

Chapter 14

Straightedge and Compass Constructions



We now apply our knowledge of field extensions in order to answer three questions posed by the ancient Greeks.

14.1 Three Ancient Problems

More than 2000 years ago, the ancient Greeks performed many geometric constructions using a straightedge and compass. For our purposes, a **straightedge** is an infinitely long ruler having no markings on it. If we have constructed two points, then we can use the straightedge to construct the line passing through those points. Furthermore, if we have constructed two points A and B , then for any point C we have constructed (which may or may not be distinct from A and B), we can use the **compass** to draw a circle centred at C with radius equal to the distance between A and B .

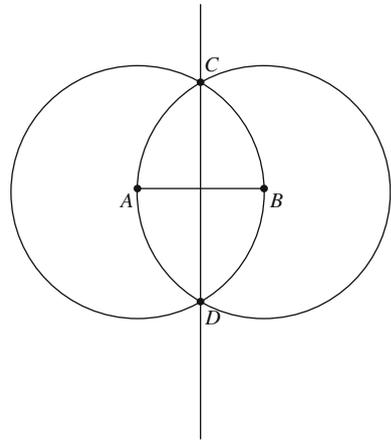
Next, we can take any two lines, any two circles, or one of each, that we have constructed, and construct their points of intersection. Then we repeat! The general question is, what can we construct in finitely many steps?

Let us discuss a few simple examples that will be of use.

Example 14.1. If we have constructed points A and B , let us construct a perpendicular bisector to the line segment AB . To this end, construct a circle centred at A with radius AB , and a circle centred at B with radius AB . Call the intersection points of these circles C and D . Then construct the line through C and D . It is a perpendicular bisector of AB , as illustrated in Figure 14.1.

Example 14.2. Suppose that we have constructed points A and B , and the line passing through them. Let us say that we have constructed point C as well, although we do not insist that $C \notin \{A, B\}$. We claim that we can construct a line through C that is perpendicular to the line through A and B . Without loss of generality, we may assume that C and A are distinct points. Construct the circle centred at C with radius

Fig. 14.1 Construction of a perpendicular bisector of AB



AC . If it intersects the line through A and B at a single point (which must necessarily be A), then the line through A and C will suffice, as illustrated in Figure 14.2.

Otherwise, suppose that the circle meets the line at points A and D . Then the line we are looking for is the perpendicular bisector of AD , which the preceding example allows us to construct. See Figure 14.3.

Example 14.3. Suppose that we have constructed three points A , B and C , that are not collinear. The three points must lie on a circle. Let us construct the centre of the circle, and hence the circle itself. Using Example 14.1, construct the perpendicular bisector of the chord AB . It must pass through the centre of the circle. Similarly, we can construct the perpendicular bisector of BC , and it too passes through the centre of the circle. Therefore, the point of intersection D of the two lines we have just constructed is the centre of the circle, and we can construct the circle itself, as it is centred at D and has radius AD . See Figure 14.4.

For all the remarkable geometric constructions that were performed in antiquity, some problems could not be solved at the time.

Question 14.1. (Squaring the Circle). Given an arbitrary circle, can we construct a square having the same area?

As we shall see, if we are given a square, we can construct another square whose area is twice that of the first square. If we extend our constructions into three dimensions, we have the following.

Question 14.2. (Doubling the Cube). Given an arbitrary cube, can we construct another cube having twice the volume of the first cube?

If we are given three distinct points A , B and C , then we can construct a bisector of the angle $\angle ABC$. That is, we can construct a point D such that $\angle DBC = \frac{1}{2}\angle ABC$. See Exercise 14.5. This naturally led to the following question.

Fig. 14.2 Construction of a perpendicular to AB passing through C (first case)

