Chapter 18
The de Rham Theorem

The topological invariance of the de Rham groups suggests that there should be some purely topological way of computing them. There is indeed, and the connection between the de Rham groups and topology was first proved by Georges de Rham himself in the 1930s. The theorem that bears his name is a major landmark in the development of smooth manifold theory. The purpose of this chapter is to give a proof of this theorem.

In the category of topological spaces, there are a number of functorial ways of associating to each space an algebraic object such as a group or a vector space, so that homeomorphic spaces have isomorphic objects. Most of these measure, in a certain sense, the existence of "holes" in different dimensions. You are already familiar with the simplest such functor: the fundamental group. In the beginning of this chapter, we describe the next most straightforward ones, called the *singular homology groups* and *singular cohomology groups*. Because a complete treatment of singular theory would be far beyond the scope of this book, we can only summarize the basic ideas here. For more details, you can consult a standard textbook on algebraic topology, such as [Hat02], [Bre93], or [Mun84]. (See also [LeeTM, Chap. 13] for a more concise treatment.) After introducing the basic definitions, we prove that singular homology can be computed by restricting attention only to smooth simplices.

At the end of the chapter we turn our attention to the de Rham theorem, which shows that integration of differential forms over smooth simplices induces isomorphisms between the de Rham groups and the singular cohomology groups.

Singular Homology

We begin with a brief summary of singular homology theory. Suppose $v_0, \dots, v_p$ are any $p+1$ points in some Euclidean space $\mathbb{R}^n$. They are said to be ***affinely independent*** (or ***in general position***) if they are not contained in any $(p-1)$-dimensional

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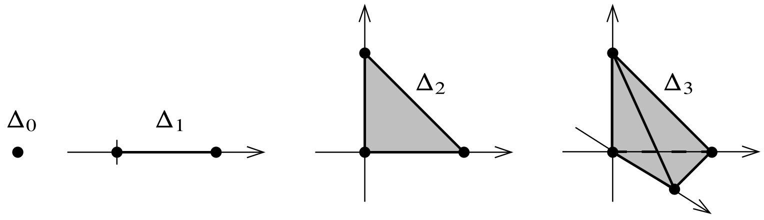

Fig. 18.1 Standard p-simplices for $p = 0, 1, 2, 3$

affine subspace. A ***geometric p-simplex*** is a subset of $\mathbb{R}^n$ of the form

$$\left\{\sum_{i=0}^{p} t_i v_i : 0 \le t_i \le 1 \text{ and } \sum_{i=0}^{p} t_i = 1\right\},$$

for some $p+1$ affinely independent points $\{v_0, \dots, v_p\}$. The integer p (one less than the number of vertices) is called the ***dimension*** of the simplex. The points $v_0, \dots, v_p$ are called its ***vertices***, and the geometric simplex with these vertices is denoted by $[v_0, \dots, v_p]$. It is a compact convex set, in fact the smallest convex set containing $\{v_0, \dots, v_p\}$. The simplices whose vertices are nonempty subsets of $\{v_0, \dots, v_p\}$ are called the ***faces*** of the simplex. The $(p-1)$-dimensional faces are called its ***boundary faces***. There are precisely $p + 1$ boundary faces, obtained by omitting each of the vertices in turn; the ith boundary face $\left[v_0, \dots, \widehat{v_i}, \dots, v_p\right]$ (with v_i omitted) is denoted by $\partial_i [v_0, \dots, v_p]$, and is called the ***face opposite v_i***.

► **Exercise 18.1.** Show that a geometric p-simplex is a p-dimensional smooth manifold with corners smoothly embedded in $\mathbb{R}^n$.

The ***standard p-simplex*** is the simplex $\Delta_p = [e_0, e_1, \dots, e_p] \subseteq \mathbb{R}^p$, where $e_0 = 0$ and e_i is the ith standard basis vector. For example, $\Delta_0 = \{0\}$, $\Delta_1 = [0, 1]$, Δ_2 is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ together with its interior, and Δ_3 is a solid tetrahedron (Fig. 18.1).

Let M be a topological space. A continuous map $\sigma\colon \Delta_p \to M$ is called a ***singular p-simplex in M***. The ***singular chain group of M in degree p***, denoted by $C_p(M)$, is the free abelian group generated by all singular p-simplices in M. An element of this group, called a ***singular p-chain***, is a finite formal linear combination of singular p-simplices in M with integer coefficients.

One special case that arises frequently is that in which the space M is a convex subset of some Euclidean space $\mathbb{R}^m$. In that case, for any ordered $(p + 1)$-tuple of points $(w_0, \dots, w_p)$ in M, not necessarily affinely independent, there is a unique affine map from $\mathbb{R}^p$ to $\mathbb{R}^m$ that takes e_i to w_i for $i = 0, \dots, p$. (The map is easily constructed by first finding a linear map that takes e_i to $w_i - w_0$ for $i = 1, \dots, p$, and then translating by w_0.) The restriction of this affine map to Δ_p is denoted by $A(w_0, \dots, w_p)$, and is called an ***affine singular simplex in M***.

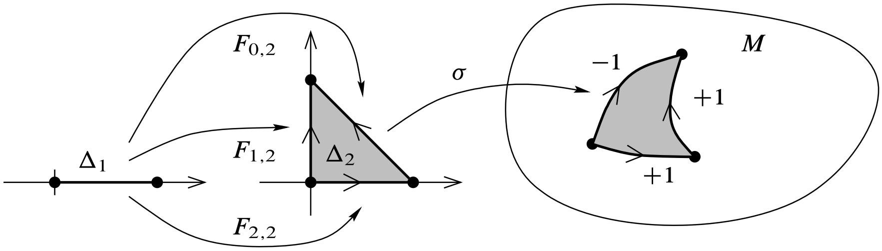

Fig. 18.2 The singular boundary operator

For each $i = 0, \dots, p$, we define the ***i th face map in*** $\boldsymbol{\Delta_p}$ to be the affine singular $(p-1)$-simplex $F_{i,p}\colon \Delta_{p-1} \to \Delta_p$ defined by

$$F_{i,p} = A\left(e_0, \dots, \widehat{e_i}, \dots, e_p\right).$$

It maps Δ_{p-1} homeomorphically onto the boundary face $\partial_i \Delta_p$. Explicitly, it is the unique affine map sending $e_0 \mapsto e_0, \dots, e_{i-1} \mapsto e_{i-1}, e_i \mapsto e_{i+1}, \dots, e_{p-1} \mapsto e_p$.

The ***boundary*** of a singular p-simplex $\sigma\colon \Delta_p \to M$ is the singular $(p-1)$-chain $\partial\sigma$ defined by

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}.$$

For example, if σ is a singular 2-simplex, its boundary is a formal sum of three singular 1-simplices with coefficients ± 1, as indicated in Fig. 18.2. This extends uniquely to a group homomorphism $\partial\colon C_p(M) \to C_{p-1}(M)$, called the ***singular boundary operator***. The basic fact about the boundary operator is the next lemma.

Lemma 18.2. *If c is any singular chain, then* $\partial(\partial c) = 0$.

Sketch of Proof. The starting point is the fact that

$$F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i-1,p-1} \tag{18.1}$$

when $i > j$, which can be verified by following what both compositions do to each of the vertices of Δ_{p-2}. Using this, the proof of the lemma is just a straightforward computation. □

A singular p-chain c is called a ***cycle*** if $\partial c = 0$, and a ***boundary*** if $c = \partial b$ for some singular $(p+1)$-chain b. Let $Z_p(M)$ denote the set of singular p-cycles in M, and $B_p(M)$ the set of singular p-boundaries. Because ∂ is a homomorphism, $Z_p(M)$ and $B_p(M)$ are subgroups of $C_p(M)$, and because $\partial \circ \partial = 0$, they satisfy $B_p(M) \subseteq Z_p(M)$. The ***pth singular homology group*** of M is the quotient group

$$H_p(M) = \frac{Z_p(M)}{B_p(M)}.$$

To put it another way, the sequence of abelian groups and homomorphisms

$$\cdots \to C_{p+1}(M) \xrightarrow{\partial} C_p(M) \xrightarrow{\partial} C_{p-1}(M) \to \cdots$$

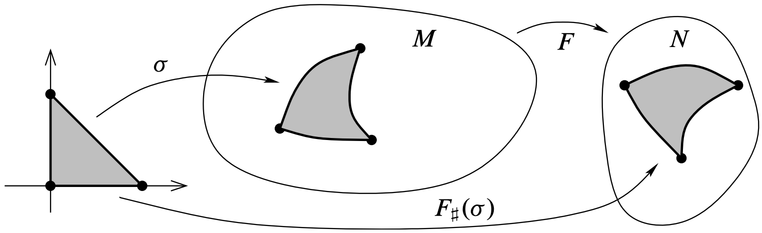

Fig. 18.3 The homology homomorphism induced by a continuous map

is a complex, called the ***singular chain complex***, and $H_p(M)$ is the pth homology group of this complex. The equivalence class in $H_p(M)$ of a singular p-cycle c is called its ***homology class***, and is denoted by $[c]$. We say that two p-cycles are ***homologous*** if they differ by a boundary.

