

Chapter 8

Vector Fields

Vector fields are familiar objects of study in multivariable calculus. In that setting, a vector field on an open subset $U \subseteq \mathbb{R}^n$ is simply a continuous map from U to \mathbb{R}^n , which can be visualized as attaching an “arrow” to each point of U . In this chapter we show how to extend this idea to smooth manifolds.

We think of a vector field on an abstract smooth manifold M as a particular kind of continuous map X from M to its tangent bundle—one that assigns to each point $p \in M$ a tangent vector $X_p \in T_p M$. After introducing the definitions, we explore the ways that vector fields behave under differentials of smooth maps.

In the next section we introduce the *Lie bracket* operation, which is a way of combining two smooth vector fields to obtain another. Then we describe the most important application of Lie brackets: the set of all smooth vector fields on a Lie group that are invariant under left multiplication is closed under Lie brackets, and thus forms an algebraic object naturally associated with the group, called the *Lie algebra of the Lie group*. We describe a few basic properties of Lie algebras, and compute the Lie algebras of some familiar groups. Then we show how Lie group homomorphisms induce homomorphisms of their Lie algebras, from which it follows that isomorphic Lie groups have isomorphic Lie algebras. Finally, at the end of the chapter we show how to identify Lie algebras of Lie subgroups.

Vector Fields on Manifolds

If M is a smooth manifold with or without boundary, a **vector field on M** is a section of the map $\pi: TM \rightarrow M$. More concretely, a vector field is a continuous map $X: M \rightarrow TM$, usually written $p \mapsto X_p$, with the property that

$$\pi \circ X = \text{Id}_M, \tag{8.1}$$

or equivalently, $X_p \in T_p M$ for each $p \in M$. (We write the value of X at p as X_p instead of $X(p)$ to be consistent with our notation for elements of the tangent bundle, as well as to avoid conflict with the notation $v(f)$ for the action of a vector on a function.) You should visualize a vector field on M in the same way as you

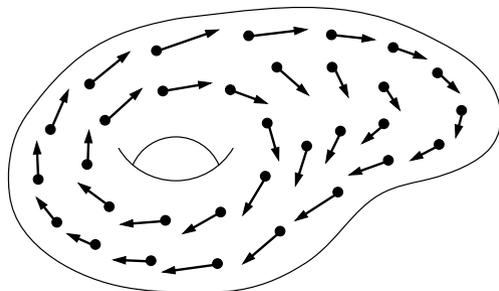


Fig. 8.1 A vector field

visualize vector fields in Euclidean space: as an arrow attached to each point of M , chosen to be tangent to M and to vary continuously from point to point (Fig. 8.1).

We are primarily interested in *smooth vector fields*, the ones that are smooth as maps from M to TM , when TM is given the smooth manifold structure described in Proposition 3.18. In addition, for some purposes it is useful to consider maps from M to TM that would be vector fields except that they might not be continuous. A *rough vector field on M* is a (not necessarily continuous) map $X: M \rightarrow TM$ satisfying (8.1). Just as for functions, if X is a vector field on M , the *support of X* is defined to be the closure of the set $\{p \in M : X_p \neq 0\}$. A vector field is said to be *compactly supported* if its support is a compact set.

Suppose M is a smooth n -manifold (with or without boundary). If $X: M \rightarrow TM$ is a rough vector field and $(U, (x^i))$ is any smooth coordinate chart for M , we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \left. \frac{\partial}{\partial x^i} \right|_p. \tag{8.2}$$

This defines n functions $X^i: U \rightarrow \mathbb{R}$, called the *component functions of X* in the given chart.

Proposition 8.1 (Smoothness Criterion for Vector Fields). *Let M be a smooth manifold with or without boundary, and let $X: M \rightarrow TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M , then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.*

Proof. Let (x^i, v^i) be the natural coordinates on $\pi^{-1}(U) \subseteq TM$ associated with the chart $(U, (x^i))$. By definition of natural coordinates, the coordinate representation of $X: M \rightarrow TM$ on U is

$$\hat{X}(x) = (x^1, \dots, x^n, X^1(x), \dots, X^n(x)),$$

where X^i is the i th component function of X in x^i -coordinates. It follows immediately that smoothness of X in U is equivalent to smoothness of its component functions. □

Example 8.2 (Coordinate Vector Fields). If $(U, (x^i))$ is any smooth chart on M , the assignment

$$p \mapsto \left. \frac{\partial}{\partial x^i} \right|_p$$

determines a vector field on U , called the ***i th coordinate vector field*** and denoted by $\partial/\partial x^i$. It is smooth because its component functions are constants. //

Example 8.3 (The Euler Vector Field). The vector field V on \mathbb{R}^n whose value at $x \in \mathbb{R}^n$ is

$$V_x = x^1 \left. \frac{\partial}{\partial x^1} \right|_x + \cdots + x^n \left. \frac{\partial}{\partial x^n} \right|_x$$

is smooth because its coordinate functions are linear. It vanishes at the origin, and points radially outward everywhere else. It is called the ***Euler vector field*** because of its appearance in Euler's homogeneous function theorem (see Problem 8-2). //

Example 8.4 (The Angle Coordinate Vector Field on the Circle). Let θ be any angle coordinate on a proper open subset $U \subseteq S^1$ (see Problem 1-8), and let $d/d\theta$ denote the corresponding coordinate vector field. Because any other angle coordinate $\tilde{\theta}$ differs from θ by an additive constant in a neighborhood of each point, the transformation law for coordinate vector fields (3.11) shows that $d/d\theta = d/d\tilde{\theta}$ on their common domain. For this reason, there is a globally defined vector field on S^1 whose coordinate representation is $d/d\theta$ with respect to any angle coordinate. It is a smooth vector field because its component function is constant in any such chart. We denote this global vector field by $d/d\theta$, even though, strictly speaking, it cannot be considered as a coordinate vector field on the entire circle at once. //

Example 8.5 (Angle Coordinate Vector Fields on Tori). On the n -dimensional torus \mathbb{T}^n , choosing an angle function θ^i for the i th circle factor, $i = 1, \dots, n$, yields local coordinates $(\theta^1, \dots, \theta^n)$ for \mathbb{T}^n . An analysis similar to that of the previous example shows that the coordinate vector fields $\partial/\partial\theta^1, \dots, \partial/\partial\theta^n$ are smooth and globally defined on \mathbb{T}^n . //

If $U \subseteq M$ is open, the fact that $T_p U$ is naturally identified with $T_p M$ for each $p \in U$ (Proposition 3.9) allows us to identify TU with the open subset $\pi^{-1}(U) \subseteq TM$. Therefore, a vector field on U can be thought of either as a map from U to TU or as a map from U to TM , whichever is more convenient. If X is a vector field on M , its restriction $X|_U$ is a vector field on U , which is smooth if X is.

The next lemma is a generalization of Lemma 2.26 to vector fields, and is proved in much the same way. If M is a smooth manifold with or without boundary and $A \subseteq M$ is an arbitrary subset, a ***vector field along A*** is a continuous map $X: A \rightarrow TM$ satisfying $\pi \circ X = \text{Id}_A$ (or in other words $X_p \in T_p M$ for each $p \in A$). We call it a ***smooth vector field along A*** if for each $p \in A$, there is a neighborhood V of p in M and a smooth vector field \tilde{X} on V that agrees with X on $V \cap A$.

Lemma 8.6 (Extension Lemma for Vector Fields). *Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth*

vector field along A . Given any open subset U containing A , there exists a smooth global vector field \tilde{X} on M such that $\tilde{X}|_A = X$ and $\text{supp } \tilde{X} \subseteq U$.

Proof. See Problem 8-1. □

As an important special case, any vector at a point can be extended to a smooth vector field on the entire manifold.

Proposition 8.7. *Let M be a smooth manifold with or without boundary. Given $p \in M$ and $v \in T_pM$, there is a smooth global vector field X on M such that $X_p = v$.*

Proof. The assignment $p \mapsto v$ is an example of a vector field along the set $\{p\}$ as defined above. It is smooth because it can be extended, say, to a constant-coefficient vector field in a coordinate neighborhood of p . Thus, the proposition follows from the extension lemma with $A = \{p\}$ and $U = M$. □

If M is a smooth manifold with or without boundary, it is standard to use the notation $\mathfrak{X}(M)$ to denote the set of all smooth vector fields on M . It is a vector space under pointwise addition and scalar multiplication:

$$(aX + bY)_p = aX_p + bY_p.$$

The zero element of this vector space is the zero vector field, whose value at each $p \in M$ is $0 \in T_pM$. In addition, smooth vector fields can be multiplied by smooth real-valued functions: if $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define $fX : M \rightarrow TM$ by

$$(fX)_p = f(p)X_p.$$

The next proposition shows that these operations yield smooth vector fields.

Proposition 8.8. *Let M be a smooth manifold with or without boundary.*

- (a) *If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.*
- (b) *$\mathfrak{X}(M)$ is a module over the ring $C^\infty(M)$.*

► **Exercise 8.9.** Prove Proposition 8.8.

For example, the basis expression (8.2) for a vector field X can also be written as an equation between vector fields instead of an equation between vectors at a point:

$$X = X^i \frac{\partial}{\partial x^i},$$

where X^i is the i th component function of X in the given coordinates.

Local and Global Frames

Coordinate vector fields in a smooth chart provide a convenient way of representing vector fields, because their values form a basis for the tangent space at each point. However, they are not the only choices.

Suppose M is a smooth n -manifold with or without boundary. An ordered k -tuple (X_1, \dots, X_k) of vector fields defined on some subset $A \subseteq M$ is said to be **linearly independent** if $(X_1|_p, \dots, X_k|_p)$ is a linearly independent k -tuple in $T_p M$ for each $p \in A$, and is said to **span the tangent bundle** if the k -tuple $(X_1|_p, \dots, X_k|_p)$ spans $T_p M$ at each $p \in A$. A **local frame for M** is an ordered n -tuple of vector fields (E_1, \dots, E_n) defined on an open subset $U \subseteq M$ that is linearly independent and spans the tangent bundle; thus the vectors $(E_1|_p, \dots, E_n|_p)$ form a basis for $T_p M$ at each $p \in U$. It is called a **global frame** if $U = M$, and a **smooth frame** if each of the vector fields E_i is smooth. We often use the shorthand notation (E_i) to denote a frame (E_1, \dots, E_n) . If M has dimension n , then to check that an ordered n -tuple of vector fields (E_1, \dots, E_n) is a local frame, it suffices to check either that it is linearly independent or that it spans the tangent bundle.

Example 8.10 (Local and Global Frames).

