Chapter 10
Vector Bundles

In Chapter 3, we saw that the tangent bundle of a smooth manifold has a natural structure as a smooth manifold in its own right. The natural coordinates we constructed on TM make it look, locally, like the Cartesian product of an open subset of M with $\mathbb{R}^n$. This kind of structure arises quite frequently—a collection of vector spaces, one for each point in M, glued together in a way that looks *locally* like the Cartesian product of M with $\mathbb{R}^n$, but globally may be "twisted." Such structures are called *vector bundles*, and are the main subject of this chapter.

The chapter begins with the definition of vector bundles and descriptions of a few examples. The most notable example, of course, is the tangent bundle of a smooth manifold. We then go on to discuss local and global sections of vector bundles (which correspond to vector fields in the case of the tangent bundle). The chapter continues with a discussion of the natural notions of maps between bundles, called *bundle homomorphisms*, and subsets of vector bundles that are themselves vector bundles, called *subbundles*. At the end of the chapter, we briefly introduce an important generalization of vector bundles, called *fiber bundles*.

There is a deep and extensive body of theory about vector bundles and fiber bundles on manifolds, which we cannot even touch. We introduce them primarily in order to have a convenient language for talking about the tangent bundle and structures like it; as you will see in the next few chapters, such structures exist in profusion on smooth manifolds.

Vector Bundles

Let M be a topological space. A ***(real) vector bundle of rank k over M*** is a topological space E together with a surjective continuous map $\pi\colon E \to M$ satisfying the following conditions:

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

J.M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics 218,
DOI 10.1007/978-1-4419-9982-5_10,

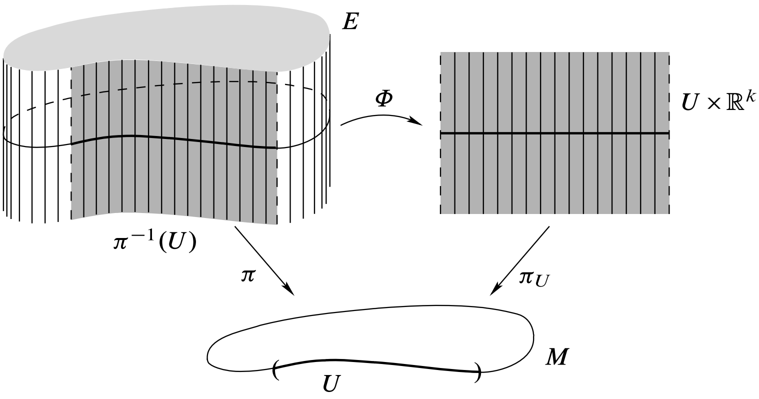


Fig. 10.1 A local trivialization of a vector bundle

(ii) For each $p \in M$, there exist a neighborhood U of p in M and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a ***local trivialization of E over U***), satisfying the following conditions (Fig. 10.1):

- $\pi_U \circ \Phi = \pi$ (where $\pi_U\colon U \times \mathbb{R}^k \to U$ is the projection);
- for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds with or without boundary, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a ***smooth vector bundle***. In this case, we call any local trivialization that is a diffeomorphism onto its image a ***smooth local trivialization***.

A rank-1 vector bundle is often called a ***(real) line bundle***. ***Complex vector bundles*** are defined similarly, with "real vector space" replaced by "complex vector space" and $\mathbb{R}^k$ replaced by $\mathbb{C}^k$ in the definition. We have no need to treat complex vector bundles in this book, so all of our vector bundles are understood without further comment to be real.

The space E is called the ***total space of the bundle***, M is called its ***base***, and π is its ***projection***. Depending on what we wish to emphasize, we sometimes omit some of the ingredients from the notation, and write "E is a vector bundle over M," or "$E \to M$ is a vector bundle," or "$\pi\colon E \to M$ is a vector bundle."

► **Exercise 10.1.** Suppose E is a smooth vector bundle over M. Show that the projection map $\pi\colon E \to M$ is a surjective smooth submersion.

If there exists a local trivialization of E over all of M (called a ***global trivialization of E***), then E is said to be a ***trivial bundle***. In this case, E itself is homeomorphic to the product space $M \times \mathbb{R}^k$. If $E \to M$ is a smooth bundle that admits a smooth global trivialization, then we say that E is ***smoothly trivial***. In this case E is *diffeomorphic* to $M \times \mathbb{R}^k$, not just homeomorphic. For brevity, when we say that a smooth bundle is trivial, we always understand this to mean smoothly trivial, not just trivial in the topological sense.

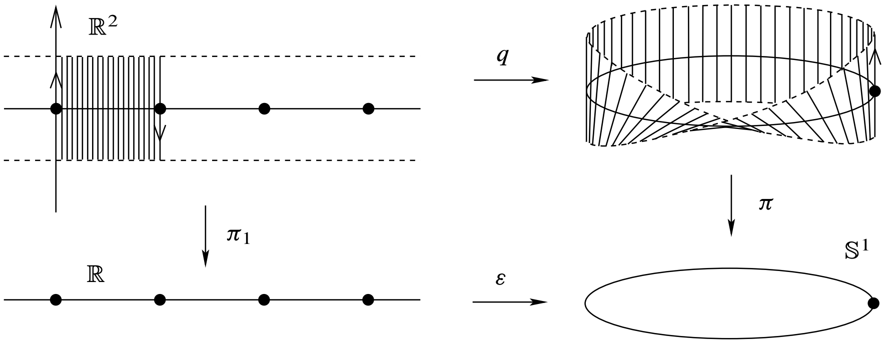


Fig. 10.2 Part of the Möbius bundle

Example 10.2 (Product Bundles). One particularly simple example of a rank-k vector bundle over any space M is the product space $E = M \times \mathbb{R}^k$ with $\pi = \pi_1 \colon M \times \mathbb{R}^k \to M$ as its projection. Any such bundle, called a ***product bundle***, is trivial (with the identity map as a global trivialization). If M is a smooth manifold with or without boundary, then $M \times \mathbb{R}^k$ is smoothly trivial. //

Although there are many vector bundles that are not trivial, the only one that is easy to visualize is the following.

Example 10.3 (The Möbius Bundle). Define an equivalence relation on $\mathbb{R}^2$ by declaring that $(x, y) \sim (x', y')$ if and only if $(x', y') = \big(x + n, (-1)^n y\big)$ for some $n \in \mathbb{Z}$. Let $E = \mathbb{R}^2/\sim$ denote the quotient space, and let $q \colon \mathbb{R}^2 \to E$ be the quotient map.

To visualize E, let S denote the strip $[0, 1] \times \mathbb{R} \subseteq \mathbb{R}^2$. The restriction of q to S is surjective and closed, so it is a quotient map. The only nontrivial identifications made by $q|_S$ are on the two boundary lines, so we can think of E as the space obtained from S by giving the right-hand edge a half-twist to turn it upside-down, and then pasting it to the left-hand edge (Fig. 10.2). For any $r > 0$, the image under the quotient map q of the rectangle $[0, 1] \times [-r, r]$ is a smooth compact manifold with boundary called a ***Möbius band***; you can make a paper model of this space by pasting the ends of a strip of paper together with a half-twist.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\;q\;} & E \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow[\;\varepsilon\;]{} & \mathbb{S}^1, \end{array}$$

where π_1 is the projection onto the first factor and $\varepsilon \colon \mathbb{R} \to \mathbb{S}^1$ is the smooth covering map $\varepsilon(x) = e^{2\pi i x}$. Because $\varepsilon \circ \pi_1$ is constant on each equivalence class, it descends to a continuous map $\pi \colon E \to \mathbb{S}^1$. A straightforward (if tedious) verification shows that E has a unique smooth manifold structure such that q is a smooth covering map and $\pi\colon E \to \mathbb{S}^1$ is a smooth real line bundle over $\mathbb{S}^1$, called the ***Möbius bundle***. (If $U \subseteq \mathbb{S}^1$ is an open subset that is evenly covered by ε, and $\widetilde{U} \subseteq \mathbb{R}$ is a component of $\varepsilon^{-1}(U)$, then q restricts to a homeomorphism from $\widetilde{U} \times \mathbb{R}$ to $\pi^{-1}(U)$. Using this, one can construct a homeomorphism from $\pi^{-1}(U)$ to $U \times \mathbb{R}$, which serves as a local trivialization of E. These local trivializations can be interpreted as coordinate charts defining the smooth structure on E. Problem 10-1 asks you to work out the details. Problem 21-9 suggests a more powerful approach.) //

The most important examples of vector bundles are tangent bundles.

