# Chapter 20
# The Exponential Map

In this chapter we apply the tools of flows, Lie derivatives, and foliations to delve deeper into the relationships between Lie groups and Lie algebras.

In the first section we define *one-parameter subgroups* of a Lie group $G$, which are just Lie group homomorphisms from $\mathbb{R}$ to $G$, and show that there is a one-to-one correspondence between elements of $\mathrm{Lie}(G)$ and one-parameter subgroups of $G$.

Next we introduce the focal point of our study, which is a canonical smooth map from the Lie algebra into the group called the *exponential map*. It maps lines through the origin in $\mathrm{Lie}(G)$ to one-parameter subgroups of $G$.

As our first major application of the exponential map, we prove the *closed subgroup theorem*, which says that every topologically closed subgroup of a Lie group is actually an embedded Lie subgroup.

Next we prove a higher-dimensional generalization of the fundamental theorem on flows. Instead of a single smooth vector field generating an action of $\mathbb{R}$, we consider a finite-dimensional family of vector fields and ask when they generate an action of some Lie group. The main theorem is that if $G$ is a simply connected Lie group, then any Lie algebra homomorphism from $\mathrm{Lie}(G)$ into the set of complete vector fields on $M$ generates a smooth action of $G$ on $M$.

Finally, in the last two sections, we bring together all of these results to deepen our understanding of the correspondence between Lie groups and Lie algebras. First, we prove that there is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups; and then we show that for any Lie group $G$, connected normal subgroups of $G$ correspond to *ideals* in the Lie algebra of $G$, which are subspaces that are stable under bracketing with arbitrary elements of the algebra. This is an excellent illustration of the fundamental philosophy of Lie theory: as much as possible, we use the Lie group/Lie algebra correspondence to translate group-theoretic questions about a Lie group into linear-algebraic questions about its Lie algebra.

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## One-Parameter Subgroups and the Exponential Map

Suppose $G$ is a Lie group. Since left-invariant vector fields are naturally defined in terms of the group structure of $G$, one might reasonably expect to find some relationship between the group law for the flow of a left-invariant vector field and group multiplication in $G$. We begin by exploring this relationship.

### *One-Parameter Subgroups*

A ***one-parameter subgroup of G*** is defined to be a Lie group homomorphism $\gamma\colon \mathbb{R} \to G$, with $\mathbb{R}$ considered as a Lie group under addition. By this definition, a one-parameter subgroup is *not* a Lie subgroup of $G$, but rather a homomorphism into $G$. (However, the *image* of a one-parameter subgroup is a Lie subgroup when endowed with a suitable smooth manifold structure; see Problem 20-1.)

**Theorem 20.1 (Characterization of One-Parameter Subgroups).** *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.*

*Proof.* First suppose $\gamma$ is the maximal integral curve of some left-invariant vector field $X \in \operatorname{Lie}(G)$ starting at the identity. Because left-invariant vector fields are complete (Theorem 9.18), $\gamma$ is defined on all of $\mathbb{R}$. Left-invariance means that $X$ is $L_g$-related to itself for every $g \in G$, so Proposition 9.6 implies that $L_g$ takes integral curves of $X$ to integral curves of $X$. Applying this with $g = \gamma(s)$ for some $s \in \mathbb{R}$, we conclude that the curve $t \mapsto L_{\gamma(s)}\big(\gamma(t)\big)$ is an integral curve starting at $\gamma(s)$. But the translation lemma (Lemma 9.4) implies that $t \mapsto \gamma(s+t)$ is also an integral curve with the same initial point, so they are equal:

$$\gamma(s)\gamma(t) = \gamma(s+t).$$

This says precisely that $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup.

Conversely, suppose $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup, and let $X = \gamma_*(d/dt) \in \operatorname{Lie}(G)$, treating $d/dt$ as a left-invariant vector field on $\mathbb{R}$. Since $\gamma(0) = e$, we just have to show that $\gamma$ is an integral curve of $X$. Recall that $\gamma_*(d/dt)$ is defined as the unique left-invariant vector field on $G$ that is $\gamma$-related to $d/dt$ (see Theorem 8.44). Therefore, for any $t_0 \in \mathbb{R}$,

$$\gamma'(t_0) = d\gamma_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) = X_{\gamma(t_0)},$$

so $\gamma$ is an integral curve of $X$. □

Given $X \in \operatorname{Lie}(G)$, the one-parameter subgroup determined by $X$ in this way is called the ***one-parameter subgroup generated by X***. Because left-invariant vector fields are uniquely determined by their values at the identity, it follows that each one-parameter subgroup is uniquely determined by its initial velocity in $T_eG$, and thus there are one-to-one correspondences

$$\{\text{one-parameter subgroups of } G\} \longleftrightarrow \operatorname{Lie}(G) \longleftrightarrow T_eG.$$

The one-parameter subgroups of $\mathrm{GL}(n,\mathbb{R})$ are not hard to compute explicitly.

**Proposition 20.2.** *For any* $A \in \mathfrak{gl}(n,\mathbb{R})$, *let*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I_n + A + \frac{1}{2} A^2 + \cdots . \tag{20.1}$$

*This series converges to an invertible matrix* $e^A \in \mathrm{GL}(n,\mathbb{R})$, *and the one-parameter subgroup of* $\mathrm{GL}(n,\mathbb{R})$ *generated by* $A \in \mathfrak{gl}(n,\mathbb{R})$ *is* $\gamma(t) = e^{tA}$.

*Proof.* First, we verify convergence. From Exercise B.48, matrix multiplication satisfies $|AB| \le |A||B|$, where the norm is the Frobenius norm on $\mathfrak{gl}(n,\mathbb{R})$. It follows by induction that $\left|A^k\right| \le |A|^k$. The Weierstrass $M$-test then shows that (20.1) converges uniformly on any bounded subset of $\mathfrak{gl}(n,\mathbb{R})$, by comparison with the series $\sum_k (1/k!)c^k = e^c$.

Fix $A \in \mathfrak{gl}(n,\mathbb{R})$. Under our identification of $\mathfrak{gl}(n,\mathbb{R})$ with $\mathrm{Lie}\big(\mathrm{GL}(n,\mathbb{R})\big)$, the matrix $A$ corresponds to the left-invariant vector field $A^{\mathrm{L}}$ given by (8.15). Thus, the one-parameter subgroup generated by $A$ is an integral curve of $A^{\mathrm{L}}$ on $\mathrm{GL}(n,\mathbb{R})$, and therefore satisfies the ODE initial value problem

$$\gamma'(t) = A^{\mathrm{L}}\big|_{\gamma(t)}, \qquad \gamma(0) = I_n.$$

Using (8.15), the condition for $\gamma$ to be an integral curve can be rewritten as

$$\dot{\gamma}^i_k(t) = \gamma^i_j(t) A^j_k,$$

or in matrix notation

$$\gamma'(t) = \gamma(t)A.$$

We will show that $\gamma(t) = e^{tA}$ satisfies this equation. Since $\gamma(0) = I_n$, this implies that $\gamma$ is the unique integral curve of $A^{\mathrm{L}}$ starting at the identity and is therefore the desired one-parameter subgroup.

To see that $\gamma$ is differentiable, we note that differentiating the series (20.1) formally term by term yields the result

$$\gamma'(t) = \sum_{k=1}^{\infty} \frac{k}{k!} t^{k-1} A^k = \left( \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} A^{k-1} \right) A = \gamma(t)A.$$

Since the differentiated series converges uniformly on bounded sets (because apart from the additional factor of $A$, it is the same series!), the term-by-term differentiation is justified. A similar computation shows that $\gamma'(t) = A\gamma(t)$. By smoothness of solutions to ODEs, $\gamma$ is a smooth curve.

It remains only to show that $\gamma(t)$ is invertible for all $t$, so that $\gamma$ actually takes its values in $\mathrm{GL}(n,\mathbb{R})$. If we let $\sigma(t) = \gamma(t)\gamma(-t) = e^{tA}e^{-tA}$, then $\sigma$ is a smooth curve in $\mathfrak{gl}(n,\mathbb{R})$, and by the previous computation and the product rule it satisfies

$$\sigma'(t) = \big(\gamma(t)A\big)\gamma(-t) - \gamma(t)\big(A\gamma(-t)\big) = 0.$$

It follows that $\sigma$ is the constant curve $\sigma(t) \equiv \sigma(0) = I_n$, which is to say that $\gamma(t)\gamma(-t) = I_n$. Substituting $-t$ for $t$, we obtain $\gamma(-t)\gamma(t) = I_n$, which shows that $\gamma(t)$ is invertible and $\gamma(t)^{-1} = \gamma(-t)$. □

Next we would like to compute the one-parameter subgroups of $\mathrm{GL}(n,\mathbb{R})$, such as $\mathrm{O}(n)$. To do so, we need the following result.

**Proposition 20.3.** *Suppose $G$ is a Lie group and $H \subseteq G$ is a Lie subgroup. The one-parameter subgroups of $H$ are precisely those one-parameter subgroups of $G$ whose initial velocities lie in $T_eH$.*

*Proof.* Let $\gamma\colon \mathbb{R} \to H$ be a one-parameter subgroup. Then the composite map

$$\mathbb{R} \xrightarrow{\gamma} H \hookrightarrow G$$

is a Lie group homomorphism and thus a one-parameter subgroup of $G$, which clearly satisfies $\gamma'(0) \in T_eH$.