A continuous map $F\colon M \to N$ induces a homomorphism $F_\sharp\colon C_p(M) \to C_p(N)$ on each singular chain group, defined by $F_\sharp(\sigma) = F \circ \sigma$ for any singular simplex σ (Fig. 18.3) and extended linearly to chains. An easy computation shows that $F_\sharp \circ \partial = \partial \circ F_\sharp$, so $F_\sharp$ is a chain map, and therefore induces a homomorphism on the singular homology groups, denoted by $F_*\colon H_p(M) \to H_p(N)$. It is immediate that $(G \circ F)_* = G_* \circ F_*$ and $(\mathrm{Id}_M)_* = \mathrm{Id}_{H_p(M)}$, so pth singular homology defines a covariant functor from the category of topological spaces and continuous maps to the category of abelian groups and homomorphisms. In particular, homeomorphic spaces have isomorphic singular homology groups.

Intuitively, you will not go too far astray if you visualize a singular p-chain in M as representing something like a compact p-dimensional submanifold of M with boundary (although, because there is no requirement that singular chains be smooth, or topological embeddings, or even injective, a chain might not *look* at all like a submanifold; hence the designation "singular"). A closed p-chain, then, is like a compact submanifold without boundary, and it represents the trivial homology class if and only if it is the boundary of a $(p+1)$-chain. Thus a nontrivial element of $H_p(M)$ is rather like a compact p-dimensional submanifold of M that does not bound a compact $(p+1)$-dimensional submanifold, and so must represent some kind of p-dimensional "hole" in M. (See Problem 18-3, which introduces *smooth triangulations* as a way of giving this intuition more substance.)

Proposition 18.3 (Properties of Singular Homology Groups).

(a) *For any one-point space* $\{q\}$, $H_0(\{q\})$ *is the infinite cyclic group generated by the homology class of the unique singular* 0*-simplex mapping* Δ_0 *to* q, *and* $H_p(\{q\}) = 0$ *for all* $p \neq 0$.
(b) *Let* $\{M_j\}$ *be any collection of topological spaces, and let* $M = \coprod_j M_j$. *The inclusion maps* $\iota_j\colon M_j \hookrightarrow M$ *induce an isomorphism* $\bigoplus_j H_p(M_j) \cong H_p(M)$.
(c) *Homotopy equivalent spaces have isomorphic singular homology groups.*

Sketch of Proof. In a one-point space $\{q\}$, there is exactly one singular p-simplex for each p, namely the constant map. The result of part (a) follows from an analysis of the boundary maps. Part (b) is immediate because the maps ι_j already induce an isomorphism on the chain level: $\bigoplus_j C_p(M_j) \cong C_p(M)$.

The main step in the proof of homotopy invariance is the construction for any space M of a linear map $h\colon C_p(M) \to C_{p+1}(M \times I)$ satisfying

$$h \circ \partial + \partial \circ h = (i_1)_\sharp - (i_0)_\sharp, \tag{18.2}$$

where $i_k\colon M \to M \times I$ is the injection $i_k(x) = (x,k)$. From this it follows just as in the proof of Proposition 17.10 that homotopic maps induce the same homology homomorphism, and then in turn that homotopy equivalent spaces have isomorphic singular homology groups. □

In addition to the properties above, singular homology satisfies the following version of the Mayer–Vietoris theorem. Suppose M is a topological space and $U, V \subseteq M$ are open subsets whose union is M. The usual diagram (17.6) of inclusions induces homology homomorphisms:

$$\begin{array}{ccccc} & & H_p(U) & & \\ & {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle k_*} & \\ H_p(U \cap V) & & & & H_p(M). \\ & {\scriptstyle j_*}\searrow & & \nearrow{\scriptstyle l_*} & \\ & & H_p(V) & & \end{array} \tag{18.3}$$

Theorem 18.4 (Mayer–Vietoris for Singular Homology). *Let M be a topological space and let U, V be open subsets of M whose union is M. For each p there is a connecting homomorphism $\partial_*\colon H_p(M) \to H_{p-1}(U \cap V)$ such that the following sequence is exact*:

$$\cdots \xrightarrow{\partial_*} H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) \xrightarrow{\beta} H_p(M) \xrightarrow{\partial_*} H_{p-1}(U \cap V) \xrightarrow{\alpha} \cdots, \tag{18.4}$$

where

$$\alpha[c] = \big(i_*[c], -j_*[c]\big), \qquad \beta\big([c],[c']\big) = k_*[c] + l_*[c'],$$

and $\partial_*[e] = [c]$, provided there exist $f \in C_p(U)$ and $f' \in C_p(V)$ such that $k_\sharp f + l_\sharp f'$ is homologous to e and $(i_\sharp c, -j_\sharp c) = (\partial f, \partial f')$.

Sketch of Proof. The basic idea, of course, is to construct a short exact sequence of complexes and use the zigzag lemma. The hardest part of the proof is showing that every homology class $[e] \in H_p(M)$ can be represented in the form $\beta([c],[c'])$, where c is a singular chain in U and c' is a singular chain in V. This is accomplished by systematically "subdividing" each chain into smaller ones, each of which maps only into U or V, and keeping careful track of the boundary operators. □

Note that the maps α and β in this Mayer–Vietoris sequence can be replaced by

$$\widetilde{\alpha}[c] = \big(i_*[c], j_*[c]\big), \qquad \widetilde{\beta}\big([c],[c']\big) = k_*[c] - l_*[c'],$$

and the same proof goes through. If you consult various algebraic topology texts, you will find both definitions in use. We are using the definition given in the statement of the theorem because it leads to a cohomology exact sequence that is compatible with the Mayer–Vietoris sequence for de Rham cohomology; see the proof of the de Rham theorem below.

Singular Cohomology

In addition to the singular homology groups, for any topological space M and any abelian group G one can define a closely related sequence of groups $H^p(M;G)$ called the ***singular cohomology groups with coefficients in G***. The precise definition is unimportant for our purposes; we are only concerned with the special case $G = \mathbb{R}$, in which case it can be shown that $H^p(M;\mathbb{R})$ is a real vector space that is naturally isomorphic to the space $\operatorname{Hom}\big(H_p(M),\mathbb{R}\big)$ of group homomorphisms from $H_p(M)$ into $\mathbb{R}$. (For simplicity, let us take this as our definition of $H^p(M,\mathbb{R})$.) Any continuous map $F\colon M \to N$ induces a linear map $F^*\colon H^p(N;\mathbb{R}) \to H^p(M;\mathbb{R})$, defined by $(F^*\gamma)[c] = \gamma(F_*[c])$ for each $\gamma \in H^p(N;\mathbb{R}) \cong \operatorname{Hom}\big(H_p(N),\mathbb{R}\big)$ and each singular p-chain c in M. The functorial properties of F_* carry over to cohomology: $(G \circ F)^* = F^* \circ G^*$ and $(\mathrm{Id}_M)^* = \mathrm{Id}_{H^p(M;\mathbb{R})}$. It follows that pth singular cohomology with coefficients in $\mathbb{R}$ defines a contravariant functor from the topological category to the category of real vector spaces and linear maps.

There is an important theorem of algebraic topology called the ***universal coefficient theorem***, which shows how the singular cohomology groups with coefficients in an arbitrary group can be recovered from the singular homology groups. Thus, the cohomology groups do not contain any new information that is not already encoded in the homology groups; but they organize it in a different way that is more convenient for many purposes. In particular, the fact that the singular cohomology groups, like the de Rham cohomology groups, define *contravariant* functors makes it much easier to compare the two.

Proposition 18.5 (Properties of Singular Cohomology).

(a) *For any one-point space* $\{q\}$, $H^p(\{q\};\mathbb{R})$ *is trivial except when* $p = 0$, *in which case it is* 1*-dimensional.*
(b) *If* $\{M_j\}$ *is any collection of topological spaces and* $M = \coprod_j M_j$, *then the inclusion maps* $\iota_j\colon M_j \hookrightarrow M$ *induce an isomorphism from* $H^p(M;\mathbb{R})$ *to* $\prod_j H^p(M_j;\mathbb{R})$.
(c) *Homotopy equivalent spaces have isomorphic singular cohomology groups.*

Sketch of Proof. These properties follow easily from the definitions and Proposition 18.3. □

The key fact about the singular cohomology groups that we need is that they, too, satisfy a Mayer–Vietoris theorem.