Proposition 10.4 (The Tangent Bundle as a Vector Bundle). *Let M be a smooth n-manifold with or without boundary, and let TM be its tangent bundle. With its standard projection map, its natural vector space structure on each fiber, and the topology and smooth structure constructed in Proposition* 3.18, *TM is a smooth vector bundle of rank n over M.*

Proof. Given any smooth chart (U,φ) for M with coordinate functions $\left(x^i\right)$, define a map $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = \left(p, \left(v^1, \dots, v^n\right)\right). \tag{10.1}$$

This is linear on fibers and satisfies $\pi_1 \circ \Phi = \pi$. The composite map

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathrm{Id}_{\mathbb{R}^n}} \varphi(U) \times \mathbb{R}^n$$

is equal to the coordinate map $\widetilde{\varphi}$ constructed in Proposition 3.18. Since both $\widetilde{\varphi}$ and $\varphi \times \mathrm{Id}_{\mathbb{R}^n}$ are diffeomorphisms, so is Φ. Thus, Φ satisfies all the conditions for a smooth local trivialization. □

Any bundle that is not trivial, of course, requires more than one local trivialization. The next lemma shows that the composition of two smooth local trivializations has a simple form where they overlap.

Lemma 10.5. *Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over M. Suppose $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \varnothing$. There exists a smooth map $\tau\colon U \cap V \to \mathrm{GL}(k,\mathbb{R})$ such that the composition $\Phi \circ \Psi^{-1}\colon (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k$ has the form*

$$\Phi \circ \Psi^{-1}(p,v) = \left(p, \tau(p)v\right),$$

where $\tau(p)v$ denotes the usual action of the $k \times k$ matrix $\tau(p)$ on the vector $v \in \mathbb{R}^k$.

Proof. The following diagram commutes:

$$\begin{array}{ccccc} (U \cap V) \times \mathbb{R}^k & \xleftarrow{\Psi} & \pi^{-1}(U \cap V) & \xrightarrow{\Phi} & (U \cap V) \times \mathbb{R}^k \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle \pi} & \swarrow {\scriptstyle \pi_1} & \\ & & U \cap V, & & \end{array} \tag{10.2}$$

where the maps on top are to be interpreted as the restrictions of Ψ and Φ to $\pi^{-1}(U \cap V)$. It follows that $\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1$, which means that

$$\Phi \circ \Psi^{-1}(p, v) = \big(p, \sigma(p, v)\big)$$

for some smooth map $\sigma\colon (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$. Moreover, for each fixed $p \in U \cap V$, the map $v \mapsto \sigma(p, v)$ from $\mathbb{R}^k$ to itself is an invertible linear map, so there is a nonsingular $k \times k$ matrix $\tau(p)$ such that $\sigma(p, v) = \tau(p)v$. It remains only to show that the map $\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R})$ is smooth. This is left to Problem 10-4. □

The smooth map $\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R})$ described in this lemma is called the ***transition function*** between the local trivializations Φ and Ψ. (This is one of the few situations in smooth manifold theory in which it is traditional to use the word "function" even though the codomain is not $\mathbb{R}$ or $\mathbb{R}^k$.) For example, if M is a smooth manifold and Φ and Ψ are the local trivializations of TM associated with two different smooth charts, then (3.12) shows that the transition function between them is the Jacobian matrix of the coordinate transition map.

Like the tangent bundle, vector bundles are often most easily described by giving a collection of vector spaces, one for each point of the base manifold. In order to make such a set into a smooth vector bundle, we would first have to construct a manifold topology and a smooth structure on the disjoint union of all the vector spaces, and then construct the local trivializations and show that they have the requisite properties. The next lemma provides a shortcut, by showing that it is sufficient to construct the local trivializations, as long as they overlap with smooth transition functions. (See also Problem 10-6 for a stronger form of this result.)

Lemma 10.6 (Vector Bundle Chart Lemma). *Let M be a smooth manifold with or without boundary, and suppose that for each $p \in M$ we are given a real vector space E_p of some fixed dimension k. Let $E = \coprod_{p \in M} E_p$, and let $\pi\colon E \to M$ be the map that takes each element of E_p to the point p. Suppose furthermore that we are given the following data:*

(i) *an open cover $\{U_\alpha\}_{\alpha \in A}$ of M*
(ii) *for each $\alpha \in A$, a bijective map $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$*
(iii) *for each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ to itself has the form*

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = \big(p, \tau_{\alpha\beta}(p)v\big) \tag{10.3}$$

Then E has a unique topology and smooth structure making it into a smooth manifold with or without boundary and a smooth rank-k vector bundle over M, with π as projection and $\{(U_\alpha, \Phi_\alpha)\}$ as smooth local trivializations.

Proof. For each point $p \in M$, choose some U_α containing p; choose a smooth chart (V_p, φ_p) for M such that $p \in V_p \subseteq U_\alpha$; and let $\widehat{V}_p = \varphi_p(V_p) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ (where n is the dimension of M). Define a map $\widetilde{\varphi}_p\colon \pi^{-1}(V_p) \to \widehat{V}_p \times \mathbb{R}^k$ by $\widetilde{\varphi}_p = (\varphi_p \times \mathrm{Id}_{\mathbb{R}^k}) \circ$

Φ_α:

$$\pi^{-1}(V_p) \xrightarrow{\Phi_\alpha} V_p \times \mathbb{R}^k \xrightarrow{\varphi_p \times \mathrm{Id}_{\mathbb{R}^k}} \widehat{V}_p \times \mathbb{R}^k.$$

We will show that the collection of all such charts $\{(\pi^{-1}(V_p), \widetilde{\varphi}_p) : p \in M\}$ satisfies the conditions of the smooth manifold chart lemma (Lemma 1.35) or its counterpart for manifolds with boundary (Exercise 1.43), and therefore gives E the structure of a smooth manifold with or without boundary.

As a composition of bijective maps, $\widetilde{\varphi}_p$ is bijective onto an open subset of either $\mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$ or $\mathbb{H}^n \times \mathbb{R}^k \approx \mathbb{H}^{n+k}$. For any $p, q \in M$, it is easy to check that

$$\widetilde{\varphi}_p\left(\pi^{-1}(V_p) \cap \pi^{-1}(V_q)\right) = \varphi_p(V_p \cap V_q) \times \mathbb{R}^k,$$

which is open because φ_p is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$. Wherever two such charts overlap, we have

$$\widetilde{\varphi}_p \circ \widetilde{\varphi}_q^{-1} = \left(\varphi_p \times \mathrm{Id}_{\mathbb{R}^k}\right) \circ \Phi_\alpha \circ \Phi_\beta^{-1} \circ \left(\varphi_q \times \mathrm{Id}_{\mathbb{R}^k}\right)^{-1}.$$

Since $\varphi_p \times \mathrm{Id}_{\mathbb{R}^k}$, $\varphi_q \times \mathrm{Id}_{\mathbb{R}^k}$, and $\Phi_\alpha \circ \Phi_\beta^{-1}$ are diffeomorphisms, the composition is a diffeomorphism. Thus, conditions (i)–(iii) of Lemma 1.35 are satisfied. Because the open cover $\{V_p : p \in M\}$ has a countable subcover, (iv) is satisfied as well.

To check the Hausdorff condition (v), just note that any two points in the same space E_p lie in one of the charts we have constructed; while if $\xi \in E_p$ and $\eta \in E_q$ with $p \neq q$, we can choose V_p and V_q to be disjoint neighborhoods of p and q, so that the sets $\pi^{-1}(V_p)$ and $\pi^{-1}(V_q)$ are disjoint coordinate neighborhoods containing ξ and η, respectively. Thus we have given E the structure of a smooth manifold with or without boundary.

With respect to this structure, each of the maps Φ_α is a diffeomorphism, because in terms of the coordinate charts $\left(\pi^{-1}(V_p), \widetilde{\varphi}_p\right)$ for E and $\left(V_p \times \mathbb{R}^k, \varphi_p \times \mathrm{Id}_{\mathbb{R}^k}\right)$ for $V_p \times \mathbb{R}^k$, the coordinate representation of Φ_α is the identity map. The coordinate representation of π, with respect to the same chart for E and the chart (V_p, φ_p) for M, is $\pi(x, v) = x$, so π is smooth as well. Because each Φ_α maps E_p to $\{p\} \times \mathbb{R}^k$, it is immediate that $\pi_1 \circ \Phi_\alpha = \pi$, and Φ_α is linear on fibers by hypothesis. Thus, Φ_α satisfies all the conditions for a smooth local trivialization.

The fact that this is the unique such smooth structure follows easily from the requirement that the maps Φ_α be diffeomorphisms onto their images: any smooth structure satisfying the same conditions must include all of the charts we constructed, so it is equal to this one. □

Here are some examples showing how the chart lemma can be used to construct new vector bundles from old ones.

