial to compute: the line integral of a differential.

Theorem 11.39 (Fundamental Theorem for Line Integrals). *Let M be a smooth manifold with or without boundary. Suppose f is a smooth real-valued function on M and $\gamma\colon [a,b] \to M$ is a piecewise smooth curve segment in M. Then*

$$\int_\gamma df = f\big(\gamma(b)\big) - f\big(\gamma(a)\big).$$

Proof. Suppose first that γ is smooth. By Propositions 11.23 and 11.38,

$$\int_\gamma df = \int_a^b df_{\gamma(t)}\big(\gamma'(t)\big)\,dt = \int_a^b (f\circ\gamma)'(t)\,dt.$$

By the fundamental theorem of calculus, this is equal to $f\circ\gamma(b) - f\circ\gamma(a)$.

If γ is merely piecewise smooth, let $a = a_0 < \dots < a_k = b$ be the endpoints of the subintervals on which γ is smooth. Applying the above argument on each subinterval and summing, we find that

$$\int_\gamma df = \sum_{i=1}^k \Big(f\big(\gamma(a_i)\big) - f\big(\gamma(a_{i-1})\big)\Big) = f\big(\gamma(b)\big) - f\big(\gamma(a)\big),$$

because the contributions from all the interior points cancel. $\square$

Conservative Covector Fields

Theorem 11.39 shows that the line integral of any covector field that can be written as the differential of a smooth function can be computed easily once the smooth function is known. For this reason, there is a special term for covector fields with this property. A smooth covector field ω on a smooth manifold M with or without boundary is said to be ***exact*** (or an ***exact differential***) on M if there is a function $f \in C^\infty(M)$ such that $\omega = df$. In this case, the function f is called a ***potential for*** $\boldsymbol{\omega}$. The potential is not uniquely determined, but by Proposition 11.22, the difference between any two potentials for ω must be constant on each component of M.

Because exact differentials are so easy to integrate, it is important to develop criteria for deciding whether a covector field is exact. Theorem 11.39 provides an important clue. It shows that the line integral of an exact covector field depends only on the endpoints $p = \gamma(a)$ and $q = \gamma(b)$: any other curve segment from p to q would give the same value for the line integral. In particular, if γ is a ***closed curve segment***, meaning that $\gamma(a) = \gamma(b)$, then the integral of df over γ is zero.

We say that a smooth covector field ω is ***conservative*** if the line integral of ω over every piecewise smooth closed curve segment is zero. This terminology comes from physics, where a force field is called conservative if the change in energy caused by the force acting along any closed path is zero ("energy is conserved"). (In elementary physics, force fields are usually thought of as vector fields rather than covector fields; see Problem 11-15 for the connection.)

Conservative covector fields can also be characterized by path independence.

Proposition 11.40. *A smooth covector field ω is conservative if and only if its line integrals are path-independent, in the sense that $\int_\gamma \omega = \int_{\tilde\gamma} \omega$ whenever γ and $\tilde\gamma$ are piecewise smooth curve segments with the same starting and ending points.*

► **Exercise 11.41.** Prove Proposition 11.40. [Remark: this would be harder to prove if we defined conservative fields in terms of smooth curves instead of piecewise smooth ones.]

Theorem 11.42. *Let M be a smooth manifold with or without boundary. A smooth covector field on M is conservative if and only if it is exact.*

Proof. If $\omega \in \mathfrak{X}^*(M)$ is exact, Theorem 11.39 shows that it is conservative, so we need only prove the converse. Suppose ω is conservative, and assume for the moment that M is connected. Because the line integrals of ω are path-independent, we can adopt the following notation: for any points $p, q \in M$, we use the notation $\int_p^q \omega$ to denote the value of any line integral of the form $\int_\gamma \omega$, where γ is a piecewise smooth curve segment from p to q. Because a backward reparametrization of a path from p to q is a path from q to p, Proposition 11.37 implies $\int_q^p \omega = -\int_p^q \omega$; and for any three points $p_1, p_2, p_3 \in M$, Proposition 11.34(c) implies that

$$\int_{p_1}^{p_2} \omega + \int_{p_2}^{p_3} \omega = \int_{p_1}^{p_3} \omega. \tag{11.19}$$