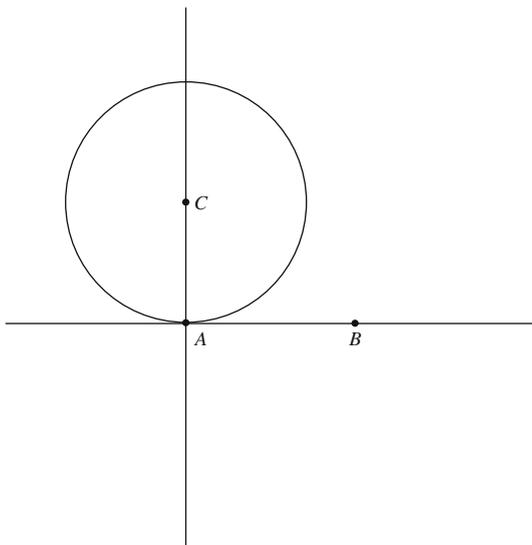
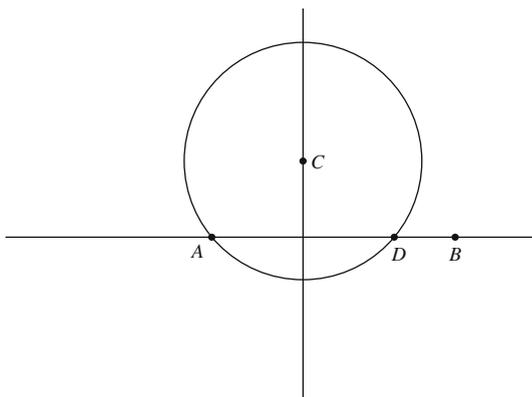


Fig. 14.3 Construction of a perpendicular to AB passing through C (second case)



Question 14.3. (Trisecting the Angle). Given any three distinct points A , B and C , can we construct a point D such that $\angle DBC = \frac{1}{3}\angle ABC$?

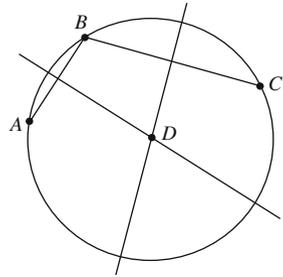
In fact, all three questions have a negative answer, but it was not until the nineteenth century that the tools of modern algebra allowed a proof to be given.

Exercises

14.1. Suppose that we have points A and B , and the distance from A to B is 1. Construct points C and D such that the distance from C to D is 1.5.

14.2. Suppose that we have points A and B , and the distance from A to B is 1. Construct points C and D such that the distance from C to D is $\sqrt{2}$.

Fig. 14.4 Construction of the circle passing through A , B and C



14.3. Given two points A and B , construct a point C such that ABC is an equilateral triangle.

14.4. Given two points A and B , construct points C , D and E all lying on the circle centred at A and passing through B , such that $BCDE$ is a square.

14.5. Given three points A , B and C , construct a point D such that $\angle DBC = \frac{1}{2}\angle ABC$ (where $\angle ABC$ is assumed to be at most 180°).

14.6. Given three points A , B and C , construct a point D such that $\angle DBC = 2\angle ABC$ (where $\angle ABC$ is assumed to be at most 90°).

14.7. Given two points A and B , construct points C and D on the circle centred at A and passing through B , such that BCD is an equilateral triangle.

14.8. Suppose we are given three points A , B and C , not collinear. Construct the inscribed circle for the triangle ABC ; that is, construct a circle that lies inside the triangle but intersects each side at a single point.

14.2 The Connection to Field Extensions

In order to tackle these problems, we need to be able to discuss them in algebraic terms. Let us formalize our procedure.

We will begin with two points. (Without those, we cannot construct any lines or circles, and so we get nowhere.) Let us identify these with the points $(0, 0)$ and $(1, 0)$ in the plane. We let $P_1 = \{(0, 0), (1, 0)\}$. Then we proceed as follows. For every positive integer i , take all pairs of distinct points A and B in P_i , and draw the line through A and B . Also, for every pair of distinct points A and B in P_i , and for every point C in P_i (where C may or may not be in $\{A, B\}$), draw the circle centred at C with radius equal to the distance between A and B . Let Q_i be the set of all lines and circles obtained in this way. Then let P_{i+1} be the set of all points of intersection of any two distinct lines, any two distinct circles, or any line and any circle in Q_i .

We note that each P_i and Q_i is a finite set, with $P_i \subseteq P_{i+1}$ and $Q_i \subseteq Q_{i+1}$ for all i .

Definition 14.1. A line or circle in the plane is **constructible** if it is in some Q_i . A point in the plane is **constructible** if it is in some P_i . A real number r is **constructible** if the point $(r, 0)$ is constructible.

Let us start by proving what numbers we can construct, and then see what limits there are upon constructibility.

Lemma 14.1. *Let $r \in \mathbb{R}$. Then the following are equivalent:*

1. r is constructible;
2. $-r$ is constructible;
3. the point $(0, r)$ is constructible; and
4. the point $(0, -r)$ is constructible.