Conversely, suppose $\gamma\colon \mathbb{R} \to G$ is a one-parameter subgroup whose initial velocity lies in $T_eH$. Let $\widetilde{\gamma}\colon \mathbb{R} \to H$ be the one-parameter subgroup of $H$ with the same initial velocity $\widetilde{\gamma}'(0) = \gamma'(0) \in T_eH \subseteq T_eG$. As in the preceding paragraph, by composing with the inclusion map, we can also consider $\widetilde{\gamma}$ as a one-parameter subgroup of $G$. Since $\gamma$ and $\widetilde{\gamma}$ are both one-parameter subgroups of $G$ with the same initial velocity, they must be equal. □

**Example 20.4.** If $H$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$, the preceding proposition shows that the one-parameter subgroups of $H$ are precisely the maps of the form $\gamma(t) = e^{tA}$ for $A \in \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{gl}(n,\mathbb{R})$ is the subalgebra corresponding to $\mathrm{Lie}(H)$ as in Theorem 8.46. For example, taking $H = \mathrm{O}(n)$, this shows that the one-parameter subgroups of $\mathrm{O}(n)$ are the maps of the form $\gamma(t) = e^{tA}$ for an arbitrary skew-symmetric matrix $A$. In particular, this shows that the exponential of any skew-symmetric matrix is orthogonal. //

## *The Exponential Map*

In the preceding section we saw that the matrix exponential maps $\mathfrak{gl}(n,\mathbb{R})$ to $\mathrm{GL}(n,\mathbb{R})$ and takes each line through the origin to a one-parameter subgroup. This has a powerful generalization to arbitrary Lie groups.

Given a Lie group $G$ with Lie algebra $\mathfrak{g}$, we define a map $\exp\colon \mathfrak{g} \to G$, called the ***exponential map of G***, as follows: for any $X \in \mathfrak{g}$, we set

$$\exp X = \gamma(1),$$

where $\gamma$ is the one-parameter subgroup generated by $X$, or equivalently the integral curve of $X$ starting at the identity (Fig. 20.1). The following proposition shows that, like the matrix exponential, this map sends the line through $X$ to the one-parameter subgroup generated by $X$.

**Proposition 20.5.** *Let $G$ be a Lie group. For any $X \in \mathrm{Lie}(G)$, $\gamma(s) = \exp sX$ is the one-parameter subgroup of $G$ generated by $X$.*

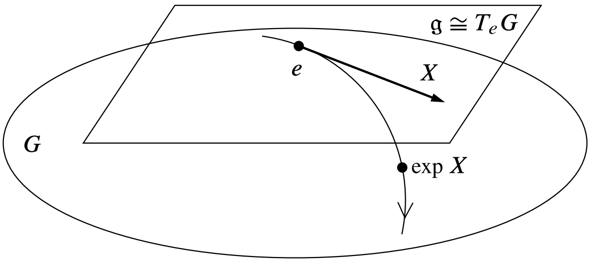


**Fig. 20.1** The exponential map

*Proof.* Let $\gamma\colon \mathbb{R} \to G$ be the one-parameter subgroup generated by $X$, which is the integral curve of $X$ starting at $e$. For any fixed $s \in \mathbb{R}$, it follows from the rescaling lemma (Lemma 9.3) that $\widetilde{\gamma}(t) = \gamma(st)$ is the integral curve of $sX$ starting at $e$, so

$$\exp sX = \widetilde{\gamma}(1) = \gamma(s). \qquad \square$$

Here are two simple but important examples.

**Example 20.6.** The results of the preceding section show that the exponential map of $\mathrm{GL}(n,\mathbb{R})$ (or any Lie subgroup of it) is given by $\exp A = e^A$. This, obviously, is the reason for the term *exponential map*. //

**Example 20.7.** If $V$ is a finite-dimensional real vector space, a choice of basis for $V$ yields isomorphisms $\mathrm{GL}(V) \cong \mathrm{GL}(n,\mathbb{R})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n,\mathbb{R})$. The analysis of the $\mathrm{GL}(n,\mathbb{R})$ case then shows that the exponential map of $\mathrm{GL}(V)$ can be written in the form

$$\exp A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k, \tag{20.2}$$

where we consider $A \in \mathfrak{gl}(V)$ as a linear map from $V$ to itself, and $A^k = A \circ \cdots \circ A$ is the $k$-fold composition of $A$ with itself. //

**Proposition 20.8 (Properties of the Exponential Map).** *Let $G$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

(a) *The exponential map is a smooth map from $\mathfrak{g}$ to $G$.*
(b) *For any $X \in \mathfrak{g}$ and $s,t \in \mathbb{R}$, $\exp(s+t)X = \exp sX \exp tX$.*
(c) *For any $X \in \mathfrak{g}$, $(\exp X)^{-1} = \exp(-X)$.*
(d) *For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp X)^n = \exp(nX)$.*
(e) *The differential $(d\exp)_0\colon T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and $T_eG$ with $\mathfrak{g}$ itself.*
(f) *The exponential map restricts to a diffeomorphism from some neighborhood of $0$ in $\mathfrak{g}$ to a neighborhood of $e$ in $G$.*

(g) *If $H$ is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi\colon G \to H$ is a Lie group homomorphism, the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\Phi_*} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[\Phi]{} & H. \end{array} \tag{20.3}$$

(h) *The flow $\theta$ of a left-invariant vector field $X$ is given by $\theta_t = R_{\exp tX}$ (right multiplication by $\exp tX$).*

*Proof.* In this proof, for any $X \in \mathfrak{g}$ we let $\theta_{(X)}$ denote the flow of $X$. To prove (a), we need to show that the expression $\theta^{(e)}_{(X)}(1)$ depends smoothly on $X$, which amounts to showing that the flow varies smoothly as the vector field varies. This is a situation not covered by the fundamental theorem on flows, but we can reduce it to that theorem by the following simple trick. Define a vector field $\Xi$ on the product manifold $G \times \mathfrak{g}$ by

$$\Xi_{(g,X)} = (X_g, 0) \in T_gG \oplus T_X\mathfrak{g} \cong T_{(g,X)}(G \times \mathfrak{g}).$$

(See Fig. 20.2.) To see that $\Xi$ is a smooth vector field, choose any basis $(X_1, \dots, X_k)$ for $\mathfrak{g}$, and let $\left(x^i\right)$ be the corresponding global coordinates for $\mathfrak{g}$, defined by $\left(x^i\right) \leftrightarrow x^i X_i$. Let $\left(w^i\right)$ be any smooth local coordinates for $G$. If $f \in C^\infty(G \times \mathfrak{g})$ is arbitrary, then locally we can write

$$\Xi f\left(w^i, x^i\right) = x^j X_j f\left(w^i, x^i\right),$$

where each vector field $X_j$ differentiates $f$ only in the $w^i$-directions. Since this depends smoothly on $\left(w^i, x^i\right)$, it follows from Proposition 8.14 that $\Xi$ is smooth. It is easy to verify that the flow $\Theta$ of $\Xi$ is given by

$$\Theta_t(g, X) = \left(\theta_{(X)}(t, g), X\right).$$

By the fundamental theorem on flows, $\Theta$ is smooth. Since $\exp X = \pi_G\left(\Theta_1(e, X)\right)$, where $\pi_G\colon G \times \mathfrak{g} \to G$ is the projection, it follows that exp is smooth.

Next, (b) and (c) follow immediately from Proposition 20.5, because $t \mapsto \exp tX$ is a group homomorphism from $\mathbb{R}$ to $G$. Then (d) for nonnegative $n$ follows from (b) by induction, and for negative $n$ it follows from (c).

To prove (e), let $X \in \mathfrak{g}$ be arbitrary, and let $\sigma\colon \mathbb{R} \to \mathfrak{g}$ be the curve $\sigma(t) = tX$. Then $\sigma'(0) = X$, and Proposition 20.5 implies

$$(d\exp)_0(X) = (d\exp)_0\left(\sigma'(0)\right) = (\exp \circ \sigma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp tX = X.$$

Part (f) then follows immediately from (e) and the inverse function theorem.

Next, to prove (g) we need to show that $\exp(\Phi_* X) = \Phi(\exp X)$ for every $X \in \mathfrak{g}$. In fact, we will show that for all $t \in \mathbb{R}$,

$$\exp(t\Phi_* X) = \Phi(\exp tX).$$

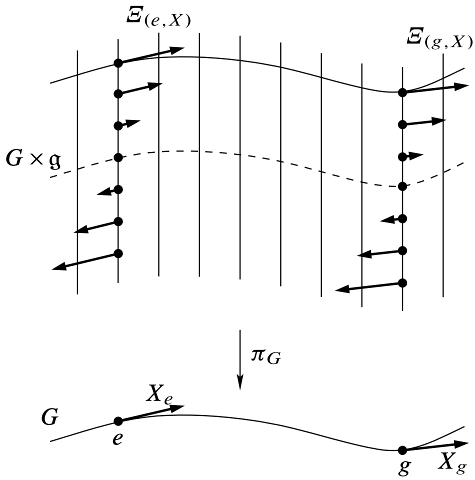


**Fig. 20.2** Proof that the exponential map is smooth

The left-hand side is, by Proposition 20.5, the one-parameter subgroup generated by $\Phi_* X$. Thus, if we put $\sigma(t) = \Phi(\exp tX)$, it suffices to show that $\sigma\colon \mathbb{R} \to H$ is a Lie group homomorphism satisfying $\sigma'(0) = (\Phi_* X)_e$. It is a Lie group homomorphism because it is the composition of the homomorphisms $\Phi$ and $t \mapsto \exp tX$. The initial velocity is computed as follows:

$$\sigma'(0) = \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp tX) = d\Phi_0 \left( \left.\frac{d}{dt}\right|_{t=0} \exp tX \right) = d\Phi_0(X_e) = (\Phi_* X)_e.$$

Finally, to show that $\left(\theta_{(X)}\right)_t = R_{\exp tX}$, we use the fact that for any $g \in G$, the left multiplication map $L_g$ takes integral curves of $X$ to integral curves of $X$. Thus, the map $t \mapsto L_g(\exp tX)$ is the integral curve starting at $g$, which means it is equal to $\theta^{(g)}_{(X)}(t)$. It follows that

$$R_{\exp tX}(g) = g \exp tX = L_g(\exp tX) = \theta^{(g)}_{(X)}(t) = \left(\theta_{(X)}\right)_t (g). \qquad \square$$

It is important to notice that Proposition 20.8(b) does not imply $\exp(X+Y) = (\exp X)(\exp Y)$ for arbitrary $X, Y$ in the Lie algebra. In fact, for connected groups, this is true only when the group is abelian (see Problem 20-8).

