Chapter 13
Adjoints of Linear Maps

In this chapter we introduce adjoints of linear maps. In some sense these represent generalizations of the (Hermitian) transposes of a matrices. A matrix is symmetric (or Hermitian) if it is equal to its (Hermitian) transpose. In an analogous way, an endomorphism is selfadjoint if it is equal to its adjoint endomorphism. The sets of symmetric (or Hermitian) matrices and of selfadjoint endomorphisms form certain vector spaces which will play a key role in our proof of the Fundamental Theorem of Algebra in Chap. 15. Special properties of selfadjoint endomorphisms will be studied in Chap. 18.

13.1 Basic Definitions and Properties

In Chap. 12 we have considered Euclidean and unitary vector spaces, and hence vector spaces over the fields $\mathbb{R}$ and $\mathbb{C}$. Now let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a general field K, and let β be a bilinear form on $\mathcal{V} \times \mathcal{W}$.

For every *fixed* vector $v \in \mathcal{V}$, the map

$$\beta_v : \mathcal{W} \to K, \quad w \mapsto \beta(v, w),$$

is a linear form on $\mathcal{W}$. Thus, we can assign to every $v \in \mathcal{V}$ a vector $\beta_v \in \mathcal{W}^*$, which defines the map

$$\beta^{(1)} : \mathcal{V} \to \mathcal{W}^*, \quad v \mapsto \beta_v. \tag{13.1}$$

Analogously, we define the map

$$\beta^{(2)} : \mathcal{W} \to \mathcal{V}^*, \quad w \mapsto \beta_w, \tag{13.2}$$

where $\beta_w : \mathcal{V} \to K$ is defined by $v \mapsto \beta(v, w)$ for every $w \in \mathcal{W}$.

J. Liesen and V. Mehrmann, *Linear Algebra*, Springer Undergraduate Mathematics Series, DOI 10.1007/978-3-319-24346-7_13

Lemma 13.1 *The maps $\beta^{(1)}$ and $\beta^{(2)}$ defined in (13.1) and (13.2), respectively, are linear, i.e., $\beta^{(1)} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^*)$ and $\beta^{(2)} \in \mathcal{L}(\mathcal{W}, \mathcal{V}^*)$. If $\dim(\mathcal{V}) = \dim(\mathcal{W}) \in \mathbb{N}$ and β is non-degenerate (cp. Definition 11.9), then $\beta^{(1)}$ and $\beta^{(2)}$ are bijective and thus isomorphisms.*

Proof We prove the assertion only for the map $\beta^{(1)}$; the proof for $\beta^{(2)}$ is analogous.

We first show the linearity. Let $v_1, v_2 \in \mathcal{V}$ and $\lambda_1, \lambda_2 \in K$. For every $w \in \mathcal{W}$ we then have

$$\begin{aligned}\beta^{(1)}(\lambda_1 v_1 + \lambda_2 v_2)(w) &= \beta(\lambda_1 v_1 + \lambda_2 v_2, w)\\ &= \lambda_1 \beta(v_1, w) + \lambda_2 \beta(v_2, w)\\ &= \lambda_1 \beta^{(1)}(v_1)(w) + \lambda_2 \beta^{(1)}(v_2)(w)\\ &= \big(\lambda_1 \beta^{(1)}(v_1) + \lambda_2 \beta^{(1)}(v_2)\big)(w),\end{aligned}$$

and hence $\beta^{(1)}(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \beta^{(1)}(v_1) + \lambda_2 \beta^{(1)}(v_2)$. Therefore, $\beta^{(1)} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^*)$.