- (a) The standard coordinate vector fields form a smooth global frame for \mathbb{R}^n .
- (b) If $(U, (x^i))$ is any smooth coordinate chart for a smooth manifold M (possibly with boundary), then the coordinate vector fields form a smooth local frame $(\partial/\partial x^i)$ on U , called a **coordinate frame**. Every point of M is in the domain of such a local frame.
- (c) The vector field $d/d\theta$ defined in Example 8.4 constitutes a smooth global frame for the circle.
- (d) The n -tuple of vector fields $(\partial/\partial\theta^1, \dots, \partial/\partial\theta^n)$ on the n -torus, defined in Example 8.4, is a smooth global frame for \mathbb{T}^n . //

The next proposition shows that local frames are easy to come by.

Proposition 8.11 (Completion of Local Frames). *Let M be a smooth n -manifold with or without boundary.*

- (a) *If (X_1, \dots, X_k) is a linearly independent k -tuple of smooth vector fields on an open subset $U \subseteq M$, with $1 \leq k < n$, then for each $p \in U$ there exist smooth vector fields X_{k+1}, \dots, X_n in a neighborhood V of p such that (X_1, \dots, X_n) is a smooth local frame for M on $U \cap V$.*
- (b) *If (v_1, \dots, v_k) is a linearly independent k -tuple of vectors in $T_p M$ for some $p \in M$, with $1 \leq k \leq n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i|_p = v_i$ for $i = 1, \dots, k$.*
- (c) *If (X_1, \dots, X_n) is a linearly independent n -tuple of smooth vector fields along a closed subset $A \subseteq M$, then there exists a smooth local frame $(\tilde{X}_1, \dots, \tilde{X}_n)$ on some neighborhood of A such that $\tilde{X}_i|_A = X_i$ for $i = 1, \dots, n$.*

Proof. See Problem 8-5. □

For subsets of \mathbb{R}^n , there is a special type of frame that is often more useful for geometric problems than arbitrary frames. A k -tuple of vector fields (E_1, \dots, E_k) defined on some subset $A \subseteq \mathbb{R}^n$ is said to be **orthonormal** if for each $p \in A$, the vectors $(E_1|_p, \dots, E_k|_p)$ are orthonormal with respect to the Euclidean dot product (where we identify $T_p \mathbb{R}^n$ with \mathbb{R}^n in the usual way). A (local or global) frame consisting of orthonormal vector fields is called an **orthonormal frame**.

Example 8.12. The standard coordinate frame is a global orthonormal frame on \mathbb{R}^n . For a less obvious example, consider the smooth vector fields defined on $\mathbb{R}^2 \setminus \{0\}$ by

$$E_1 = \frac{x}{r} \frac{\partial}{\partial x} + \frac{y}{r} \frac{\partial}{\partial y}, \quad E_2 = -\frac{y}{r} \frac{\partial}{\partial x} + \frac{x}{r} \frac{\partial}{\partial y}, \quad (8.3)$$

where $r = \sqrt{x^2 + y^2}$. A straightforward computation shows that (E_1, E_2) is an orthonormal frame for \mathbb{R}^2 over the open subset $\mathbb{R}^2 \setminus \{0\}$. Geometrically, E_1 and E_2 are unit vector fields tangent to radial lines and circles centered at the origin, respectively. //

The next lemma describes a useful method for creating orthonormal frames.

Lemma 8.13 (Gram–Schmidt Algorithm for Frames). *Suppose (X_j) is a smooth local frame for $T\mathbb{R}^n$ over an open subset $U \subseteq \mathbb{R}^n$. Then there is a smooth orthonormal frame (E_j) over U such that $\text{span}(E_1|_p, \dots, E_j|_p) = \text{span}(X_1|_p, \dots, X_j|_p)$ for each $j = 1, \dots, n$ and each $p \in U$.*

Proof. Applying the Gram–Schmidt algorithm to the vectors $(X_j|_p)$ at each $p \in U$, we obtain an n -tuple of rough vector fields (E_1, \dots, E_n) given inductively by

$$E_j = \frac{X_j - \sum_{i=1}^{j-1} (X_j \cdot E_i) E_i}{|X_j - \sum_{i=1}^{j-1} (X_j \cdot E_i) E_i|}.$$

For each $j = 1, \dots, n$ and each $p \in U$, $X_j|_p \notin \text{span}(E_1|_p, \dots, E_{j-1}|_p)$ (which is equal to $\text{span}(X_1|_p, \dots, X_{j-1}|_p)$), so the denominator above is a nowhere-vanishing smooth function on U . Therefore, this formula defines (E_j) as a smooth orthonormal frame on U that satisfies the conclusion of the lemma. \square

Although smooth local frames are plentiful, global ones are not. A smooth manifold with or without boundary is said to be **parallelizable** if it admits a smooth global frame. Example 8.10 shows that \mathbb{R}^n , \mathbb{S}^1 , and \mathbb{T}^n are all parallelizable. Problems 8-6 and 8-7 show that \mathbb{S}^3 and \mathbb{S}^7 are parallelizable. Later in this chapter, we will see that all Lie groups are parallelizable (see Corollary 8.39 below). However, despite the evidence of these examples, most smooth manifolds are not parallelizable. (As we will see in Chapter 10, parallelizability of M is intimately connected to the question of whether its tangent bundle is diffeomorphic to the product $M \times \mathbb{R}^n$.)

The simplest example of a nonparallelizable manifold is \mathbb{S}^2 , but the proof of this fact will have to wait until we have developed more machinery (see Problem 16-6). In fact, using more advanced methods from algebraic topology, it was shown in 1958 by Raoul Bott and John Milnor [MB58] and independently by Michel Ker-vaire [Ker58] that \mathbb{S}^1 , \mathbb{S}^3 , and \mathbb{S}^7 are the *only* spheres that are parallelizable. Thus these are the only positive-dimensional spheres that can possibly admit Lie group structures. The first two do (see Example 7.3(j) and Problem 7-16), but it turns out that \mathbb{S}^7 has no Lie group structure (see [Bre93, p. 301]).

Vector Fields as Derivations of $C^\infty(M)$

An essential property of vector fields is that they define operators on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$ and f is a smooth real-valued function defined on an open subset $U \subseteq M$, we obtain a new function $Xf: U \rightarrow \mathbb{R}$, defined by

$$(Xf)(p) = X_p f.$$

(Be careful not to confuse the notations fX and Xf : the former is the smooth *vector field* on U obtained by multiplying X by f , while the latter is the real-valued *function* on U obtained by applying the vector field X to the smooth function f .) Because the action of a tangent vector on a function is determined by the values of the function in an arbitrarily small neighborhood, it follows that Xf is locally determined. In particular, for any open subset $V \subseteq U$,

$$(Xf)|_V = X(f|_V). \quad (8.4)$$

This construction yields another useful smoothness criterion for vector fields.

Proposition 8.14. *Let M be a smooth manifold with or without boundary, and let $X: M \rightarrow TM$ be a rough vector field. The following are equivalent:*

- (a) X is smooth.
- (b) For every $f \in C^\infty(M)$, the function Xf is smooth on M .
- (c) For every open subset $U \subseteq M$ and every $f \in C^\infty(U)$, the function Xf is smooth on U .

Proof. We will prove that (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).

To prove (a) \Rightarrow (b), assume X is smooth, and let $f \in C^\infty(M)$. For any $p \in M$, we can choose smooth coordinates (x^i) on a neighborhood U of p . Then for $x \in U$, we can write

$$Xf(x) = \left(X^i(x) \frac{\partial}{\partial x^i} \Big|_x \right) f = X^i(x) \frac{\partial f}{\partial x^i}(x).$$

Since the component functions X^i are smooth on U by Proposition 8.1, it follows that Xf is smooth in U . Since the same is true in a neighborhood of each point, Xf is smooth on M .

To prove (b) \Rightarrow (c), suppose $U \subseteq M$ is open and $f \in C^\infty(U)$. For any $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighborhood of p and supported in U , and define $\tilde{f} = \psi f$, extended to be zero on $M \setminus \text{supp } \psi$. Then $X\tilde{f}$ is smooth by assumption, and is equal to Xf in a neighborhood of p by (8.4). This shows that Xf is smooth in a neighborhood of each point of U .

Finally, to prove (c) \Rightarrow (a), suppose Xf is smooth whenever f is smooth on an open subset of M . If (x^i) are any smooth local coordinates on $U \subseteq M$, we can think of each coordinate x^i as a smooth function on U . Applying X to one of these functions, we obtain

$$Xx^i = X^j \frac{\partial}{\partial x^j} (x^i) = X^i.$$

Because Xx^i is smooth by assumption, it follows that the component functions of X are smooth, so X is smooth. \square

One consequence of the preceding proposition is that a smooth vector field $X \in \mathfrak{X}(M)$ defines a map from $C^\infty(M)$ to itself by $f \mapsto Xf$. This map is clearly linear over \mathbb{R} . Moreover, the product rule (3.4) for tangent vectors translates into the following product rule for vector fields:

$$X(fg) = fXg + gXf, \quad (8.5)$$

as you can easily check by evaluating both sides at an arbitrary point $p \in M$. In general, a map $X: C^\infty(M) \rightarrow C^\infty(M)$ is called a **derivation** (as distinct from a *derivation at p* , defined in Chapter 3) if it is linear over \mathbb{R} and satisfies (8.5) for all $f, g \in C^\infty(M)$.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields.

Proposition 8.15. *Let M be a smooth manifold with or without boundary. A map $D: C^\infty(M) \rightarrow C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field $X \in \mathfrak{X}(M)$.*

Proof. We just showed that every smooth vector field induces a derivation. Conversely, suppose $D: C^\infty(M) \rightarrow C^\infty(M)$ is a derivation. We need to concoct a vector field X such that $Df = Xf$ for all f . From the discussion above, it is clear that if there is such a vector field, its value at $p \in M$ must be the derivation at p whose action on any smooth real-valued function f is given by

$$X_p f = (Df)(p).$$

The linearity of D guarantees that this expression depends linearly on f , and the fact that D is a derivation yields the product rule (3.4) for tangent vectors. Thus, the map $X_p: C^\infty(M) \rightarrow \mathbb{R}$ so defined is indeed a tangent vector, that is, a derivation of $C^\infty(M)$ at p . This defines X as a rough vector field. Because $Xf = Df$ is smooth whenever $f \in C^\infty(M)$, this vector field is smooth by Proposition 8.14. \square

Because of this result, we sometimes *identify* smooth vector fields on M with derivations of $C^\infty(M)$, using the same letter for both the vector field (thought of as a smooth map from M to TM) and the derivation (thought of as a linear map from $C^\infty(M)$ to itself).