Example 10.7 (Whitney Sums). Given a smooth manifold M and smooth vector bundles $E' \to M$ and $E'' \to M$ of ranks k' and k'', respectively, we will construct a new vector bundle over M called the ***Whitney sum of E' and E''***, whose fiber at each $p \in M$ is the direct sum $E'_p \oplus E''_p$. The total space is defined as $E' \oplus E'' = \coprod_{p \in M}\left(E'_p \oplus E''_p\right)$, with the obvious projection $\pi : E' \oplus E'' \to M$. For each $p \in M$, choose a neighborhood U of p small enough that there exist local trivializations (U,Φ') of E' and (U,Φ'') of E'', and define $\Phi\colon \pi^{-1}(U)\to U\times\mathbb{R}^{k'+k''}$ by

$$\Phi(v',v'')=\left(\pi'(v'),\left(\pi_{\mathbb{R}^{k'}}\circ\Phi'(v'),\pi_{\mathbb{R}^{k''}}\circ\Phi''(v'')\right)\right).$$

Suppose we are given another such pair of local trivializations $\left(\widetilde{U},\widetilde{\Phi}'\right)$ and $\left(\widetilde{U},\widetilde{\Phi}''\right)$. Let $\tau'\colon \left(U\cap\widetilde{U}\right)\to \mathrm{GL}(k',\mathbb{R})$ and $\tau''\colon \left(U\cap\widetilde{U}\right)\to \mathrm{GL}(k'',\mathbb{R})$ be the corresponding transition functions. Then the transition function for $E'\oplus E''$ has the form

$$\widetilde{\Phi}\circ\Phi^{-1}\left(p,(v',v'')\right)=\left(p,\tau(p)(v',v'')\right),$$

where $\tau(p)=\tau'(p)\oplus\tau''(p)\in \mathrm{GL}(k'+k'',\mathbb{R})$ is the block diagonal matrix

$$\begin{pmatrix}\tau'(p) & 0\\ 0 & \tau''(p)\end{pmatrix}.$$

Because this depends smoothly on p, it follows from the chart lemma that $E'\oplus E''$ is a smooth vector bundle over M. //

Example 10.8 (Restriction of a Vector Bundle). Suppose $\pi\colon E\to M$ is a rank-k vector bundle and $S\subseteq M$ is any subset. We define the ***restriction of E to S*** to be the set $E|_S=\bigcup_{p\in S}E_p$, with the projection $E|_S\to S$ obtained by restricting π. If $\Phi\colon \pi^{-1}(U)\to U\times\mathbb{R}^k$ is a local trivialization of E over $U\subseteq M$, it restricts to a bijective map $\Phi|_U\colon \left(\pi|_S\right)^{-1}(U\cap S)\to (U\cap S)\times\mathbb{R}^k$, and it is easy to check that these form local trivializations for a vector bundle structure on $E|_S$. If E is a smooth vector bundle and $S\subseteq M$ is an immersed or embedded submanifold, it follows easily from the chart lemma that $E|_S$ is a smooth vector bundle. In particular, if $S\subseteq M$ is a smooth (embedded or immersed) submanifold, then the restricted bundle $TM|_S$ is called the ***ambient tangent bundle*** over M. //

Local and Global Sections of Vector Bundles

Let $\pi\colon E\to M$ be a vector bundle. A ***section of E*** (sometimes called a ***cross section***) is a section of the map π, that is, a continuous map $\sigma\colon M\to E$ satisfying $\pi\circ\sigma=\mathrm{Id}_M$. This means that $\sigma(p)$ is an element of the fiber E_p for each $p\in M$.

More generally, a ***local section of E*** is a continuous map $\sigma\colon U\to E$ defined on some open subset $U\subseteq M$ and satisfying $\pi\circ\sigma=\mathrm{Id}_U$ (see Fig. 10.3). To emphasize the distinction, a section defined on all of M is sometimes called a ***global section***. Note that a local section of E over $U\subseteq M$ is the same as a global section of the restricted bundle $E|_U$. If M is a smooth manifold with or without boundary and E is a smooth vector bundle, a ***smooth (local or global) section of E*** is one that is a smooth map from its domain to E.

Just as with vector fields, for some purposes it is useful also to consider maps that would be sections except that they might not be continuous. Thus, we define a ***rough (local or global) section of E*** over a set $U\subseteq M$ to be a map $\sigma\colon U\to E$

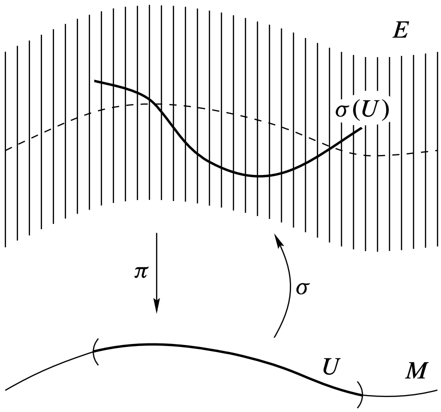


Fig. 10.3 A local section of a vector bundle

(not necessarily continuous) such that $\pi \circ \sigma = \mathrm{Id}_U$. A "section" without further qualification always means a continuous section.

The ***zero section of E*** is the global section $\zeta: M \to E$ defined by

$$\zeta(p) = 0 \in E_p \text{ for each } p \in M.$$

As in the case of vector fields, the ***support*** of a section σ is the closure of the set $\{p \in M : \sigma(p) \neq 0\}$.

► **Exercise 10.9.** Show that the zero section of every vector bundle is continuous, and the zero section of every smooth vector bundle is smooth. [Hint: consider $\Phi \circ \zeta$, where Φ is a local trivialization.]

Example 10.10 (Sections of Vector Bundles). Suppose M is a smooth manifold with or without boundary.

(a) Sections of TM are vector fields on M.
(b) Given an immersed submanifold $S \subseteq M$ with or without boundary, a section of the ambient tangent bundle $TM|_S \to S$ is called a ***vector field along S***. It is a continuous map $X: S \to TM$ such that $X_p \in T_pM$ for each $p \in S$. This is different from a vector field *on* S, which satisfies $X_p \in T_pS$ at each point.
(c) If $E = M \times \mathbb{R}^k$ is a product bundle, there is a natural one-to-one correspondence between sections of E and continuous functions from M to $\mathbb{R}^k$: a continuous function $F: M \to \mathbb{R}^k$ determines a section $\widetilde{F}: M \to M \times \mathbb{R}^k$ by $\widetilde{F}(x) = (x, F(x))$, and vice versa. If M is a smooth manifold with or without boundary, then the section $\widetilde{F}$ is smooth if and only if F is.
(d) The correspondence in the preceding paragraph yields a natural identification between the space $C^\infty(M)$ and the space of smooth sections of the trivial line bundle $M \times \mathbb{R} \to M$. //

If $E \to M$ is a smooth vector bundle, the set of all smooth global sections of E is a vector space under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p).$$

This vector space is usually denoted by $\Gamma(E)$. (For particular vector bundles, we will often introduce specialized notations for their spaces of sections, such as the notation $\mathfrak{X}(M)$ introduced in Chapter 8 for the space of smooth sections of TM.)

Just like smooth vector fields, smooth sections of a smooth bundle $E \to M$ can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, we obtain a new section $f\sigma$ defined by

$$(f\sigma)(p) = f(p)\sigma(p).$$

► **Exercise 10.11.** Let $E \to M$ be a smooth vector bundle.

(a) Show that if $\sigma, \tau \in \Gamma(E)$ and $f, g \in C^\infty(M)$, then $f\sigma + g\tau \in \Gamma(E)$.
(b) Show that $\Gamma(E)$ is a module over the ring $C^\infty(M)$.

Lemma 10.12 (Extension Lemma for Vector Bundles). *Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M with or without boundary. Suppose A is a closed subset of M, and $\sigma\colon A \to E$ is a section of $E|_A$ that is smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each point. For each open subset $U \subseteq M$ containing A, there exists a global smooth section $\widetilde{\sigma} \in \Gamma(E)$ such that $\widetilde{\sigma}|_A = \sigma$ and* supp $\widetilde{\sigma} \subseteq U$.

► **Exercise 10.13.** Prove the preceding lemma.

► **Exercise 10.14.** Let $\pi\colon E \to M$ be a smooth vector bundle. Show that each element of E is in the image of a smooth global section.

Local and Global Frames

The concept of local frames that we introduced in Chapter 8 extends readily to vector bundles. Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be ***linearly independent*** if their values $\big(\sigma_1(p), \dots, \sigma_k(p)\big)$ form a linearly independent k-tuple in E_p for each $p \in U$. Similarly, they are said to ***span E*** if their values span E_p for each $p \in U$. A ***local frame for E over U*** is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections over U that span E; thus $\big(\sigma_1(p), \dots, \sigma_k(p)\big)$ is a basis for the fiber E_p for each $p \in U$. It is called a ***global frame*** if $U = M$. If $E \to M$ is a smooth vector bundle, a local or global frame is a ***smooth frame*** if each σ_i is a smooth section. We often denote a frame $(\sigma_1, \dots, \sigma_k)$ by (σ_i).