Theorem 18.6 (Mayer–Vietoris for Singular Cohomology). *Suppose M, U, and V satisfy the hypotheses of Theorem 18.4. The following sequence is exact*:

$$\cdots \xrightarrow{\partial^*} H^p(M;\mathbb{R}) \xrightarrow{k^*\oplus l^*} H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) \xrightarrow{i^*-j^*} H^p(U\cap V;\mathbb{R}) \xrightarrow{\partial^*} H^{p+1}(M;\mathbb{R}) \xrightarrow{k^*\oplus l^*} \cdots, \quad (18.5)$$

where the maps $k^*\oplus l^*$ and i^*-j^* are defined as in (17.8), *and ∂^* is defined by $\partial^*(\gamma)=\gamma\circ\partial_*$, with ∂_* as in Theorem 18.4.*

Sketch of Proof. For any homomorphism $F\colon A\to B$ between abelian groups, there is a ***dual homomorphism*** $F^*\colon \operatorname{Hom}(B,\mathbb{R})\to\operatorname{Hom}(A,\mathbb{R})$ given by $F^*(\gamma)=\gamma\circ F$. Applying this to the Mayer–Vietoris sequence (18.4) for singular homology, we obtain the cohomology sequence (18.5). Exactness of (18.5) is a consequence of the fact that the assignments $A\mapsto\operatorname{Hom}(A,\mathbb{R})$ and $F\mapsto F^*$ define an ***exact functor***, meaning that it takes exact sequences to exact sequences. This in turn follows from the fact that $\mathbb{R}$ is an ***injective group***: this means that whenever H is a subgroup of an abelian group G, every homomorphism from H into $\mathbb{R}$ extends to all of G. □

Smooth Singular Homology

The connection between the singular and de Rham cohomology groups will be established by integrating differential forms over singular chains. More precisely, given a singular p-simplex σ in a manifold M and a p-form ω on M, we would like to pull ω back by σ and integrate the resulting form over Δ_p. However, there is an immediate problem with this approach, because forms can be pulled back only by *smooth* maps, while singular simplices are in general only continuous. (Actually, since only first derivatives of the map appear in the formula for the pullback, it would be sufficient to consider C^1 maps, but merely continuous ones definitely will not do.) In this section we overcome this problem by showing that singular homology can be computed equally well with smooth simplices.

If M is a smooth manifold, a ***smooth p-simplex in M*** is a map $\sigma\colon\Delta_p\to M$ that is smooth in the sense that it has a smooth extension to a neighborhood of each point. The subgroup of $C_p(M)$ generated by smooth simplices is denoted by $C_p^\infty(M)$ and called the ***smooth chain group in degree p***. Elements of this group, which are finite formal linear combinations of smooth simplices, are called ***smooth chains***. Because the boundary of a smooth simplex is a smooth chain, we can define the ***pth smooth singular homology group of M*** to be the quotient group

$$H_p^\infty(M)=\frac{\operatorname{Ker}\big(\partial\colon C_p^\infty(M)\to C_{p-1}^\infty(M)\big)}{\operatorname{Im}\big(\partial\colon C_{p+1}^\infty(M)\to C_p^\infty(M)\big)}.$$

The inclusion map $\iota\colon C_p^\infty(M)\hookrightarrow C_p(M)$ commutes with the boundary operator, and so induces a map on homology: $\iota_*\colon H_p^\infty(M)\to H_p(M)$ by $\iota_*[c]=[\iota(c)]$.

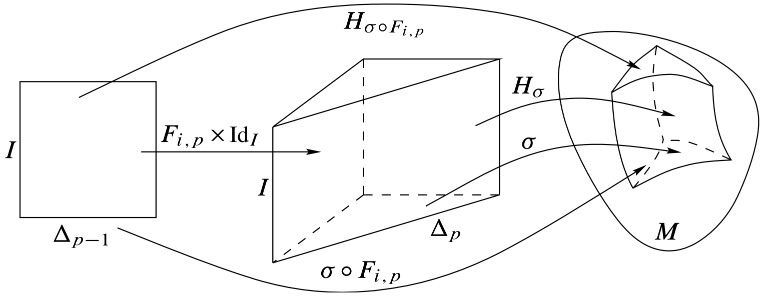

Fig. 18.4 The homotopy H_σ

Theorem 18.7 (Smooth Singular vs. Singular Homology). *For any smooth manifold M, the map $\iota_*\colon H_p^\infty(M)\to H_p(M)$ induced by inclusion is an isomorphism.*

The basic idea of the proof is to construct, with the help of the Whitney approximation theorem, two operators: first, a smoothing operator $s\colon C_p(M)\to C_p^\infty(M)$ such that $s\circ\partial=\partial\circ s$ and $s\circ\iota$ is the identity on $C_p^\infty(M)$; and second, a homotopy operator that shows that $\iota\circ s$ induces the identity map on $H_p(M)$. The details are highly technical, so unless algebraic topology is your primary interest, you may wish to skim the rest of this section on first reading.

The key to the proof is a systematic construction of a homotopy from each continuous simplex to a smooth one, in a way that respects the restriction to each boundary face of Δ_p. This is summarized in the following lemma.

Lemma 18.8. *Let M be a smooth manifold. For each integer $p\geq 0$ and each singular p-simplex $\sigma\colon\Delta_p\to M$, there exists a continuous map $H_\sigma\colon\Delta_p\times I\to M$ such that the following properties hold*:

(i) *H_σ is a homotopy from $\sigma(x)=H_\sigma(x,0)$ to a smooth p-simplex $\widetilde{\sigma}(x)=H_\sigma(x,1)$.*
(ii) *For each face map $F_{i,p}\colon\Delta_{p-1}\to\Delta_p$,*

$$H_{\sigma\circ F_{i,p}}=H_\sigma\circ\big(F_{i,p}\times\mathrm{Id}_I\big), \tag{18.6}$$

or more explicitly,

$$H_{\sigma\circ F_{i,p}}(x,t)=H_\sigma\big(F_{i,p}(x),t\big),\quad (x,t)\in\Delta_{p-1}\times I. \tag{18.7}$$

(iii) *If σ is a smooth p-simplex, then H_σ is the constant homotopy $H_\sigma(x,t)=\sigma(x)$.*

Proof. We will construct the homotopies H_σ (see Fig. 18.4) by induction on the dimension of σ. To get started, for each 0-simplex $\sigma\colon\Delta_0\to M$, we just define $H_\sigma(x,t)=\sigma(x)$. Since each 0-simplex is smooth and there are no face maps, conditions (i)–(iii) are automatically satisfied.

Now suppose by induction that for each $p'<p$ and for each p'-simplex σ' we have defined $H_{\sigma'}$ in such a way that the primed analogues of (i)–(iii) are satisfied.

Let $\sigma\colon \Delta_p \to M$ be an arbitrary singular p-simplex in M. If σ is smooth, we just let $H_\sigma(x,t) = \sigma(x)$, and (i)–(iii) are easily verified (using the fact that the restriction of σ to each boundary face is also smooth).

Assume that σ is not smooth, and let S be the subset

$$S = \big(\Delta_p \times \{0\}\big) \cup \big(\partial\Delta_p \times I\big) \subseteq \Delta_p \times I$$

(the bottom and side faces of the "prism" $\Delta_p \times I$). Recall that $\partial\Delta_p$ is the union of the boundary faces $\partial_i\Delta_p$ for $i = 0,\dots,p$, and for each i, the face map $F_{i,p}$ is a homeomorphism from Δ_{p-1} onto $\partial_i\Delta_p$. Define $H_0\colon S \to M$ by

$$H_0(x,t) = \begin{cases} \sigma(x), & x \in \Delta_p,\ t = 0; \\ H_{\sigma\circ F_{i,p}}\big(F_{i,p}^{-1}(x),t\big), & x \in \partial_i\Delta_p,\ t \in I. \end{cases}$$

We need to check that the various definitions agree where they overlap, which implies that H_0 is continuous by the gluing lemma.

When $t = 0$, the inductive hypothesis (i) applied to the singular $(p-1)$-simplex $\sigma\circ F_{i,p}$ implies that $H_{\sigma\circ F_{i,p}}(x,0) = \sigma\circ F_{i,p}(x)$. It follows that

$$H_{\sigma\circ F_{i,p}}\big(F_{i,p}^{-1}(x),0\big) = \sigma(x),$$

so the different definitions of H_0 agree at points where $t = 0$.

Suppose now that x is a point in the intersection of two boundary faces $\partial_i\Delta_p$ and $\partial_j\Delta_p$, and assume without loss of generality that $i > j$. Since $F_{i,p}\circ F_{j,p-1}$ is a homeomorphism from Δ_{p-2} onto $\partial_i\Delta_p \cap \partial_j\Delta_p$, we can write $x = F_{i,p}\circ F_{j,p-1}(y)$ for some point $y \in \Delta_{p-2}$. Then (18.7) applied with $\sigma\circ F_{i,p}$ in place of σ and $F_{j-1,p}$ in place of $F_{i,p}$ implies that

$$H_{\sigma\circ F_{i,p}}\big(F_{i,p}^{-1}(x),t\big) = H_{\sigma\circ F_{i,p}}\big(F_{j,p-1}(y),t\big) = H_{\sigma\circ F_{i,p}\circ F_{j,p-1}}(y,t).$$

On the other hand, thanks to (18.1), we can also write $x = F_{j,p}\circ F_{i-1,p-1}(y)$, and then the same argument applied to $\sigma\circ F_{j,p}$ yields

$$H_{\sigma\circ F_{j,p}}\big(F_{j,p}^{-1}(x),t\big) = H_{\sigma\circ F_{j,p}}\big(F_{i-1,p-1}(y),t\big) = H_{\sigma\circ F_{j,p}\circ F_{i-1,p-1}}(y,t).$$

Because of (18.1), this shows that the two definitions of $H_0(x,t)$ agree.