The exponential map yields the following alternative characterization of the Lie subalgebra of a subgroup. We will use this later in the chapter when we study normal subgroups.

**Proposition 20.9.** *Let $G$ be a Lie group, and let $H \subseteq G$ be a Lie subgroup. With* $\mathrm{Lie}(H)$ *considered as a subalgebra of* $\mathrm{Lie}(G)$ *in the usual way, the exponential map of $H$ is the restriction to* $\mathrm{Lie}(H)$ *of the exponential map of $G$, and*

$$\mathrm{Lie}(H) = \left\{ X \in \mathrm{Lie}(G) : \exp tX \in H \textit{ for all } t \in \mathbb{R} \right\}.$$

*Proof.* The fact that the exponential map of $H$ is the restriction of that of $G$ is an immediate consequence of Proposition 20.3. To prove the second assertion, by the way we have identified $\mathrm{Lie}(H)$ as a subalgebra of $\mathrm{Lie}(G)$, we need to establish the following equivalence for every $X \in \mathrm{Lie}(G)$:

$$\exp tX \in H \quad \text{for all } t \in \mathbb{R} \quad \Leftrightarrow \quad X_e \in T_e H.$$

Assume first that $\exp tX \in H$ for all $t$. Since $H$ is weakly embedded in $G$ by Theorem 19.25, it follows that the curve $t \mapsto \exp tX$ is smooth as a map into $H$, and thus $X_e = \gamma'(0) \in T_e H$. Conversely, if $X_e \in T_e H$, then Proposition 20.3 implies that $\exp tX \in H$ for all $t$. □

## The Closed Subgroup Theorem

Recall that in Theorem 7.21 we showed that a Lie subgroup is embedded if and only if it is closed. In this section, we use the exponential map to prove a much stronger form of that theorem, showing that if a subgroup of a Lie group is topologically a closed subset, then it is actually an embedded Lie subgroup.

We begin with a simple result that shows how group multiplication in $G$ is reflected "to first order" in the vector space structure of its Lie algebra.

**Proposition 20.10.** *Let $G$ be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For any $X, Y \in \mathfrak{g}$, there is a smooth function $Z\colon (-\varepsilon, \varepsilon) \to \mathfrak{g}$ for some $\varepsilon > 0$ such that the following identity holds for all $t \in (-\varepsilon, \varepsilon)$:*

$$(\exp tX)(\exp tY) = \exp\big(t(X + Y) + t^2 Z(t)\big). \tag{20.4}$$

*Proof.* Since the exponential map is a diffeomorphism on some neighborhood of the origin in $\mathfrak{g}$, there is some $\varepsilon > 0$ such that the map $\varphi\colon (-\varepsilon, \varepsilon) \to \mathfrak{g}$ defined by

$$\varphi(t) = \exp^{-1}(\exp tX \exp tY)$$

is smooth. It obviously satisfies $\varphi(0) = 0$ and

$$\exp tX \exp tY = \exp \varphi(t).$$

Observe that we can write $\varphi$ as the composition

$$\mathbb{R} \xrightarrow{e_X \times e_Y} G \times G \xrightarrow{m} G \xrightarrow{\exp^{-1}} \mathfrak{g},$$

where $e_X(t) = \exp tX$ and $e_Y(t) = \exp tY$. The result of Problem 7-2 shows that $dm_{(e,e)}(X, Y) = X + Y$ for $X, Y \in T_e G$, which implies

$$\varphi'(0) = \big((d\exp)_0\big)^{-1}\big(e_X'(0) + e_Y'(0)\big) = X + Y.$$

Therefore, Taylor's theorem yields

$$\varphi(t) = t(X + Y) + t^2 Z(t)$$

for some smooth function $Z$. □

**Corollary 20.11.** *Under the hypotheses of the preceding proposition,*

$$\lim_{n\to\infty}\left(\left(\exp\frac{t}{n}X\right)\left(\exp\frac{t}{n}Y\right)\right)^n = \exp t(X+Y). \tag{20.5}$$

*Proof.* Formula (20.4) implies that for any $t \in \mathbb{R}$ and any sufficiently large $n \in \mathbb{Z}$,

$$\left(\exp\frac{t}{n}X\right)\left(\exp\frac{t}{n}Y\right) = \exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right),$$

and then Proposition 20.8(d) yields

$$\begin{aligned}\left(\left(\exp\frac{t}{n}X\right)\left(\exp\frac{t}{n}Y\right)\right)^n &= \left(\exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right)\right)^n \\ &= \exp\left(t(X+Y) + \frac{t^2}{n}Z\left(\frac{t}{n}\right)\right).\end{aligned}$$

Fixing $t$ and taking the limit as $n \to \infty$, we obtain (20.5). □

**Theorem 20.12 (Closed Subgroup Theorem).** *Suppose $G$ is a Lie group and $H \subseteq G$ is a subgroup that is also a closed subset of $G$. Then $H$ is an embedded Lie subgroup.*

*Proof.* By Proposition 7.11, it suffices to show that $H$ is an embedded submanifold of $G$. We begin by identifying a subspace of $\operatorname{Lie}(G)$ that will turn out to be the Lie algebra of $H$.

Let $\mathfrak{g} = \operatorname{Lie}(G)$, and define a subset $\mathfrak{h} \subseteq \mathfrak{g}$ by

$$\mathfrak{h} = \{X \in \mathfrak{g} : \exp tX \in H \text{ for all } t \in \mathbb{R}\}.$$

We need to show that $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$. It is obvious from the definition that $\mathfrak{h}$ is closed under scalar multiplication: if $X \in \mathfrak{h}$, then $tX \in \mathfrak{h}$ for all $t \in \mathbb{R}$. Suppose $X, Y \in \mathfrak{h}$, and let $t \in \mathbb{R}$ be arbitrary. Then $\exp\big((t/n)X\big)$ and $\exp\big((t/n)Y\big)$ are in $H$ for each positive integer $n$, and because $H$ is a closed subgroup of $G$, (20.5) implies that $\exp t(X+Y) \in H$. Thus $X + Y \in \mathfrak{h}$, so $\mathfrak{h}$ is a subspace.

Next we show that there is a neighborhood $U$ of the origin in $\mathfrak{g}$ on which the exponential map of $G$ is a diffeomorphism, and which has the property that

$$\exp(U \cap \mathfrak{h}) = (\exp U) \cap H. \tag{20.6}$$

(See Fig. 20.3.) This will enable us to construct a slice chart for $H$ in a neighborhood of the identity, and we will then use left translation to get a slice chart in a neighborhood of any point of $H$.

If $U \subseteq \mathfrak{g}$ is any neighborhood of 0 on which exp is a diffeomorphism, then $\exp(U \cap \mathfrak{h}) \subseteq (\exp U) \cap H$ by definition of $\mathfrak{h}$. So to find a neighborhood satisfying (20.6), all we need to do is to show that $U$ can be chosen small enough that $(\exp U) \cap H \subseteq \exp(U \cap \mathfrak{h})$. Assume this is not possible.

Choose a linear subspace $\mathfrak{b} \subseteq \mathfrak{g}$ that is complementary to $\mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$ as vector spaces. By the result of Problem 20-3, the map $\Phi\colon \mathfrak{h} \oplus \mathfrak{b} \to G$ given

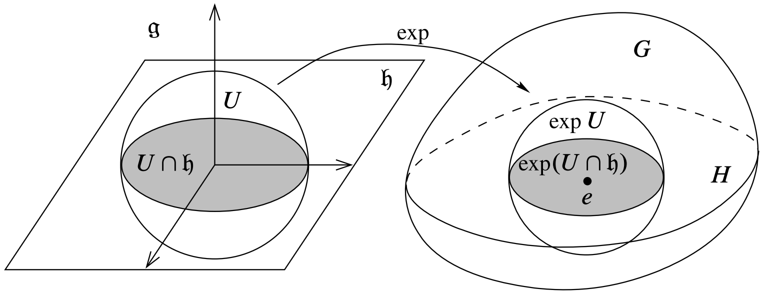


**Fig. 20.3** A neighborhood used to construct a slice chart for $H$

by $\Phi(X,Y) = \exp X \exp Y$ is a diffeomorphism in some neighborhood of $(0,0)$. Choose neighborhoods $U_0$ of 0 in $\mathfrak{g}$ and $\widetilde{U}_0$ of $(0,0)$ in $\mathfrak{h} \oplus \mathfrak{b}$ such that both $\exp|_{U_0}$ and $\Phi|_{\widetilde{U}_0}$ are diffeomorphisms onto their images. Let $\{U_i\}$ be a countable neighborhood basis for $\mathfrak{g}$ at 0 (e.g., a countable sequence of coordinate balls whose radii approach zero). If we set $V_i = \exp(U_i)$ and $\widetilde{U}_i = \Phi^{-1}(V_i)$, then $\{V_i\}$ and $\{\widetilde{U}_i\}$ are neighborhood bases for $G$ at $e$ and $\mathfrak{h} \oplus \mathfrak{b}$ at $(0,0)$, respectively. By discarding finitely many terms at the beginning of the sequence, we may assume that $U_i \subseteq U_0$ and $\widetilde{U}_i \subseteq \widetilde{U}_0$ for each $i$.

Our assumption implies that for each $i$, there exists $h_i \in (\exp U_i) \cap H$ such that $h_i \notin \exp(U_i \cap \mathfrak{h})$. This means $h_i = \exp Z_i$ for some $Z_i \in U_i$. Because $\exp(U_i) = \Phi(\widetilde{U}_i)$, we can also write

$$h_i = \exp X_i \exp Y_i$$

for some $(X_i, Y_i) \in \widetilde{U}_i$. If $Y_i$ were zero, then we would have $\exp Z_i = \exp X_i \in \exp(\mathfrak{h})$; but because exp is injective on $U_0$, this implies $X_i = Z_i \in U_i \cap \mathfrak{h}$, which contradicts our assumption that $h_i \notin \exp(U_i \cap \mathfrak{h})$ (Fig. 20.4). Since $\{\widetilde{U}_i\}$ is a neighborhood basis, $Y_i \to 0$ as $i \to \infty$. Observe that $\exp X_i \in H$ by definition of $\mathfrak{h}$, so it follows that $\exp Y_i = (\exp X_i)^{-1} h_i \in H$ as well.