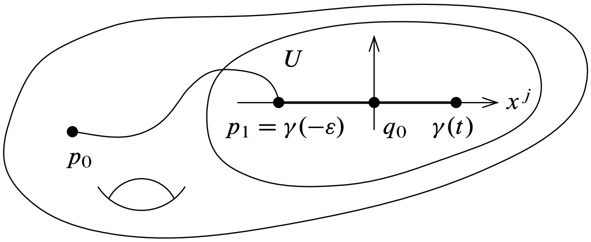

Fig. 11.5 Proof that a conservative covector field is exact

Now choose any base point $p_0 \in M$, and define a function $f\colon M \to \mathbb{R}$ by $f(q) = \int_{p_0}^{q} \omega$. We need to show that f is smooth and $df = \omega$. To accomplish this, let $q_0 \in M$ be arbitrary, let $\left(U, \left(x^i\right)\right)$ be a smooth chart centered at q_0, and write the coordinate representation of ω in U as $\omega = \omega_i \, dx^i$. We need to show that

$$\frac{\partial f}{\partial x^j}(q_0) = \omega_j(q_0)$$

for $j = 1, \dots, n$, which implies $df_{q_0} = \omega_{q_0}$.

First suppose $q_0 \in \operatorname{Int} M$. Fix j, and let $\gamma\colon [-\varepsilon, \varepsilon] \to U$ be the smooth curve segment defined in coordinates by $\gamma(t) = (0, \dots, t, \dots, 0)$, with t in the jth place, and with ε chosen small enough that $\gamma[-\varepsilon, \varepsilon] \subseteq U$ (Fig. 11.5). Let $p_1 = \gamma(-\varepsilon)$, and define a new function $\widetilde{f}\colon M \to \mathbb{R}$ by $\widetilde{f}(q) = \int_{p_1}^{q} \omega$. Note that (11.19) implies that for all $q \in M$,

$$f(q) - \widetilde{f}(q) = \int_{p_0}^{q} \omega - \int_{p_1}^{q} \omega = \int_{p_0}^{q} \omega + \int_{q}^{p_1} \omega = \int_{p_0}^{p_1} \omega,$$

which does not depend on q. Thus $\widetilde{f}$ and f differ by a constant, so it suffices to show that $\partial \widetilde{f} / \partial x^j (q_0) = \omega_j(q_0)$.

Now $\gamma'(t) = \partial/\partial x^j|_{\gamma(t)}$ by construction, so

$$\omega_{\gamma(t)}\big(\gamma'(t)\big) = \omega_i\big(\gamma(t)\big) dx^i \left(\left. \frac{\partial}{\partial x^j} \right|_{\gamma(t)} \right) = \omega_j\big(\gamma(t)\big).$$

Since the restriction of γ to $[-\varepsilon, t]$ is a smooth curve from p_1 to $\gamma(t)$, we have

$$\widetilde{f} \circ \gamma(t) = \int_{p_1}^{\gamma(t)} \omega = \int_{-\varepsilon}^{t} \omega_{\gamma(s)}\big(\gamma'(s)\big) \, ds = \int_{-\varepsilon}^{t} \omega_j\big(\gamma(s)\big) \, ds.$$

Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \frac{\partial \widetilde{f}}{\partial x^j}(q_0) &= \gamma'(0) \widetilde{f} = \left. \frac{d}{dt} \right|_{t=0} \widetilde{f} \circ \gamma(t) \\ &= \left. \frac{d}{dt} \right|_{t=0} \int_{-\varepsilon}^{t} \omega_j\big(\gamma(s)\big) \, ds = \omega_j\big(\gamma(0)\big) = \omega_j(q_0). \end{aligned}$$

This shows that $df_{q_0} = \omega_{q_0}$ when $q_0 \in \operatorname{Int} M$.

For $q_0 \in \partial M$, the chart $\left(U, \left(x^i\right)\right)$ is a boundary chart centered at q_0. The proof above shows that $\partial f / \partial x^j (q_0) = \omega_j(q_0)$ except in the case $j = n = \dim M$; but that case requires a special argument because x^n takes on only nonnegative values in a boundary chart. In that case we simply set $p_1 = \gamma(0)$ instead of $p_1 = \gamma(-\varepsilon)$, and proceed as before. Then the same argument shows that $df_{q_0} = \omega_{q_0}$ in this case as well. This completes the proof that $df = \omega$ when M is connected. Since the component functions of ω are smooth and equal to the partial derivatives of f in coordinates, this also shows that f is smooth.