Proof. If $r = 0$, there is nothing to do, so assume that $r \neq 0$.

Suppose that (1) holds. Then let $A = (0, 0)$ and $B = (r, 0)$. We can construct the circle centred at A with radius AB and the line through A and B (namely, the x -axis). They intersect at $C = (-r, 0)$, giving (2). As in Example 14.1, construct the perpendicular bisector of BC , which is the y -axis. The circle we constructed above intersects it at $(0, r)$ and $(0, -r)$, giving (3) and (4). By symmetry, (2) implies (1) as well.

If we assume (3), then again, we can construct a circle centred at $(0, 0)$ with radius $|r|$. As we are given $(0, 0)$ and $(1, 0)$, we can construct the x -axis, which intersects the circle at $(r, 0)$, giving (1). By symmetry, (4) implies (1) as well. \square

Note from the proof that since $(0, 0)$ and $(1, 0)$ are constructible, so are the x - and y -axes.

Lemma 14.2. *Let $a, b \in \mathbb{R}$. Then the point (a, b) is constructible if and only if the numbers a and b are constructible.*

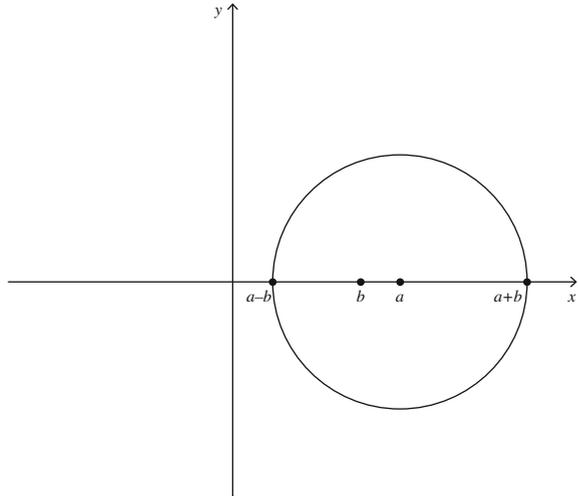
Proof. Suppose that (a, b) is constructible. As in Example 14.2, construct a line through (a, b) perpendicular to the x -axis. It intersects the x -axis at $(a, 0)$, and so a is constructible. Then construct the line through (a, b) perpendicular to the y -axis. It intersects the y -axis at $(0, b)$. Hence, by the preceding lemma, b is constructible.

Conversely, let a and b be constructible. Then the points $(a, 0)$ and $(0, b)$ are constructible. Construct the line perpendicular to the x -axis through $(a, 0)$. Similarly, construct the line perpendicular to the y -axis through $(0, b)$. These two lines meet at (a, b) . \square

Theorem 14.1. *The constructible numbers form a subfield of \mathbb{R} .*

Proof. By definition, 1 is constructible. Suppose that a and b are constructible. We would like to show that $a + b$ and $a - b$ are constructible. If $b = 0$, there is nothing to do. Otherwise, construct the circle centred at $(a, 0)$, the radius of which is the distance from $(0, 0)$ to $(b, 0)$. It intersects the x -axis at the points $(a + b, 0)$ and $(a - b, 0)$. See Figure 14.5.

Fig. 14.5 Construction of $a + b$ and $a - b$



If $b \neq 0$, we also need to construct ab^{-1} . As it is sufficient to construct $-ab^{-1}$, we may assume that $a \geq 0$ and $b > 0$. But in view of the preceding lemma, we can construct the points $(-b, 0)$, $(0, a)$ and $(0, -1)$. As in Example 14.3, we can construct the circle passing through these points. Either geometrically or through algebraic manipulation (see Exercise 14.10), we can prove that this circle intersects the x -axis at $(ab^{-1}, 0)$. See Figure 14.6.

Thus, ab^{-1} is constructible. Theorem 8.12 completes the proof. □

As \mathbb{Q} is the prime subfield of \mathbb{R} , we now know that every rational number is constructible. But we can say more.