To extend H_0 to all of $\Delta_p \times I$, we use the fact that there is a retraction from $\Delta_p \times I$ onto S. For example, if q_0 is any point in the interior of Δ_p, then the map $R\colon \Delta_p \times I \to S$ obtained by radially projecting from the point $(q_0,2) \in \mathbb{R}^p \times \mathbb{R}$ is such a retraction (see Fig. 18.5). Extend H_0 to a continuous map $H\colon \Delta_p \times I \to M$ by setting $H(x,t) = H_0\big(R(x,t)\big)$. Because H agrees with H_0 on S, it is a homotopy from σ to some other (continuous) singular simplex $\sigma'(x) = H(x,1)$, and it satisfies (18.7) by construction. Our only remaining task is to modify H so that it becomes a homotopy from σ to a *smooth* simplex.

Before we do so, we need to observe that the restriction of H to each boundary face $\partial_i\Delta_p \times \{1\}$ is smooth: since these faces lie in S, H agrees with H_0 on each of these sets, and hypothesis (i) applied to $\sigma\circ F_{i,p}$ shows that H_0 is smooth there. By virtue of Lemma 18.9 below, this implies that the restriction of H to the entire set

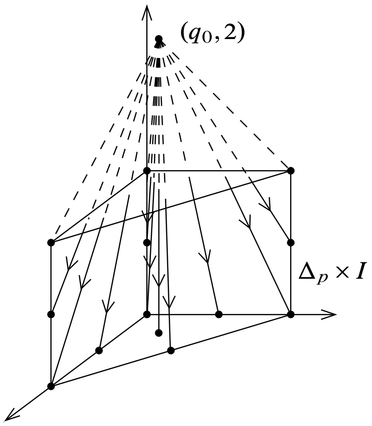

Fig. 18.5 A retraction from $\Delta_p \times I$ onto S

$\partial\Delta_p \times \{1\}$ is smooth. Let σ'' be any continuous extension of σ' to an open subset $U \subseteq \mathbb{R}^p$ containing Δ_p. (For example, σ'' could be defined by projecting points outside Δ_p to $\partial\Delta_p$ along radial lines from some point in the interior of Δ_p, and then applying σ'.) By the Whitney approximation theorem, σ'' is homotopic relative to $\partial\Delta_p$ to a smooth map, and restricting the homotopy to $\Delta_p \times I$ we obtain a homotopy $G\colon \sigma' \simeq \widetilde{\sigma}$ from σ' to some smooth singular p-simplex $\widetilde{\sigma}$, again relative to $\partial\Delta_p$.

Now let $u\colon \Delta_p \to \mathbb{R}$ be any continuous function that is equal to 1 on $\partial\Delta_p$ and satisfies $0 < u(x) < 1$ for $u \in \operatorname{Int}\Delta_p$. (For example, we could take $u\big(\sum_{0\le i\le p} t_i e_i\big) = 1 - t_0 t_1 \cdots t_p$, where $\sum_{0\le i\le p} t_i = 1$ and $e_0, \dots, e_p$ are the vertices of Δ_p.) We combine the two homotopies H and G into a single homotopy $H_\sigma\colon \Delta_p \times I \to M$ by

$$H_\sigma(x,t) = \begin{cases} H\left(x, \dfrac{t}{u(x)}\right), & x \in \Delta_p,\ 0 \le t \le u(x), \\ G\left(x, \dfrac{t-u(x)}{1-u(x)}\right), & x \in \operatorname{Int}\Delta_p,\ u(x) \le t \le 1. \end{cases}$$

Because $H(x,1) = \sigma'(x) = G(x,0)$, the gluing lemma shows that H_σ is continuous in $\operatorname{Int}\Delta_p \times I$. Also, $H_\sigma(x,t) = H\big(x, t/u(x)\big)$ in a neighborhood of $\partial\Delta_p \times [0,1)$, and thus is continuous there. It remains only to show that H_σ is continuous on $\partial\Delta_p \times \{1\}$. Let $x_0 \in \partial\Delta_p$ be arbitrary, and let $U \subseteq M$ be any neighborhood of $H_\sigma(x_0,1) = H(x_0,1)$. By continuity of H and u, there exists $\delta_1 > 0$ such that $H\big(x, t/u(x)\big) \in U$ whenever $|(x,t) - (x_0,1)| < \delta_1$ and $0 \le t \le u(x)$. Since $G(x_0,t) = G(x_0,0) = H(x_0,1) = H_\sigma(x_0,1) \in U$ for all $t \in I$, a simple compactness argument shows that there exists $\delta_2 > 0$ such that $|x - x_0| < \delta_2$ implies $G(x,t) \in U$ for all $t \in I$. Thus, if $|(x,t) - (x_0,1)| < \min(\delta_1,\delta_2)$, we have $H_\sigma(x,t) \in U$ in both cases, showing that $(x_0,1)$ has a neighborhood mapped into U by H_σ. Thus H_σ is continuous.

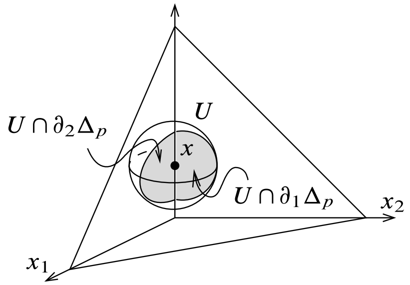

Fig. 18.6 Showing that f is smooth on $\partial \Delta_p$

It follows from the definition that $H_\sigma = H$ on $\partial \Delta_p \times I$, so (ii) is satisfied. For any $x \in \Delta_p$, $H_\sigma(x,0) = H(x,0) = \sigma(x)$. Moreover, when $x \in \operatorname{Int} \Delta_p$, $H_\sigma(x,1) = G(x,1) = \widetilde{\sigma}(x)$, and when $x \in \partial \Delta_p$, $H_\sigma(x,1) = H(x,1) = \sigma'(x) = \widetilde{\sigma}(x)$ (because G is a homotopy relative to $\partial \Delta_p$). Thus, H_σ is a homotopy from σ to the smooth simplex $\widetilde{\sigma}$, and (i) is satisfied as well. □

Here is the lemma used in the preceding proof.

Lemma 18.9. *Let M be a smooth manifold, let Δ be a geometric p-simplex in $\mathbb{R}^n$, and let $f : \partial \Delta \to M$ be a continuous map whose restriction to each individual boundary face of Δ is smooth. Then f is smooth when considered as a map from the entire boundary $\partial \Delta$ to M.*

Proof. Let $(v_0, \dots, v_p)$ denote the vertices of Δ in some order, and for each $i = 0, \dots, p$, let $\partial_i \Delta = \left[v_0, \dots, \widehat{v_i}, \dots, v_p\right]$ be the boundary face opposite v_i. The hypothesis means that for each i and each $x \in \partial_i \Delta$, there exist an open subset $U_x \subseteq \mathbb{R}^n$ and a smooth map $\widetilde{f} : U_x \to M$ whose restriction to $U_x \cap \partial_i \Delta$ agrees with f. We need to show that a single smooth extension can be chosen simultaneously for all the boundary faces containing x.

Suppose $x \in \partial \Delta$. Note that x is in one or more boundary faces of Δ, but cannot be in all of them. By reordering the vertices, we may assume that $x \in \partial_1 \Delta \cap \dots \cap \partial_k \Delta$ for some $1 \le k \le p$, but $x \notin \partial_0 \Delta$. After composing with an affine diffeomorphism that takes v_i to e_i for $i = 0, \dots, p$, we may assume without loss of generality that $\Delta = \Delta_p$ and $x \notin \partial_0 \Delta_p$. Then the boundary faces containing x are precisely the intersections with Δ_p of the coordinate hyperplanes $x^1 = 0, \dots, x^k = 0$. For each i, there are a neighborhood U_i of x in $\mathbb{R}^n$ (which can be chosen disjoint from $\partial_0 \Delta_p$) and a smooth map $\widetilde{f}_i : U_i \to M$ whose restriction to $U_i \cap \partial_i \Delta_p$ agrees with f.

Let $U = U_1 \cap \dots \cap U_k$ (see Fig. 18.6). We show by induction on k that there is a smooth map $\widetilde{f} : U \to M$ whose restriction to $U \cap \partial_i \Delta_p$ agrees with f for $i = 1, \dots, k$. Because the argument is local from this point on, after shrinking U if necessary we may replace M with a coordinate neighborhood of $f(x)$ that is diffeomorphic to $\mathbb{R}^m$; thus we henceforth identify M with $\mathbb{R}^m$.