Finally, if M is not connected, let $\{M_i\}$ be the components of M. The argument above shows that for each i there is a function $f_i \in C^\infty(M_i)$ such that $df_i = \omega$ on M_i. Letting $f\colon M \to \mathbb{R}$ be the function that is equal to f_i on M_i for each i, we have $df = \omega$, thus completing the proof. □

It would be nice if every smooth covector field were exact, for then the evaluation of any line integral would just be a matter of finding a potential function and evaluating it at the endpoints, a process analogous to evaluating an ordinary integral by finding an indefinite integral (also called a *primitive* or *antiderivative*). However, this is too much to hope for.

Example 11.43. The covector field ω of Example 11.36 cannot be exact on $\mathbb{R}^2 \smallsetminus \{0\}$, because it is not conservative: the computation in that example showed that $\int_\gamma \omega = 2\pi \neq 0$, where γ is the unit circle traversed counterclockwise. //

Because exactness has such important consequences for the evaluation of line integrals, we would like to have an easy way to check whether a given covector field is exact. Fortunately, there is a very simple necessary condition, which follows from the fact that partial derivatives of smooth functions can be taken in any order.

To see what this condition is, suppose $\omega \in \mathfrak{X}^*(M)$ is exact. Let f be any potential function for ω, and let $\left(U, \left(x^i\right)\right)$ be any smooth chart on M. Because f is smooth, it satisfies the following identity on U:

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}. \tag{11.20}$$

Writing $\omega = \omega_i dx^i$ in coordinates, we see that $\omega = df$ is equivalent to $\omega_i = \partial f / \partial x^i$. Substituting this into (11.20), we find that the component functions of ω satisfy the following identity for each pair of indices i and j:

$$\frac{\partial \omega_j}{\partial x^i} = \frac{\partial \omega_i}{\partial x^j}. \tag{11.21}$$

We say that a smooth covector field ω is ***closed*** if its components in every smooth chart satisfy (11.21). The following proposition summarizes the computation above.

Proposition 11.44. *Every exact covector field is closed.* □

One technical difficulty in checking directly from the definition that a covector field is closed is that it would require checking that (11.21) holds in *every* coordinate chart. The next proposition gives an alternative characterization of closed covector fields that is coordinate independent, and incidentally shows that it suffices to check (11.21) in *some* coordinate chart around each point.

Proposition 11.45. *Let ω be a smooth covector field on a smooth manifold M with or without boundary. The following are equivalent*:

(a) *ω is closed.*
(b) *ω satisfies* (11.21) *in some smooth chart around every point.*
(c) *For any open subset $U \subseteq M$ and smooth vector fields $X, Y \in \mathfrak{X}(U)$,*

$$X\big(\omega(Y)\big) - Y\big(\omega(X)\big) = \omega\big([X,Y]\big). \tag{11.22}$$

Proof. We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a). The implication (a) $\Rightarrow$ (b) is immediate from the definition of closed covector fields.

To prove (b) $\Rightarrow$ (c), assume (b) holds, and suppose $U \subseteq M$ and $X, Y \in \mathfrak{X}(U)$ as in the statement of (c). It suffices to verify that (11.22) holds in a neighborhood of each point of U. In any coordinate chart $\big(V,\big(x^i\big)\big)$ with $V \subseteq U$, we can write $\omega = \omega_i\, dx^i$, $X = X^j \partial/\partial x^j$, and $Y = Y^k \partial/\partial x^k$, and compute

$$X\big(\omega(Y)\big) = X\big(\omega_i Y^i\big) = Y^i X\omega_i + \omega_i XY^i = Y^i X^j \frac{\partial \omega_i}{\partial x^j} + \omega_i XY^i.$$

If we repeat the same computation with X and Y reversed and subtract, the terms involving derivatives of ω_i cancel by virtue of (11.21). Thus we get

$$X\big(\omega(Y)\big) - Y\big(\omega(X)\big) = \omega_i\big(XY^i - YX^i\big).$$

Formula (8.9) shows that this last expression is equal to $\omega\big([X,Y]\big)$.

Finally, if ω satisfies (c), in any coordinate chart we can apply (11.22) with $X = \partial/\partial x^i$ and $Y = \partial/\partial x^j$, noting that $[X,Y] = 0$ in that case, to obtain (11.21). □

One consequence of this proposition is that closedness can be easily checked using criterion (b), so many covector fields can be shown quickly not to be exact because they are not closed. Another is the following corollary.