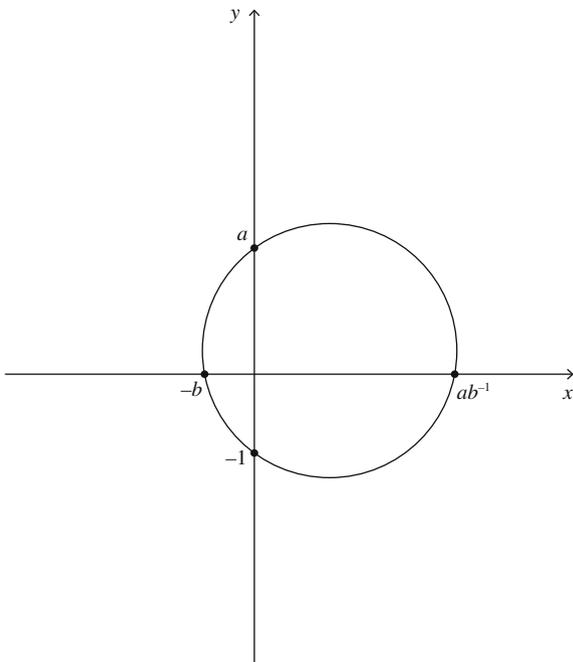
Theorem 14.2. *If a is a positive constructible number, then so is \sqrt{a} .*

Proof. As a and 1 are constructible, Lemma 14.1 tells us that we can construct the points $A = (0, a)$ and $B = (0, -1)$. By Example 14.1, we can construct the perpendicular bisector of AB and, hence, its midpoint $C = (0, \frac{a-1}{2})$. Now construct the circle with centre C and radius AC . This circle intersects the x -axis at $(d, 0)$ and $(-d, 0)$ for some $d > 0$. Again using Exercise 14.10, we see that $d = \sqrt{a}$. Thus, \sqrt{a} is constructible. □

Corollary 14.1. *Suppose there exist fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \dots \subseteq F_k$, where each F_{i+1} is a quadratic extension of F_i and $F_k \subseteq \mathbb{R}$. Then every element of F_k is constructible.*

Proof. We noted above that every element of F_0 is constructible. Thus, by induction, it suffices to show that if every element of a field F is constructible, and $[K : F] = 2$, then every element of K is constructible. Now, if $a \in K$, but $a \notin F$, then $\{1, a\}$ is linearly independent over F and hence, in view of Theorem 12.6, a basis for K . In

Fig. 14.6 Construction of ab^{-1}



particular, $K = F(a)$. By Theorem 12.9, a is algebraic over F and, in particular, Theorem 12.11 tells us that the minimal polynomial has degree 2. Say that it is $x^2 + bx + c$, with $b, c \in F$. But then $a = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$. By the preceding theorem, $\sqrt{b^2 - 4c}$ is constructible. But now Theorem 14.1 tells us that a is constructible, and hence so is every element of $F(a) = K$, as required. \square

Example 14.4. The number $\sqrt{3} + \sqrt[4]{2 + 5\sqrt{15}}$ is constructible. Let $F_0 = \mathbb{Q}$, $F_1 = F_0(\sqrt{3})$, $F_2 = F_1(\sqrt{15})$, $F_3 = F_2(\sqrt{2 + 5\sqrt{15}})$ and $F_4 = F_3(\sqrt[4]{2 + 5\sqrt{15}})$. It is clear that each extension is of degree at most 2, since if $a \in F$, then either $\sqrt{a} \in F$ or the minimal polynomial of a over F is $x^2 - a$. Furthermore, $\sqrt{3} + \sqrt[4]{2 + 5\sqrt{15}} \in F_4$.

Now, let us try to restrict the sorts of numbers that can be constructed.

Lemma 14.3. *Let F be a subfield of \mathbb{R} . Suppose that we have two distinct points A and B such that the coordinates of both points lie in F . Then the line through A and B has an equation of the form $ax + by = c$, for some $a, b, c \in F$. If C is any point with coordinates in F , then the circle centred at C with radius equal to the distance between A and B has an equation of the form $(x - d)^2 + (y - e)^2 = f$, for some $d, e, f \in F$.*

Proof. Let us say that $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$. Then the equation of the line is $(b_2 - a_2)x + (a_1 - b_1)y = a_1b_2 - a_2b_1$, and we can see that the

coefficients are in F . Similarly, the equation of the circle is $(x - c_1)^2 + (y - c_2)^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2$, which is of the correct form. \square

Readers familiar with linear algebra will not find the next lemma surprising, as the solution to a system of linear equations can be found using only addition, subtraction, multiplication and division.