The (local or global) frames for M that we defined in Chapter 8 are, in our new terminology, frames for the tangent bundle. We use both terms interchangeably depending on context: "frame for M" and "frame for TM" mean the same thing.

The next proposition is an analogue for vector bundles of Proposition 8.11.

Proposition 10.15 (Completion of Local Frames for Vector Bundles). *Suppose $\pi\colon E \to M$ is a smooth vector bundle of rank k.*

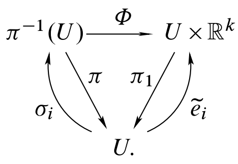
(a) *If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, with $1 \le m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.*
(b) *If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of E_p for some $p \in M$, with $1 \le m \le k$, then there exists a smooth local frame (σ_i) for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for $i = 1, \dots, m$.*
(c) *If $A \subseteq M$ is a closed subset and $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ that are smooth in the sense described in Lemma 10.12, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for $i = 1, \dots, k$.*

► **Exercise 10.16.** Prove the preceding proposition.

Local frames for a vector bundle are intimately connected with local trivializations, as the next two examples show.

Example 10.17 (A Global Frame for a Product Bundle). If $E = M \times \mathbb{R}^k \to M$ is a product bundle, the standard basis $(e_1, \dots, e_k)$ for $\mathbb{R}^k$ yields a global frame $(\widetilde{e}_i)$ for E, defined by $\widetilde{e}_i(p) = (p, e_i)$. If M is a smooth manifold with or without boundary, then this global frame is smooth. //

Example 10.18 (Local Frames Associated with Local Trivializations). Suppose $\pi\colon E \to M$ is a smooth vector bundle. If $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a smooth local trivialization of E, we can use the same idea as in the preceding example to construct a local frame for E over U. Define maps $\sigma_1, \dots, \sigma_k\colon U \to E$ by $\sigma_i(p) = \Phi^{-1}(p, e_i) = \Phi^{-1} \circ \widetilde{e}_i(p)$:



Then σ_i is smooth because Φ is a diffeomorphism, and the fact that $\pi_1 \circ \Phi = \pi$ implies that

$$\pi \circ \sigma_i(p) = \pi \circ \Phi^{-1}(p, e_i) = \pi_1(p, e_i) = p,$$

so σ_i is a section. To see that $\big(\sigma_i(p)\big)$ forms a basis for E_p, just note that Φ restricts to an isomorphism from E_p to $\{p\} \times \mathbb{R}^k$, and $\Phi\big(\sigma_i(p)\big) = (p, e_i)$, so Φ takes $\big(\sigma_i(p)\big)$ to the standard basis for $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$. We say that this local frame (σ_i) is ***associated with Φ***. //

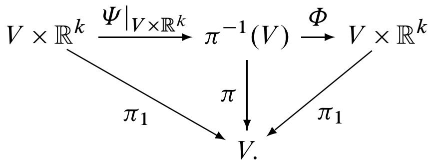
Proposition 10.19. *Every smooth local frame for a smooth vector bundle is associated with a smooth local trivialization as in Example 10.18.*

Proof. Suppose $E \to M$ is a smooth vector bundle and (σ_i) is a smooth local frame for E over an open subset $U \subseteq M$. We define a map $\Psi\colon U \times \mathbb{R}^k \to \pi^{-1}(U)$ by

$$\Psi\big(p, \big(v^1, \dots, v^k\big)\big) = v^i \sigma_i(p). \tag{10.4}$$

The fact that $\big(\sigma_i(p)\big)$ forms a basis for E_p at each $p \in U$ implies that Ψ is bijective, and an easy computation shows that $\sigma_i = \Psi \circ \widetilde{e}_i$. Thus, if we can show that Ψ is a diffeomorphism, then Ψ^{-1} will be a smooth local trivialization whose associated local frame is (σ_i).

Since Ψ is bijective, to show that it is a diffeomorphism it suffices to show that it is a local diffeomorphism. Given $q \in U$, we can choose a neighborhood V of q in M over which there exists a smooth local trivialization $\Phi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$, and by shrinking V if necessary we may assume that $V \subseteq U$. Since Φ is a diffeomorphism, if we can show that $\Phi \circ \Psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism from $V \times \mathbb{R}^k$ to itself, it follows that Ψ restricts to a diffeomorphism from $V \times \mathbb{R}^k$ to $\pi^{-1}(V)$:



For each of our smooth sections σ_i, the composite map $\Phi \circ \sigma_i|_V\colon V \to V \times \mathbb{R}^k$ is smooth, and thus there are smooth functions $\sigma_i^1, \dots, \sigma_i^k\colon V \to \mathbb{R}$ such that

$$\Phi \circ \sigma_i(p) = \big(p, \big(\sigma_i^1(p), \dots, \sigma_i^k(p)\big)\big).$$

On $V \times \mathbb{R}^k$, therefore,

$$\Phi \circ \Psi\big(p, \big(v^1, \dots, v^k\big)\big) = \big(p, \big(v^i \sigma_i^1(p), \dots, v^i \sigma_i^k(p)\big)\big),$$

which is clearly smooth.

To show that $(\Phi \circ \Psi)^{-1}$ is smooth, note that the matrix $\big(\sigma_i^j(p)\big)$ is invertible for each p, because $\big(\sigma_i(p)\big)$ is a basis for E_p. Let $\big(\tau_i^j(p)\big)$ denote the inverse matrix. Because matrix inversion is a smooth map from $\mathrm{GL}(k, \mathbb{R})$ to itself, the functions τ_i^j are smooth. It follows from the computations in the preceding paragraph that

$$(\Phi \circ \Psi)^{-1}\big(p, \big(w^1, \dots, w^k\big)\big) = \big(p, \big(w^i \tau_i^1(p), \dots, w^i \tau_i^k(p)\big)\big),$$

which is also smooth. □

Corollary 10.20. *A smooth vector bundle is smoothly trivial if and only if it admits a smooth global frame.*

Proof. Example 10.18 and Proposition 10.19 show that there is a smooth local trivialization over an open subset $U \subseteq M$ if and only if there is a smooth local frame over U. The corollary is just the special case of this statement when $U = M$. □

When applied to the tangent bundle of a smooth manifold M, this corollary says that TM is trivial if and only if M is parallelizable. (Recall that in Chapter 8 we defined a *parallelizable manifold* to be one that admits a smooth global frame for its tangent bundle.)

Corollary 10.21. *Let $\pi\colon E \to M$ be a smooth vector bundle of rank k, let (V, φ) be a smooth chart on M with coordinate functions $\left(x^i\right)$, and suppose there exists a smooth local frame (σ_i) for E over V. Define $\widetilde{\varphi}\colon \pi^{-1}(V) \to \varphi(V) \times \mathbb{R}^k$ by*

$$\widetilde{\varphi}\left(v^i \sigma_i(p)\right) = \left(x^1(p), \dots, x^n(p), v^1, \dots, v^k\right).$$

Then $\left(\pi^{-1}(V), \widetilde{\varphi}\right)$ is a smooth coordinate chart for E.

Proof. Just check that $\widetilde{\varphi}$ is equal to the composition $(\varphi \times \mathrm{Id}_{\mathbb{R}^k}) \circ \Phi$, where Φ is the local trivialization associated with (σ_i). As a composition of diffeomorphisms, it is a diffeomorphism. □

Just as smoothness of vector fields can be characterized in terms of their component functions in any smooth chart, smoothness of sections of vector bundles can be characterized in terms of local frames. Suppose (σ_i) is a smooth local frame for E over some open subset $U \subseteq M$. If $\tau\colon M \to E$ is a rough section, the value of τ at an arbitrary point $p \in U$ can be written $\tau(p) = \tau^i(p)\sigma_i(p)$ for some uniquely determined numbers $\left(\tau^1(p), \dots, \tau^n(p)\right)$. This defines k functions $\tau^i\colon U \to \mathbb{R}$, called the ***component functions of τ*** with respect to the given local frame.

Proposition 10.22 (Local Frame Criterion for Smoothness). *Let $\pi\colon E \to M$ be a smooth vector bundle, and let $\tau\colon M \to E$ be a rough section. If (σ_i) is a smooth local frame for E over an open subset $U \subseteq M$, then τ is smooth on U if and only if its component functions with respect to (σ_i) are smooth.*

Proof. Let $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the local trivialization associated with the local frame (σ_i). Because Φ is a diffeomorphism, τ is smooth on U if and only if the composite map $\Phi \circ \tau$ is smooth on U. It is straightforward to check that $\Phi \circ \tau(p) = \left(p, \left(\tau^1(p), \dots, \tau^k(p)\right)\right)$, where $\left(\tau^i\right)$ are the component functions of τ with respect to (σ_i), so $\Phi \circ \tau$ is smooth if and only if the component functions τ^i are smooth. □

► **Exercise 10.23.** Let $E \to M$ be a vector bundle. Show that a rough section of E is continuous if and only if its component functions in each local frame are continuous.