Choose an inner product on $\mathfrak{b}$ and let $|\cdot|$ denote the norm associated with this inner product. If we define $c_i = |Y_i|$, then we have $c_i \to 0$ as $i \to \infty$. The sequence $\left(c_i^{-1} Y_i\right)$ lies on the unit sphere in $\mathfrak{b}$, so replacing it by a subsequence we may assume that $c_i^{-1} Y_i \to Y \in \mathfrak{b}$, with $|Y| = 1$ by continuity. In particular, $Y \neq 0$. We will show that $\exp tY \in H$ for all $t \in \mathbb{R}$, which implies that $Y \in \mathfrak{h}$. Since $\mathfrak{h} \cap \mathfrak{b} = \{0\}$, this is a contradiction.

Let $t \in \mathbb{R}$ be arbitrary, and for each $i$, let $n_i$ be the greatest integer less than or equal to $t/c_i$. Then

$$\left| n_i - \frac{t}{c_i} \right| \le 1,$$

which implies

$$|n_i c_i - t| \le c_i \to 0,$$

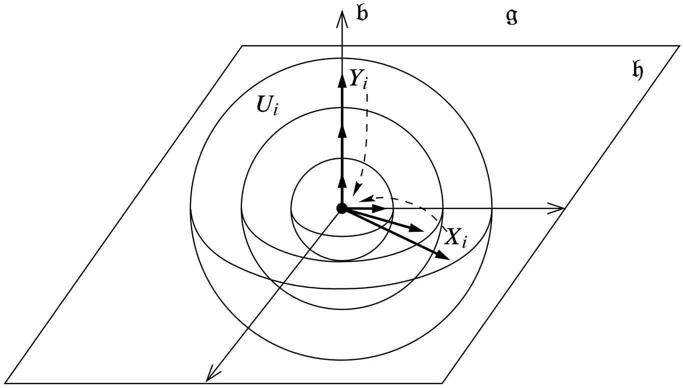


**Fig. 20.4** Proof of the closed subgroup theorem

so $n_i c_i \to t$. Thus,

$$n_i Y_i = (n_i c_i)\left(c_i^{-1} Y_i\right) \to tY,$$

which implies $\exp n_i Y_i \to \exp tY$ by continuity. But $\exp n_i Y_i = (\exp Y_i)^{n_i} \in H$, so the fact that $H$ is closed implies $\exp tY \in H$. This completes the proof of the existence of $U$ satisfying (20.6).

Choose any linear isomorphism $E\colon \mathfrak{g} \to \mathbb{R}^m$ that sends $\mathfrak{h}$ to $\mathbb{R}^k$. The composite map $\varphi = E \circ \exp^{-1}\colon \exp U \to \mathbb{R}^m$ is then a smooth chart for $G$, and $\varphi\big((\exp U) \cap H\big) = E(U \cap \mathfrak{h})$ is the slice obtained by setting the last $m-k$ coordinates equal to zero. Moreover, if $h \in H$ is arbitrary, the left translation map $L_h$ is a diffeomorphism from $\exp U$ to a neighborhood of $h$. Since $H$ is a subgroup, $L_h(H) = H$, and so

$$L_h\big((\exp U) \cap H\big) = L_h(\exp U) \cap H,$$

and $\varphi \circ L_h^{-1}$ is easily seen to be a slice chart for $H$ in a neighborhood of $h$. Thus, $H$ is an embedded submanifold of $G$, hence a Lie subgroup. □

The following corollary summarizes the results of the closed subgroup theorem, Proposition 7.11, and Theorem 7.21.

**Corollary 20.13.** *If $G$ is a Lie group and $H$ is any subgroup of $G$, the following are equivalent:*

(a) *$H$ is closed in $G$.*
(b) *$H$ is an embedded submanifold of $G$.*
(c) *$H$ is an embedded Lie subgroup of $G$.* □

## Infinitesimal Generators of Group Actions

In Chapter 9, we showed that a complete vector field on a manifold generates an action of $\mathbb{R}$ on the manifold. In this section, using the Frobenius theorem and properties of the exponential map, we show how to generalize this notion to actions of higher-dimensional groups.

To begin, we need to specify what we mean by an "infinitesimal generator" of a Lie group action. For reasons that will become apparent, in this section we work primarily with right actions. Afterwards, we will show how the theory has to be modified in the case of left actions (see Theorem 20.18). Because $\mathbb{R}$ is abelian, global flows can be considered either as left actions or as right actions, so everything in this section applies to global flows without modification.

Suppose we are given a smooth right action of a Lie group $G$ on a smooth manifold $M$, which we denote either by $\theta\colon M \times G \to M$ or by $(p,g) \mapsto p \cdot g$, depending on context. Each element $X \in \operatorname{Lie}(G)$ determines a smooth global flow on $M$:

$$(t,p) \mapsto p \cdot \exp tX.$$

(It is a flow because $\exp(0X) = e$ and $(\exp sX)(\exp tX) = \exp(s+t)X$.) Let $\widehat{X} \in \mathfrak{X}(M)$ be the infinitesimal generator of this flow, so for each $p \in M$,

$$\widehat{X}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp tX. \tag{20.7}$$

Thus we obtain a map $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$, defined by $\widehat{\theta}(X) = \widehat{X}$.

There is a useful alternative characterization of $\widehat{X}$ in terms of the orbit map $\theta^{(p)}\colon G \to M$ defined by $\theta^{(p)}(g) = p \cdot g$. Since $\gamma(t) = \exp tX$ is a smooth curve in $G$ whose initial velocity is $\gamma'(0) = X_e$, it follows from Corollary 3.25 that for each $p \in M$ we have

$$d\left(\theta^{(p)}\right)_e(X_e) = \left(\theta^{(p)} \circ \gamma\right)'(0) = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp tX = \widehat{X}_p. \tag{20.8}$$

**Lemma 20.14.** *Suppose $G$ is a Lie group and $\theta$ is a smooth right action of $G$ on a smooth manifold $M$. For any $X \in \operatorname{Lie}(G)$ and $p \in M$, the vector fields $X$ and $\widehat{\theta}(X)$ are $\theta^{(p)}$-related.*

*Proof.* Let $X \in \operatorname{Lie}(G)$ and $p \in M$ be arbitrary, and write $\widehat{X} = \widehat{\theta}(X)$. Note that the group law $p \cdot gg' = (p \cdot g) \cdot g'$ translates to

$$\theta^{(p)} \circ L_g(g') = \theta^{(p \cdot g)}(g'). \tag{20.9}$$

Let $g \in G$ be arbitrary, and write $q = p \cdot g = \theta^{(p)}(g)$. Then (20.9) yields $\theta^{(p)} \circ L_g = \theta^{(q)}$. Using this together with (20.8) and the fact that $X$ is left-invariant, we obtain

$$\widehat{X}_q = d\left(\theta^{(q)}\right)_e(X_e) = d\left(\theta^{(p)}\right)_g \circ d(L_g)_e(X_e) = d\left(\theta^{(p)}\right)_g(X_g),$$

which proves the claim. □

**Theorem 20.15.** *Suppose $G$ is a Lie group and $\theta$ is a smooth right action of $G$ on a smooth manifold $M$. Then the map $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$ defined above is a Lie algebra homomorphism.*

*Proof.* For each $p \in M$, it follows from (20.8) that $\widehat{X}_p$ depends linearly on $X$, so $\widehat{\theta}$ is a linear map. Given $p \in M$, Lemma 20.14 together with the naturality of Lie brackets implies that $[X, Y]$ is $\theta^{(p)}$-related to $\big[\widehat{X}, \widehat{Y}\big]$. This means, in particular, that

$$\big[\widehat{X}, \widehat{Y}\big]_p = d\big(\theta^{(p)}\big)_e\big([X,Y]_e\big) = \widehat{[X,Y]}_p.$$

Since every point of $M$ is in the image of some orbit map, we conclude that $\big[\widehat{\theta}(X), \widehat{\theta}(Y)\big] = \widehat{\theta}\big([X,Y]\big)$ as claimed. □

The Lie algebra homomorphism $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$ defined above is known as the ***infinitesimal generator of*** $\boldsymbol{\theta}$. More generally, if $\mathfrak{g}$ is an arbitrary finite-dimensional Lie algebra, any Lie algebra homomorphism $\widehat{\theta}\colon \mathfrak{g} \to \mathfrak{X}(M)$ is called a **(*right*)** $\boldsymbol{\mathfrak{g}}$***-action on*** $\boldsymbol{M}$. A $\mathfrak{g}$-action $\widehat{\theta}$ is said to be ***complete*** if for every $X \in \mathfrak{g}$, the vector field $\widehat{\theta}(X)$ is complete.

Just as every complete vector field generates an $\mathbb{R}$-action, the next theorem shows that, at least for simply connected groups, every complete Lie algebra action generates a Lie group action.

**Theorem 20.16 (Fundamental Theorem on Lie Algebra Actions).** *Let $M$ be a smooth manifold, let $G$ be a simply connected Lie group, and let $\mathfrak{g} = \operatorname{Lie}(G)$. Suppose $\widehat{\theta}\colon \mathfrak{g} \to \mathfrak{X}(M)$ is a complete $\mathfrak{g}$-action on $M$. Then there is a unique smooth right $G$-action on $M$ whose infinitesimal generator is $\widehat{\theta}$.*

*Proof.* We begin by defining a distribution $D$ on $G \times M$; we will show that $D$ is involutive, and then each leaf will turn out to be the graph of an orbit map $\theta^{(p)}\colon G \to M$. For brevity, given $X \in \mathfrak{g}$, we use the notation $\widehat{X}$ for $\widehat{\theta}(X) \in \mathfrak{X}(M)$.