Let now $\dim(\mathcal{V}) = \dim(\mathcal{W}) \in \mathbb{N}$ and let β be non-degenerate. We show that $\beta^{(1)} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^*)$ is injective. By (5) in Lemma 10.7, this holds if and only if $\ker(\beta^{(1)}) = \{0\}$. If $v \in \ker(\beta^{(1)})$, then $\beta^{(1)}(v) = \beta_v = 0 \in \mathcal{W}^*$, and thus

$$\beta_v(w) = \beta(v, w) = 0 \quad \text{for all } w \in \mathcal{W}.$$

Since β is non-degenerate, we have $v = 0$. Finally, $\dim(\mathcal{V}) = \dim(\mathcal{W})$ and $\dim(\mathcal{W}) = \dim(\mathcal{W}^*)$ imply that $\dim(\mathcal{V}) = \dim(\mathcal{W}^*)$ so that $\beta^{(1)}$ is bijective (cp. Corollary 10.11). □

We next discuss the existence of the adjoint map.

Theorem 13.2 *If $\mathcal{V}$ and $\mathcal{W}$ are K-vector spaces with $\dim(\mathcal{V}) = \dim(\mathcal{W}) \in \mathbb{N}$ and β is a non-degenerate bilinear form on $\mathcal{V} \times \mathcal{W}$, then the following assertions hold:*

(1) For every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ there exists a uniquely determined $g \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ with

$$\beta(f(v), w) = \beta(v, g(w)) \quad \textit{for all } v \in \mathcal{V} \textit{ and } w \in \mathcal{W}.$$

The map g is called the right adjoint *of f with respect to β.*

(2) For every $h \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ there exists a uniquely determined $k \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ with

$$\beta(v, h(w)) = \beta(k(v), w) \quad \textit{for all } v \in \mathcal{V} \textit{ and } w \in \mathcal{W}.$$

The map k is called the left adjoint *of h with respect to β.*

Proof We only show (1); the proof of (2) is analogous.

Let $\mathcal{V}^*$ be the dual space of $\mathcal{V}$, let $f^* \in \mathcal{L}(\mathcal{V}^*, \mathcal{V}^*)$ be the dual map of f, and let $\beta^{(2)} \in \mathcal{L}(\mathcal{W}, \mathcal{V}^*)$ be as in (13.2). Since β is non-degenerate, $\beta^{(2)}$ is bijective by Lemma 13.1. Define

$$g := (\beta^{(2)})^{-1} \circ f^* \circ \beta^{(2)} \in \mathcal{L}(\mathcal{W}, \mathcal{W}).$$

Then, for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$,

$$\begin{aligned}\beta(v, g(w)) &= \beta(v, ((\beta^{(2)})^{-1} \circ f^* \circ \beta^{(2)})(w)) \\ &= \beta^{(2)}\big(((\beta^{(2)})^{-1} \circ f^* \circ \beta^{(2)})(w)\big)(v) \\ &= \beta^{(2)}\big((\beta^{(2)})^{-1}(f^*(\beta^{(2)}(w)))\big)(v) \\ &= \big(\beta^{(2)} \circ (\beta^{(2)})^{-1} \circ \beta^{(2)}(w) \circ f\big)(v) \\ &= \beta^{(2)}(w)(f(v)) \\ &= \beta(f(v), w).\end{aligned}$$

(Recall that the dual map satisfies $f^*(\beta^{(2)}(w)) = \beta^{(2)}(w) \circ f$.)

It remains to show the uniqueness of g. Let $\widetilde{g} \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ with $\beta(v, \widetilde{g}(w)) = \beta(f(v), w)$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Then $\beta(v, \widetilde{g}(w)) = \beta(v, g(w))$, and hence

$$\beta(v, (\widetilde{g} - g)(w)) = 0 \quad \text{for all } v \in \mathcal{V} \text{ and } w \in \mathcal{W}.$$

Since β is non-degenerate in the second variable, we have $(\widetilde{g} - g)(w) = 0$ for all $w \in \mathcal{W}$, so that $g = \widetilde{g}$. □