Proposition 10.22 applies equally well to local sections, since a local section of E over an open subset $V \subseteq M$ is a global section of the restricted bundle $E|_V$.

The correspondence between local frames and local trivializations leads to the following uniqueness result characterizing the smooth structure on the tangent bundle of a smooth manifold.

Proposition 10.24 (Uniqueness of the Smooth Structure on *TM*). *Let M be a smooth n-manifold with or without boundary. The topology and smooth structure on TM constructed in Proposition* 3.18 *are the unique ones with respect to which $\pi\colon TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.*

Proof. Suppose TM is endowed with some topology and smooth structure making it into a smooth vector bundle with the given properties. If (U, φ) is any smooth chart for M, the corresponding coordinate frame $\left(\partial/\partial x^i\right)$ is a smooth local frame over U, so by Proposition 10.19 there is a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ associated with this local frame. Referring back to the construction of Example 10.18, we see that this local trivialization is none other than the map Φ constructed in Proposition 10.4. It follows from Corollary 10.21 that the natural coordinate chart $\widetilde{\varphi} = (\varphi \times \mathrm{Id}_{\mathbb{R}^n}) \circ \Phi$ belongs to the given smooth structure. Thus, the given smooth structure is equal to the one constructed in Proposition 3.18. □

Bundle Homomorphisms

If $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ are vector bundles, a continuous map $F\colon E \to E'$ is called a ***bundle homomorphism*** if there exists a map $f\colon M \to M'$ satisfying $\pi' \circ F = f \circ \pi$,

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M', \end{array}$$

with the property that for each $p \in M$, the restricted map $F|_{E_p}\colon E_p \to E'_{f(p)}$ is linear. The relationship between F and f is expressed by saying that ***F covers f***.

Proposition 10.25. *Suppose $\pi\colon E \to M$ and $\pi'\colon E \to M'$ are vector bundles and $F\colon E \to E'$ is a bundle homomorphism covering $f\colon M \to M'$. Then f is continuous and is uniquely determined by F. If the bundles and F are all smooth, then f is smooth as well.*

Proof. All of the conclusions follow from the easily verified fact that $f = \pi' \circ F \circ \zeta$, where $\zeta\colon M \to E$ is the zero section. □

A bijective bundle homomorphism $F\colon E \to E'$ whose inverse is also a bundle homomorphism is called a ***bundle isomorphism***; if F is also a diffeomorphism, it is called a ***smooth bundle isomorphism***. If there exists a (smooth) bundle isomorphism between E and E', the two bundles are said to be **(*smoothly*) *isomorphic***.

In the special case in which both E and E' are vector bundles over the same base space M, a slightly more restrictive notion of bundle homomorphism is usually more useful. A ***bundle homomorphism over M*** is a bundle homomorphism covering the identity map of M, or in other words, a continuous map $F\colon E \to E'$ such that $\pi' \circ F = \pi$,

$$\begin{array}{ccc} E & \xrightarrow{\quad F \quad} & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & M, & \end{array}$$

and whose restriction to each fiber is linear. If there exists a bundle homomorphism $F\colon E \to E'$ over M that is also a (smooth) bundle isomorphism, then we say that

E and E' are ***(smoothly) isomorphic over M***. The next proposition shows that it is not necessary to check smoothness of the inverse.

Proposition 10.26. *Suppose E and E' are smooth vector bundles over a smooth manifold M with or without boundary, and $F: E \to E'$ is a bijective smooth bundle homomorphism over M. Then F is a smooth bundle isomorphism.*

Proof. Problem 10-11. □

► **Exercise 10.27.** Show that a smooth rank-k vector bundle over M is smoothly trivial if and only if it is smoothly isomorphic over M to the product bundle $M \times \mathbb{R}^k$.

Example 10.28 (Bundle Homomorphisms).

(a) If $F: M \to N$ is a smooth map, the global differential $dF: TM \to TN$ is a smooth bundle homomorphism covering F.
(b) If $E \to M$ is a smooth vector bundle and $S \subseteq M$ is an immersed submanifold with or without boundary, then the inclusion map $E|_S \hookrightarrow E$ is a smooth bundle homomorphism covering the inclusion of S into M. //

Suppose $E \to M$ and $E' \to M$ are smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth global sections. If $F: E \to E'$ is a smooth bundle homomorphism over M, then composition with F induces a map $\widetilde{F}: \Gamma(E) \to \Gamma(E')$ as follows:

$$\widetilde{F}(\sigma)(p) = (F \circ \sigma)(p) = F\big(\sigma(p)\big). \tag{10.5}$$

It is easy to check that $\widetilde{F}(\sigma)$ is a section of E', and it is smooth by composition.

Because a bundle homomorphism is linear on fibers, the resulting map $\widetilde{F}$ on sections is linear over $\mathbb{R}$. In fact, it satisfies a stronger linearity property. A map $\mathcal{F}: \Gamma(E) \to \Gamma(E')$ is said to be ***linear over $C^\infty(M)$*** if for any smooth functions $u_1, u_2 \in C^\infty(M)$ and smooth sections $\sigma_1, \sigma_2 \in \Gamma(E)$,

$$\mathcal{F}(u_1\sigma_1 + u_2\sigma_2) = u_1\mathcal{F}(\sigma_1) + u_2\mathcal{F}(\sigma_2).$$

It follows easily from the definition (10.5) that the map on sections induced by a smooth bundle homomorphism is linear over $C^\infty(M)$. The next lemma shows that the converse is true as well.

Lemma 10.29 (Bundle Homomorphism Characterization Lemma). *Let $\pi: E \to M$ and $\pi': E' \to M$ be smooth vector bundles over a smooth manifold M with or without boundary, and let $\Gamma(E)$, $\Gamma(E')$ denote their spaces of smooth sections. A map $\mathcal{F}: \Gamma(E) \to \Gamma(E')$ is linear over $C^\infty(M)$ if and only if there is a smooth bundle homomorphism $F: E \to E'$ over M such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

Proof. We noted above that the map on sections induced by a smooth bundle homomorphism is linear over $C^\infty(M)$. Conversely, suppose $\mathcal{F}: \Gamma(E) \to \Gamma(E')$ is linear over $C^\infty(M)$. First, we show that $\mathcal{F}$ acts locally: if $\sigma_1 \equiv \sigma_2$ in some open subset $U \subseteq M$, then $\mathcal{F}(\sigma_1) \equiv \mathcal{F}(\sigma_2)$ in U. Write $\tau = \sigma_1 - \sigma_2$; then by linearity of $\mathcal{F}$, it suffices to assume that τ vanishes in U and show that $\mathscr{F}(\tau)$ does too. Given $p \in U$, let $\psi \in C^\infty(M)$ be a smooth bump function supported in U and equal to 1 at p. Because $\psi\tau$ is identically zero on M, the fact that $\mathscr{F}$ is linear over $C^\infty(M)$ implies

$$0 = \mathscr{F}(\psi\tau) = \psi\mathscr{F}(\tau).$$

Evaluating at p shows that $\mathscr{F}(\tau)(p) = \psi(p)\mathscr{F}(\tau)(p) = 0$; since the same is true for every $p \in U$, the claim follows.

Next we show that $\mathscr{F}$ actually acts pointwise: if $\sigma_1(p) = \sigma_2(p)$, then $\mathscr{F}(\sigma_1)(p) = \mathscr{F}(\sigma_2)(p)$. Once again, it suffices to assume that $\tau(p) = 0$ and show that $\mathscr{F}(\tau)(p) = 0$. Let $(\sigma_1, \dots, \sigma_k)$ be a smooth local frame for E in some neighborhood U of p, and write τ in terms of this frame as $\tau = u^i\sigma_i$ for some smooth functions u^i defined in U. The fact that $\tau(p) = 0$ means that $u^1(p) = \dots = u^k(p) = 0$. By the extension lemmas for vector bundles and for functions, there exist smooth global sections $\widetilde{\sigma}_i \in \Gamma(E)$ that agree with σ_i in a neighborhood of p, and smooth functions $\widetilde{u}^i \in C^\infty(M)$ that agree with u^i in some neighborhood of p. Then since $\tau = \widetilde{u}^i\widetilde{\sigma}_i$ on a neighborhood of p, we have

$$\mathscr{F}(\tau)(p) = \mathscr{F}\big(\widetilde{u}^i\widetilde{\sigma}_i\big)(p) = \widetilde{u}^i(p)\mathscr{F}(\widetilde{\sigma}_i)(p) = 0.$$

Define a bundle homomorphism $F\colon E \to E'$ as follows. For any $p \in M$ and $v \in E_p$, let $F(v) = \mathscr{F}(\widetilde{v})(p) \in E'_p$, where $\widetilde{v}$ is any global smooth section of E such that $\widetilde{v}(p) = v$. The discussion above shows that the resulting element of E'_p is independent of the choice of section. This map F clearly satisfies $\pi' \circ F = \pi$, and it is linear on each fiber because of the linearity of $\mathscr{F}$. It also satisfies $F \circ \sigma(p) = \mathscr{F}(\sigma)(p)$ for each $\sigma \in \Gamma(E)$ by definition. It remains only to show that F is smooth. It suffices to show that it is smooth in a neighborhood of each point.