Define $D$ as follows: for each $X \in \mathfrak{g}$, define a smooth vector field $\widetilde{X}$ on $G \times M$ by

$$\widetilde{X}_{(g,p)} = \big(X_g, \widehat{X}_p\big) \in T_gG \oplus T_pM \cong T_{(g,p)}(G \times M).$$

In the notation of Problem 8-17, this is $\widetilde{X} = X \oplus \widehat{X}$. Then for each $(g,p) \in G \times M$, let $D_{(g,p)}$ be the set of all vectors of the form $\widetilde{X}_{(g,p)}$ as $X$ ranges over $\mathfrak{g}$. If $X_1, \dots, X_k$ is a basis for $\mathfrak{g}$, then the smooth vector fields $\widetilde{X}_1, \dots, \widetilde{X}_k$ are independent and span $D$, so $D$ is a smooth distribution whose rank is equal to the dimension of $G$. To see that it is involutive, note that Problem 8-17 and the fact that $\widehat{\theta}$ is a Lie algebra homomorphism imply

$$\big[\widetilde{X}_i, \widetilde{X}_j\big] = [X_i, X_j] \oplus \big[\widehat{X}_i, \widehat{X}_j\big] = [X_i, X_j] \oplus \widehat{[X_i, X_j]} = \widetilde{[X_i, X_j]}.$$

Let $\mathcal{S}$ denote the foliation determined by $D$, and for each $(g,p) \in G \times M$, let $\mathcal{S}_{(g,p)}$ denote the leaf of $\mathcal{S}$ containing $(g,p)$.

Next we show that $D$ is invariant under a certain $G$-action on $G \times M$. Combining the natural action of $G$ on itself by left translation with the trivial action of $G$ on $M$, we get a left action of $G$ on $G \times M$ given by

$$\psi_g(g', p) = (gg', p).$$

A straightforward computation shows

$$d(\psi_g)_{(g',p)}\big(\widetilde{X}_{(g',p)}\big) = d(\psi_g)_{(g',p)}\big(X_{g'}, \widehat{X}_p\big) = \big(d(L_g)_{g'}(X_{g'}), \widehat{X}_p\big)$$
$$= \big(X_{gg'}, \widehat{X}_p\big) = \widetilde{X}_{(gg',p)},$$

so $D$ is invariant under $\psi_g$ for each $g \in G$. It follows that $\psi_g$ takes leaves of $\mathcal{S}$ to leaves of $\mathcal{S}$ (Proposition 19.23).

Let $\pi_G \colon G \times M \to G$ and $\pi_M \colon G \times M \to M$ denote the projections. Let $p \in M$ be arbitrary, let $S_p = \mathcal{S}_{(e,p)} \subseteq G \times M$ denote the leaf containing $(e,p)$, and let $\Pi_p = \pi_G|_{S_p} \colon S_p \to G$. We will show that $\Pi_p$ is a smooth covering map. To begin with, at each point $(g,q) \in S_p$, $d(\Pi_p)_{(g,p)}\big(\widetilde{X}_{(g,p)}\big) = X_g$ for all $X \in \mathfrak{g}$, so $\Pi_p$ is a smooth submersion, and for dimensional reasons it is a local diffeomorphism.

To show that $\Pi_p$ is a covering map, choose a connected neighborhood $U$ of $e$ in $G$ small enough that the exponential map of $G$ is a diffeomorphism from some neighborhood $V$ of $0$ in $\mathfrak{g}$ onto $U$, and for any $g \in G$, consider the neighborhood $gU = \{gh : h \in U\}$ of $g$. We will show that $gU$ is evenly covered by constructing local sections. For each $q \in M$ such that $(g,q)$ is in the fiber $\Pi_p^{-1}(g)$, define a map $\sigma_q \colon gU \to G \times M$ by

$$\sigma_q(g \exp X) = \Big(g \exp X, \eta_{(\widehat{X})}(1,q)\Big),$$

where $X \in V$ and $\eta_{(\widehat{X})}$ denotes the flow of $\widehat{X}$. It follows immediately from the definition that $\sigma_q$ is smooth and satisfies $\pi_G \circ \sigma_q = \mathrm{Id}_{gU}$, so to show that $\sigma_q$ is a local section of $\Pi_p$, it suffices to show that it takes its values in $S_p$. A straightforward computation shows that $\gamma(t) = \big(g \exp tX, \eta_{(\widehat{X})}(t,q)\big)$ is an integral curve of $\widetilde{X}$ starting at $(g,q)$, from which it follows easily that $\sigma_q(g \exp X) = \gamma(1) \in S_p$. It is smooth because it is a local section of the local diffeomorphism $\Pi_p$.

For each $(g,q) \in \Pi_p^{-1}(g)$, the set $\sigma_q(gU)$ is a connected open subset of $S_p$, which is mapped diffeomorphically onto $gU$ by $\Pi_p$. To complete the proof that $\Pi_p$ is a covering map, we need only prove that every point in $\Pi_p^{-1}(gU)$ is in exactly one such set. First suppose $(g',q') \in \Pi_p^{-1}(gU)$. Then $\Pi_p(g',q') \in gU$ means that $g' = g \exp X$ for some $X \in V$. If we let $q = \eta_{(\widehat{X})}(-1,q')$, then the group law for $\eta_{(\widehat{X})}$ implies that $q' = \eta_{(\widehat{X})}(1,q)$ and therefore $(g',q') = \sigma_q(g \exp X)$. On the other hand, suppose two such sets $\sigma_q(gU)$ and $\sigma_{q'}(gU)$ intersect nontrivially. Then for some $X, X' \in V$, we have $\big(g \exp X, \eta_{(\widehat{X})}(1,q)\big) = \big(g \exp X', \eta_{(\widehat{X}')}(1,q')\big)$, which implies that $X = X'$ and therefore $\eta_{(\widehat{X})}(1,q) = \eta_{(\widehat{X})}(1,q')$; then flowing back along the integral curve of $\widehat{X}$ for time $-1$ shows that $q = q'$. This completes the proof that $\Pi_p$ is a smooth covering map. Because we are assuming $G$ is simply connected, $\Pi_p$ is actually a diffeomorphism.

Now for each $p \in M$, define $\theta^{(p)} \colon G \to M$ by $\theta^{(p)} = \pi_M \circ \Pi_p^{-1}$ (so $S_p$ is the graph of $\theta^{(p)}$), and define an action of $G$ on $M$ by $p \cdot g = \theta^{(p)}(g)$. This is equivalent to declaring that $p \cdot g = q$ if and only if $\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)}$. To show that this is an action, assuming $p \cdot g = q$ and $q \cdot g' = r$, we need to show that $p \cdot gg' = r$.

Equivalently, assuming that $\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)}$ and $\mathcal{S}_{(e,q)} = \mathcal{S}_{(g',r)}$, we need to show that $\mathcal{S}_{(e,p)} = \mathcal{S}_{(gg',r)}$. This follows from $\psi$-invariance:

$$\mathcal{S}_{(e,p)} = \mathcal{S}_{(g,q)} = \psi_g\big(\mathcal{S}_{(e,q)}\big) = \psi_g\big(\mathcal{S}_{(g',r)}\big) = \mathcal{S}_{(gg',r)}.$$

It remains to show that the action is smooth, that $\widehat{\theta}$ is its infinitesimal generator, and that it is the unique such action. For $g = \exp X$ near the identity, the discussion above shows that the action can be expressed as

$$p \cdot g = \theta^{(p)}(\exp X) = \pi_M \circ \sigma_p(\exp X) = \eta_{(\widehat{X})}(1, p). \tag{20.10}$$

An argument analogous to the one we used to prove smoothness of the exponential map (with $\Xi_{(p,X)} = \big(\widehat{X}_g, 0\big)$ on $M \times \mathfrak{g}$) shows that this depends smoothly on $X$ and $p$ and thus on $g$ and $p$. But since any neighborhood of the identity generates $G$ (Proposition 7.14), every element of $G$ can be expressed as a finite product of elements of the form $\exp X$ for $X \in V$, so it follows that $(p, g) \mapsto p \cdot g$ can be written as a finite composition of smooth maps. The fact that the infinitesimal generator of the action is $\widehat{\theta}$ is an immediate consequence of (20.10). Uniqueness is left as an exercise. □

► **Exercise 20.17.** Prove that the action constructed in the previous proof is the unique one that has $\widehat{\theta}$ as its infinitesimal generator.

## *Left Actions*

The situation for left actions is similar, but with a slight twist. Let $G$ be a Lie group and $M$ be a smooth manifold. If $\theta\colon G \times M \to M$ is a smooth left action of $G$ on $M$, define the ***infinitesimal generator of $\theta$*** as the map $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$ given by $\widehat{\theta}(X) = \widehat{X}$, where

$$\widehat{X}_p = \frac{d}{dt}\bigg|_{t=0} \big((\exp tX) \cdot p\big) = d\big(\theta^{(p)}\big)_e(X_e), \tag{20.11}$$

and $\theta^{(p)}\colon G \to M$ is the orbit map $\theta^{(p)}(g) = g \cdot p$.

We have the following analogue of Theorems 20.15 and 20.16 for left actions.

**Theorem 20.18.** *Suppose $G$ is a Lie group and $M$ is a smooth manifold.*

(a) *If $\theta$ is a smooth left action of $G$ on $M$, the map $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$ defined by (20.11) is an* ***antihomomorphism*** *(a linear map satisfying $\widehat{\theta}\big([X,Y]\big) = -\big[\widehat{\theta}(X), \widehat{\theta}(Y)\big]$ for all $X, Y \in \operatorname{Lie}(G)$).*
(b) *Conversely, if $G$ is simply connected, every antihomomorphism $\widehat{\theta}\colon \operatorname{Lie}(G) \to \mathfrak{X}(M)$ such that $\widehat{\theta}(X)$ is complete for each $X \in \operatorname{Lie}(G)$ is the infinitesimal generator of a unique left $G$-action.*

*Proof.* Problem 20-15. □

Because of this theorem, for a finite-dimensional Lie algebra $\mathfrak{g}$ and a smooth manifold $M$, a ***left $\mathfrak{g}$-action on $M$*** is defined as an antihomomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M)$.