Lemma 14.4. *Let F be a subfield of \mathbb{R} . Suppose that we have two lines, $ax + by = c$ and $dx + ey = f$, where $a, b, c, d, e, f \in F$, and that the two lines intersect at a single point. Then that point has coordinates in F .*

Proof. If $ae = bd$, then the lines are parallel (or identical), which is not permitted. Assume otherwise. Then the point of intersection is $\left(\frac{ce-bf}{ae-bd}, \frac{af-cd}{ae-bd}\right)$, and these coordinates lie in F . \square

Lemma 14.5. *Let F be a subfield of \mathbb{R} . Suppose that we have a line $ax + by = c$ and a circle $(x - d)^2 + (y - e)^2 = f$, with $a, b, c, d, e, f \in F$. If the line and circle intersect, then there is a nonnegative number $g \in F$ such that the coordinates of the intersection point(s) lie in $F(\sqrt{g})$.*

Proof. As a and b cannot both be 0, without loss of generality, say $a \neq 0$. Then $x = \frac{c-by}{a}$. Substituting into the equation of the circle, we obtain

$$\left(\frac{c-by}{a} - d\right)^2 + (y - e)^2 = f.$$

Simplifying, we obtain an equation of the form $uy^2 + vy + w = 0$, for some $u, v, w \in F$. Furthermore, $u = \frac{b^2}{a^2} + 1 > 0$. Then $y = \frac{-v \pm \sqrt{v^2 - 4uw}}{2u}$. If $v^2 - 4uw < 0$, then the line and circle do not intersect, contradicting our assumption. Thus, let $g = v^2 - 4uw \geq 0$. Then $y \in F(\sqrt{g})$, and as $x = \frac{c-by}{a}$, we see that $x \in F(\sqrt{g})$ as well. \square

Lemma 14.6. *Let F be a subfield of \mathbb{R} . Suppose that we have two distinct circles $(x - a)^2 + (y - b)^2 = c$ and $(x - d)^2 + (y - e)^2 = f$, with $a, b, c, d, e, f \in F$. If these circles intersect, then there is a nonnegative $g \in F$ such that the coordinates of the intersection point(s) lie in $F(\sqrt{g})$.*

Proof. Subtracting one equation from the other, we obtain

$$(2a - 2d)x + (2b - 2e)y = f - c + a^2 + b^2 - d^2 - e^2.$$

This is the equation of a line unless $2a - 2d = 2b - 2e = 0$. But in the latter case, the circles have the same centre, meaning that they are identical or do not intersect, so we may assume otherwise. Thus, we are now looking at the intersection of a circle and a line (with coefficients in F), and Lemma 14.5 applies. \square

Time to put it all together!

Theorem 14.3. *A real number a is constructible if and only if there exist subfields F_i of \mathbb{R} such that $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$, where each F_{i+1} is a quadratic extension of F_i and $a \in F_k$.*

Proof. One direction of the theorem is given by Corollary 14.1. Let us prove the other. Suppose that a is constructible. Referring to the sets P_i and Q_i from the definition of constructibility, let K_i be the intersection of all subfields of \mathbb{R} containing all of the coordinates of the points in P_i . Then each K_{i+1} is an extension field of K_i .

We claim that for each i , there exist fields F_j with $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = K_i$, where each F_{j+1} is a quadratic extension of F_j . Our proof is by induction on i . If $i = 1$, then $P_i = \{(0, 0), (1, 0)\}$ and hence $K_i = \mathbb{Q}$ and there is nothing to do. Assume that our claim holds for i . Then we have $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = K_i$, where each F_{j+1} is a quadratic extension of F_j . Now, by Lemma 14.3, every line and circle in Q_i has coefficients in K_i . Furthermore, Lemmas 14.4, 14.5 and 14.6 tell us that every possible intersection point of two distinct lines, two distinct circles or a line and a circle in P_i has coordinates either in K_i or in $K_i(\sqrt{b})$, for some nonnegative $b \in K_i$. But P_i is a finite set, so there are only finitely many values of b being used. Let them be b_1, \dots, b_n . For each k , $1 \leq k \leq n$, let $F_{m+k} = F_{m+k-1}(\sqrt{b_k})$. If $\sqrt{b_k} \in F_{m+k-1}$, then $F_{m+k} = F_{m+k-1}$, and we can discard F_{m+k} . Otherwise, $x^2 - b_k$ is irreducible over F_{m+k-1} , and so F_{m+k} is a quadratic extension of F_{m+k-1} , by Theorem 12.11. But the last of the F_{m+k} is, by definition, K_{i+1} , establishing the claim.