Given $p \in M$, let (σ_i) be a smooth local frame for E on some neighborhood of p. By the extension lemma, there are global sections $\widetilde{\sigma}_i$ that agree with σ_i in a (smaller) neighborhood U of p. Shrinking U further if necessary, we may also assume that there exists a smooth local frame (σ'_j) for E' over U. Because $\mathscr{F}$ maps smooth global sections of E to smooth global sections of E', there are smooth functions $A_i^j \in C^\infty(U)$ such that $\mathscr{F}(\widetilde{\sigma}_i)|_U = A_i^j\sigma'_j$.

For any $q \in U$ and $v \in E_q$, we can write $v = v^i\sigma_i(q)$ for some real numbers $\big(v^1, \dots, v^k\big)$, and then

$$F\big(v^i\sigma_i(q)\big) = \mathscr{F}\big(v^i\widetilde{\sigma}_i\big)(q) = v^i\mathscr{F}(\widetilde{\sigma}_i)(q) = v^iA_i^j(q)\sigma'_j(q),$$

because $v^i\widetilde{\sigma}_i$ is a global smooth section of E whose value at q is v. If Φ and Φ' denote the local trivializations of E and E' associated with the frames (σ_i) and (σ'_i), respectively, it follows that the composite map $\Phi' \circ F \circ \Phi^{-1}\colon U \times \mathbb{R}^k \to U \times \mathbb{R}^m$ has the form

$$\Phi' \circ F \circ \Phi^{-1}\big(q, \big(v^1, \dots, v^k\big)\big) = \big(q, \big(A_i^1(q)v^i, \dots, A_i^m(q)v^i\big)\big),$$

which is smooth. Because Φ and Φ' are diffeomorphisms, this shows that F is smooth on $\pi^{-1}(U)$. □

Later, after we have developed more tools, we will see many examples of smooth bundle homomorphisms. For now, here are some elementary examples.

Example 10.30 (Bundle Homomorphisms Over Manifolds).

(a) If M is a smooth manifold and $f \in C^\infty(M)$, the map from $\mathfrak{X}(M)$ to itself defined by $X \mapsto fX$ is linear over $C^\infty(M)$ because $f(u_1X_1 + u_2X_2) = u_1fX_1 + u_2fX_2$, and thus defines a smooth bundle homomorphism over M from TM to itself.

(b) If Z is a smooth vector field on $\mathbb{R}^3$, the cross product with Z defines a map from $\mathfrak{X}\left(\mathbb{R}^3\right)$ to itself: $X \mapsto X \times Z$. Since it is linear over $C^\infty\left(\mathbb{R}^3\right)$ in X, it determines a smooth bundle homomorphism over $\mathbb{R}^3$ from $T\mathbb{R}^3$ to $T\mathbb{R}^3$.

(c) Given $Z \in \mathfrak{X}(\mathbb{R}^n)$, the Euclidean dot product defines a map $X \mapsto X \cdot Z$ from $\mathfrak{X}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$, which is linear over $C^\infty(\mathbb{R}^n)$ and thus determines a smooth bundle homomorphism over $\mathbb{R}^n$ from $T\mathbb{R}^n$ to the trivial line bundle $\mathbb{R}^n \times \mathbb{R}$. //

Because of Lemma 10.29, we usually dispense with the notation $\widetilde{F}$ and use the same symbol for both a bundle homomorphism $F\colon E \to E'$ over M and the linear map $F\colon \Gamma(E) \to \Gamma(E')$ that it induces on sections, and we refer to a map of either of these types as a bundle homomorphism. Because the action on sections is obtained simply by applying the bundle homomorphism pointwise, this should cause no confusion. In fact, we have been doing the same thing all along in certain circumstances. For example, if $a \in \mathbb{R}$, we use the same notation $X \mapsto aX$ to denote both the operation of multiplying vectors in each tangent space T_pM by a, and the operation of multiplying vector fields by a. Because multiplying by a is a bundle homomorphism from TM to itself, there is no ambiguity about what is meant.

It should be noted that most maps that involve differentiation are *not* bundle homomorphism. For example, if X is a smooth vector field on a smooth manifold M, the Lie derivative operator $\mathcal{L}_X\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ is not a bundle homomorphism from the tangent bundle to itself, because it is not linear over $C^\infty(M)$. As a rule of thumb, a linear map that takes smooth sections of one bundle to smooth sections of another is likely to be a bundle homomorphism if it acts pointwise, but not if it involves differentiation.

Subbundles

Given a vector bundle $\pi_E\colon E \to M$, a ***subbundle of E*** (see Fig. 10.4) is a vector bundle $\pi_D\colon D \to M$, in which D is a topological subspace of E and π_D is the restriction of π_E to D, such that for each $p \in M$, the subset $D_p = D \cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p. Note that the condition that D be a vector bundle over M implies that all of the fibers D_p must be nonempty and have the same dimension. If $E \to M$ is a smooth bundle, then a subbundle of E is called a ***smooth subbundle*** if it is a smooth vector bundle and an embedded submanifold with or without boundary in E.

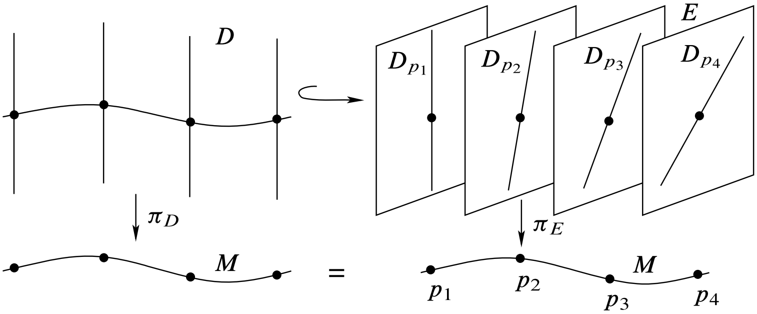


Fig. 10.4 A subbundle of a vector bundle

► **Exercise 10.31.** Given a smooth vector bundle $E \to M$ and a smooth subbundle $D \subseteq E$, show that the inclusion map $\iota\colon D \hookrightarrow E$ is a smooth bundle homomorphism over M.

The following lemma gives a convenient condition for checking that a union of subspaces $\{D_p \subseteq E_p : p \in M\}$ is a smooth subbundle.

Lemma 10.32 (Local Frame Criterion for Subbundles). *Let $\pi\colon E \to M$ be a smooth vector bundle, and suppose that for each $p \in M$ we are given an m-dimensional linear subspace $D_p \subseteq E_p$. Then $D = \bigcup_{p \in M} D_p \subseteq E$ is a smooth subbundle of E if and only if the following condition is satisfied:*

Each point of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \dots, \sigma_m\colon U \to E$ with the property that $\sigma_1(q), \dots, \sigma_m(q)$ form a basis for D_q at each $q \in U$. (10.6)

Proof. If D is a smooth subbundle, then by definition each $p \in M$ has a neighborhood U over which there exists a smooth local trivialization of D, and Example 10.18 shows that there exists a smooth local frame for D over each such set U. Such a local frame is by definition a collection of smooth sections $\tau_1, \dots, \tau_m\colon U \to D$ whose images form a basis for D_p at each point $p \in U$. The smooth sections of E that we seek are obtained by composing with the inclusion map $\iota\colon D \hookrightarrow E$: $\sigma_j = \iota \circ \tau_j$.

Conversely, suppose $E \to M$ is a smooth bundle of rank k, and $D \subseteq E$ satisfies (10.6). Each set $D \cap E_p$ is a linear subspace of E_p by hypothesis, so we need to show that D is an embedded submanifold with or without boundary in E and that the restriction of π makes it into a smooth vector bundle over M.