For $k = 1$ there is nothing to prove, because $\widetilde{f}_1$ is already such an extension. So suppose $k \ge 2$, and we have shown that there is a smooth map $\widetilde{f}_0 : U \to M$ whose restriction to $U \cap \partial_i \Delta_p$ agrees with f for $i = 1, \ldots, k-1$. Define $\widetilde{f}: U \to M$ by

$$\widetilde{f}(x^1, \ldots, x^n) = \widetilde{f}_0(x^1, \ldots, x^n) - \widetilde{f}_0(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ + \widetilde{f}_k(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n).$$

For $i = 1, \ldots, k-1$, the restriction of $\widetilde{f}$ to $U \cap \partial_i \Delta_p$ is given by

$$\begin{aligned} &\widetilde{f}(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \\ &\quad = \widetilde{f}_0(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n) \\ &\qquad - \widetilde{f}_0(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ &\qquad + \widetilde{f}_k(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ &\quad = f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n), \end{aligned}$$

since $\widetilde{f}_0$ agrees with f when $x \in \Delta_p$ and $x^i = 0$, as does $\widetilde{f}_k$ when $x \in \Delta_p$ and $x^k = 0$. Similarly, the restriction to $U \cap \partial_k \Delta_p$ is

$$\begin{aligned} \widetilde{f}(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) &= \widetilde{f}_0(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ &\quad - \widetilde{f}_0(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ &\quad + \widetilde{f}_k(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n) \\ &= f(x^1, \ldots, x^{k-1}, 0, x^{k+1}, \ldots, x^n). \end{aligned}$$

This completes the inductive step and thus the proof. □

Proof of Theorem 18.7. Let $i_0, i_1: \Delta_p \to \Delta_p \times I$ be the smooth embeddings $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$. Define a homomorphism $s: C_p(M) \to C_p^\infty(M)$ by setting

$$s\sigma = H_\sigma \circ i_1$$

for each singular p-simplex σ (where H_σ is the homotopy whose existence is proved in Lemma 18.8) and extending linearly to p-chains. Because of property (i) in Lemma 18.8, $s\sigma$ is a smooth p-simplex homotopic to σ.

Using (18.6), we can verify that s is a chain map: for each singular p-simplex σ,

$$\begin{aligned} s\partial\sigma &= s\sum_{i=0}^{p}(-1)^i \sigma \circ F_{i,p} = \sum_{i=0}^{p}(-1)^i H_{\sigma \circ F_{i,p}} \circ i_1 \\ &= \sum_{i=0}^{p}(-1)^i H_\sigma \circ (F_{i,p} \times \mathrm{Id}_I) \circ i_1 = \sum_{i=0}^{p}(-1)^i H_\sigma \circ i_1 \circ F_{i,p} \\ &= \partial(H_\sigma \circ i_1) = \partial s\sigma. \end{aligned}$$

(In the fourth equality we used the fact that $(F_{i,p} \times \mathrm{Id}_I) \circ i_1(x) = \big(F_{i,p}(x), 1\big) = i_1 \circ F_{i,p}(x)$.) Therefore, s descends to a homomorphism $s_*\colon H_p(M) \to H_p^\infty(M)$. We will show that s_* is an inverse for $\iota_*\colon H_p^\infty(M) \to H_p(M)$.

First, observe that condition (iii) in Lemma 18.8 guarantees that $s \circ \iota$ is the identity map of $C_p^\infty(M)$, so clearly $s_* \circ \iota_*$ is the identity on $H_p^\infty(M)$. To show that $\iota_* \circ s_*$ is also the identity, we construct for each $p \geq 0$ a homotopy operator $h\colon C_p(M) \to C_{p+1}(M)$ satisfying

$$\partial \circ h + h \circ \partial = \iota \circ s - \mathrm{Id}_{C_p(M)}\,. \tag{18.8}$$

Once the existence of such an operator is known, it follows just as in the proof of Proposition 17.10 that $\iota_* \circ s_* = \mathrm{Id}_{H_p(M)}$: for any cycle $c \in C_p(M)$,

$$\iota_* \circ s_*[c] - [c] = [\iota \circ s(c) - c] = \big[\partial(hc) + h(\partial c)\big] = 0,$$

because $\partial c = 0$ and $\partial(hc)$ is a boundary.

To define the homotopy operator h, we need to introduce a family of affine singular simplices in the convex set $\Delta_p \times I \subseteq \mathbb{R}^p \times \mathbb{R}$. For each $i = 0, \ldots, p$, let $E_i = (e_i, 0) \in \mathbb{R}^p \times \mathbb{R}$ and $E'_i = (e_i, 1) \in \mathbb{R}^p \times \mathbb{R}$, so that $E_0, \ldots, E_p$ are the vertices of the geometric p-simplex $\Delta_p \times \{0\}$, and $E'_0 \ldots, E'_p$ are those of $\Delta_p \times \{1\}$. For each $i = 0, \ldots, p$, let $G_{i,p}\colon \Delta_{p+1} \to \Delta_p \times I$ be the affine singular $(p+1)$-simplex

$$G_{i,p} = A\big(E_0, \ldots, E_i, E'_i, \ldots, E'_p\big).$$

Thus, $G_{i,p}$ is the unique affine map that sends $e_0 \mapsto E_0, \ldots, e_i \mapsto E_i, e_{i+1} \mapsto E'_i, \ldots$, and $e_{p+1} \mapsto E'_p$. A routine computation shows that these maps compose with the face maps as follows:

$$G_{j,p} \circ F_{j,p+1} = G_{j-1,p} \circ F_{j,p+1} = A\big(E_0, \ldots, E_{j-1}, E'_j, \ldots, E'_p\big). \tag{18.9}$$

In particular, this implies that

$$G_{p,p} \circ F_{p+1,p+1} = A(E_0, \ldots, E_p) = i_0, \tag{18.10}$$

$$G_{0,p} \circ F_{0,p+1} = A(E'_0, \ldots, E'_p) = i_1. \tag{18.11}$$

A similar computation shows that

$$(F_{j,p} \times \mathrm{Id}_I) \circ G_{i,p-1} = \begin{cases} G_{i+1,p} \circ F_{j,p+1}, & i \geq j, \\ G_{i,p} \circ F_{j+1,p+1}, & i < j. \end{cases} \tag{18.12}$$

We define $h\colon C_p(M) \to C_{p+1}(M)$ as follows:

$$h\sigma = \sum_{i=0}^{p} (-1)^i H_\sigma \circ G_{i,p}.$$

The proof that it satisfies the homotopy formula (18.8) is just a laborious computation using (18.7), (18.9), and (18.12):

$$\begin{aligned} h(\partial\sigma) =& h\sum_{j=0}^{p}(-1)^j \sigma \circ F_{j,p} = \sum_{i=0}^{p-1}\sum_{j=0}^{p}(-1)^{i+j} H_{\sigma\circ F_{j,p}} \circ G_{i,p-1} \\ &= \sum_{i=0}^{p-1}\sum_{j=0}^{p}(-1)^{i+j} H_\sigma \circ (F_{j,p} \times \mathrm{Id}_I) \circ G_{i,p-1} \\ &= \sum_{0\le j\le i\le p-1} (-1)^{i+j} H_\sigma \circ G_{i+1,p} \circ F_{j,p+1} \\ &\quad + \sum_{0\le i<j\le p} (-1)^{i+j} H_\sigma \circ G_{i,p} \circ F_{j+1,p+1}, \end{aligned} \tag{18.13}$$

while

$$\partial(h\sigma) = \partial\sum_{i=0}^{p}(-1)^i H_\sigma \circ G_{i,p} = \sum_{j=0}^{p+1}\sum_{i=0}^{p}(-1)^{i+j} H_\sigma \circ G_{i,p} \circ F_{j,p+1}.$$

Writing separately the terms in $\partial(h\sigma)$ for which $i < j - 1$, $i = j - 1$, $i = j$, and $i > j$, we get

$$\begin{aligned} \partial(h\sigma) = &\sum_{\substack{0\le i<j-1\\ j\le p+1}} (-1)^{i+j} H_\sigma \circ G_{i,p} \circ F_{j,p+1} - \sum_{1\le j\le p+1} H_\sigma \circ G_{j-1,p} \circ F_{j,p+1} \\ &+ \sum_{0\le j\le p} H_\sigma \circ G_{j,p} \circ F_{j,p+1} + \sum_{0\le j<i\le p} (-1)^{i+j} H_\sigma \circ G_{i,p} \circ F_{j,p+1}. \end{aligned}$$

After substituting $j = j' + 1$ in the first of these four sums and $i = i' + 1$ in the last, we see that the first and last sums exactly cancel the two sums in the expression (18.13) for $h(\partial\sigma)$. Using (18.9), all the terms in the middle two sums cancel each other except those in which $j = 0$ and $j = p + 1$. Thanks to (18.10) and (18.11), these two terms simplify to

$$\begin{aligned} h(\partial\sigma) + \partial(h\sigma) &= -H_\sigma \circ G_{p,p} \circ F_{p+1,p+1} + H_\sigma \circ G_{0,p} \circ F_{0,p+1} \\ &= -H_\sigma \circ i_0 + H_\sigma \circ i_1 = -\sigma + s\sigma. \end{aligned}$$

Since ι is an inclusion map, $s\sigma = \iota \circ s\sigma$ for any singular p-simplex σ, so this completes the proof. □

The de Rham Theorem

In this section we state and prove the de Rham theorem. Before getting to the theorem itself, we need one more algebraic lemma. Its proof is another diagram chase like the proof of the zigzag lemma.

Lemma 18.10 (The Five Lemma). *Consider the following commutative diagram of modules and linear maps*:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4 & \xrightarrow{\beta_4} & B_5. \end{array}$$

If the horizontal rows are exact and f_1, f_2, f_4, *and* f_5 *are isomorphisms*, *then* f_3 *is also an isomorphism.*

► **Exercise 18.11.** Prove (or look up) the five lemma.