## The Lie Correspondence

Many of our results about Lie groups show how essential properties of a Lie group are reflected in its Lie algebra, and vice versa. This raises a natural question: To what extent is the correspondence between Lie groups and Lie algebras (or at least between their isomorphism classes) one-to-one?

We have already seen in Chapter 8 that the assignment that sends a Lie group to its Lie algebra and a Lie group homomorphism to its induced Lie algebra homomorphism is a functor from the category of Lie groups to the category of finite-dimensional Lie algebras. Because functors take isomorphisms to isomorphisms, it follows that isomorphic Lie groups have isomorphic Lie algebras. The converse is easily seen to be false: both $\mathbb{R}^n$ and $\mathbb{T}^n$ have $n$-dimensional abelian Lie algebras, which are obviously isomorphic to each other, but $\mathbb{R}^n$ and $\mathbb{T}^n$ are certainly not isomorphic Lie groups. However, as we will see in this section, if we restrict our attention to simply connected Lie groups, then we do obtain a one-to-one correspondence.

In order to prove this correspondence, we need a way to construct an isomorphism between simply connected Lie groups when we are given an isomorphism between their algebras. Theorem 8.44 showed that every Lie group homomorphism gives rise to a Lie algebra homomorphism. Using the fundamental theorem on Lie algebra actions, we can prove the following partial converse.

**Theorem 20.19.** *Suppose $G$ and $H$ are Lie groups with $G$ simply connected, and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. For any Lie algebra homomorphism $\varphi\colon \mathfrak{g}\to\mathfrak{h}$, there is a unique Lie group homomorphism $\Phi\colon G\to H$ such that $\Phi_*=\varphi$.*

*Proof.* The Lie algebra homomorphism $\varphi\colon \mathfrak{g}\to\mathfrak{h}\subseteq\mathfrak{X}(H)$ is, in particular, a complete $\mathfrak{g}$-action on $H$ (since every left-invariant vector field is complete). Thus, by Theorem 20.16, there is a unique smooth right $G$-action $\theta\colon H\times G\to H$ for which $\varphi$ is the infinitesimal generator. Let us use the notation $\widehat{X}=\varphi(X)$ for $X\in\mathfrak{g}$, and the notation $h\cdot g=\theta(h,g)$ for $h\in H$ and $g\in G$. Define a smooth map $\Phi\colon G\to H$ by $\Phi(g)=e\cdot g$ (where $e$ is the identity in $H$). We will show that $\Phi$ is the desired homomorphism.

Lemma 20.14 shows that for each $h\in H$ and each $X\in\mathfrak{g}$, the vector fields $X$ and $\widehat{X}$ are $\theta^{(h)}$-related. By Proposition 9.6, $\theta^{(h)}$ takes integral curves of $X$ to integral curves of $\widehat{X}$. Therefore, $t\mapsto h\cdot\exp tX$ is the integral curve of $\widehat{X}$ starting at $h$.

On the other hand, because $\widehat{X}$ is a left-invariant vector field on $H$, left translation in $H$ takes integral curves of $\widehat{X}$ to integral curves of $\widehat{X}$. For any $h\in H$ and $X\in\mathfrak{g}$, therefore,

$$L_h(e\cdot\exp tX)=h\cdot\exp tX. \tag{20.12}$$

Applying this with $h = e \cdot g$ for some $g \in G$, we get

$$(e \cdot g)(e \cdot \exp tX) = (e \cdot g) \cdot \exp tX = e \cdot g \exp tX.$$

(The last equality follows from the fact that $\theta$ is an action.) Rewritten in terms of $\Phi$, this says

$$\Phi(g)\Phi(\exp tX) = \Phi(g \exp tX).$$

Since $G$ is connected, it is generated by the image of the exponential map by Proposition 7.14, so this implies that $\Phi$ is a homomorphism.

To see that $\Phi_* = \varphi$, let $X \in \mathfrak{g}$ be arbitrary. The fact that $\varphi$ is the infinitesimal generator of $\theta$ means

$$\varphi(X)\big|_e = \frac{d}{dt}\bigg|_{t=0} (e \cdot \exp tX) = \frac{d}{dt}\bigg|_{t=0} \Phi(\exp tX) = d\Phi_e(X_e).$$

Since $\Phi_*$ is determined by the action of $d\Phi_e$, this implies $\Phi_* X = \varphi(X)$.

The proof is completed by invoking Problem 20-17, which shows that $\Phi$ is the unique homomorphism with this property. □

**Corollary 20.20.** *If $G$ and $H$ are simply connected Lie groups with isomorphic Lie algebras, then $G$ and $H$ are isomorphic.*

*Proof.* Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of $G$ and $H$, respectively, and let $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra isomorphism between them. By the preceding theorem, there are Lie group homomorphisms $\Phi\colon G \to H$ and $\Psi\colon H \to G$ satisfying $\Phi_* = \varphi$ and $\Psi_* = \varphi^{-1}$. Both the identity map of $G$ and the composition $\Psi \circ \Phi$ are Lie group homomorphisms from $G$ to itself whose induced Lie algebra homomorphisms are equal to the identity, so the uniqueness part of Theorem 20.19 implies that $\Psi \circ \Phi = \mathrm{Id}_G$. Similarly, $\Phi \circ \Psi = \mathrm{Id}_H$, so $\Phi$ is a Lie group isomorphism. □

Now we are ready for our main theorem.

**Theorem 20.21 (The Lie Correspondence).** *There is a one-to-one correspondence between isomorphism classes of finite-dimensional Lie algebras and isomorphism classes of simply connected Lie groups, given by associating each simply connected Lie group with its Lie algebra.*

*Proof.* We need to show that the functor that sends a simply connected Lie group to its Lie algebra is both surjective and injective up to isomorphism. Injectivity is precisely the content of Corollary 20.20.

To prove surjectivity, suppose $\mathfrak{g}$ is any finite-dimensional Lie algebra. By Corollary 8.50 to Ado's theorem, we may replace $\mathfrak{g}$ by an isomorphic Lie subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{gl}(n,\mathbb{R})$. By Theorem 19.26, there is a connected Lie subgroup $G_0 \subseteq \mathrm{GL}(n,\mathbb{R})$ that has $\mathfrak{g}_0$ as its Lie algebra. If $G$ is the universal covering group of $G_0$, Problem 8-27 shows that $\mathrm{Lie}(G) \cong \mathrm{Lie}(G_0) \cong \mathfrak{g}_0 \cong \mathfrak{g}$. □

In the next chapter, we will see what happens when we remove the restriction to simply connected groups (see Theorem 21.32).

## *Lie's Fundamental Theorems*

As the name of the previous theorem suggests, a version of the Lie correspondence theorem was proved in the nineteenth century by Sophus Lie. However, since global topological notions such as manifolds and simple connectivity had not yet been formulated, what he was able to prove was essentially a local version of the theorem.

Instead of considering Lie groups as abstract objects, Lie worked with vector fields on open subsets of Euclidean space, and the (local) group actions they generate. Define a ***local Lie group*** to be an open subset $U$ in some finite-dimensional vector space $V$, together with an element $e \in U$ and smooth maps $m\colon U \times U \to V$ (multiplication) and $i\colon U \to V$ (inversion), satisfying the following identities for all $g, h, k$ sufficiently close to $e$ that both sides are defined:

$$\begin{aligned} m\big(g, m(h,k)\big) &= m\big(m(g,h), k\big) && \text{(associativity);}\\ m(e,g) = g &= m(g,e) && \text{(identity);}\\ m\big(i(g), g\big) = e &= m\big(g, i(g)\big) && \text{(inverses).}\end{aligned}$$

The left translation map $L_g\colon U \to V$ is defined just as for ordinary Lie groups, and a vector field $X \in \mathfrak{X}(U)$ is said to be ***left-invariant*** if $d(L_g)_{g'}(X_{g'}) = X_{m(g,g')}$ for all $g, g' \in U$ such that $m(g,g') \in U$. Two local Lie groups $(U, e, m, i)$ and $(U', e', m', i')$ are said to be ***locally isomorphic*** if there is a diffeomorphism from a neighborhood of $e$ in $U$ to a neighborhood of $e'$ in $U'$ that takes $e$ to $e'$, $m$ to $m'$, and $i$ to $i'$, whenever the respective operations are defined. A ***local (left or right) action of a local Lie group*** on an open subset $W \subseteq \mathbb{R}^n$ is defined like an ordinary action, except that $g \cdot x$ (or $x \cdot g$) is required to be defined only for $(g, x)$ in a neighborhood of $\{e\} \times W$ in $U \times W$. A coordinate neighborhood of the identity in any Lie group is a local Lie group, and any smooth action of a Lie group on a smooth manifold restricts to a local action on any sufficiently small coordinate neighborhood.

**Theorem 20.22 (The Fundamental Theorems of Sophus Lie).**

(i) FIRST FUNDAMENTAL THEOREM: *The set of left-invariant vector fields on a local Lie group is a finite-dimensional Lie algebra under Lie bracket, and two local Lie groups with isomorphic Lie algebras are locally isomorphic.*

(ii) SECOND FUNDAMENTAL THEOREM: *Given an open subset $W \subseteq \mathbb{R}^n$, there is a one-to-one correspondence between smooth right actions of local Lie groups on $W$ and finite-dimensional Lie subalgebras of $\mathfrak{X}(W)$.*

(iii) THIRD FUNDAMENTAL THEOREM: *Given any finite-dimensional abstract Lie algebra $\mathfrak{g}$, there exists a local Lie group whose algebra of left-invariant vector fields is isomorphic to $\mathfrak{g}$.*

It is an interesting exercise to see if you can adapt the techniques of this chapter to prove these theorems. (See Problem 20-19.)

## Normal Subgroups

Normal subgroups (those that are invariant under conjugation) play a central role in abstract group theory: they are the only subgroups whose quotients have group structures, and the only subgroups that are kernels of group homomorphisms.

For Lie groups, the following criterion for normality is useful. It says that for a connected Lie group, normality need only be checked for elements that are in the image of the exponential map, because such elements generate the group.