Corollary 11.46. *Suppose $F\colon M \to N$ is a local diffeomorphism. Then the pullback $F^*\colon \mathfrak{X}^*(N) \to \mathfrak{X}^*(M)$ takes closed covector fields to closed covector fields, and exact ones to exact ones.*

Proof. The result for exact covector fields follows immediately from (11.15). For closed covector fields, if (U,φ) is any smooth chart for N, then $\varphi \circ F$ is a smooth chart for M in a neighborhood of each point of $F^{-1}(U)$. In these coordinates, the coordinate representation of F is the identity, so if ω satisfies (11.21) in U, then $F^*\omega$ satisfies (11.21) in $F^{-1}(U)$. □

Example 11.47. Consider the following covector field on $\mathbb{R}^2$:

$$\omega = y\cos xy\, dx + x\cos xy\, dy.$$

It is easy to check that

$$\frac{\partial(y\cos xy)}{\partial y} = \frac{\partial(x\cos xy)}{\partial x} = \cos xy - xy\sin xy,$$

so ω is closed. In fact, you might guess that $\omega = d(\sin xy)$. On the other hand, the covector field

$$\eta = x \cos xy\, dx + y \cos xy\, dy$$

is not closed, because

$$\frac{\partial(x \cos xy)}{\partial y} = -x^2 \sin xy, \quad \frac{\partial(y \cos xy)}{\partial x} = -y^2 \sin xy.$$

Thus η is not exact. //

The question then naturally arises whether the converse of Proposition 11.44 is true: Is every closed covector field exact? The answer is *almost* yes, but there is an important restriction. It turns out that the answer to the question depends in a subtle way on the shape of the domain, as the next example illustrates.

Example 11.48. Look once again at the covector field ω of Example 11.36. A straightforward computation shows that ω is closed; but as we observed above, it is not exact on $\mathbb{R}^2 \smallsetminus \{0\}$. On the other hand, if we restrict the domain to the right half-plane $U = \{(x, y) : x > 0\}$, a computation shows that $\omega = d\left(\tan^{-1} y/x\right)$ there. This can be seen more clearly in polar coordinates, where $\omega = d\theta$. The problem, of course, is that there is no smooth (or even continuous) angle function on all of $\mathbb{R}^2 \smallsetminus \{0\}$, which is a consequence of the "hole" in the center. //

This last example illustrates a key principle: the question of whether a particular closed covector field is exact is a global one, depending on the shape of the domain in question. This observation is the starting point for *de Rham cohomology*, which expresses a deep relationship between smooth structures and topology. We will define de Rham cohomology and study this relationship in Chapter 17, but for now we can prove the following result. If V is a finite-dimensional vector space, a subset $U \subseteq V$ is said to be ***star-shaped*** if there is a point $c \in U$ such that for every $x \in U$, the line segment from c to x is entirely contained in U (Fig. 11.6). For example, every convex subset is star-shaped.

Theorem 11.49 (Poincaré Lemma for Covector Fields). *If U is a star-shaped open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, then every closed covector field on U is exact.*

Proof. Suppose U is star-shaped with respect to $c \in U$, and let $\omega = \omega_i dx^i$ be a closed covector field on U. As in the proof of Theorem 11.42, we will construct a potential function for ω by integrating along smooth curve segments from c. However, in this case we do not know a priori that the line integrals are path-independent, so we must integrate along specific paths.

Because diffeomorphisms take closed forms to closed forms and exact ones to exact ones, we can apply a translation to U to arrange that $c = 0$. For any point $x \in U$, let $\gamma_x \colon [0, 1] \to U$ denote the line segment from 0 to x, parametrized as $\gamma_x(t) = tx$. The hypothesis guarantees that the image of γ_x lies entirely in U for each $x \in U$. Define a function $f : U \to \mathbb{R}$ by

$$f(x) = \int_{\gamma_x} \omega. \tag{11.23}$$

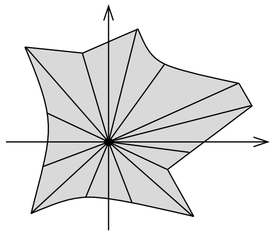

Fig. 11.6 A star-shaped subset of $\mathbb{R}^2$

We need to show that f is a potential for ω, or equivalently that $\partial f/\partial x^j = \omega_j$ for $j = 1, \dots, n$. To begin, we compute

$$f(x) = \int_0^1 \omega_{\gamma_x(t)}\big(\gamma_x'(t)\big)\,dt = \int_0^1 \omega_i(tx)x^i\,dt. \tag{11.24}$$

(The summation convention is in effect.) To compute the partial derivatives of f, we note that the integrand is smooth in all variables, so it is permissible to differentiate under the integral sign to obtain

$$\frac{\partial f}{\partial x^j}(x) = \int_0^1 \left(t\frac{\partial \omega_i}{\partial x^j}(tx)x^i + \omega_j(tx)\right) dt.$$