Since a is constructible, $(a, 0) \in P_i$, for some i , hence $a \in K_i$ and the proof is complete. \square

Corollary 14.2. *Let $a \in \mathbb{R}$. If a is constructible, then a is algebraic over \mathbb{Q} and, in fact, its minimal polynomial over \mathbb{Q} has degree 2^m for some nonnegative integer m .*

Proof. Using F_i as in the statement of the preceding theorem, we have $a \in F_k$, and by Theorem 12.8,

$$[F_k : \mathbb{Q}] = [F_k : F_{k-1}][F_{k-1} : F_{k-2}] \cdots [F_1 : \mathbb{Q}] = 2^k.$$

Thus, by Lemma 12.1, the numbers $1, a, a^2, \dots, a^{2^k}$ are linearly dependent over \mathbb{Q} . That is, a satisfies a nonzero polynomial over \mathbb{Q} . By Theorem 12.11, $[\mathbb{Q}(a) : \mathbb{Q}] < \infty$ and therefore, by Theorem 12.8,

$$2^k = [F_k : \mathbb{Q}] = [F_k : \mathbb{Q}(a)][\mathbb{Q}(a) : \mathbb{Q}].$$

That is, $[\mathbb{Q}(a) : \mathbb{Q}]$ divides 2^k , and so it is 2^m , for some m . By Theorem 12.11, the degree of the minimal polynomial of a over \mathbb{Q} is 2^m . \square

Please note that while the condition given in Theorem 14.3 is necessary and sufficient for a to be constructible, the condition given in the corollary is not. It is possible to find a real number a whose minimal polynomial over \mathbb{Q} has degree 4, but such that a is not constructible.

Exercises

14.9. Let a and b be nonzero real numbers. If a is constructible and b is not, show that neither $a + b$ nor ab is constructible. If c is not constructible, show by example that $b + c$ may or may not be constructible.

14.10. Let a, b, c and d be positive real numbers and suppose that a circle in the plane passes through the points $(a, 0)$, $(-b, 0)$, $(0, c)$ and $(0, -d)$. Show that $ab = cd$.

14.11. Are the following numbers constructible?

1. $\sqrt[4]{2 + \sqrt{5} - \sqrt{3}}$
2. $\sqrt{3} + \sqrt[3]{3}$

14.12. Is $\sqrt[3]{2} + \sqrt[3]{4}$ constructible?

14.13. Let a be a real root of the polynomial $x^6 - 15x^5 + 27x^4 - 12x^3 + 30x^2 - 21x + 87$. Is a constructible?

14.14. Let a be a real root of the polynomial $x^6 - 6x^4 + 12x^2 - 8$. Is a constructible?

14.3 Proof of the Impossibility of the Problems

We now have the machinery necessary to answer the three questions from Section 14.1. First, let us look at squaring the circle. We may as well assume that our two initial points are the centre of the circle, $(0, 0)$, and a point on the circle, $(1, 0)$. Thus, we can construct the unit circle immediately. Its area is π . If we were to construct a square with area π , we would need to construct an edge of length $\sqrt{\pi}$. The following theorem tells us that we cannot.

Theorem 14.4. *The number $\sqrt{\pi}$ is not constructible.*

Proof. If $\sqrt{\pi}$ were constructible, then by Theorem 14.1, π would be constructible as well. But as we mentioned in Section 12.3, π is transcendental over \mathbb{Q} . This contradicts Corollary 14.2. \square

As we discussed in Section 14.1, doubling a square is possible. Indeed, if we can construct a side with length s , then by Theorems 14.1 and 14.2, the number $s\sqrt{2}$ is also constructible, and this will be the side length of a square with twice the area. To deal with the problem of doubling the cube, without worrying about going into the third dimension, we may simply suppose that one edge extends between our two initial points, and thus has length 1. This would lead to a cube with volume 1. To obtain a cube with volume 2, we would need an edge with length $\sqrt[3]{2}$. This is not going to happen.