Vector Fields and Smooth Maps

If $F: M \rightarrow N$ is a smooth map and X is a vector field on M , then for each point $p \in M$, we obtain a vector $dF_p(X_p) \in T_{F(p)}N$ by applying the differential of F to X_p . However, this does not in general define a *vector field* on N . For example, if F is not surjective, there is no way to decide what vector to assign to a point $q \in N \setminus F(M)$ (Fig. 8.2). If F is not injective, then for some points of N there

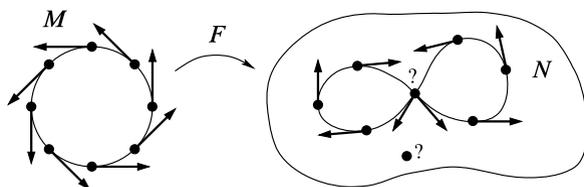


Fig. 8.2 The differential might not take vector fields to vector fields

may be several different vectors obtained by applying dF to X at different points of M .

Suppose $F: M \rightarrow N$ is smooth and X is a vector field on M , and suppose there happens to be a vector field Y on N with the property that for each $p \in M$, $dF_p(X_p) = Y_{F(p)}$. In this case, we say the vector fields X and Y are **F -related** (see Fig. 8.3). The next proposition shows how F -related vector fields act on smooth functions.

Proposition 8.16. *Suppose $F: M \rightarrow N$ is a smooth map between manifolds with or without boundary, $X \in \mathfrak{X}(M)$, and $Y \in \mathfrak{X}(N)$. Then X and Y are F -related if and only if for every smooth real-valued function f defined on an open subset of N ,*

$$X(f \circ F) = (Yf) \circ F. \quad (8.6)$$

Proof. For any $p \in M$ and any smooth real-valued f defined in a neighborhood of $F(p)$,

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)f,$$

while

$$(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)}f.$$

Thus, (8.6) is true for all f if and only if $dF_p(X_p) = Y_{F(p)}$ for all p , i.e., if and only if X and Y are F -related. \square

Example 8.17. Let $F: \mathbb{R} \rightarrow \mathbb{R}^2$ be the smooth map $F(t) = (\cos t, \sin t)$. Then $d/dt \in \mathfrak{X}(\mathbb{R})$ is F -related to the vector field $Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$Y = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}. \quad //$$

► **Exercise 8.18.** Prove the claim in the preceding example in two ways: directly from the definition, and by using Proposition 8.16.

It is important to remember that for a given smooth map $F: M \rightarrow N$ and vector field $X \in \mathfrak{X}(M)$, there may not be *any* vector field on N that is F -related to X . There is one special case, however, in which there is always such a vector field, as the next proposition shows.

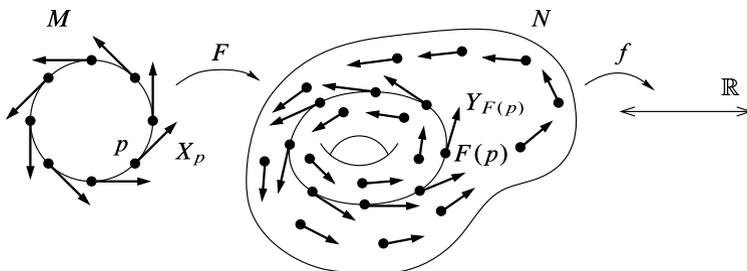


Fig. 8.3 F -related vector fields

Proposition 8.19. *Suppose M and N are smooth manifolds with or without boundary, and $F : M \rightarrow N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F -related to X .*

Proof. For $Y \in \mathfrak{X}(N)$ to be F -related to X means that $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$. If F is a diffeomorphism, therefore, we define Y by

$$Y_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}).$$

It is clear that Y , so defined, is the unique (rough) vector field that is F -related to X . Note that $Y : N \rightarrow TN$ is the composition of the following smooth maps:

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN.$$

It follows that Y is smooth. □

In the situation of the preceding proposition we denote the unique vector field that is F -related to X by F_*X , and call it the **pushforward of X by F** . Remember, it is only when F is a diffeomorphism that F_*X is defined. The proof of Proposition 8.19 shows that F_*X is defined explicitly by the formula

$$(F_*X)_q = dF_{F^{-1}(q)}(X_{F^{-1}(q)}). \tag{8.7}$$

As long as the inverse map F^{-1} can be computed explicitly, the pushforward of a vector field can be computed directly from this formula.

Example 8.20 (Computing the Pushforward of a Vector Field). Let M and N be the following open submanifolds of \mathbb{R}^2 :

$$M = \{(x, y) : y > 0 \text{ and } x + y > 0\},$$

$$N = \{(u, v) : u > 0 \text{ and } v > 0\},$$

and define $F : M \rightarrow N$ by $F(x, y) = (x + y, x/y + 1)$. Then F is a diffeomorphism because its inverse is easily computed: just solve $(u, v) = (x + y, x/y + 1)$ for x and y to obtain the formula $(x, y) = F^{-1}(u, v) = (u - u/v, u/v)$. Let us

compute the pushforward F_*X , where X is the following smooth vector field on M :

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x} \Big|_{(x,y)}.$$

The differential of F at a point $(x, y) \in M$ is represented by its Jacobian matrix,

$$DF(x, y) = \begin{pmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF\left(u - \frac{u}{v}, \frac{u}{v}\right) = \begin{pmatrix} 1 & 1 \\ \frac{v}{u} & \frac{v-v^2}{u} \end{pmatrix}.$$

For any $(u, v) \in N$,

$$X_{F^{-1}(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial x} \Big|_{F^{-1}(u,v)}.$$

Therefore, applying (8.7) with $p = (u, v)$ yields the formula for F_*X :

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u} \Big|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v} \Big|_{(u,v)}. \quad //$$

The next corollary follows directly from Proposition 8.16.

Corollary 8.21. *Suppose $F: M \rightarrow N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. For any $f \in C^\infty(N)$,*

$$((F_*X)f) \circ F = X(f \circ F). \quad \square$$

Vector Fields and Submanifolds

If $S \subseteq M$ is an immersed or embedded submanifold (with or without boundary), a vector field X on M does not necessarily restrict to a vector field on S , because X_p may not lie in the subspace $T_pS \subseteq T_pM$ at a point $p \in S$. Given a point $p \in S$, a vector field X on M is said to be **tangent to S at p** if $X_p \in T_pS \subseteq T_pM$. It is **tangent to S** if it is tangent to S at every point of S (Fig. 8.4).

Proposition 8.22. *Let M be a smooth manifold, $S \subseteq M$ be an embedded submanifold with or without boundary, and X be a smooth vector field on M . Then X is tangent to S if and only if $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ such that $f|_S \equiv 0$.*

Proof. This is an immediate consequence of Proposition 5.37. □

Suppose $S \subseteq M$ is an immersed submanifold with or without boundary, and Y is a smooth vector field on M . If there is a vector field $X \in \mathfrak{X}(S)$ that is ι -related to Y , where $\iota: S \hookrightarrow M$ is the inclusion map, then clearly Y is tangent to S , because

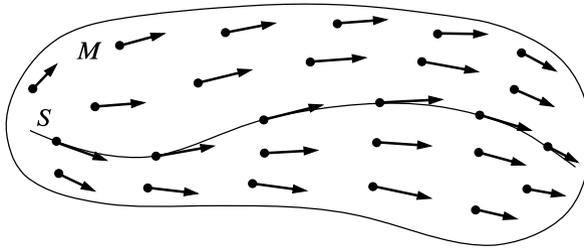


Fig. 8.4 A vector field tangent to a submanifold

$Y_p = d\iota_p(X_p)$ is in the image of $d\iota_p$ for each $p \in S$. The next proposition shows that the converse is true.

Proposition 8.23 (Restricting Vector Fields to Submanifolds). *Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold with or without boundary, and let $\iota: S \hookrightarrow M$ denote the inclusion map. If $Y \in \mathfrak{X}(M)$ is tangent to S , then there is a unique smooth vector field on S , denoted by $Y|_S$, that is ι -related to Y .*

Proof. The fact that Y is tangent to S means by definition that Y_p is in the image of $d\iota_p$ for each p . Thus, for each p there is a vector $X_p \in T_p S$ such that $Y_p = d\iota_p(X_p)$. Since $d\iota_p$ is injective, X_p is unique, so this defines X as a rough vector field on S . If we can show that X is smooth, it is the unique vector field that is ι -related to Y . It suffices to show that it is smooth in a neighborhood of each point.

Let p be any point in S . Since an immersed submanifold (with or without boundary) is locally embedded, there is a neighborhood V of p in S that is embedded in M . Let $(U, (x^i))$ be a slice chart (or boundary slice chart) for V in M centered at p , so that $V \cap U$ is the subset where $x^{k+1} = \dots = x^n = 0$ (and $x^k \geq 0$ if $p \in \partial S$), and (x^1, \dots, x^k) form local coordinates for S in $V \cap U$. If $Y = Y^1 \partial/\partial x^1 + \dots + Y^n \partial/\partial x^n$ in these coordinates, it follows from our construction that X has the coordinate representation $Y^1 \partial/\partial x^1 + \dots + Y^k \partial/\partial x^k$, which is clearly smooth on $V \cap U$. □

Lie Brackets

In this section we introduce an important way of combining two smooth vector fields to obtain another vector field.

Let X and Y be smooth vector fields on a smooth manifold M . Given a smooth function $f: M \rightarrow \mathbb{R}$, we can apply X to f and obtain another smooth function Xf (see Proposition 8.14). In turn, we can apply Y to this function, and obtain yet another smooth function $YXf = Y(Xf)$. The operation $f \mapsto YXf$, however, does not in general satisfy the product rule and thus cannot be a vector field, as the following example shows.

Example 8.24. Define vector fields $X = \partial/\partial x$ and $Y = x\partial/\partial y$ on \mathbb{R}^2 , and let $f(x, y) = x$, $g(x, y) = y$. Then direct computation shows that $XY(fg) = 2x$, while $fXYg + gXYf = x$, so XY is not a derivation of $C^\infty(\mathbb{R}^2)$. //

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) function XYf . Applying both of these operators to f and subtracting, we obtain an operator $[X, Y]: C^\infty(M) \rightarrow C^\infty(M)$, called the **Lie bracket of X and Y** , defined by

$$[X, Y]f = XYf - YXf.$$

The key fact is that this operator *is* a vector field.