Because ω is closed, this reduces to

$$\begin{aligned}\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \left(t\frac{\partial \omega_j}{\partial x^i}(tx)x^i + \omega_j(tx)\right) dt\\ &= \int_0^1 \frac{d}{dt}\big(t\omega_j(tx)\big)\,dt = \Big[t\omega_j(tx)\Big]_{t=0}^{t=1} = \omega_j(x).\end{aligned}$$ □

Corollary 11.50 (Local Exactness of Closed Covector Fields). *Let ω be a closed covector field on a smooth manifold M with or without boundary. Then every point of M has a neighborhood on which ω is exact.*

Proof. Let $p \in M$ be arbitrary. The hypothesis implies that ω satisfies (11.21) in some smooth coordinate ball or half-ball (U, φ) containing p. Because balls and half-balls are convex, we can apply Theorem 11.49 to the coordinate representation of ω and conclude that there is a function $f \in C^\infty(U)$ such that $\omega|_U = df$. □

The key to constructing a potential function in Theorem 11.49 is that we can reach every point $x \in M$ by a definite path from c to x, chosen to vary smoothly as x varies. That is what fails in the case of the closed covector field ω on the punctured plane (Example 11.43): because of the hole, it is impossible to choose a smoothly varying family of paths starting at a fixed base point and reaching every point of the domain exactly once. In Chapter 16 we will generalize Theorem 11.49 to show that every closed covector field is exact on any simply connected manifold.

When you actually have to *compute* a potential function for a given covector field that is known to be exact, there is a much simpler procedure that almost always works. It is a straightforward generalization of the method introduced in calculus texts for computing a potential function for a vector field that is known to be a gradient. Rather than describe it in complete generality, we illustrate it with an example.

Example 11.51. Let ω be the following covector field on $\mathbb{R}^3$:

$$\omega = e^{y^2}dx + 2xye^{y^2}dy - 2z\,dz.$$

You can check that ω is closed. For f to be a potential for ω, it must satisfy

$$\frac{\partial f}{\partial x} = e^{y^2}, \quad \frac{\partial f}{\partial y} = 2xye^{y^2}, \quad \frac{\partial f}{\partial z} = -2z. \tag{11.25}$$

Holding y and z fixed and integrating the first equation with respect to x, we obtain

$$f(x,y,z) = \int e^{y^2}dx = xe^{y^2} + C_1(y,z),$$

where the "constant" of integration $C_1(y,z)$ may depend on the choice of (y,z). Now the second equation of (11.25) implies

$$2xye^{y^2} = \frac{\partial}{\partial y}\Big(xe^{y^2} + C_1(y,z)\Big) = 2xye^{y^2} + \frac{\partial C_1}{\partial y},$$

which forces $\partial C_1/\partial y = 0$, so C_1 is actually a function of z only. Finally, the third equation implies

$$-2z = \frac{\partial}{\partial z}\Big(xe^{y^2} + C_1(z)\Big) = \frac{dC_1}{dz},$$

from which we conclude that $C_1(z) = -z^2 + C$, where C is an arbitrary constant. Thus a potential function for ω is given by $f(x,y,z) = xe^{y^2} - z^2$. Any other potential differs from this one by a constant. //

You should convince yourself that the formal procedure we followed in this example is equivalent to choosing an arbitrary base point $c \in \mathbb{R}^3$ and defining $f(x,y,z)$ by integrating ω along a path from c to (x,y,z) consisting of three straight line segments parallel to the axes. This works for any closed covector field on an open rectangle in $\mathbb{R}^n$ (which we know must be exact, because a rectangle is convex). In practice, once a formula is found for f on some open rectangle, the same formula typically works for the entire domain. (This is because most of the covector fields for which one can explicitly compute the integrals are real-analytic, and real-analytic functions are determined by their behavior in any open subset.)

Problems

11-1. (a) Suppose V and W are finite-dimensional vector spaces and $A\colon V \to W$ is any linear map. Show that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ {\scriptstyle \xi_V}\downarrow & & \downarrow{\scriptstyle \xi_W} \\ V^{**} & \xrightarrow[(A^*)^*]{} & W^{**}, \end{array}$$

where ξ_V and ξ_W denote the isomorphisms defined by (11.3) for V and W, respectively.