Suppose M is a smooth manifold, ω is a closed p-form on M, and σ is a smooth p-simplex in M. We define the ***integral of ω over σ*** to be

$$\int_\sigma \omega = \int_{\Delta_p} \sigma^*\omega.$$

This makes sense because Δ_p is a smooth p-submanifold with corners embedded in $\mathbb{R}^p$, and it inherits the orientation of $\mathbb{R}^p$. (Or we could just consider Δ_p as a domain of integration in $\mathbb{R}^p$.) Observe that when $p = 1$, this is the same as the line integral of ω over the smooth curve segment $\sigma\colon [0,1] \to M$. If $c = \sum_{i=1}^k c_i\sigma_i$ is a smooth p-chain, the integral of ω over c is defined as

$$\int_c \omega = \sum_{i=1}^k c_i \int_{\sigma_i} \omega.$$

Theorem 18.12 (Stokes's Theorem for Chains). *If c is a smooth p-chain in a smooth manifold M, and ω is a smooth $(p-1)$-form on M, then*

$$\int_{\partial c} \omega = \int_c d\omega.$$

Proof. It suffices to prove the theorem when c is just a smooth simplex σ. Since Δ_p is a manifold with corners, Stokes's theorem says that

$$\int_\sigma d\omega = \int_{\Delta_p} \sigma^* d\omega = \int_{\Delta_p} d\sigma^*\omega = \int_{\partial\Delta_p} \sigma^*\omega.$$

The maps $\{F_{i,p} : 0 = 1,\dots,p\}$ are parametrizations of the boundary faces of Δ_p satisfying the conditions of Proposition 16.21, except possibly that they might not be orientation-preserving. To check the orientations, note that $F_{i,p}$ is the restriction to $\Delta_p \cap \partial\mathbb{H}^p$ of the affine diffeomorphism sending the simplex $[e_0,\dots,e_p]$ to $\left[e_0,\dots,\widehat{e_i},\dots,e_p,e_i\right]$. This is easily seen to be orientation-preserving if and only if $\left(e_0,\dots,\widehat{e_i},\dots,e_p,e_i\right)$ is an even permutation of $(e_0,\dots,e_p)$, which is the case if and only if $p-i$ is even. Since the standard coordinates on $\partial\mathbb{H}^p$ are positively oriented if and only if p is even, the upshot is that $F_{i,p}$ is orientation-preserving for $\partial\Delta_p$ if and only if i is even. Thus, by Proposition 16.21,

$$\begin{aligned}\int_{\partial\Delta_p} \sigma^*\omega &= \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} F_{i,p}^*\sigma^*\omega = \sum_{i=0}^{p}(-1)^i \int_{\Delta_{p-1}} (\sigma\circ F_{i,p})^*\omega \\ &= \sum_{i=0}^{p}(-1)^i \int_{\sigma\circ F_{i,p}} \omega.\end{aligned}$$

By definition of the singular boundary operator, this is equal to $\int_{\partial\sigma}\omega$. □

Using this theorem, we define a natural linear map $\mathscr{I}\colon H^p_{\mathrm{dR}}(M)\to H^p(M;\mathbb{R})$, called the ***de Rham homomorphism***, as follows. For any $[\omega]\in H^p_{\mathrm{dR}}(M)$ and $[c]\in H_p(M)\cong H_p^\infty(M)$, we define

$$\mathscr{I}[\omega][c] = \int_{\widetilde{c}} \omega, \tag{18.14}$$

where $\widetilde{c}$ is any smooth p-cycle representing the homology class $[c]$. This is well defined, because if $\widetilde{c}$, $\widetilde{c}'$ are smooth cycles representing the same homology class, then Theorem 18.7 guarantees that $\widetilde{c}-\widetilde{c}'=\partial\widetilde{b}$ for some smooth $(p+1)$-chain $\widetilde{b}$, which implies

$$\int_{\widetilde{c}}\omega - \int_{\widetilde{c}'}\omega = \int_{\partial\widetilde{b}}\omega = \int_{\widetilde{b}} d\omega = 0,$$

while if $\omega = d\eta$ is exact, then

$$\int_{\widetilde{c}}\omega = \int_{\widetilde{c}} d\eta = \int_{\partial\widetilde{c}}\eta = 0.$$

(Note that $\partial\widetilde{c}=0$ because $\widetilde{c}$ represents a homology class, and $d\omega=0$ because ω represents a cohomology class.) Clearly, $\mathscr{I}[\omega][c+c'] = \mathscr{I}[\omega][c]+\mathscr{I}[\omega][c']$, and the resulting homomorphism $\mathscr{I}[\omega]\colon H_p(M)\to\mathbb{R}$ depends linearly on ω. Thus, $\mathscr{I}[\omega]$ is a well-defined element of $\mathrm{Hom}(H_p(M),\mathbb{R})\cong H^p(M;\mathbb{R})$.

Proposition 18.13 (Naturality of the de Rham Homomorphism). *For a smooth manifold M and nonnegative integer p, let $\mathscr{I}\colon H^p_{\mathrm{dR}}(M)\to H^p(M;\mathbb{R})$ denote the de Rham homomorphism.*

(a) *If $F\colon M\to N$ is a smooth map, then the following diagram commutes:*

$$\begin{array}{ccc} H^p_{\mathrm{dR}}(N) & \xrightarrow{F^*} & H^p_{\mathrm{dR}}(M) \\ \Big\downarrow{\scriptstyle\mathscr{I}} & & \Big\downarrow{\scriptstyle\mathscr{I}} \\ H^p(N;\mathbb{R}) & \xrightarrow[F^*]{} & H^p(M;\mathbb{R}). \end{array}$$

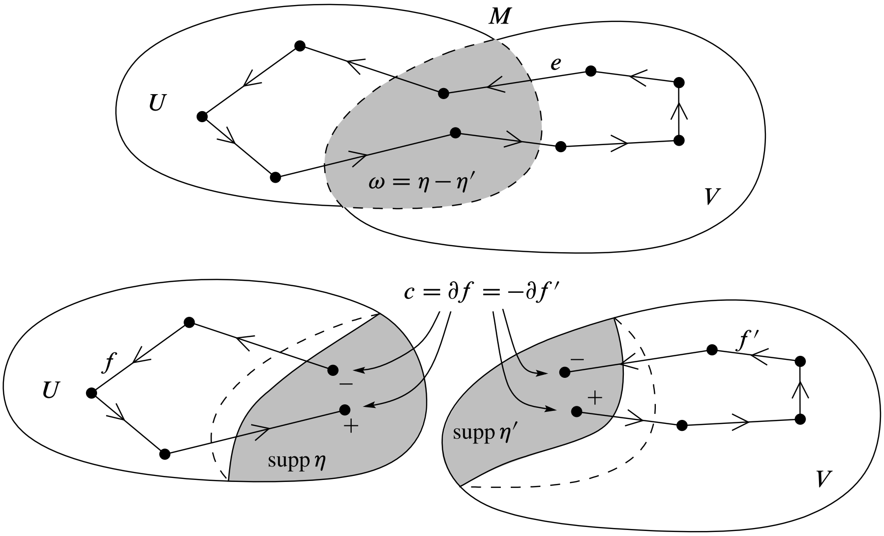

Fig. 18.7 Naturality of $\mathscr{I}$ with respect to connecting homomorphisms

(b) *If M is a smooth manifold and U, V are open subsets of M whose union is M, then the following diagram commutes*:

$$\begin{array}{ccc} H^{p-1}_{\mathrm{dR}}(U \cap V) & \xrightarrow{\ \delta\ } & H^p_{\mathrm{dR}}(M) \\ \mathscr{I}\downarrow & & \downarrow\mathscr{I} \\ H^{p-1}(U \cap V; \mathbb{R}) & \xrightarrow[\partial^*]{} & H^p(M; \mathbb{R}), \end{array} \tag{18.15}$$

where δ and ∂^* are the connecting homomorphisms of the Mayer–Vietoris sequences for de Rham and singular cohomology, *respectively.*

Proof. Directly from the definitions, if σ is a smooth p-simplex in M and ω is a smooth p-form on N,

$$\int_\sigma F^*\omega = \int_{\Delta_p} \sigma^* F^* \omega = \int_{\Delta_p} (F \circ \sigma)^* \omega = \int_{F \circ \sigma} \omega.$$

This implies

$$\mathscr{I}\big(F^*[\omega]\big)[\sigma] = \mathscr{I}[\omega][F \circ \sigma] = \mathscr{I}[\omega]\big(F_*[\sigma]\big) = F^*\big(\mathscr{I}[\omega]\big)[\sigma],$$

which proves (a).