**Lemma 20.23.** *Let $G$ be a connected Lie group, and let $H \subseteq G$ be a connected Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of $G$ and $H$, respectively. Then $H$ is normal in $G$ if and only if*

$$(\exp X)(\exp Y)\big(\exp(-X)\big) \in H \quad \textit{for all } X \in \mathfrak{g} \textit{ and } Y \in \mathfrak{h}. \tag{20.13}$$

*Proof.* Note that $\exp(-X) = (\exp X)^{-1}$. Thus if $H$ is normal, then (20.13) holds by definition. Conversely, suppose (20.13) holds, and choose open subsets $V \subseteq \mathfrak{g}$ containing 0 and $U \subseteq G$ containing the identity such that $\exp\colon V \to U$ is a diffeomorphism. Since the exponential map of $H$ is the restriction of that of $G$, after shrinking $V$ if necessary, we may assume that the restriction of exp to $V \cap \mathfrak{h}$ is a diffeomorphism from $V \cap \mathfrak{h}$ to a neighborhood $U_0$ of the identity in $H$. Shrinking $V$ still further, we may assume also that $X \in V$ if and only if $-X \in V$. Then (20.13) implies that $ghg^{-1} \in H$ whenever $g \in U$ and $h \in U_0$.

Since every element of $H$ can be written as a finite product $h = h_1 \cdots h_m$ with $h_1, \dots, h_m \in U_0$ (Proposition 7.14), it follows that for any $g \in U$ and $h \in H$ we have

$$ghg^{-1} = gh_1 \cdots h_m g^{-1} = \big(gh_1 g^{-1}\big) \cdots \big(gh_m g^{-1}\big) \in H. \tag{20.14}$$

Similarly, any $g \in G$ can be written $g = g_1 \cdots g_k$ with $g_1, \dots, g_k \in U$, so it follows by induction on $k$ that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. □

Our next goal is a theorem that expresses a deep relationship between Lie groups and their Lie algebras. If $\mathfrak{g}$ is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called an ***ideal in*** $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ (see Problem 8-31). Because ideals are kernels of Lie algebra homomorphisms and normal subgroups are kernels of Lie group homomorphisms, it should not be surprising that there is a connection between ideals and normal subgroups. The key to analyzing this connection is the adjoint representation, which we study next.

### *The Adjoint Representation*

Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. For any $g \in G$, the conjugation map $C_g\colon G \to G$ given by $C_g(h) = ghg^{-1}$ is a Lie group homomorphism (see Example 7.4(f)). We let $\operatorname{Ad}(g) = (C_g)_*\colon \mathfrak{g} \to \mathfrak{g}$ denote its induced Lie algebra homomorphism.

**Proposition 20.24 (The Adjoint Representation).** *If $G$ is a Lie group with Lie algebra $\mathfrak{g}$, the map $\operatorname{Ad}\colon G \to \operatorname{GL}(\mathfrak{g})$ is a Lie group representation, called the* ***adjoint representation of $G$****.*

*Proof.* Because $C_{g_1 g_2} = C_{g_1} \circ C_{g_2}$ for any $g_1, g_2 \in G$, it follows immediately that $\operatorname{Ad}(g_1 g_2) = \operatorname{Ad}(g_1) \circ \operatorname{Ad}(g_2)$, and $\operatorname{Ad}(g)$ is invertible with inverse $\operatorname{Ad}\big(g^{-1}\big)$.

To see that Ad is smooth, let $C\colon G \times G \to G$ be the smooth map defined by $C(g,h) = ghg^{-1}$. Let $X \in \mathfrak{g}$ and $g \in G$ be arbitrary. Then $\operatorname{Ad}(g)X$ is the left-invariant vector field whose value at $e \in G$ is

$$\big((C_g)_* X\big)_e = \left.\frac{d}{dt}\right|_{t=0} C_g(\exp tX) = \left.\frac{d}{dt}\right|_{t=0} C(g, \exp tX) = dC_{(g,e)}(0, X_e),$$

where we are regarding $(0, X_e)$ as an element of $T_{(g,e)}(G \times G) \cong T_g G \oplus T_e G$. Because $dC\colon T(G \times G) \to TG$ is a smooth bundle homomorphism by Example 10.28(a), this expression depends smoothly on $g$ and $X$. Smooth coordinates on $\operatorname{GL}(\mathfrak{g})$ are obtained by choosing a basis $(E_i)$ for $\mathfrak{g}$ and using matrix entries with respect to this basis as coordinates. If $(\varepsilon^j)$ is the dual basis, the matrix entries of $\operatorname{Ad}(g)\colon \mathfrak{g} \to \mathfrak{g}$ are given by $\big(\operatorname{Ad}(g)\big)_i^j = \varepsilon^j\big(\operatorname{Ad}(g)E_i\big)$. The computation above with $X = E_i$ shows that these are smooth functions of $g$. □

There is also an adjoint representation for Lie algebras. Given a finite-dimensional Lie algebra $\mathfrak{g}$, for each $X \in \mathfrak{g}$, define a map $\operatorname{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ by $\operatorname{ad}(X)Y = [X, Y]$.

**Proposition 20.25.** *For any Lie algebra $\mathfrak{g}$, the map $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation, called the* ***adjoint representation of $\mathfrak{g}$****.*

► **Exercise 20.26.** Prove the preceding proposition.

Using the exponential map, we can show that these two representations are intimately related.

**Theorem 20.27.** *Let $G$ be a Lie group, let $\mathfrak{g}$ be its Lie algebra, and let $\operatorname{Ad}\colon G \to \operatorname{GL}(\mathfrak{g})$ be the adjoint representation of $G$. The induced Lie algebra representation $\operatorname{Ad}_*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $\operatorname{Ad}_* = \operatorname{ad}$.*

*Proof.* Let $X \in \mathfrak{g}$ be arbitrary. Then $\operatorname{Ad}_* X$ is determined by its value at the identity, which we can interpret as an element of $\mathfrak{gl}(\mathfrak{g})$, the set of all linear maps from $\mathfrak{g}$ to itself. Because $t \mapsto \exp tX$ is a smooth curve in $G$ whose velocity vector at $t = 0$ is $X_e$, we can compute the action of $\operatorname{Ad}_* X$ on an element $Y \in \mathfrak{g}$ by

$$(\operatorname{Ad}_* X)Y = \left(\left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}(\exp tX)\right) Y = \left.\frac{d}{dt}\right|_{t=0} \big(\operatorname{Ad}(\exp tX)Y\big).$$

As an element of $\mathfrak{g}$, $\operatorname{Ad}(\exp tX)Y$ is a left-invariant vector field on $G$, and thus is itself determined by its value at the identity. Using the fact that $\operatorname{Ad}(g) = (C_g)_* =$

$(R_{g^{-1}})_* \circ (L_g)_*$, its value at $e \in G$ can be computed as

$$\begin{aligned}\big(\mathrm{Ad}(\exp tX)Y\big)_e &= d(R_{\exp(-tX)}) \circ d(L_{\exp tX})(Y_e) \\ &= d(R_{\exp(-tX)})(Y_{\exp tX}).\end{aligned} \tag{20.15}$$

Recall from Proposition 20.8(h) that the flow of $X$ is given by $\theta_t(g) = R_{\exp tX}(g)$. Therefore, (20.15) can be rewritten as

$$\big(\mathrm{Ad}(\exp tX)Y\big)_e = d(\theta_{-t})(Y_{\theta_t(e)}).$$

Taking the derivative with respect to $t$ and setting $t = 0$, we obtain

$$\big((\mathrm{Ad}_* X)Y\big)_e = \left.\frac{d}{dt}\right|_{t=0} d(\theta_{-t})(Y_{\theta_t(e)}) = (\mathscr{L}_X Y)_e = [X,Y]_e.$$

Since $(\mathrm{Ad}_* X)Y$ is determined by its value at $e$, this completes the proof. □

## *Ideals and Normal Subgroups*

Now we are in a position to prove the main theorem of this section.

**Theorem 20.28 (Ideals and Normal Subgroups).** *Let $G$ be a connected Lie group, and suppose $H \subseteq G$ is a connected Lie subgroup. Then $H$ is a normal subgroup of $G$ if and only if* $\mathrm{Lie}(H)$ *is an ideal in* $\mathrm{Lie}(G)$.

*Proof.* Write $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$, considering $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g}$. For any $g \in G$, the commutative diagram (20.3) applied to the Lie group homomorphism $C_g(h) = ghg^{-1}$ yields

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}(g)} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[C_g]{} & G. \end{array} \tag{20.16}$$

Suppose that $\mathfrak{h}$ is an ideal. Applying (20.16) to $Y \in \mathfrak{h}$ with $g = \exp X$, we obtain

$$\exp\big(\mathrm{Ad}(\exp X)Y\big) = C_{\exp X}(\exp Y) = (\exp X)(\exp Y)\big(\exp(-X)\big). \tag{20.17}$$

On the other hand, applying (20.3) to the homomorphism $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ and noting that $\mathrm{Ad}_* = \mathrm{ad}$ by Theorem 20.27, we obtain

$$\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}\, X). \tag{20.18}$$

Formula (20.2) for the exponential map of the group $\mathrm{GL}(\mathfrak{g})$ reads

$$\mathrm{Ad}(\exp X)Y = \big(\exp(\mathrm{ad}\, X)\big)Y = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}\, X)^k Y. \tag{20.19}$$

Whenever $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, we have $(\operatorname{ad} X)Y = [X, Y] \in \mathfrak{h}$, and by induction $(\operatorname{ad} X)^k Y \in \mathfrak{h}$ for all $k$. Therefore, (20.19) implies that $\operatorname{Ad}(\exp X)Y \in \mathfrak{h}$, and so (20.17) implies that $(\exp X)(\exp Y)\big(\exp(-X)\big) \in \exp \mathfrak{h} \subseteq H$. By Lemma 20.23, $H$ is normal.