(b) Show that there does not exist a way to assign to each finite-dimensional vector space V an isomorphism $\beta_V\colon V \to V^*$ such that for every linear map $A\colon V \to W$, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ {\scriptstyle \beta_V}\downarrow & & \downarrow{\scriptstyle \beta_W} \\ V^{*} & \xleftarrow[A^*]{} & W^{*}. \end{array}$$

11-2. Suppose V is an infinite-dimensional real vector space, and let V^* denote the vector space of all linear functionals from V to $\mathbb{R}$. (This is often called the ***algebraic dual space of V*** to distinguish it from the space of continuous linear functionals when V is endowed with a norm or a topology.)

(a) Prove that there is an injective linear map from V to V^*. [Hint: see Exercise B.5.]

(b) Prove that there is no injective linear map from V^* to V. [Hint: if S is a basis for V, prove that there is a linearly independent subset of V^* with the same cardinality as the power set of S.]

(c) Use (a) and (b) to prove there is no isomorphism between V and V^{**}.

(*Used on p. 489.*)

11-3. Let Vec_1 be the category of finite-dimensional vector spaces and linear isomorphisms as in Problem 10-8. Define a functor $\mathcal{F}\colon \mathsf{Vec}_1 \to \mathsf{Vec}_1$ by setting $\mathcal{F}(V) = V^*$ for a vector space V, and $\mathcal{F}(A) = (A^{-1})^*$ for an isomorphism A. Show that $\mathcal{F}$ is a smooth covariant functor, and that for every M, $\mathcal{F}(TM)$ and T^*M are canonically smoothly isomorphic vector bundles.

11-4. Let M be a smooth manifold with or without boundary and p be a point of M. Let $\mathcal{I}_p$ denote the subspace of $C^\infty(M)$ consisting of smooth functions that vanish at p, and let $\mathcal{I}_p^2$ be the subspace of $\mathcal{I}_p$ spanned by functions of the form fg for some $f, g \in \mathcal{I}_p$.

(a) Show that $f \in \mathcal{I}_p^2$ if and only if in any smooth local coordinates, its first-order Taylor polynomial at p is zero. (Because of this, a function in $\mathcal{I}_p^2$ is said to ***vanish to second order***.)

(b) Define a map $\Phi\colon \mathcal{I}_p \to T_p^*M$ by setting $\Phi(f) = df_p$. Show that the restriction of Φ to $\mathcal{I}_p^2$ is zero, and that Φ descends to a vector space isomorphism from $\mathcal{I}_p/\mathcal{I}_p^2$ to T_p^*M.

[Remark: Problem 3-8 showed that tangent vectors at p can be viewed as equivalence classes of smooth curves, which are smooth maps from (subsets of) $\mathbb{R}$ to M. This problem shows that covectors at p can be viewed dually as equivalence classes of smooth functions from M to $\mathbb{R}$: a covector is an equivalence class of smooth functions that vanish at p, with two such functions considered equivalent if they differ by a function that vanishes to second order. In some treatments of smooth manifold theory, T_p^*M is defined first in this way, and then T_pM is defined as the dual space $(\mathcal{I}_p/\mathcal{I}_p^2)^*$.]

11-5. For any smooth manifold M, show that T^*M is a trivial vector bundle if and only if TM is trivial.

11-6. Suppose M is a smooth n-manifold, $p \in M$, and $y^1,\dots,y^k$ are smooth real-valued functions defined on a neighborhood of p in M. Prove the following statements.

(a) If $k = n$ and $\left(dy^1|_p,\dots,dy^n|_p\right)$ is a basis for T_p^*M, then $\left(y^1,\dots,y^n\right)$ are smooth coordinates for M in some neighborhood of p.

(b) If $\left(dy^1|_p,\dots,dy^k|_p\right)$ is a linearly independent k-tuple of covectors and $k < n$, then there are smooth functions $y^{k+1},\dots,y^n$ such that $\left(y^1,\dots,y^n\right)$ are smooth coordinates for M in a neighborhood of p.

(c) If $\left(dy^1|_p,\dots,dy^k|_p\right)$ span T_p^*M, there are indices $i_1,\dots,i_n$ such that $\left(y^{i_1},\dots,y^{i_n}\right)$ are smooth coordinates for M in a neighborhood of p.

(*Used on p. 584.*)

11-7. In the following problems, M and N are smooth manifolds, $F\colon M \to N$ is a smooth map, and $\omega \in \mathfrak{X}^*(N)$. Compute $F^*\omega$ in each case.

(a) $M = N = \mathbb{R}^2$;
$F(s,t) = (st, e^t)$;
$\omega = x\,dy - y\,dx$.