Now consider (b). Commutativity of this diagram means

$$\mathscr{I}\big(\delta[\omega]\big)[e] = \big(\partial^* \mathscr{I}[\omega]\big)[e]$$

for any $[\omega] \in H^{p-1}_{\mathrm{dR}}(U \cap V)$ and any $[e] \in H_p(M)$. Using our identification of $H^p(M;\mathbb{R})$ with $\operatorname{Hom}(H_p(M),\mathbb{R})$, we can rewrite this as

$$\mathscr{I}\big(\delta[\omega]\big)[e] = \mathscr{I}\big([\omega]\big)\big(\partial_*[e]\big).$$

If σ is a smooth p-form representing $\delta[\omega]$ and c is a smooth $(p-1)$-chain representing $\partial_*[e]$, this is the same as $\int_e \sigma = \int_c \omega$. By the characterization of ∂_* given in Theorem 18.4, we can let $c = \partial f$, where f, f' are smooth p-chains in U and V, respectively, such that $f + f'$ represents the same homology class as e (Fig. 18.7). Similarly, by Corollary 17.42, we can choose $\eta \in \Omega^{p-1}(U)$ and $\eta' \in \Omega^{p-1}(V)$ such that $\omega = \eta|_{U\cap V} - \eta'|_{U\cap V}$, and then let σ be the p-form that is equal to $d\eta$ on U and to $d\eta'$ on V. Then, because $\partial f + \partial f' = \partial e = 0$ and $d\eta|_{U\cap V} - d\eta'|_{U\cap V} = d\omega = 0$, we have

$$\begin{aligned}\int_c \omega = \int_{\partial f} \omega = \int_{\partial f} \eta - \int_{\partial f} \eta' = \int_{\partial f} \eta + \int_{\partial f'} \eta' \\ = \int_f d\eta + \int_{f'} d\eta' = \int_f \sigma + \int_{f'} \sigma = \int_e \sigma.\end{aligned}$$

Thus the diagram commutes. □

Theorem 18.14 (de Rham). *For every smooth manifold M and nonnegative integer p, the de Rham homomorphism $\mathscr{I}\colon H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$ is an isomorphism.*

Proof. Let us say that a smooth manifold M is a ***de Rham manifold*** if the homomorphism $\mathscr{I}\colon H^p_{\mathrm{dR}}(M) \to H^p(M;\mathbb{R})$ is an isomorphism for each p. Since $\mathscr{I}$ commutes with the cohomology maps induced by smooth maps (Proposition 18.13), any manifold that is diffeomorphic to a de Rham manifold is also de Rham. The theorem will be proved once we show that every smooth manifold is de Rham.

If M is any smooth manifold, let us call an open cover $\{U_i\}$ of M a ***de Rham cover*** if each subset U_i is a de Rham manifold, and every finite intersection $U_{i_1} \cap \cdots \cap U_{i_k}$ is de Rham. A de Rham cover that is also a basis for the topology of M is called a ***de Rham basis*** for M.

STEP 1: *If $\{M_j\}$ is any countable collection of de Rham manifolds, then their disjoint union is de Rham.* By Propositions 17.5 and 18.5(b), for both de Rham and singular cohomology the inclusions $\iota_j\colon M_j \hookrightarrow \coprod_j M_j$ induce isomorphisms between the cohomology groups of the disjoint union and the direct product of the cohomology groups of the manifolds M_j. By Proposition 18.13, $\mathscr{I}$ commutes with these isomorphisms.

STEP 2: *Every convex open subset of $\mathbb{R}^n$ is de Rham*. Let U be such a subset. By the Poincaré lemma, $H^p_{\mathrm{dR}}(U)$ is trivial when $p \neq 0$. Since U is homotopy equivalent to a one-point space, Proposition 18.5 implies that the singular cohomology groups of U are also trivial for $p \neq 0$. In the $p = 0$ case, $H^0_{\mathrm{dR}}(U)$ is the 1-dimensional space consisting of the constant functions, and $H^0(U;\mathbb{R}) = \operatorname{Hom}\big(H_0(U),\mathbb{R}\big)$ is also 1-dimensional because $H_0(U)$ is generated by any singular 0-simplex. If $\sigma\colon \Delta_0 \to M$ is a singular 0-simplex (which is smooth because any map from a 0-manifold is smooth), and f is the constant function equal to 1, then

$$\mathcal{I}[f][\sigma] = \int_{\Delta_0} \sigma^* f = (f \circ \sigma)(0) = 1.$$

Thus $\mathcal{I}\colon H^0_{\mathrm{dR}}(U) \to H^0(U;\mathbb{R})$ is not the zero map, so it is an isomorphism.

STEP 3: *If M has a finite de Rham cover, then M is de Rham.* This is the heart of the proof. Suppose $M = U_1 \cup \cdots \cup U_k$, where the open subsets U_i and their finite intersections are de Rham. We prove the result by induction on k. For $k = 1$, the result is obvious. Suppose next that M has a de Rham cover consisting of two sets $\{U, V\}$. Putting together the Mayer–Vietoris sequences for de Rham and singular cohomology, we obtain the following commutative diagram, in which the horizontal rows are exact and the vertical maps are all de Rham homomorphisms:

$$\begin{array}{ccccccc}
H^{p-1}_{\mathrm{dR}}(U) \oplus H^{p-1}_{\mathrm{dR}}(V) & \longrightarrow & H^{p-1}_{\mathrm{dR}}(U \cap V) & \longrightarrow & H^p_{\mathrm{dR}}(M) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
H^{p-1}(U;\mathbb{R}) \oplus H^{p-1}(V;\mathbb{R}) & \longrightarrow & H^{p-1}(U \cap V;\mathbb{R}) & \longrightarrow & H^p(M;\mathbb{R}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
H^p_{\mathrm{dR}}(U) \oplus H^p_{\mathrm{dR}}(V) & \longrightarrow & H^p_{\mathrm{dR}}(U \cap V) \\
\downarrow & & \downarrow \\
H^p(U;\mathbb{R}) \oplus H^p(V;\mathbb{R}) & \longrightarrow & H^p(U \cap V;\mathbb{R}).
\end{array}$$

The commutativity of the diagram is an immediate consequence of Proposition 18.13. By hypothesis the first, second, fourth, and fifth vertical maps are all isomorphisms, so by the five lemma the middle map is an isomorphism, which proves that M is de Rham.

Now assume the claim is true for smooth manifolds admitting a de Rham cover with $k \geq 2$ sets, and suppose $\{U_1, \ldots, U_{k+1}\}$ is a de Rham cover of M. Define $U = U_1 \cup \cdots \cup U_k$ and $V = U_{k+1}$. The hypothesis implies that U and V are de Rham, and $U \cap V$ is also de Rham because it has a k-fold de Rham cover given by $\{U_1 \cap U_{k+1}, \ldots, U_k \cap U_{k+1}\}$. Therefore, $M = U \cup V$ is also de Rham by the argument above.

STEP 4: *If M has a de Rham basis, then M is de Rham.* Suppose $\{U_\alpha\}$ is a de Rham basis for M. Let $f\colon M \to \mathbb{R}$ be an exhaustion function (see Proposition 2.28). For each integer m, define subsets A_m and A'_m of M by

$$A_m = \left\{q \in M : m \leq f(q) \leq m+1\right\},$$
$$A'_m = \left\{q \in M : m - \tfrac{1}{2} < f(q) < m + \tfrac{3}{2}\right\}.$$

(See Fig. 18.8.) For each point $q \in A_m$, there is a basis open subset containing q and contained in A'_m. The collection of all such basis sets is an open cover of A_m. Since f is an exhaustion function, A_m is compact, and therefore it is covered by finitely many of these basis sets. Let B_m be the union of this finite collection of sets. This is a finite de Rham cover of B_m, so by Step 3, B_m is de Rham.

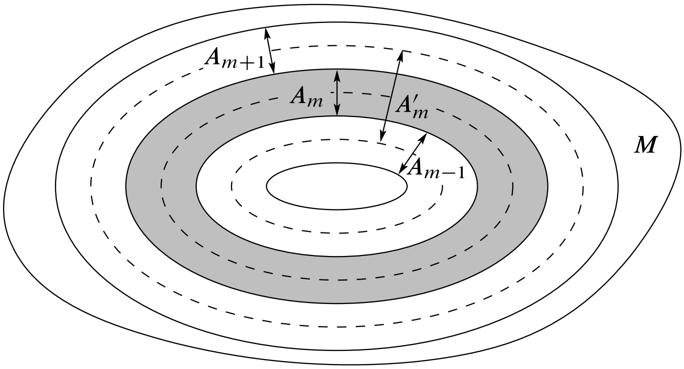

Fig. 18.8 Proof of the de Rham theorem, Step 4

Observe that $B_m \subseteq A'_m$, so B_m can have nonempty intersection with $B_{\widetilde{m}}$ only when $\widetilde{m} = m-1$, m, or $m+1$. Therefore, if we define

$$U = \bigcup_{m \text{ odd}} B_m, \qquad V = \bigcup_{m \text{ even}} B_m,$$

then U and V are disjoint unions of de Rham manifolds, and so they are both de Rham by Step 1. Finally, $U \cap V$ is de Rham because it is the disjoint union of the sets $B_m \cap B_{m+1}$ for $m \in \mathbb{Z}$, each of which has a finite de Rham cover consisting of sets of the form $U_\alpha \cap U_\beta$, where U_α and U_β are basis sets used to define B_m and B_{m+1}, respectively. Thus $M = U \cup V$ is de Rham by Step 3.

STEP 5: *Every open subset of* $\mathbb{R}^n$ *is de Rham.* If $U \subseteq \mathbb{R}^n$ is such a subset, then U has a basis consisting of Euclidean balls. Because each ball is convex, it is de Rham, and because any finite intersection of balls is again convex, finite intersections are also de Rham. Thus, U has a de Rham basis, so it is de Rham by Step 4.