To prove that D is an embedded submanifold with or without boundary, it suffices to show that each $p \in M$ has a neighborhood U such that $D \cap \pi^{-1}(U)$ is an embedded submanifold (possibly with boundary) in $\pi^{-1}(U) \subseteq E$. Given $p \in M$, let $\sigma_1, \dots, \sigma_m$ be smooth local sections of E satisfying (10.6) on a neighborhood of p. By Proposition 10.15, we can complete these to a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood U of p. By Proposition 10.19, this local frame is associated with a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$, defined by

$$\Phi\big(s^1\sigma_1(q) + \cdots + s^k\sigma_k(q)\big) = \big(q, \big(s^1, \ldots, s^k\big)\big).$$

This map Φ takes $D \cap \pi^{-1}(U)$ to the subset $\big\{\big(q, \big(s^1, \ldots, s^m, 0, \ldots, 0\big)\big)\big\} \subseteq U \times \mathbb{R}^k$, which is an embedded submanifold (with boundary if U has a boundary). Moreover, the map $\Psi\colon D \cap \pi^{-1}(U) \to U \times \mathbb{R}^m$ defined by

$$\Psi\left(s^1\sigma_1(q) + \cdots + s^m\sigma_m(q)\right) = \left(q, \left(s^1, \ldots, s^m\right)\right)$$

is a smooth local trivialization of D, so D is itself a smooth vector bundle. □

Example 10.33 (Subbundles).

(a) If M is a smooth manifold and V is a nowhere-vanishing smooth vector field on M, then the set $D \subseteq TM$ whose fiber at each $p \in M$ is the linear span of V_p is a smooth 1-dimensional subbundle of TM.
(b) Suppose $E \to M$ is any trivial bundle, and let $(E_1, \ldots, E_k)$ be a smooth global frame for E. If $0 \le m \le k$, the subset $D \subseteq E$ defined by $D_p = \operatorname{span}\left(E_1|_p, \ldots, E_m|_p\right)$ for each $p \in M$ is a smooth subbundle of E.
(c) Suppose M is a smooth manifold with or without boundary and $S \subseteq M$ is an immersed k-submanifold with or without boundary. Problem 10-14 asks you to prove that TS is a smooth rank-k subbundle of the ambient tangent bundle $TM|_S$. //

The next theorem shows how to obtain many more subbundles. Suppose $E \to M$ and $E' \to M$ are vector bundles and $F\colon E \to E'$ is a bundle homomorphism over M. For each $p \in M$, the rank of the linear map $F|_{E_p}$ is called the ***rank of F at p***. We say that F has ***constant rank*** if its rank is the same for all $p \in M$.

Theorem 10.34. *Let E and E' be smooth vector bundles over a smooth manifold M, and let $F\colon E \to E'$ be a smooth bundle homomorphism over M. Define subsets* $\operatorname{Ker} F \subseteq E$ *and* $\operatorname{Im} F \subseteq E'$ *by*

$$\operatorname{Ker} F = \bigcup_{p \in M} \operatorname{Ker}\left(F|_{E_p}\right), \qquad \operatorname{Im} F = \bigcup_{p \in M} \operatorname{Im}\left(F|_{E_p}\right).$$

Then $\operatorname{Ker} F$ *and* $\operatorname{Im} F$ *are smooth subbundles of E and E', respectively, if and only if F has constant rank.*

Proof. One direction is obvious: since the fibers of a bundle have the same dimension everywhere, the constant-rank condition is certainly necessary for $\operatorname{Ker} F$ and $\operatorname{Im} F$ to be subbundles. To prove sufficiency, suppose F has constant rank r, and let k and k' be the ranks of the bundles E and E', respectively. Let $p \in M$ be arbitrary, and choose a smooth local frame $(\sigma_1, \ldots, \sigma_k)$ for E over a neighborhood U of p. For each i, the map $F \circ \sigma_i\colon U \to E'$ is a smooth local section of E', and these sections span $(\operatorname{Im} F)|_U$. After rearranging the indices if necessary, we can assume that the elements $\{F \circ \sigma_1(p), \ldots, F \circ \sigma_r(p)\}$ form a basis for $\operatorname{Im}\left(F|_{E_p}\right)$, and by continuity they remain linearly independent in some neighborhood U_0 of p. Since F has constant rank, this means that $(F \circ \sigma_1, \dots, F \circ \sigma_r)$ forms a smooth local frame for $\operatorname{Im} F$ over U_0. Since we can do the same in a neighborhood of each point, the local frame criterion shows that $\operatorname{Im} F$ is a smooth subbundle of E'.

To prove that $\operatorname{Ker} F$ is also a smooth subbundle, let U_0 and (σ_i) be as above, and let $V \subseteq E|_{U_0}$ be the smooth subbundle spanned by $\sigma_1, \dots, \sigma_r$. The smooth bundle homomorphism $F|_V : V \to (\operatorname{Im} F)|_{U_0}$ is bijective, and is thus a smooth bundle isomorphism by Proposition 10.26. Define a smooth bundle homomorphism $\Psi : E|_{U_0} \to E|_{U_0}$ by $\Psi(v) = v - (F|_V)^{-1} \circ F(v)$. If $v \in V$, then $F(v) = (F|_V)(v)$, so $F(\Psi(v)) = F(v) - F \circ (F|_V)^{-1} \circ (F|_V)(v) = 0$. On the other hand, if $v \in \operatorname{Ker} F$, then $\Psi(v) = v$, so again $F(\Psi(v)) = F(v) = 0$. Since V and $(\operatorname{Ker} F)|_{U_0}$ together span $E|_{U_0}$, it follows that Ψ takes its values in $(\operatorname{Ker} F)|_{U_0}$, and since it restricts to the identity on $(\operatorname{Ker} F)|_{U_0}$, its image is exactly $(\operatorname{Ker} F)|_{U_0}$. Thus Ψ has constant rank, and by the argument in the preceding paragraph, $(\operatorname{Ker} F)|_{U_0} = \operatorname{Im} \Psi$ is a smooth subbundle of $E|_{U_0}$. Since we can do the same thing in a neighborhood of each point, $\operatorname{Ker} F$ is a smooth subbundle of E. □

The next proposition illustrates another method for constructing interesting subbundles of the tangent bundle over submanifolds of $\mathbb{R}^n$.

Lemma 10.35 (Orthogonal Complement Bundles). *Let M be an immersed submanifold with or without boundary in $\mathbb{R}^n$, and D be a smooth rank-k subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, let $D_p^\perp$ denote the orthogonal complement of D_p in $T_p\mathbb{R}^n$ with respect to the Euclidean dot product, and let $D^\perp \subseteq T\mathbb{R}^n|_M$ be the subset*

$$D^\perp = \{(p, v) \in T\mathbb{R}^n : p \in M,\ v \in D_p^\perp\}.$$

Then $D^\perp$ is a smooth rank-$(n-k)$ subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, there is a smooth orthonormal frame for $D^\perp$ on a neighborhood of p.

Proof. Let $p \in M$ be arbitrary, and let $(X_1, \dots, X_k)$ be a smooth local frame for D over some neighborhood V of p in M. Because immersed submanifolds are locally embedded, by shrinking V if necessary, we may assume that it is a single slice in some coordinate ball or half-ball $U \subseteq \mathbb{R}^n$. Since V is closed in U, Proposition 8.11(c) shows that we can complete $(X_1, \dots, X_k)$ to a smooth local frame $(\widetilde{X}_1, \dots, \widetilde{X}_n)$ for $T\mathbb{R}^n$ over U, and then Lemma 8.13 yields a smooth *orthonormal* frame (E_j) over U such that $\operatorname{span}(E_1|_p, \dots, E_k|_p) = \operatorname{span}(X_1|_p, \dots, X_k|_p) = D_p$ for each $p \in U$. It follows that $(E_{k+1}, \dots, E_n)$ restricts to a smooth orthonormal frame for $D^\perp$ over V. Thus $D^\perp$ satisfies the local frame criterion, and is therefore a smooth subbundle of $T\mathbb{R}^n|_M$. □

Corollary 10.36 (The Normal Bundle to a Submanifold of $\mathbb{R}^n$). *If $M \subseteq \mathbb{R}^n$ is an immersed m-dimensional submanifold with or without boundary, its normal bundle NM is a smooth rank-$(n-m)$ subbundle of $T\mathbb{R}^n|_M$. For each $p \in M$, there exists a smooth orthonormal frame for NM on a neighborhood of p.*

Proof. Apply Lemma 10.35 to the smooth subbundle $TM \subseteq T\mathbb{R}^n|_M$. □

Fiber Bundles

We conclude this chapter by giving a brief introduction to an important generalization of vector bundles, in which the fibers are allowed to be arbitrary topological spaces instead of vector spaces. We can only touch on the subject here; but fiber bundles appear in many applications of manifold theory, so it is important to be at least familiar with the definitions.

Let M and F be topological spaces. A ***fiber bundle over M with model fiber F*** is a topological space E together with a surjective continuous map $\pi\colon E \to M$ with the property that for each $x \in M$, there exist a neighborhood U of x in M and a homeomorphism $\Phi\colon \pi^{-1}(U) \to U \times F$, called a ***local trivialization of E over U***, such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \Phi\ } & U \times F \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & U. & \end{array}$$

The space E is called the ***total space of the bundle***, M is its ***base***, and π is its ***projection***. If E, M, and F are smooth manifolds with or without boundary, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then it is called a ***smooth fiber bundle***.