Lemma 8.25. *The Lie bracket of any pair of smooth vector fields is a smooth vector field.*

Proof. By Proposition 8.15, it suffices to show that $[X, Y]$ is a derivation of $C^\infty(M)$. For arbitrary $f, g \in C^\infty(M)$, we compute

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(fYg + gYf) - Y(fXg + gXf) \\ &= XfYg + fXYg + XgYf + gXYf \\ &\quad - YfXg - fYXg - YgXf - gYXf \\ &= fXYg + gXYf - fYXg - gYXf \\ &= f[X, Y]g + g[X, Y]f. \end{aligned} \quad \square$$

We will describe one significant application of Lie brackets later in this chapter, and we will see many others in later chapters. Unfortunately, we are not yet in a position to give Lie brackets a geometric interpretation, but we will do so in Chapter 9. For now, we develop some of their basic properties.

The value of the vector field $[X, Y]$ at a point $p \in M$ is the derivation at p given by the formula

$$[X, Y]_p f = X_p(Yf) - Y_p(Xf).$$

However, this formula is of limited usefulness for computations, because it requires one to compute terms involving second derivatives of f that will always cancel each other out. The next proposition gives an extremely useful coordinate formula for the Lie bracket, in which the cancellations have already been accounted for.

Proposition 8.26 (Coordinate Formula for the Lie Bracket). *Let X, Y be smooth vector fields on a smooth manifold M with or without boundary, and let $X = X^i \partial/\partial x^i$ and $Y = Y^j \partial/\partial x^j$ be the coordinate expressions for X and Y in terms of some smooth local coordinates (x^i) for M . Then $[X, Y]$ has the following coordinate expression:*

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}, \quad (8.8)$$

or more concisely,

$$[X, Y] = (XY^j - YX^j) \frac{\partial}{\partial x^j}. \quad (8.9)$$

Proof. Because we know already that $[X, Y]$ is a smooth vector field, its action on a function is determined locally: $([X, Y]f)|_U = [X, Y](f|_U)$. Thus it suffices to compute in a single smooth chart, where we have

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x^i} \left(Y^j \frac{\partial f}{\partial x^j} \right) - Y^j \frac{\partial}{\partial x^j} \left(X^i \frac{\partial f}{\partial x^i} \right) \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i}, \end{aligned}$$

where in the last step we have used the fact that mixed partial derivatives of a smooth function can be taken in any order. Interchanging the roles of the dummy indices i and j in the second term, we obtain (8.8). \square

One trivial application of (8.8) is to compute the Lie brackets of the coordinate vector fields $(\partial/\partial x^i)$ in any smooth chart: because the component functions of the coordinate vector fields are all constants, it follows that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] \equiv 0 \quad \text{for all } i \text{ and } j. \quad (8.10)$$

(This also follows from the definition of the Lie bracket, and is essentially a restatement of the fact that mixed partial derivatives of smooth functions commute.) Here is a slightly less trivial computation.

Example 8.27. Define smooth vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ by

$$\begin{aligned} X &= x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1) \frac{\partial}{\partial z}, \\ Y &= \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}. \end{aligned}$$

Then (8.9) yields

$$\begin{aligned} [X, Y] &= X(1) \frac{\partial}{\partial x} + X(y) \frac{\partial}{\partial z} - Y(x) \frac{\partial}{\partial x} - Y(1) \frac{\partial}{\partial y} - Y(x(y+1)) \frac{\partial}{\partial z} \\ &= 0 \frac{\partial}{\partial x} + 1 \frac{\partial}{\partial z} - 1 \frac{\partial}{\partial x} - 0 \frac{\partial}{\partial y} - (y+1) \frac{\partial}{\partial z} \\ &= -\frac{\partial}{\partial x} - y \frac{\partial}{\partial z}. \quad // \end{aligned}$$

Proposition 8.28 (Properties of the Lie Bracket). *The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:*

(a) **BILINEARITY:** For $a, b \in \mathbb{R}$,

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z], \\ [Z, aX + bY] &= a[Z, X] + b[Z, Y]. \end{aligned}$$

(b) **ANTISYMMETRY:**

$$[X, Y] = -[Y, X].$$

(c) **JACOBI IDENTITY:**

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

(d) For $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X. \quad (8.11)$$

Proof. Bilinearity and antisymmetry are obvious consequences of the definition. The proof of the Jacobi identity is just a computation:

$$\begin{aligned} & [X, [Y, Z]]f + [Y, [Z, X]]f + [Z, [X, Y]]f \\ &= X[Y, Z]f - [Y, Z]Xf + Y[Z, X]f \\ &\quad - [Z, X]Yf + Z[X, Y]f - [X, Y]Zf \\ &= XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf \\ &\quad - ZXyf + XZYf + ZXYf - ZYXf - XYZf + YXZf. \end{aligned}$$

In this last expression all the terms cancel in pairs. Part (d) is a direct computation from the definition of the Lie bracket, and is left as an exercise. \square

► **Exercise 8.29.** Prove part (d) of the preceding proposition.

The significance of part (d) of this proposition might not be evident at this point, but it will become clearer in the next chapter, where we will see that it expresses the fact that the Lie bracket satisfies product rules with respect to both of its arguments (see Corollary 9.39).

Proposition 8.30 (Naturality of the Lie Bracket). *Let $F: M \rightarrow N$ be a smooth map between manifolds with or without boundary, and let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be vector fields such that X_i is F -related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ is F -related to $[Y_1, Y_2]$.*

Proof. Using Proposition 8.16 and the fact that X_i and Y_i are F -related,

$$X_1X_2(f \circ F) = X_1((Y_2f) \circ F) = (Y_1Y_2f) \circ F.$$

Similarly,

$$X_2X_1(f \circ F) = (Y_2Y_1f) \circ F.$$

Therefore,

$$[X_1, X_2](f \circ F) = X_1X_2(f \circ F) - X_2X_1(f \circ F)$$

$$\begin{aligned}
 &= (Y_1 Y_2 f) \circ F - (Y_2 Y_1 f) \circ F \\
 &= ([Y_1, Y_2] f) \circ F. \quad \square
 \end{aligned}$$

See Problem 11-18 for an indication of why this property is called “naturality.” When applied in special cases, this result has the following important corollaries. First we consider the case in which the map is a diffeomorphism.

Corollary 8.31 (Pushforwards of Lie Brackets). *Suppose $F: M \rightarrow N$ is a diffeomorphism and $X_1, X_2 \in \mathfrak{X}(M)$. Then $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.*

Proof. This is just the special case of Proposition 8.30 in which F is a diffeomorphism and $Y_i = F_*X_i$. \square

The second special case is that of the inclusion of a submanifold.

Corollary 8.32 (Brackets of Vector Fields Tangent to Submanifolds). *Let M be a smooth manifold and let S be an immersed submanifold with or without boundary in M . If Y_1 and Y_2 are smooth vector fields on M that are tangent to S , then $[Y_1, Y_2]$ is also tangent to S .*

Proof. By Proposition 8.23, there exist smooth vector fields X_1 and X_2 on S such that X_i is ι -related to Y_i for $i = 1, 2$ (where $\iota: S \rightarrow M$ is the inclusion). By Proposition 8.30, $[X_1, X_2]$ is ι -related to $[Y_1, Y_2]$, which is therefore tangent to S . \square

The Lie Algebra of a Lie Group

One of the most important applications of Lie brackets occurs in the context of Lie groups. Suppose G is a Lie group. Recall that G acts smoothly and transitively on itself by left translation: $L_g(h) = gh$. (See Example 7.22(c).) A vector field X on G is said to be **left-invariant** if it is invariant under all left translations, in the sense that it is L_g -related to itself for every $g \in G$. More explicitly, this means

$$d(L_g)_{g'}(X_{g'}) = X_{gg'}, \quad \text{for all } g, g' \in G. \tag{8.12}$$

Since L_g is a diffeomorphism, this can be abbreviated by writing $(L_g)_*X = X$ for every $g \in G$.

Because $(L_g)_*(aX + bY) = a(L_g)_*X + b(L_g)_*Y$, the set of all smooth left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$. But it is much more than that. The central fact is that it is closed under Lie brackets.

Proposition 8.33. *Let G be a Lie group, and suppose X and Y are smooth left-invariant vector fields on G . Then $[X, Y]$ is also left-invariant.*

Proof. Let $g \in G$ be arbitrary. Since $(L_g)_*X = X$ and $(L_g)_*Y = Y$ by definition of left-invariance, it follows from Corollary 8.31 that

$$(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y].$$

Thus, $[X, Y]$ is L_g -related to itself for each g , which is to say it is left-invariant. \square

A **Lie algebra** (over \mathbb{R}) is a real vector space \mathfrak{g} endowed with a map called the **bracket** from $\mathfrak{g} \times \mathfrak{g}$ to \mathfrak{g} , usually denoted by $(X, Y) \mapsto [X, Y]$, that satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(i) **BILINEARITY**: For $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(ii) **ANTISYMMETRY**:

$$[X, Y] = -[Y, X].$$

(iii) **JACOBI IDENTITY**:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Notice that the Jacobi identity is a substitute for associativity, which does not hold in general for brackets in a Lie algebra. It is useful in some circumstances to define Lie algebras over \mathbb{C} or other fields, but we do not have any reason to consider such Lie algebras; thus all of our Lie algebras are assumed without further comment to be real.

If \mathfrak{g} is a Lie algebra, a linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a **Lie subalgebra of \mathfrak{g}** if it is closed under brackets. In this case \mathfrak{h} is itself a Lie algebra with the restriction of the same bracket.

If \mathfrak{g} and \mathfrak{h} are Lie algebras, a linear map $A: \mathfrak{g} \rightarrow \mathfrak{h}$ is called a **Lie algebra homomorphism** if it preserves brackets: $A[X, Y] = [AX, AY]$. An invertible Lie algebra homomorphism is called a **Lie algebra isomorphism**. If there exists a Lie algebra isomorphism from \mathfrak{g} to \mathfrak{h} , we say that they are **isomorphic** as Lie algebras.

► **Exercise 8.34.** Verify that the kernel and image of a Lie algebra homomorphism are Lie subalgebras.

► **Exercise 8.35.** Suppose \mathfrak{g} and \mathfrak{h} are finite-dimensional Lie algebras and $A: \mathfrak{g} \rightarrow \mathfrak{h}$ is a linear map. Show that A is a Lie algebra homomorphism if and only if $A[E_i, E_j] = [AE_i, AE_j]$ for some basis (E_1, \dots, E_n) of \mathfrak{g} .

Example 8.36 (Lie Algebras).

- The space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket by Proposition 8.28.
- If G is a Lie group, the set of all smooth left-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$ and is therefore a Lie algebra.
- The vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices becomes an n^2 -dimensional Lie algebra under the **commutator bracket**:

$$[A, B] = AB - BA.$$

Bilinearity and antisymmetry are obvious from the definition, and the Jacobi identity follows from a straightforward calculation. When we are regarding $M(n, \mathbb{R})$ as a Lie algebra with this bracket, we denote it by $\mathfrak{gl}(n, \mathbb{R})$.