(b) $M = \mathbb{R}^2$ and $N = \mathbb{R}^3$;
$F(\theta,\varphi) = \big((\cos\varphi + 2)\cos\theta,\ (\cos\varphi + 2)\sin\theta,\ \sin\varphi\big)$;
$\omega = z^2\,dx$.

(c) $M = \left\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 < 1\right\}$ and $N = \mathbb{R}^3 \smallsetminus \{0\}$;
$F(s,t) = \left(s, t, \sqrt{1 - s^2 - t^2}\right)$;
$\omega = (1 - x^2 - y^2)\,dz$.

11-8. (a) Suppose $F\colon M \to N$ is a diffeomorphism, and let $dF^*\colon T^*N \to T^*M$ be the map whose restriction to each cotangent space T_q^*N is equal to $dF^*_{F^{-1}(q)}$. Show that dF^* is a smooth bundle homomorphism.

(b) Let Diff_1 be the category whose objects are smooth manifolds, but whose only morphisms are diffeomorphisms; and let VB be the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphisms. Show that the assignment $M \mapsto$

T^*M, $F \mapsto dF^*$ defines a contravariant functor from Diff_1 to VB, called the ***cotangent functor***.

(*Used on p. 592.*)

11-9. Let $f: \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x,y,z) = x^2 + y^2 + z^2$, and let $F: \mathbb{R}^2 \to \mathbb{R}^3$ be the following map (the inverse of the stereographic projection):

$$F(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

Compute F^*df and $d(f \circ F)$ separately, and verify that they are equal.

11-10. In each of the cases below, M is a smooth manifold and $f: M \to \mathbb{R}$ is a smooth function. Compute the coordinate representation for df, and determine the set of all points $p \in M$ at which $df_p = 0$.

(a) $M = \{(x,y) \in \mathbb{R}^2 : x > 0\}$; $f(x,y) = x/(x^2+y^2)$. Use standard coordinates (x,y).

(b) M and f are as in (a); this time use polar coordinates (r,θ).

(c) $M = S^2 \subseteq \mathbb{R}^3$; $f(p) = z(p)$ (the z-coordinate of p as a point in $\mathbb{R}^3$). Use north and south stereographic coordinates (Problem 1-7).

(d) $M = \mathbb{R}^n$; $f(x) = |x|^2$. Use standard coordinates.

11-11. Let M be a smooth manifold, and $C \subseteq M$ be an embedded submanifold. Let $f \in C^\infty(M)$, and suppose $p \in C$ is a point at which f attains a local maximum or minimum value among points in C. Given a smooth local defining function $\Phi: U \to \mathbb{R}^k$ for C on a neighborhood U of p in M, show that there are real numbers $\lambda_1, \dots, \lambda_k$ (called ***Lagrange multipliers***) such that

$$df_p = \lambda_1 d\Phi^1\big|_p + \cdots + \lambda_k d\Phi^k\big|_p.$$

11-12. Show that any two points in a connected smooth manifold can be joined by a smooth curve segment.

11-13. The ***length*** of a smooth curve segment $\gamma: [a,b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral

$$\mathrm{L}(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Show that there is no smooth covector field $\omega \in \mathfrak{X}^*(\mathbb{R}^n)$ with the property that $\int_\gamma \omega = \mathrm{L}(\gamma)$ for every smooth curve γ.

11-14. Consider the following two covector fields on $\mathbb{R}^3$:

$$\omega = -\frac{4z\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2x\,dz}{x^2+1},$$

$$\eta = -\frac{4xz\,dx}{(x^2+1)^2} + \frac{2y\,dy}{y^2+1} + \frac{2\,dz}{x^2+1}.$$

(a) Set up and evaluate the line integral of each covector field along the straight line segment from $(0,0,0)$ to $(1,1,1)$.
(b) Determine whether either of these covector fields is exact.
(c) For each one that is exact, find a potential function and use it to recompute the line integral.

11-15. LINE INTEGRALS OF VECTOR FIELDS: Let X be a smooth vector field on an open subset $U \subseteq \mathbb{R}^n$. Given a piecewise smooth curve segment $\gamma\colon [a,b] \to U$, define the ***line integral of X over γ***, denoted by $\int_\gamma X \cdot ds$, as

$$\int_\gamma X \cdot ds = \int_a^b X_{\gamma(t)} \cdot \gamma'(t)\, dt,$$

where the dot on the right-hand side denotes the Euclidean dot product between tangent vectors at $\gamma(t)$, identified with elements of $\mathbb{R}^n$. A ***conservative vector field*** is one whose line integral around every piecewise smooth closed curve is zero.