Theorem 14.5. *The number $\sqrt[3]{2}$ is not constructible.*

Proof. By Example 12.27, the minimal polynomial of $\sqrt[3]{2}$ over \mathbb{Q} is $x^3 - 2$. But then Corollary 14.2 tells us that $\sqrt[3]{2}$ is not constructible. \square

Finally, what about trisecting an angle? Some angles can be trisected (see Exercise 14.15). But not all. Indeed, we will show that an angle of 60° (or $\frac{\pi}{3}$, as we will be doing a bit of trigonometry) cannot be trisected. In view of Theorems 14.1 and 14.2, the numbers 0 , 1 , $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ are all constructible. Thus, by Lemma 14.2, we can construct the points $A = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $B = (0, 0)$ and $C = (1, 0)$. Then $\angle ABC = \frac{\pi}{3}$. Thus, we do not need to assume anything extra to obtain the angle. If we could find a point D such that $\angle DBC = \frac{\pi}{9}$, then we could draw the line through D and B , and then intersect with the unit circle centred at B . An intersection point would be $(\cos(\frac{\pi}{9}), \sin(\frac{\pi}{9}))$. By Lemma 14.2, this would require $\cos(\frac{\pi}{9})$ to be constructible. However, the following theorem dashes any hopes of that.

Theorem 14.6. *The number $\cos(\frac{\pi}{9})$ is not constructible.*

Proof. For any $\theta \in \mathbb{R}$, note that

$$\begin{aligned} \cos(3\theta) &= \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta) \\ &= (\cos^2(\theta) - \sin^2(\theta))\cos(\theta) - 2\sin^2(\theta)\cos(\theta) \\ &= \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta) \\ &= \cos^3(\theta) - 3(1 - \cos^2(\theta))\cos(\theta) \\ &= 4\cos^3(\theta) - 3\cos(\theta). \end{aligned}$$

Let $\theta = \frac{\pi}{9}$. Then $\cos(3\theta) = \frac{1}{2}$, and so

$$\frac{1}{2} = 4\cos^3(\theta) - 3\cos(\theta).$$

That is, $\cos(\frac{\pi}{9})$ satisfies the polynomial $8x^3 - 6x - 1$. If this polynomial were reducible over \mathbb{Q} , then by Corollary 11.2, it would have a rational root. By Theorem 11.3, the only possible roots are $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$. But none of these work. Thus, $8x^3 - 6x - 1$ is irreducible over \mathbb{Q} . Therefore, the minimal polynomial of $\cos(\frac{\pi}{9})$ over \mathbb{Q} is $x^3 - \frac{3}{4}x - \frac{1}{8}$. Corollary 14.2 completes the proof. \square

Exercises

All angles are expressed in radians.

14.15. Show that it is possible to trisect a right angle using straightedge and compass.

14.16. Show that the angles $\pi/6$ and $2\pi/3$ cannot be trisected using straightedge and compass.

14.17. In the next two problems, we will show that it is impossible to construct an angle of $\theta = 2\pi/7$. If it were possible, then as we are given the points $(0, 0)$ and $(1, 0)$, we would also be able to construct $(\cos(\theta), \sin(\theta))$. In particular, the number $\cos(\theta)$ would be constructible. Let us show that it is not. To this end, for each n , $1 \leq n \leq 6$, express $\cos(n\theta)$ as a linear combination over \mathbb{Q} of $\cos^k(\theta)$, $0 \leq k \leq 3$.

14.18. Let $\theta = 2\pi/7$.

1. Show that $\cos(\theta) + \sin(\theta)i$ is a complex root of $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$.
2. Show that $1 + \cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cos(4\theta) + \cos(5\theta) + \cos(6\theta) = 0$.
3. Use the answer to the preceding exercise to show that $\cos(\theta)$ is a root of $8x^3 + 4x^2 - 4x - 1$.
4. Conclude that $\cos(\theta)$ is not constructible.

14.19. Suppose we are given the vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (c_1, c_2)$ of a triangle. Show that we can construct points D , E and F such that the triangles ABC and DEF are similar, but DEF has twice the area of ABC .

14.20. Suppose that we are forced to perform our constructions using a straightedge and a collapsing compass. That is, any time we lift the compass, it collapses. In particular, we cannot directly use it to construct a circle centred at A with radius equal to the distance between B and C . All we can do is take two points A and B that we have constructed, and construct a circle centred at A and passing through B . Show that this does not change the set of constructible numbers in any way. That is, show that any number that was constructible before is still constructible using a straightedge and collapsing compass.