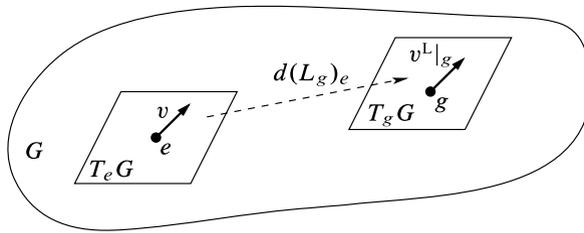


Fig. 8.5 Defining a left-invariant vector field

- (d) Similarly, $\mathfrak{gl}(n, \mathbb{C})$ is the $2n^2$ -dimensional (real) Lie algebra obtained by endowing $M(n, \mathbb{C})$ with the commutator bracket.
- (e) If V is a vector space, the vector space of all linear maps from V to itself becomes a Lie algebra, which we denote by $\mathfrak{gl}(V)$, with the commutator bracket:

$$[A, B] = A \circ B - B \circ A.$$

Under our usual identification of $n \times n$ matrices with linear maps from \mathbb{R}^n to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

- (f) Any vector space V becomes a Lie algebra if we define all brackets to be zero. Such a Lie algebra is said to be **abelian**. (The name reflects the fact that brackets in most Lie algebras, as in the preceding examples, are defined as commutators in terms of underlying associative products, so all brackets are zero precisely when the underlying product is commutative; it also reflects the connection between abelian Lie algebras and abelian Lie groups, which you will explore in Problems 8-25 and 20-7.) //

Example (b) is the most important one. The Lie algebra of all smooth left-invariant vector fields on a Lie group G is called the **Lie algebra of G** , and is denoted by $\text{Lie}(G)$. (We will see below that the assumption of smoothness is redundant; see Corollary 8.38.) The fundamental fact is that $\text{Lie}(G)$ is finite-dimensional, and in fact has the same dimension as G itself, as the following theorem shows.

Theorem 8.37. *Let G be a Lie group. The evaluation map $\varepsilon: \text{Lie}(G) \rightarrow T_e G$, given by $\varepsilon(X) = X_e$, is a vector space isomorphism. Thus, $\text{Lie}(G)$ is finite-dimensional, with dimension equal to $\dim G$.*

Proof. It is clear from the definition that ε is linear over \mathbb{R} . It is easy to prove that it is injective: if $\varepsilon(X) = X_e = 0$ for some $X \in \text{Lie}(G)$, then left-invariance of X implies that $X_g = d(L_g)_e(X_e) = 0$ for every $g \in G$, so $X = 0$.

To show that ε is surjective, let $v \in T_e G$ be arbitrary, and define a (rough) vector field v^L on G by

$$v^L|_g = d(L_g)_e(v). \tag{8.13}$$

(See Fig. 8.5.) If there is a left-invariant vector field on G whose value at the identity is v , clearly it has to be given by this formula.

First we need to check that v^L is smooth. By Proposition 8.14, it suffices to show that $v^L f$ is smooth whenever $f \in C^\infty(G)$. Choose a smooth curve $\gamma: (-\delta, \delta) \rightarrow G$ such that $\gamma(0) = e$ and $\gamma'(0) = v$. Then for all $g \in G$,

$$\begin{aligned} (v^L f)(g) &= v^L|_g f = d(L_g)_e(v) f = v(f \circ L_g) = \gamma'(0)(f \circ L_g) \\ &= \left. \frac{d}{dt} \right|_{t=0} (f \circ L_g \circ \gamma)(t). \end{aligned}$$

If we define $\varphi: (-\delta, \delta) \times G \rightarrow \mathbb{R}$ by $\varphi(t, g) = f \circ L_g \circ \gamma(t) = f(g\gamma(t))$, the computation above shows that $(v^L f)(g) = \partial\varphi/\partial t(0, g)$. Because φ is a composition of group multiplication, f , and γ , it is smooth. It follows that $\partial\varphi/\partial t(0, g)$ depends smoothly on g , so $v^L f$ is smooth.

Next we show that v^L is left-invariant, which is to say that $d(L_h)_g(v^L|_g) = v^L|_{hg}$ for all $g, h \in G$. This follows from the definition of v^L and the fact that $L_h \circ L_g = L_{hg}$:

$$d(L_h)_g(v^L|_g) = d(L_h)_g \circ d(L_g)_e(v) = d(L_h \circ L_g)_e(v) = d(L_{hg})_e(v) = v^L|_{hg}.$$

Thus $v^L \in \text{Lie}(G)$. Since L_e (left translation by the identity) is the identity map of G , it follows that $\varepsilon(v^L) = v^L|_e = v$, so ε is surjective. \square

Given any vector $v \in T_e G$, we continue to use the notation v^L to denote the smooth left-invariant vector field defined by (8.13).

It is worth observing that the preceding proof also shows that the assumption of smoothness in the definition of $\text{Lie}(G)$ is unnecessary.

Corollary 8.38. *Every left-invariant rough vector field on a Lie group is smooth.*

Proof. Let X be a left-invariant rough vector field on a Lie group G , and let $v = X_e$. The fact that X is left-invariant implies that $X = v^L$, which is smooth. \square

The existence of global left-invariant vector fields also yields the following important property of Lie groups. Recall that a smooth manifold is said to be *parallelizable* if it admits a smooth global frame. If G is a Lie group, a local or global frame consisting of left-invariant vector fields is called a **left-invariant frame**.

Corollary 8.39. *Every Lie group admits a left-invariant smooth global frame, and therefore every Lie group is parallelizable.*

Proof. If G is a Lie group, every basis for $\text{Lie}(G)$ is a left-invariant smooth global frame for G . \square

Example 8.40. Let us determine the Lie algebras of some familiar Lie groups.

- (a) **EUCLIDEAN SPACE \mathbb{R}^n :** If we consider \mathbb{R}^n as a Lie group under addition, left translation by an element $b \in \mathbb{R}^n$ is given by the affine map $L_b(x) = b + x$,

whose differential $d(L_b)$ is represented by the identity matrix in standard coordinates. Thus a vector field $X^i \partial/\partial x^i$ is left-invariant if and only if its coefficients X^i are constants. Because the Lie bracket of two constant-coefficient vector fields is zero by (8.8), the Lie algebra of \mathbb{R}^n is abelian, and is isomorphic to \mathbb{R}^n itself with the trivial bracket. In brief, $\text{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.

- (b) THE CIRCLE GROUP \mathbb{S}^1 : In terms of appropriate angle coordinates, each left translation has a local coordinate representation of the form $\theta \mapsto \theta + c$. Since the differential of this map is the 1×1 identity matrix, it follows that the vector field $d/d\theta$ defined in Example 8.4 is left-invariant, and is therefore a basis for the Lie algebra of \mathbb{S}^1 . This Lie algebra is 1-dimensional and abelian, and therefore $\text{Lie}(\mathbb{S}^1) \cong \mathbb{R}$.
- (c) THE n -TORUS $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$: An analysis similar to the one above shows that $(\partial/\partial\theta^1, \dots, \partial/\partial\theta^n)$ is a basis for $\text{Lie}(\mathbb{T}^n)$, where $\partial/\partial\theta^i$ is the angle coordinate vector field on the i th \mathbb{S}^1 factor. Since the Lie brackets of these coordinate vector fields are all zero, $\text{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n$. //

The Lie groups \mathbb{R}^n , \mathbb{S}^1 , and \mathbb{T}^n are abelian, and as the discussion above shows, their Lie algebras turn out also to be abelian. This is no accident: every abelian Lie group has an abelian Lie algebra (see Problem 8-25). Later, we will see that the converse is true provided that the group is connected (Problem 20-7).

Just as we can view the tangent space as a “linear model” of a smooth manifold near a point, the Lie algebra of a Lie group provides a “linear model” of the group, which reflects many of the properties of the group. Because Lie groups have more structure than ordinary smooth manifolds, it should come as no surprise that their linear models have more structure than ordinary vector spaces. Since a finite-dimensional Lie algebra is a purely linear-algebraic object, it is in many ways simpler to understand than the group itself. Much of the progress in the theory of Lie groups has come from a careful analysis of Lie algebras.

We conclude this section by analyzing the Lie algebra of the most important non-abelian Lie group of all, the general linear group. Theorem 8.37 gives a vector space isomorphism between $\text{Lie}(\text{GL}(n, \mathbb{R}))$ and the tangent space to $\text{GL}(n, \mathbb{R})$ at the identity matrix I_n . Because $\text{GL}(n, \mathbb{R})$ is an open subset of the vector space $\mathfrak{gl}(n, \mathbb{R})$, its tangent space is naturally isomorphic to $\mathfrak{gl}(n, \mathbb{R})$ itself. The composition of these two isomorphisms gives a vector space isomorphism $\text{Lie}(\text{GL}(n, \mathbb{R})) \cong \mathfrak{gl}(n, \mathbb{R})$.

The vector spaces $\text{Lie}(\text{GL}(n, \mathbb{R}))$ and $\mathfrak{gl}(n, \mathbb{R})$ have independently defined Lie algebra structures—the first coming from Lie brackets of vector fields, and the second from commutator brackets of matrices. The next proposition shows that the natural vector space isomorphism between these spaces is in fact a Lie algebra isomorphism.

Proposition 8.41 (Lie Algebra of the General Linear Group). *The composition of the natural maps*

$$\text{Lie}(\text{GL}(n, \mathbb{R})) \rightarrow T_{I_n} \text{GL}(n, \mathbb{R}) \rightarrow \mathfrak{gl}(n, \mathbb{R}) \tag{8.14}$$

gives a Lie algebra isomorphism between $\text{Lie}(\text{GL}(n, \mathbb{R}))$ and the matrix algebra $\mathfrak{gl}(n, \mathbb{R})$.

Proof. Using the matrix entries X_j^i as global coordinates on $\mathrm{GL}(n, \mathbb{R}) \subseteq \mathfrak{gl}(n, \mathbb{R})$, the natural isomorphism $T_{I_n} \mathrm{GL}(n, \mathbb{R}) \longleftrightarrow \mathfrak{gl}(n, \mathbb{R})$ takes the form

$$A_j^i \frac{\partial}{\partial X_j^i} \Big|_{I_n} \longleftrightarrow (A_j^i).$$

(Because of the dual role of the indices i, j as coordinate indices and matrix row and column indices, in this case it is impossible to maintain our convention that all coordinates have upper indices. However, we continue to observe the summation convention and the other index conventions associated with it. In particular, in the expression above, an upper index “in the denominator” is to be regarded as a lower index, and vice versa.)