STEP 6: *Every smooth manifold is de Rham.* Any smooth manifold has a basis of smooth coordinate domains. Since every smooth coordinate domain is diffeomorphic to an open subset of $\mathbb{R}^n$, as are their finite intersections, this is a de Rham basis. The claim therefore follows from Step 4. □

This result expresses a deep connection between the topological and analytic properties of a smooth manifold, and plays a central role in differential geometry. If one has some information about the topology of a manifold M, the de Rham theorem can be used to draw conclusions about solutions to differential equations such as $d\eta = \omega$ on M. Conversely, if one can prove that such solutions do or do not exist, then one can draw conclusions about the topology.

As befits so fundamental a theorem, the de Rham theorem has many and varied proofs. The elegant proof given here is due to Glen E. Bredon [Bre93]. Another common approach is via the theory of sheaves; for example, a proof using this technique can be found in [War83]. The sheaf-theoretic proof is extremely powerful and lends itself to countless generalizations, but it has two significant disadvantages that prevent it from being useful for our purposes: it requires the entire technical apparatus of sheaf theory and sheaf cohomology, which would take us too far afield; and although it produces an isomorphism between de Rham and singular cohomology, it is not easy to see that the isomorphism is given specifically by integration. Nonetheless, because the technique leads to other important applications in such fields as differential geometry, algebraic geometry, algebraic topology, and complex analysis, it is worth taking some time and effort to study it if you get the opportunity.

Problems

18-1. Suppose M is an oriented smooth manifold and ω is a closed p-form on M.
  (a) Show that ω is exact if and only if the integral of ω over every smooth p-cycle is zero.
  (b) Suppose $H_p(M)$ is generated by the homology classes of finitely many smooth p-cycles $\{c_1, \dots, c_m\}$. Define real numbers $P_1(\omega), \dots, P_m(\omega)$, called the ***periods of ω*** with respect to this set of generators, by $P_i(\omega) = \int_{c_i} \omega$. Show that ω is exact if and only if all of its periods are zero. [Remark: if you look back now at Problem 11-17, you will see that it is essentially proving the same theorem in the special case of a 1-form on $\mathbb{T}^n$.]

18-2. If G is a group, the ***commutator subgroup of G***, denoted by $[G,G]$, is the smallest normal subgroup containing all elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1, g_2 \in G$; and the ***abelianization of G***, denoted by $\operatorname{Ab}(G)$, is the quotient group $G/[G,G]$. Suppose M is a connected smooth manifold and $q \in M$. It can be shown that there is a group homomorphism from $\pi_1(M,q)$ to $H_1(M)$ that sends the homotopy class of a loop γ to the homology class of the 1-cycle determined by γ, and this map descends to an isomorphism from $\operatorname{Ab}\bigl(\pi_1(M,q)\bigr)$ to $H_1(M)$ (see [LeeTM, Thm. 13.14]). Use this result together with the de Rham theorem to prove that the map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \operatorname{Hom}\bigl(\pi_1(M,q), \mathbb{R}\bigr)$ of Theorem 17.17 is an isomorphism.

18-3. Let M be a smooth n-manifold and suppose $S \subseteq M$ is an oriented compact embedded p-dimensional submanifold. A ***smooth triangulation of S*** is a smooth p-chain $c = \sum_i \sigma_i$ in M with the following properties:
  (i) Each $\sigma_i \colon \Delta_p \to S$ is a smooth orientation-preserving embedding.
  (ii) If $i \neq j$, then $\sigma_i(\operatorname{Int} \Delta_p) \cap \sigma_j(\operatorname{Int} \Delta_p) = \varnothing$.
  (iii) $S = \bigcup_i \sigma_i(\Delta_p)$.
  (iv) $\partial c = 0$.

  (It can be shown that every oriented compact embedded submanifold admits a smooth triangulation, but we will not use that fact; see [Mun66] for a proof.) Two oriented compact embedded p-dimensional submanifolds $S, S' \subseteq M$ are said to be ***homologous*** if there exist smooth triangulations c for S and c' for S' such that $c - c'$ is a boundary.
  (a) Show that for any smooth triangulation c of S and any smooth p-form ω on M, we have $\int_c \omega = \int_S \omega$.

(b) Show that if ω is closed and S, S' are homologous, then $\int_S \omega = \int_{S'} \omega$.

18-4. Suppose (M, g) is a Riemannian n-manifold. A smooth p-form ω on M is called a ***calibration*** if ω is closed and $\omega_x(v_1, \dots, v_p) \leq 1$ whenever $(v_1, \dots, v_p)$ are orthonormal vectors in some tangent space $T_x M$. An oriented embedded p-dimensional submanifold $S \subseteq M$ is said to be ***calibrated*** if there is a calibration ω such that the pullback $\iota_S^* \omega$ is the volume form for the induced Riemannian metric on S. Suppose $S \subseteq M$ is a smoothly triangulated calibrated compact submanifold. Prove that the volume of S (with respect to the induced Riemannian metric) is less than or equal to that of any other submanifold homologous to S (see Problem 18-3). [Remark: calibrations were invented in 1982 by Reese Harvey and Blaine Lawson [HL82]; they have become increasingly important in recent years because in many situations a calibration is the only known way of proving that a given submanifold is volume-minimizing in its homology class.]

18-5. Let $D \subseteq \mathbb{R}^3$ be the surface obtained by revolving the circle $(r-2)^2 + z^2 = 1$ around the z-axis, with the induced Riemannian metric (see Example 13.18(a)), and let $C \subseteq D$ be the "inner circle" defined by $C = \{(x, y, z) : z = 0,\ x^2 + y^2 = 1\}$. Show that C is calibrated, and therefore is the shortest curve in its homology class.

18-6. For any smooth manifold M, let $H_c^p(M)$ denote the pth compactly supported de Rham cohomology group of M.

(a) Given an open subset $U \subseteq M$, let $\iota\colon U \hookrightarrow M$ denote the inclusion map, and define a linear map $\iota_\sharp\colon \Omega_c^p(U) \to \Omega_c^p(M)$ by extending each compactly supported form to be zero on $M \smallsetminus U$. Show that $d \circ \iota_\sharp = \iota_\sharp \circ d$, and so $\iota_\sharp$ induces a linear map on compactly supported cohomology, denoted by $\iota_*\colon H_c^p(U) \to H_c^p(M)$.

(b) Mayer–Vietoris with Compact Supports: Suppose M is a smooth manifold and $U, V \subseteq M$ are open subsets whose union is M. Prove that for each nonnegative integer p, there is a linear map $\delta_*\colon H_c^p(M) \to H_c^{p+1}(U \cap V)$ such that the following sequence is exact:

$$\cdots \xrightarrow{\delta_*} H_c^p(U \cap V) \xrightarrow{i_* \oplus (-j_*)} H_c^p(U) \oplus H_c^p(V) \xrightarrow{k_* + l_*} H_c^p(M) \xrightarrow{\delta_*} H_c^{p+1}(U \cap V) \xrightarrow{i_* \oplus (-j_*)} \cdots,$$

where i, j, k, l are the inclusion maps as in (17.6).

(c) Let $H_c^p(M)^*$ denote the algebraic dual space to $H_c^p(M)$, that is, the vector space of all linear maps from $H_c^p(M)$ to $\mathbb{R}$. Show that the following sequence is also exact:

$$\cdots \xrightarrow{(\delta_*)^*} H_c^p(M)^* \xrightarrow{(k_*)^* \oplus (l_*)^*} H_c^p(U)^* \oplus H_c^p(V)^* \xrightarrow{(i_*)^* - (j_*)^*} H_c^p(U \cap V)^* \xrightarrow{(\delta_*)^*} H_c^{p-1}(M)^* \xrightarrow{(k_*)^* \oplus (l_*)^*} \cdots. \tag{18.16}$$

18-7. THE POINCARÉ DUALITY THEOREM: Let M be an oriented smooth n-manifold. Define a map $\mathrm{PD}\colon \Omega^p(M) \to \Omega_c^{n-p}(M)^*$ by

$$\mathrm{PD}(\omega)(\eta) = \int_M \omega \wedge \eta.$$

(a) Show that PD descends to a linear map (still denoted by the same symbol) $\mathrm{PD}\colon H^p_{\mathrm{dR}}(M) \to H_c^{n-p}(M)^*$.

(b) Show that PD is an isomorphism for each p. [Hint: imitate the proof of the de Rham theorem, with "de Rham manifold" replaced by "PD manifold." You will need Lemma 17.27 and Problem 18-6.]

18-8. Let M be a compact smooth n-manifold.

(a) Show that all de Rham groups of M are finite-dimensional. [Hint: for the orientable case, use Poincaré duality to show that $H^p_{\mathrm{dR}}(M) \cong H^p_{\mathrm{dR}}(M)^{**}$, and use the result of Problem 11-2. For the nonorientable case, use Lemma 17.33.]

(b) Show that if M is orientable, then $\dim H^p_{\mathrm{dR}}(M) = \dim H^{n-p}_{\mathrm{dR}}(M)$ for all p.

18-9. Let M be a smooth n-manifold all of whose de Rham groups are finite-dimensional. (Problem 18-8 shows that this is always the case when M is compact.) The ***Euler characteristic of M*** is the number

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p_{\mathrm{dR}}(M).$$

Show that $\chi(M)$ is a homotopy invariant of M, and $\chi(M) = 0$ when M is compact, orientable, and odd-dimensional.