(a) Show that X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \operatorname{grad} f$. [Hint: consider the covector field ω defined by $\omega_x(v) = X_x \cdot v$.]
(b) Suppose $n = 3$. Show that if X is conservative, then $\operatorname{curl} X = 0$, where

$$\begin{aligned}\operatorname{curl} X = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right)\frac{\partial}{\partial x^1} + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right)\frac{\partial}{\partial x^2}\\ + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right)\frac{\partial}{\partial x^3}.\end{aligned} \tag{11.26}$$

(c) Show that if $U \subseteq \mathbb{R}^3$ is star-shaped, then X is conservative on U if and only if $\operatorname{curl} X = 0$.

11-16. Let M be a compact manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.

11-17. Let $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \subseteq \mathbb{C}^n$ denote the n-torus. For each $j = 1,\dots,n$, let $\gamma_j\colon [0,1] \to \mathbb{T}^n$ be the curve segment

$$\gamma_j(t) = \left(1,\dots,e^{2\pi i t},\dots,1\right) \quad \left(\text{with } e^{2\pi i t} \text{ in the } j\text{th place}\right).$$

Show that a closed covector field ω on $\mathbb{T}^n$ is exact if and only if $\int_{\gamma_j} \omega = 0$ for $j = 1,\dots,n$. [Hint: consider first $(\varepsilon^n)^*\omega$, where $\varepsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n$ is the smooth covering map $\varepsilon^n\left(x^1,\dots,x^n\right) = \left(e^{2\pi i x^1},\dots,e^{2\pi i x^n}\right)$.]

11-18. This problem shows how to give a rigorous meaning to words like "natural" and "canonical" that are so often used informally in mathematics. Suppose C and D are categories, and $\mathcal{F},\mathcal{G}$ are (covariant or contravariant) functors from C to D. A ***natural transformation*** λ from $\mathcal{F}$ to $\mathcal{G}$ is a rule that assigns to each object $X \in \operatorname{Ob}(\mathsf{C})$ a morphism $\lambda_X \in \operatorname{Hom}_{\mathsf{D}}\left(\mathcal{F}(X),\mathcal{G}(X)\right)$ in such a way that for every pair of objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\lambda_X"'] & \mathcal{F}(Y) \arrow[d, "\lambda_Y"] \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(Y).
\end{tikzcd}$$

(If either $\mathcal{F}$ or $\mathcal{G}$ is contravariant, the corresponding horizontal arrow should be reversed.)

(a) Let $\mathsf{Vec}_{\mathbb{R}}$ denote the category of real vector spaces and linear maps, and let $\mathcal{D}$ be the contravariant functor from $\mathsf{Vec}_{\mathbb{R}}$ to itself that sends each vector space to its dual space and each linear map to its dual map. Show that the assignment $V \mapsto \xi_V$, where $\xi_V : V \to V^{**}$ is the map defined by $\xi_V(v)\omega = \omega(v)$, is a natural transformation from the identity functor of $\mathsf{Vec}_{\mathbb{R}}$ to $\mathcal{D} \circ \mathcal{D}$.

(b) Show that there does not exist a natural transformation from the identity functor of $\mathsf{Vec}_{\mathbb{R}}$ to $\mathcal{D}$.

(c) Let Diff_1 be the category of smooth manifolds and diffeomorphisms and VB the category of smooth vector bundles and smooth bundle homomorphisms, and let $T, T^* : \mathsf{Diff}_1 \to \mathsf{VB}$ be the tangent and cotangent functors, respectively (see Problems 10-3 and 11-8). Show that there does not exist a natural transformation from T to T^*.

(d) Let $\mathfrak{X} : \mathsf{Diff}_1 \to \mathsf{Vec}_{\mathbb{R}}$ be the covariant functor given by $M \mapsto \mathfrak{X}(M)$, $F \mapsto F_*$; and let $\mathfrak{X} \times \mathfrak{X} : \mathsf{Diff}_1 \to \mathsf{Vec}_{\mathbb{R}}$ be the covariant functor given by $M \mapsto \mathfrak{X}(M) \times \mathfrak{X}(M)$, $F \mapsto F_* \times F_*$. Show that the Lie bracket is a natural transformation from $\mathfrak{X} \times \mathfrak{X}$ to $\mathfrak{X}$.

(*Used on pp. 189, 347, 376.*)