Let \mathfrak{g} denote the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$. Any matrix $A = (A_j^i) \in \mathfrak{gl}(n, \mathbb{R})$ determines a left-invariant vector field $A^L \in \mathfrak{g}$ defined by (8.13), which in this case becomes

$$A^L|_X = d(L_X)_{I_n}(A) = d(L_X)_{I_n} \left(A_j^i \frac{\partial}{\partial X_j^i} \Big|_{I_n} \right).$$

Since L_X is the restriction to $\mathrm{GL}(n, \mathbb{R})$ of the linear map $A \mapsto XA$ on $\mathfrak{gl}(n, \mathbb{R})$, its differential is represented in coordinates by exactly the same linear map. In other words, the left-invariant vector field A^L determined by A is the one whose value at $X \in \mathrm{GL}(n, \mathbb{R})$ is

$$A^L|_X = X_j^i A_k^j \frac{\partial}{\partial X_k^i} \Big|_X. \quad (8.15)$$

Given two matrices $A, B \in \mathfrak{gl}(n, \mathbb{R})$, the Lie bracket of the corresponding left-invariant vector fields is given by

$$\begin{aligned} [A^L, B^L] &= \left[X_j^i A_k^j \frac{\partial}{\partial X_k^i}, X_q^p B_r^q \frac{\partial}{\partial X_r^p} \right] \\ &= X_j^i A_k^j \frac{\partial}{\partial X_k^i} \left(X_q^p B_r^q \right) \frac{\partial}{\partial X_r^p} - X_q^p B_r^q \frac{\partial}{\partial X_r^p} \left(X_j^i A_k^j \right) \frac{\partial}{\partial X_k^i} \\ &= X_j^i A_k^j B_r^k \frac{\partial}{\partial X_r^i} - X_q^p B_r^q A_k^r \frac{\partial}{\partial X_k^p} \\ &= \left(X_j^i A_k^j B_r^k - X_j^i B_k^j A_r^k \right) \frac{\partial}{\partial X_r^i}, \end{aligned}$$

where we have used the fact that $\partial X_q^p / \partial X_k^i$ is equal to 1 if $p = i$ and $q = k$, and 0 otherwise, and A_j^i and B_j^i are constants. Evaluating this last expression when X is equal to the identity matrix, we get

$$[A^L, B^L]_{I_n} = \left(A_k^i B_r^k - B_k^i A_r^k \right) \frac{\partial}{\partial X_r^i} \Big|_{I_n}.$$

This is the vector corresponding to the matrix commutator bracket $[A, B]$. Since the left-invariant vector field $[A^L, B^L]$ is determined by its value at the identity, this implies that

$$[A^L, B^L] = [A, B]^L,$$

which is precisely the statement that the composite map (8.14) is a Lie algebra isomorphism. \square

There is an analogue of this result for abstract vector spaces. If V is any finite-dimensional real vector space, recall that we have defined $\mathrm{GL}(V)$ as the Lie group of invertible linear transformations of V , and $\mathfrak{gl}(V)$ as the Lie algebra of all linear transformations. Just as in the case of $\mathrm{GL}(n, \mathbb{R})$, we can regard $\mathrm{GL}(V)$ as an open submanifold of $\mathfrak{gl}(V)$, and thus there are canonical vector space isomorphisms

$$\mathrm{Lie}(\mathrm{GL}(V)) \rightarrow T_{\mathrm{Id}} \mathrm{GL}(V) \rightarrow \mathfrak{gl}(V). \quad (8.16)$$

Corollary 8.42. *If V is any finite-dimensional real vector space, the composition of the canonical isomorphisms in (8.16) yields a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{GL}(V))$ and $\mathfrak{gl}(V)$.*

► **Exercise 8.43.** Prove the preceding corollary by choosing a basis for V and applying Proposition 8.41.

Induced Lie Algebra Homomorphisms

The importance of the Lie algebra of a Lie group stems, in large part, from the fact that each Lie group homomorphism induces a Lie algebra homomorphism, as the next theorem shows.

Theorem 8.44 (Induced Lie Algebra Homomorphisms). *Let G and H be Lie groups, and let \mathfrak{g} and \mathfrak{h} be their Lie algebras. Suppose $F: G \rightarrow H$ is a Lie group homomorphism. For every $X \in \mathfrak{g}$, there is a unique vector field in \mathfrak{h} that is F -related to X . With this vector field denoted by F_*X , the map $F_*: \mathfrak{g} \rightarrow \mathfrak{h}$ so defined is a Lie algebra homomorphism.*

Proof. If there is any vector field $Y \in \mathfrak{h}$ that is F -related to X , it must satisfy $Y_e = dF_e(X_e)$, and thus it must be uniquely determined by

$$Y = (dF_e(X_e))^L.$$

To show that this Y is F -related to X , we note that the fact that F is a homomorphism implies

$$\begin{aligned} F(gg') &= F(g)F(g') \Rightarrow F(L_g g') = L_{F(g)}F(g') \\ &\Rightarrow F \circ L_g = L_{F(g)} \circ F \\ &\Rightarrow dF \circ d(L_g) = d(L_{F(g)}) \circ dF. \end{aligned}$$

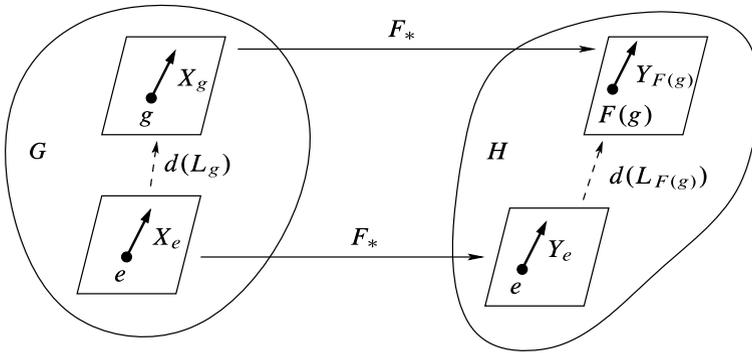


Fig. 8.6 The induced Lie algebra homomorphism

Thus,

$$dF(X_g) = dF(d(L_g)(X_e)) = d(L_{F(g)})(dF(X_e)) = d(L_{F(g)})(Y_e) = Y_{F(g)}.$$

(See Fig. 8.6.) This says precisely that X and Y are F -related.

For each $X \in \mathfrak{g}$, let F_*X denote the unique vector field in \mathfrak{h} that is F -related to X . It then follows immediately from the naturality of Lie brackets that $F_*[X, Y] = [F_*X, F_*Y]$, so F_* is a Lie algebra homomorphism. \square

The map $F_*: \mathfrak{g} \rightarrow \mathfrak{h}$ whose existence is asserted in this theorem is called the **induced Lie algebra homomorphism**. Note that the theorem implies that for any left-invariant vector field $X \in \mathfrak{g}$, F_*X is a well-defined smooth vector field on H , even though F may not be a diffeomorphism.

Proposition 8.45 (Properties of Induced Homomorphisms).

- (a) The homomorphism $(\text{Id}_G)_*: \text{Lie}(G) \rightarrow \text{Lie}(G)$ induced by the identity map of G is the identity of $\text{Lie}(G)$.
- (b) If $F_1: G \rightarrow H$ and $F_2: H \rightarrow K$ are Lie group homomorphisms, then

$$(F_2 \circ F_1)_* = (F_2)_* \circ (F_1)_*: \text{Lie}(G) \rightarrow \text{Lie}(K).$$

- (c) Isomorphic Lie groups have isomorphic Lie algebras.

Proof. Both of the relations $d(\text{Id}_G)_e = \text{Id}_{T_e G}$ and $d(F_2 \circ F_1)_e = d(F_2)_e \circ d(F_1)_e$ hold for differentials. Since the induced homomorphism is determined by the differential at the identity, this proves (a) and (b). If $F: G \rightarrow H$ is an isomorphism, (a) and (b) together imply that $F_* \circ (F^{-1})_* = (F \circ F^{-1})_* = \text{Id} = (F^{-1})_* \circ F_*$, so $F_*: \text{Lie}(G) \rightarrow \text{Lie}(H)$ is an isomorphism. \square

The preceding proposition has a categorical interpretation, as you might have guessed. Let Lie denote the category whose objects are Lie groups and whose morphisms are Lie group homomorphisms, and lie the one whose objects are finite-dimensional Lie algebras and whose morphisms are Lie algebra homomorphisms. Proposition 8.45 can be interpreted as showing that the assignments $G \mapsto \text{Lie}(G)$, $F \mapsto F_*$ define a covariant functor from Lie to lie .

The Lie Algebra of a Lie Subgroup

If G is a Lie group and $H \subseteq G$ is a Lie subgroup, we might hope that the Lie algebra of H would be a Lie subalgebra of that of G . However, elements of $\text{Lie}(H)$ are vector fields on H , not G , and so, strictly speaking, are not elements of $\text{Lie}(G)$. Nonetheless, the next proposition gives us a way to view $\text{Lie}(H)$ as a subalgebra of $\text{Lie}(G)$.

Theorem 8.46 (The Lie Algebra of a Lie Subgroup). *Suppose $H \subseteq G$ is a Lie subgroup, and $\iota: H \hookrightarrow G$ is the inclusion map. There is a Lie subalgebra $\mathfrak{h} \subseteq \text{Lie}(G)$ that is canonically isomorphic to $\text{Lie}(H)$, characterized by either of the following descriptions:*

$$\begin{aligned} \mathfrak{h} &= \iota_*(\text{Lie}(H)) \\ &= \{X \in \text{Lie}(G) : X_e \in T_e H\}. \end{aligned} \tag{8.17}$$

Proof. Because the inclusion map $\iota: H \hookrightarrow G$ is a Lie group homomorphism, $\iota_*(\text{Lie}(H))$ is a Lie subalgebra of $\text{Lie}(G)$. By the way we defined the induced Lie algebra homomorphism, this subalgebra is precisely the set of left-invariant vector fields on G whose values at the identity are of the form $d\iota_e(v)$ for some $v \in T_e H$. Since the differential $d\iota_e: T_e H \rightarrow T_e G$ is the inclusion of $T_e H$ as a subspace in $T_e G$, the two characterizations of \mathfrak{h} given in (8.17) are equal. Since $d\iota_e$ is injective on $T_e H$, it follows that ι_* is injective on $\text{Lie}(H)$; since it is surjective by definition on \mathfrak{h} , it is an isomorphism between $\text{Lie}(H)$ and \mathfrak{h} . \square

Using this proposition, whenever H is a Lie subgroup of G , we often *identify* $\text{Lie}(H)$ as a subalgebra of $\text{Lie}(G)$. As we mentioned above, elements of $\text{Lie}(H)$ are not themselves left-invariant vector fields on G . But the preceding proposition shows that every element of $\text{Lie}(H)$ corresponds to a unique element of $\text{Lie}(G)$, determined by its value at the identity, and the injection of $\text{Lie}(H)$ into $\text{Lie}(G)$ thus determined respects Lie brackets; so by thinking of $\text{Lie}(H)$ as a subalgebra of $\text{Lie}(G)$ we are not committing a grave error.