A ***trivial fiber bundle*** is one that admits a local trivialization over the entire base space (a ***global trivialization***). It is said to be ***smoothly trivial*** if it is a smooth bundle and the global trivialization is a diffeomorphism.

Example 10.37 (Fiber Bundles).

(a) Every product space $M \times F$ is a fiber bundle with projection $\pi_1\colon M \times F \to M$, called a ***product fiber bundle***. It has a global trivialization given by the identity map $M \times F \to M \times F$, so every product bundle is trivial.
(b) Every rank-k vector bundle is a fiber bundle with model fiber $\mathbb{R}^k$.
(c) If $E \to \mathbb{S}^1$ is the Möbius bundle of Example 10.3, then the image of $\mathbb{R} \times [-1, 1]$ under the quotient map $q\colon \mathbb{R}^2 \to E$ is a fiber bundle over $\mathbb{S}^1$ with model fiber $[-1, 1]$. It is not a trivial bundle. (Can you prove it?)
(d) Every covering map $\pi\colon E \to M$ is a fiber bundle whose model fiber is discrete. To construct local trivializations, let S be a discrete space with the same cardinality as the fibers of π. For each evenly covered open subset $U \subseteq M$, define a map $\Phi\colon \pi^{-1}(U) \to U \times S$ by choosing a bijection between the set of components of $\pi^{-1}(U)$ and S, and letting $\Phi(x) = \big(\pi(x), c(x)\big)$, where $c(x)$ is the element of S corresponding to the component containing x. //

We will see a few more examples of fiber bundles as we go along.

Problems

10-1. Let E be the total space of the Möbius bundle constructed in Example 10.3.

(a) Show that E has a unique smooth structure such that the quotient map $q\colon \mathbb{R}^2 \to E$ is a smooth covering map.
(b) Show that $\pi\colon E \to \mathbb{S}^1$ is a smooth rank-1 vector bundle.
(c) Show that it is not a trivial bundle.

10-2. Let E be a vector bundle over a topological space M. Show that the projection map $\pi\colon E \to M$ is a homotopy equivalence.

10-3. Let VB denote the category whose objects are smooth vector bundles and whose morphisms are smooth bundle homomorphism, and let Diff denote the category whose objects are smooth manifolds and whose morphisms are smooth maps. Show that the assignment $M \mapsto TM$, $F \mapsto dF$ defines a covariant functor from Diff to VB, called the ***tangent functor***. (*Used on p. 303.*)

10-4. Complete the proof of Lemma 10.5 by showing that $\tau\colon U \cap V \to \mathrm{GL}(k,\mathbb{R})$ is smooth. [Hint: use the same idea as in the proof of Proposition 7.37.]

10-5. Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M with or without boundary. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, and for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. For each $\alpha,\beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, let $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R})$ be the transition function defined by (10.3). Show that the following identity is satisfied for all $\alpha,\beta,\gamma \in A$:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p), \qquad p \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{10.7}$$

(The juxtaposition on the left-hand side represents matrix multiplication.)

10-6. Vector Bundle Construction Theorem: Let M be a smooth manifold with or without boundary, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose for each $\alpha,\beta \in A$ we are given a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R})$ such that (10.7) is satisfied for all $\alpha,\beta,\gamma \in A$. Show that there is a smooth rank-k vector bundle $E \to M$ with smooth local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose transition functions are the given maps $\tau_{\alpha\beta}$. [Hint: define an appropriate equivalence relation on $\coprod_{\alpha\in A}\left(U_\alpha \times \mathbb{R}^k\right)$, and use the vector bundle chart lemma.]

10-7. Compute the transition function for $T\mathbb{S}^2$ associated with the two local trivializations determined by stereographic coordinates (Problem 1-7).

10-8. Let Vec_1 be the category whose objects are finite-dimensional real vector spaces and whose morphisms are linear isomorphisms. If $\mathcal{F}$ is a covariant functor from Vec_1 to itself, for each finite-dimensional vector space V we get a map $\mathcal{F}\colon \mathrm{GL}(V) \to \mathrm{GL}(\mathcal{F}(V))$ sending each isomorphism $A\colon V \to V$ to the induced isomorphism $\mathcal{F}(A)\colon \mathcal{F}(V) \to \mathcal{F}(V)$. We say $\mathcal{F}$ is a ***smooth functor*** if this map is smooth for every V. Given a smooth vector bundle $E \to M$ and a smooth functor $\mathcal{F}\colon \mathsf{Vec}_1 \to \mathsf{Vec}_1$, show that there is a smooth vector bundle $\mathcal{F}(E) \to M$ whose fiber at each point $p \in M$ is $\mathcal{F}(E_p)$. (*Used on p. 299.*)

10-9. Extension Lemma for Sections of Restricted Bundles: Suppose M is a smooth manifold, $E \to M$ is a smooth vector bundle, and $S \subseteq M$ is an embedded submanifold with or without boundary. For any smooth section σ of the restricted bundle $E|_S \to S$, show that there exist a neighborhood U of S in M and a smooth section $\widetilde{\sigma}$ of $E|_U$ such that $\sigma = \widetilde{\sigma}|_S$. If E has positive rank, show that every smooth section of $E|_S$ extends smoothly to all of M if and only if S is properly embedded.

10-10. Suppose M is a compact smooth manifold and $E \to M$ is a smooth vector bundle of rank k. Use transversality to prove that E admits a smooth section σ with the following property: if $k > \dim M$, then σ is nowhere vanishing; while if $k \le \dim M$, then the set of points where σ vanishes is a smooth compact codimension-k submanifold of M. Use this to show that M admits a smooth vector field with only finitely many singular points.

10-11. Prove Proposition 10.26 (a bijective bundle homomorphism is a bundle isomorphism).

10-12. Let $\pi\colon E \to M$ and $\widetilde{\pi}\colon \widetilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M with or without boundary. Suppose $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit smooth local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}$ and $\{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$, respectively. Show that E and $\widetilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha\colon U_\alpha \to \mathrm{GL}(k,\mathbb{R})$ such that

$$\widetilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \qquad p \in U_\alpha \cap U_\beta.$$

10-13. Let $U = \mathbb{S}^1 \smallsetminus \{1\}$ and $V = \mathbb{S}^1 \smallsetminus \{-1\}$, and define $\tau\colon U \cap V \to \mathrm{GL}(1,\mathbb{R})$ by

$$\tau(z) = \begin{cases} (1), & \operatorname{Im} z > 0, \\ (-1), & \operatorname{Im} z < 0. \end{cases}$$

By the result of Problem 10-6, there is a smooth real line bundle $F \to \mathbb{S}^1$ that is trivial over U and V, and has τ as transition function. Show that F is smoothly isomorphic over $\mathbb{S}^1$ to the Möbius bundle of Example 10.3.

10-14. Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. Identifying T_pS as a subspace of T_pM for each $p \in S$ in the usual way, show that TS is a smooth subbundle of $TM|_S$. (See Example 10.33.)

10-15. Let V be a finite-dimensional real vector space, and let $\mathrm{G}_k(V)$ be the Grassmannian of k-dimensional subspaces of V (see Example 1.36). Let T be the subset of $\mathrm{G}_k(V) \times V$ defined by

$$T = \big\{(S,v) \in \mathrm{G}_k(V) \times V : v \in S\big\}.$$

Show that T is a smooth rank-k subbundle of the product bundle $\mathrm{G}_k(V) \times V \to \mathrm{G}_k(V)$, and is thus a smooth rank-k vector bundle over $\mathrm{G}_k(V)$.

[Remark: T is called the ***tautological vector bundle*** over $G_k(V)$, because the fiber over each point $S \in G_k(V)$ is S itself.]

10-16. Show that the tautological vector bundle over $G_1(\mathbb{R}^2)$ is smoothly isomorphic to the Möbius bundle. (See Problems 10-1, 10-13, and 10-15.)

10-17. Suppose $M \subseteq \mathbb{R}^n$ is an immersed submanifold. Prove that the ambient tangent bundle $T\mathbb{R}^n|_M$ is isomorphic to the Whitney sum $TM \oplus NM$, where $NM \to M$ is the normal bundle.

10-18. Suppose S is a properly embedded codimension-k submanifold of $\mathbb{R}^n$. Show that the following are equivalent:

(a) There exists a smooth defining function for S on some neighborhood U of S in $\mathbb{R}^n$, that is, a smooth function $\Phi: U \to \mathbb{R}^k$ such that S is a regular level set of Φ.

(b) The normal bundle NS is a trivial vector bundle.

10-19. Suppose $\pi: E \to M$ is a fiber bundle with fiber F. Prove the following:

(a) π is an open quotient map.

(b) If the bundle is smooth, then π is a smooth submersion.

(c) π is a proper map if and only if F is compact.

(d) E is compact if and only if both M and F are compact.

(*Used on p. 560.*)