Conversely, suppose $H$ is normal. Given $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$, note that (20.16) applied to $sY$ with $g = \exp tX$ implies

$$\exp\big(\operatorname{Ad}(\exp tX)sY\big) = (\exp tX)(\exp sY)(\exp tX)^{-1} \in H.$$

Since $\operatorname{Ad}(\exp tX)$ is linear over $\mathbb{R}$, it follows that

$$\exp\big(s \operatorname{Ad}(\exp tX)Y\big) = \exp\big(\operatorname{Ad}(\exp tX)sY\big),$$

which we have just shown to be in $H$ for all $s$, so $\operatorname{Ad}(\exp tX)Y \in \mathfrak{h}$ by Proposition 20.9. From the proof of Theorem 20.27, we have

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Ad}(\exp tX)Y = [X, Y],$$

and therefore $[X, Y] \in \mathfrak{h}$, so $\mathfrak{h}$ is an ideal. □

## Problems

20-1. Let $G$ be a Lie group.
(a) Show that the images of one-parameter subgroups in $G$ are precisely the connected Lie subgroups of dimension less than or equal to 1.
(b) Show that the image of every one-parameter subgroup is isomorphic as a Lie group to one of the following: $\mathbb{R}$, $\mathbb{S}^1$, or the trivial group $\{e\}$.

20-2. Compute the exponential maps of the abelian Lie groups $\mathbb{R}^n$ and $\mathbb{T}^n$.

20-3. Let $G$ be a Lie group, and suppose $A, B \subseteq \mathfrak{g}$ are complementary linear subspaces of $\operatorname{Lie}(G)$. Show that the map $A \oplus B \to G$ given by $(X, Y) \mapsto \exp X \exp Y$ is a diffeomorphism from some neighborhood of $(0, 0)$ in $A \oplus B$ to a neighborhood of $e$ in $G$. (*Used on p. 523.*)

20-4. Show that the matrix exponential satisfies the identity

$$\det e^A = e^{\operatorname{tr} A}.$$

[Hint: apply Proposition 20.8(g) to $\det\colon \operatorname{GL}(n, \mathbb{R}) \to \mathbb{R}^*$.]

20-5. Let $a, b, c$ be real numbers, and let $A$, $B$, and $C$ be the following elements of $\mathfrak{gl}(3, \mathbb{R})$:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}; \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Give explicit formulas (not infinite series) for the one-parameter subgroups of $\operatorname{GL}(3, \mathbb{R})$ generated by $A$, $B$, and $C$.

20-6. This problem shows that the exponential map of a connected Lie group need not be surjective.

(a) Suppose $A \in \mathrm{SL}(n,\mathbb{R})$ is of the form $e^B$ for some $B \in \mathfrak{gl}(n,\mathbb{R})$. Show that $A$ has a square root in $\mathrm{SL}(n,\mathbb{R})$, i.e., a matrix $C \in \mathrm{SL}(n,\mathbb{R})$ such that $C^2 = A$.

(b) Let

$$A = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}.$$

Show that $\exp\colon \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ is not surjective, by showing that $A$ is not in its image. [Remark: in the next chapter, Problem 21-25 will show that $\mathrm{SL}(2,\mathbb{R})$ is connected.]

20-7. Let $G$ be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra.

(a) For any $X, Y \in \mathfrak{g}$, show that $[X,Y] = 0$ if and only if

$$\exp tX \exp sY = \exp sY \exp tX \quad \text{for all } s,\ t \in \mathbb{R}.$$

(b) Show that $G$ is abelian if and only if $\mathfrak{g}$ is abelian.

(c) Give a counterexample to (b) when $G$ is not connected.

20-8. Suppose $G$ is a Lie group. Prove that $\exp(X+Y) = (\exp X)(\exp Y)$ for all $X, Y \in \mathrm{Lie}(G)$ if and only if the identity component of $G$ is abelian. [Hint: for the "if" direction, prove that $t \mapsto (\exp tX)(\exp tY)$ is a 1-parameter subgroup. For the "only if" direction, use Problem 20-7.]

20-9. Extend the result of Proposition 20.10 by showing that under the same hypotheses there is a smooth function $\widehat{Z}\colon (-\varepsilon,\varepsilon) \to \mathfrak{g}$ such that

$$(\exp tX)(\exp tY) = \exp\bigl(t(X+Y) + \tfrac{1}{2}t^2[X,Y] + t^3\widehat{Z}(t)\bigr). \tag{20.20}$$

[Remark: there is an explicit formula, known as the *Baker–Campbell–Hausdorff formula*, for all of the terms in the Taylor series of the map $\varphi\colon (-\varepsilon,\varepsilon) \to \mathfrak{g}$ that satisfies $\exp tX \exp tY = \exp\varphi(t)$. Formula (20.20) gives the first two terms in this series. See [Var84] for the full formula.]

20-10. Suppose $G$ is a Lie group and $S$ is a Lie subgroup of $G$. Show that the closure of $S$ is also a Lie subgroup. Conclude that every Lie subgroup of $G$ is either a properly embedded submanifold of $G$, or a dense subset of a properly embedded submanifold. [Remark: this shows that the subgroup $S \subseteq \mathbb{T}^3$ of Exercise 7.20—a dense subgroup of a properly embedded subgroup—is typical of nonembedded Lie subgroups.]

20-11. Let $G$ and $H$ be Lie groups.

(a) Show that every continuous homomorphism $\gamma\colon \mathbb{R} \to H$ is smooth. [Hint: let $V \subseteq \mathrm{Lie}(H)$ be a neighborhood of 0 such that the exponential map is a diffeomorphism from $2V = \{2X : X \in V\}$ to $\exp(2V)$. Choose $t_0$ small enough that $\gamma(t) \in \exp(V)$ whenever $|t| \le t_0$, and let $X_0$ be the element of $V$ such that $\gamma(t_0) = \exp X_0$. Show that $\gamma(qt_0) = \exp(qX_0)$ whenever $q$ is a dyadic rational, i.e., a number of the form $m/2^n$ for $m, n \in \mathbb{Z}$.]

(b) Show that every continuous homomorphism $F: G \to H$ is smooth. [Hint: show that there is a map $\varphi$: $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Lie}(G) & \xrightarrow{\varphi} & \mathrm{Lie}(H) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[F]{} & H. \end{array}$$

Then use Corollary 20.11 to show that $\varphi$ is linear.]

(c) Show that with the given topology on $G$, there is only one smooth structure that makes $G$ into a Lie group.

(*Used on p. 556.*)

20-12. Let $G$ be a Lie group. Show that the infinitesimal generator of the action of $G$ on itself by right translation is the inclusion map $\mathrm{Lie}(G) \hookrightarrow \mathfrak{X}(G)$.

20-13. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and let $M$ be a smooth manifold. A Lie algebra action $\widehat{\theta}: \mathfrak{g} \to \mathfrak{X}(M)$ is said to be ***transitive*** if for every $p \in M$, the vectors of the form $\widehat{X}_p$ for $\widehat{X} \in \widehat{\theta}(\mathfrak{g})$ span $T_p M$. Show that a smooth right action of a Lie group $G$ on a connected smooth manifold $M$ is transitive if and only if its infinitesimal generator is transitive. [Hint: show that if the Lie algebra action is transitive, then every orbit is open.]

20-14. Let $M$ be a smooth manifold, and suppose $\mathfrak{g}$ is a finite-dimensional Lie subalgebra of $\mathfrak{X}(M)$ consisting only of complete vector fields. Show that there is a smooth right action of a Lie group $G$ on $M$ such that $\mathfrak{g}$ is the image of its infinitesimal generator. Determine such an action for the Lie subalgebra of $\mathfrak{X}\left(\mathbb{R}^3\right)$ described in Problem 8-20.

20-15. Prove Theorem 20.18 (the fundamental theorem on left Lie algebra actions). [Hint: use the one-to-one correspondence between left actions and right actions given by $g \cdot p = p \cdot g^{-1}$.]

20-16. Let $G$ be a simply connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Show that every representation of $\mathfrak{g}$ is of the form $\rho_*: \mathfrak{g} \to \mathfrak{gl}(V)$ for some representation $\rho: G \to \mathrm{GL}(V)$ of $G$.

20-17. Suppose $G$ is a connected Lie group, $H$ is any Lie group, and $\Phi, \Psi: G \to H$ are Lie group homomorphisms such that $\Phi_* = \Psi_*: \mathrm{Lie}(G) \to \mathrm{Lie}(H)$. Prove that $\Phi = \Psi$. (*Used on p. 531.*)

20-18. If $\mathsf{C}$ and $\mathsf{D}$ are categories, a covariant functor $\mathcal{F}: \mathsf{C} \to \mathsf{D}$ is called an ***equivalence of categories*** if every object of $\mathsf{D}$ is isomorphic to $\mathcal{F}(X)$ for some $X \in \mathrm{Ob}(\mathsf{C})$, and the map $\mathcal{F}: \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ is bijective for each pair of objects $X, Y \in \mathrm{Ob}(\mathsf{C})$. Show that the assignment $G \mapsto \mathrm{Lie}(G)$, $\varphi \mapsto \varphi_*$ is an equivalence of categories between the category $\mathsf{SLie}$ of simply connected Lie groups and the category $\mathsf{lie}$ of finite-dimensional Lie algebras.

20-19. Prove Theorem 20.22 (Lie's fundamental theorems).

20-20. Let $G$ be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Prove that the kernel of $\operatorname{Ad}: G \to \operatorname{GL}(\mathfrak{g})$ is the ***center of G***, that is, the set of elements of $G$ that commute with every element of $G$.

20-21. Show that the adjoint representation of $\operatorname{GL}(n,\mathbb{R})$ is given by $\operatorname{Ad}(A)Y = AYA^{-1}$ for $A \in \operatorname{GL}(n,\mathbb{R})$ and $Y \in \mathfrak{gl}(n,\mathbb{R})$. Show that it is not faithful.

20-22. If $\mathfrak{g}$ is a Lie algebra, the ***center of*** $\boldsymbol{\mathfrak{g}}$ is the set of all $X \in \mathfrak{g}$ such that $[X,Y] = 0$ for all $Y \in \mathfrak{g}$. Suppose $G$ is a connected Lie group. Show that the center of $\operatorname{Lie}(G)$ is the Lie algebra of the center of $G$.