This identification is especially illuminating in the case of Lie subgroups of $\text{GL}(n, \mathbb{R})$.

Example 8.47 (The Lie Algebra of $\text{O}(n)$). The orthogonal group $\text{O}(n)$ is a Lie subgroup of $\text{GL}(n, \mathbb{R})$. By Example 7.27, it is equal to the level set $\Phi^{-1}(I_n)$, where $\Phi: \text{GL}(n, \mathbb{R}) \rightarrow \text{M}(n, \mathbb{R})$ is the map $\Phi(A) = A^T A$. By the result of Exercise 5.40, $T_{I_n} \text{O}(n)$ is equal to the kernel of $d\Phi_{I_n}: T_{I_n} \text{GL}(n, \mathbb{R}) \rightarrow T_{I_n} \text{M}(n, \mathbb{R})$. By the computation in Example 7.27, this differential is $d\Phi_{I_n}(B) = B^T + B$, so

$$\begin{aligned} T_{I_n} \text{O}(n) &= \{B \in \mathfrak{gl}(n, \mathbb{R}) : B^T + B = 0\} \\ &= \{\text{skew-symmetric } n \times n \text{ matrices}\}. \end{aligned}$$

We denote this subspace of $\mathfrak{gl}(n, \mathbb{R})$ by $\mathfrak{o}(n)$. Theorem 8.46 then implies that $\mathfrak{o}(n)$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ that is canonically isomorphic to $\text{Lie}(\text{O}(n))$. Notice that we did not even have to verify directly that $\mathfrak{o}(n)$ is a subalgebra. \parallel

We showed above that the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is naturally isomorphic to the matrix algebra $\mathfrak{gl}(n, \mathbb{R})$. We can now prove a similar result for $\mathrm{GL}(n, \mathbb{C})$. Just as in the real case, our usual identification of $\mathrm{GL}(n, \mathbb{C})$ as an open subset of $\mathfrak{gl}(n, \mathbb{C})$ yields a sequence of vector space isomorphisms

$$\mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) \xrightarrow{\varepsilon} T_{I_n} \mathrm{GL}(n, \mathbb{C}) \xrightarrow{\varphi} \mathfrak{gl}(n, \mathbb{C}), \quad (8.18)$$

where ε is the evaluation map and φ is the usual identification between the tangent space to an open subset of a vector space and the vector space itself. (Note that we are considering these as real vector spaces, not complex ones.)

Proposition 8.48 (The Lie Algebra of $\mathrm{GL}(n, \mathbb{C})$). *The composition of the maps in (8.18) yields a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{GL}(n, \mathbb{C}))$ and the matrix algebra $\mathfrak{gl}(n, \mathbb{C})$.*

Proof. The Lie group homomorphism $\beta: \mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{GL}(2n, \mathbb{R})$ that we constructed in Example 7.18(d) induces a Lie algebra homomorphism

$$\beta_*: \mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) \rightarrow \mathrm{Lie}(\mathrm{GL}(2n, \mathbb{R})).$$

Composing β_* with our canonical isomorphisms yields a commutative diagram

$$\begin{array}{ccccc} \mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) & \xrightarrow{\varepsilon} & T_{I_n} \mathrm{GL}(n, \mathbb{C}) & \xrightarrow{\varphi} & \mathfrak{gl}(n, \mathbb{C}) \\ \downarrow \beta_* & & \downarrow d\beta_{I_n} & & \downarrow \alpha \\ \mathrm{Lie}(\mathrm{GL}(2n, \mathbb{R})) & \xrightarrow{\varepsilon} & T_{I_{2n}} \mathrm{GL}(2n, \mathbb{R}) & \xrightarrow{\varphi} & \mathfrak{gl}(2n, \mathbb{R}), \end{array} \quad (8.19)$$

in which $\alpha = \varphi \circ d\beta_{I_n} \circ \varphi^{-1}$. Proposition 8.41 showed that the composition of the isomorphisms in the bottom row is a Lie algebra isomorphism; we need to show the same thing for the top row.

It is easy to see from the formula in Example 7.18(d) that β is (the restriction of) a linear map. It follows that $d\beta_{I_n}: T_{I_n} \mathrm{GL}(n, \mathbb{C}) \rightarrow T_{I_{2n}} \mathrm{GL}(2n, \mathbb{R})$ is given by exactly the same formula as β , as is $\alpha: \mathfrak{gl}(n, \mathbb{C}) \rightarrow \mathfrak{gl}(2n, \mathbb{R})$. Because $\beta(AB) = \beta(A)\beta(B)$, it follows that α preserves matrix commutators:

$$\alpha[A, B] = \alpha(AB - BA) = \alpha(A)\alpha(B) - \alpha(B)\alpha(A) = [\alpha(A), \alpha(B)].$$

Thus α is an injective Lie algebra homomorphism from $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{gl}(2n, \mathbb{R})$ (considering both as matrix algebras). Replacing the bottom row in (8.19) by the images of the vertical maps, we obtain a commutative diagram of vector space isomorphisms

$$\begin{array}{ccc} \mathrm{Lie}(\mathrm{GL}(n, \mathbb{C})) & \xrightarrow{\cong} & \mathfrak{gl}(n, \mathbb{C}) \\ \downarrow \beta_* & & \downarrow \alpha \\ \beta_*(\mathrm{Lie}(\mathrm{GL}(n, \mathbb{C}))) & \xrightarrow{\cong} & \alpha(\mathfrak{gl}(n, \mathbb{C})), \end{array}$$

in which the bottom map and the two vertical maps are Lie algebra isomorphisms; it follows that the top map is also a Lie algebra isomorphism. \square

Parallel to the notion of representations of Lie groups, there is also a notion of representations of Lie algebras. If \mathfrak{g} is a finite-dimensional Lie algebra, a **(finite-dimensional) representation of \mathfrak{g}** is a Lie algebra homomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ for some finite-dimensional vector space V , where $\mathfrak{gl}(V)$ denotes the Lie algebra of linear maps from V to itself. If φ is injective, it is said to be **faithful**, in which case \mathfrak{g} is isomorphic to the Lie subalgebra $\varphi(\mathfrak{g}) \subseteq \mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$.

There is a close connection between representations of Lie groups and representations of their Lie algebras. Suppose G is a Lie group and \mathfrak{g} is its Lie algebra. If $\rho: G \rightarrow \text{GL}(V)$ is any representation of G , then $\rho_*: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is easily seen to be a representation of \mathfrak{g} .

We close this section by stating a deep algebraic result about Lie algebras, which we will use in Chapter 20. The proof of the following theorem requires far more algebra than we have at our disposal, so we refer the interested reader to the proof in [Var84].

Theorem 8.49 (Ado’s Theorem). *Every finite-dimensional real Lie algebra admits a faithful finite-dimensional representation.*

Corollary 8.50. *Every finite-dimensional real Lie algebra is isomorphic to a Lie subalgebra of some matrix algebra $\mathfrak{gl}(n, \mathbb{R})$ with the commutator bracket.*

Proof. Let \mathfrak{g} be a finite-dimensional real Lie algebra. By Ado’s theorem, \mathfrak{g} has a faithful representation $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ for some finite-dimensional real vector space V . Choosing a basis for V yields an isomorphism of $\mathfrak{gl}(V)$ with $\mathfrak{gl}(n, \mathbb{R})$ for some n , and composing ρ with this isomorphism yields an injective Lie algebra homomorphism from \mathfrak{g} into $\mathfrak{gl}(n, \mathbb{R})$. Its image is a Lie subalgebra isomorphic to \mathfrak{g} . \square

As we mentioned in the previous chapter, it is important to remember that the analogous result for Lie groups is false: there are Lie groups that are not isomorphic to Lie subgroups of $\text{GL}(n, \mathbb{R})$ (see Problem 21-26 for an example).

Problems

- 8-1. Prove Lemma 8.6 (the extension lemma for vector fields).
- 8-2. EULER’S HOMOGENEOUS FUNCTION THEOREM: Let c be a real number, and let $f: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be a smooth function that is **positively homogeneous of degree c** , meaning that $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$, where V is the Euler vector field defined in Example 8.3. (Used on p. 248.)
- 8-3. Let M be a nonempty positive-dimensional smooth manifold with or without boundary. Show that $\mathfrak{X}(M)$ is infinite-dimensional.

- 8-4. Let M be a smooth manifold with boundary. Show that there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing. (Used on pp. 223, 386.)
- 8-5. Prove Proposition 8.11 (completion of local frames).
- 8-6. Let \mathbb{H} be the algebra of quaternions and let $\mathcal{S} \subseteq \mathbb{H}$ be the group of unit quaternions (see Problems 7-22 and 7-23).
- (a) Show that if $p \in \mathbb{H}$ is imaginary, then qp is tangent to \mathcal{S} at each $q \in \mathcal{S}$. (Here we are identifying each tangent space to \mathbb{H} with \mathbb{H} itself in the usual way.)
- (b) Define vector fields X_1, X_2, X_3 on \mathbb{H} by

$$X_1|_q = q\mathbf{i}, \quad X_2|_q = q\mathbf{j}, \quad X_3|_q = q\mathbf{k}.$$

Show that these vector fields restrict to a smooth left-invariant global frame on \mathcal{S} .

- (c) Under the isomorphism $(x^1, x^2, x^3, x^4) \leftrightarrow x^1\mathbf{1} + x^2\mathbf{i} + x^3\mathbf{j} + x^4\mathbf{k}$ between \mathbb{R}^4 and \mathbb{H} , show that these vector fields have the following coordinate representations:

$$X_1 = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4},$$

$$X_2 = -x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4},$$

$$X_3 = -x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}.$$

(Used on pp. 179, 562.)

- 8-7. The algebra of **octonions** (also called **Cayley numbers**) is the 8-dimensional real vector space $\mathbb{O} = \mathbb{H} \times \mathbb{H}$ (where \mathbb{H} is the space of quaternions defined in Problem 7-22) with the following bilinear product:

$$(p, q)(r, s) = (pr - sq^*, p^*s + rq), \quad \text{for } p, q, r, s \in \mathbb{H}. \quad (8.20)$$

Show that \mathbb{O} is a noncommutative, nonassociative algebra over \mathbb{R} , and prove that there exists a smooth global frame on \mathbb{S}^7 by imitating as much of Problem 8-6 as you can. [Hint: it might be helpful to prove that $(PQ^*)Q = P(Q^*Q)$ for all $P, Q \in \mathbb{O}$, where $(p, q)^* = (p^*, -q)$. For more about the octonions, see [Bae02].] (Used on p. 179.)

- 8-8. The algebra of **sedonions** is the 16-dimensional real vector space $\mathbb{S} = \mathbb{O} \times \mathbb{O}$ with the product defined by (8.20), but with p, q, r , and s interpreted as elements of \mathbb{O} . Why does sedenionic multiplication not yield a global frame for \mathbb{S}^{15} ? [Remark: the name “sedonion” comes from the Latin *sedecim*, meaning sixteen. A **division algebra** is an algebra with a multiplicative identity element and no zero divisors (i.e., $ab = 0$ if and only if $a = 0$ or $b = 0$).

It follows from the work of Bott, Milnor, and Kervaire on parallelizability of spheres [MB58, Ker58] that a finite-dimensional division algebra over \mathbb{R} must have dimension 1, 2, 4, or 8.]

- 8-9. Show by finding a counterexample that Proposition 8.19 is false if we replace the assumption that F is a diffeomorphism by the weaker assumption that it is smooth and bijective.
- 8-10. Let M be the open submanifold of \mathbb{R}^2 where both x and y are positive, and let $F: M \rightarrow M$ be the map $F(x, y) = (xy, y/x)$. Show that F is a diffeomorphism, and compute F_*X and F_*Y , where

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad Y = y \frac{\partial}{\partial x}.$$

- 8-11. For each of the following vector fields on the plane, compute its coordinate representation in polar coordinates on the right half-plane $\{(x, y) : x > 0\}$.

(a) $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$.

(b) $Y = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$.

(c) $Z = (x^2 + y^2) \frac{\partial}{\partial x}$.

- 8-12. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}P^2$ be the smooth map $F(x, y) = [x, y, 1]$, and let $X \in \mathfrak{X}(\mathbb{R}^2)$ be defined by $X = x\partial/\partial y - y\partial/\partial x$. Prove that there is a vector field $Y \in \mathfrak{X}(\mathbb{R}P^2)$ that is F -related to X , and compute its coordinate representation in terms of each of the charts defined in Example 1.5.

- 8-13. Show that there is a smooth vector field on \mathbb{S}^2 that vanishes at exactly one point. [Hint: try using stereographic projection; see Problem 1-7.]

- 8-14. Let M be a smooth manifold with or without boundary, let N be a smooth manifold, and let $f: M \rightarrow N$ be a smooth map. Define $F: M \rightarrow M \times N$ by $F(x) = (x, f(x))$. Show that for every $X \in \mathfrak{X}(M)$, there is a smooth vector field on $M \times N$ that is F -related to X .

- 8-15. **EXTENSION LEMMA FOR VECTOR FIELDS ON SUBMANIFOLDS:** Suppose M is a smooth manifold and $S \subseteq M$ is an embedded submanifold with or without boundary. Given $X \in \mathfrak{X}(S)$, show that there is a smooth vector field Y on a neighborhood of S in M such that $X = Y|_S$. Show that every such vector field extends to all of M if and only if S is properly embedded.

- 8-16. For each of the following pairs of vector fields X, Y defined on \mathbb{R}^3 , compute the Lie bracket $[X, Y]$.

(a) $X = y \frac{\partial}{\partial z} - 2xy^2 \frac{\partial}{\partial y}; \quad Y = \frac{\partial}{\partial y}$.

(b) $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}; \quad Y = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}$.

$$(c) \quad X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}; \quad Y = x \frac{\partial}{\partial y} + y \frac{\partial}{\partial x}.$$

- 8-17. Let M and N be smooth manifolds. Given vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, we can define a vector field $X \oplus Y$ on $M \times N$ by

$$(X \oplus Y)_{(p,q)} = (X_p, Y_q),$$

where we think of the right-hand side as an element of $T_p M \oplus T_q N$, which is naturally identified with $T_{p,q}(M \times N)$ as in Proposition 3.14. Prove that $X \oplus Y$ is smooth if X and Y are smooth, and $[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, X_2] \oplus [Y_1, Y_2]$. (Used on pp. 346, 527.)

- 8-18. Suppose $F: M \rightarrow N$ is a smooth submersion, where M and N are positive-dimensional smooth manifolds. Given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, we say that X is a **lift of Y** if X and Y are F -related. A vector field $V \in \mathfrak{X}(M)$ is said to be **vertical** if V is everywhere tangent to the fibers of F (or, equivalently, if V is F -related to the zero vector field on N).

- (a) Show that if $\dim M = \dim N$, then every smooth vector field on N has a unique lift.
 (b) Show that if $\dim M \neq \dim N$, then every smooth vector field on N has a lift, but that it is not unique.
 (c) Assume in addition that F is surjective. Given $X \in \mathfrak{X}(M)$, show that X is a lift of a smooth vector field on N if and only if $dF_p(X_p) = dF_q(X_q)$ whenever $F(p) = F(q)$. Show that if this is the case, then X is a lift of a *unique* smooth vector field.
 (d) Assume in addition that F is surjective with connected fibers. Show that a vector field $X \in \mathfrak{X}(M)$ is a lift of a smooth vector field on N if and only if $[V, X]$ is vertical whenever $V \in \mathfrak{X}(M)$ is vertical.

(Used on p. 434.)

- 8-19. Show that \mathbb{R}^3 with the cross product is a Lie algebra.

- 8-20. Let $A \subseteq \mathfrak{X}(\mathbb{R}^3)$ be the subspace spanned by $\{X, Y, Z\}$, where

$$X = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad Y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad Z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$

Show that A is a Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$, which is isomorphic to \mathbb{R}^3 with the cross product. (Used on p. 538.)

- 8-21. Prove that up to isomorphism, there are exactly one 1-dimensional Lie algebra and two 2-dimensional Lie algebras. Show that all three algebras are isomorphic to Lie subalgebras of $\mathfrak{gl}(2, \mathbb{R})$.

- 8-22. Let A be any algebra over \mathbb{R} . A **derivation of A** is a linear map $D: A \rightarrow A$ satisfying $D(xy) = (Dx)y + x(Dy)$ for all $x, y \in A$. Show that if D_1 and D_2 are derivations of A , then $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation. Show that the set of derivations of A is a Lie algebra with this bracket operation.

- 8-23. (a) Given Lie algebras \mathfrak{g} and \mathfrak{h} , show that the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ is a Lie algebra with the bracket defined by

$$[(X, Y), (X', Y')] = ([X, X'], [Y, Y']).$$

- (b) Suppose G and H are Lie groups. Prove that $\text{Lie}(G \times H)$ is isomorphic to $\text{Lie}(G) \oplus \text{Lie}(H)$.

- 8-24. Suppose G is a Lie group and \mathfrak{g} is its Lie algebra. A vector field $X \in \mathfrak{X}(G)$ is said to be **right-invariant** if it is invariant under all right translations.

- (a) Show that the set $\bar{\mathfrak{g}}$ of right-invariant vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$.

- (b) Let $i: G \rightarrow G$ denote the inversion map $i(g) = g^{-1}$. Show that the push-forward $i_*: \mathfrak{X}(G) \rightarrow \mathfrak{X}(G)$ restricts to a Lie algebra isomorphism from \mathfrak{g} to $\bar{\mathfrak{g}}$.

- 8-25. Prove that if G is an abelian Lie group, then $\text{Lie}(G)$ is abelian. [Hint: show that the inversion map $i: G \rightarrow G$ is a group homomorphism, and use Problem 7-2.]

- 8-26. Suppose $F: G \rightarrow H$ is a Lie group homomorphism. Show that the kernel of $F_*: \text{Lie}(G) \rightarrow \text{Lie}(H)$ is the Lie algebra of $\text{Ker } F$ (under the identification of the Lie algebra of a subgroup with a Lie subalgebra as in Theorem 8.46).

- 8-27. Let G and H be Lie groups, and suppose $F: G \rightarrow H$ is a Lie group homomorphism that is also a local diffeomorphism. Show that the induced homomorphism $F_*: \text{Lie}(G) \rightarrow \text{Lie}(H)$ is an isomorphism of Lie algebras. (Used on pp. 531, 557.)

- 8-28. Considering $\det: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^*$ as a Lie group homomorphism, show that its induced Lie algebra homomorphism is $\text{tr}: \mathfrak{gl}(n, \mathbb{R}) \rightarrow \mathbb{R}$. [Hint: see Problem 7-4.]

- 8-29. Theorem 8.46 implies that the Lie algebra of any Lie subgroup of $\text{GL}(n, \mathbb{R})$ is canonically isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, with a similar statement for Lie subgroups of $\text{GL}(n, \mathbb{C})$. Under this isomorphism, show that

$$\text{Lie}(\text{SL}(n, \mathbb{R})) \cong \mathfrak{sl}(n, \mathbb{R}),$$

$$\text{Lie}(\text{SO}(n)) \cong \mathfrak{o}(n),$$

$$\text{Lie}(\text{SL}(n, \mathbb{C})) \cong \mathfrak{sl}(n, \mathbb{C}),$$

$$\text{Lie}(\text{U}(n)) \cong \mathfrak{u}(n),$$

$$\text{Lie}(\text{SU}(n)) \cong \mathfrak{su}(n),$$

where

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) : \text{tr } A = 0\},$$

$$\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) : A^T + A = 0\},$$

$$\mathfrak{sl}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : \text{tr } A = 0\},$$

$$\mathfrak{u}(n) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : A^* + A = 0\},$$
$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C}).$$

- 8-30. Show by giving an explicit isomorphism that $\mathfrak{su}(2)$ and $\mathfrak{o}(3)$ are isomorphic Lie algebras, and that both are isomorphic to \mathbb{R}^3 with the cross product.
- 8-31. Let \mathfrak{g} be a Lie algebra. A linear subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called an **ideal in \mathfrak{g}** if $[X, Y] \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.
- (a) Show that if \mathfrak{h} is an ideal in \mathfrak{g} , then the quotient space $\mathfrak{g}/\mathfrak{h}$ has a unique Lie algebra structure such that the projection $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ is a Lie algebra homomorphism.
- (b) Show that a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal if and only if it is the kernel of a Lie algebra homomorphism.
- (Used on p. 533.)