Example 13.3 Let $\mathcal{V} = \mathcal{W} = K^{n,1}$ and $\beta(v, w) = w^T Bv$ with a matrix $B \in GL_n(K)$, so that β is non-degenerate (cp. (1) in Example 11.10). We consider the linear map $f : \mathcal{V} \to \mathcal{V}$, $v \mapsto Fv$, with a matrix $F \in K^{n,n}$, and the linear map $h : \mathcal{W} \to \mathcal{W}$, $w \mapsto Hw$, with a matrix $H \in K^{n,n}$. Then

$$\begin{aligned}\beta_v &: \mathcal{W} \to K, \quad w \mapsto w^T(Bv), \\ \beta^{(1)} &: \mathcal{V} \to \mathcal{W}^*, \quad v \mapsto (Bv)^T, \\ \beta^{(2)} &: \mathcal{W} \to \mathcal{V}^*, \quad w \mapsto w^T B,\end{aligned}$$

where we have identified the isomorphic vector spaces $\mathcal{W}^*$ and $K^{1,n}$, respectively $\mathcal{V}^*$ and $K^{1,n}$, with each other. If $g \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is the right adjoint of f with respect to β, then

$$\beta(f(v), w) = w^T Bf(v) = w^T BFv = \beta(v, g(w)) = g(w)^T Bv$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. If we represent the linear map g via the multiplication with a matrix $G \in K^{n,n}$, i.e., $g(w) = Gw$, then $w^T BFv = w^T G^T Bv$ for all $v, w \in K^{n,1}$. Hence $BF = G^T B$. Since B is invertible, the unique right adjoint is given by $G = (BFB^{-1})^T = B^{-T} F^T B^T$.

Analogously, for the left adjoint $k \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ of h with respect to β we obtain the equation

$$\beta(v, h(w)) = (h(w))^T Bv = w^T H^T Bv = \beta(k(v), w) = w^T Bk(v)$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. With $k(v) = Lv$ for a matrix $L \in K^{n,n}$, we obtain $H^T B = BL$ and hence $L = B^{-1} H^T B$.

If $\mathcal{V}$ is finite dimensional and β is a non-degenerate bilinear form on $\mathcal{V}$, then by Theorem 13.2 every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ has a unique right adjoint g and a unique left adjoint k, such that

$$\beta(f(v), w) = \beta(v, g(w)) \quad \text{and} \quad \beta(v, f(w)) = \beta(k(v), w) \tag{13.3}$$

for all $v, w \in \mathcal{V}$. If β is symmetric, i.e., if $\beta(v, w) = \beta(w, v)$ holds for all $v, w \in \mathcal{V}$, then (13.3) yields

$$\beta(v, g(w)) = \beta(f(v), w) = \beta(w, f(v)) = \beta(k(w), v) = \beta(v, k(w)).$$

Therefore, $\beta(v, (g - k)(w)) = 0$ for all $v, w \in \mathcal{V}$, and hence $g = k$, since β is non-degenerate. Thus, we have proved the following result.

Corollary 13.4 *If β is a symmetric and non-degenerate bilinear form on a finite dimensional K-vector space $\mathcal{V}$, then for every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ there exists a unique $g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ with*

$$\beta(f(v), w) = \beta(v, g(w)) \quad \textit{and} \quad \beta(v, f(w)) = \beta(g(v), w)$$

for all $v, w \in \mathcal{V}$.

By definition, a scalar product on a Euclidean vector space is a symmetric and non-degenerate bilinear form (cp. Definition 12.1). This leads to the following corollary.

Corollary 13.5 *If $\mathcal{V}$ is a finite dimensional Euclidean vector space with the scalar product $\langle \cdot, \cdot \rangle$, then for every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ there exists a unique $f^{ad} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ with*

$$\langle f(v), w \rangle = \langle v, f^{ad}(w) \rangle \quad \textit{and} \quad \langle v, f(w) \rangle = \langle f^{ad}(v), w \rangle \tag{13.4}$$

for all $v, w \in \mathcal{V}$. The map f^{ad} is called the adjoint *of f (with respect to $\langle \cdot, \cdot \rangle$).*

In order to determine whether a given map $g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is the unique adjoint of $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, only one of the two conditions in (13.4) have to be verified: If for $f, g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ the equation

$$\langle f(v), w \rangle = \langle v, g(w) \rangle$$

holds for all $v, w \in \mathcal{V}$, then also

$$\langle v, f(w) \rangle = \langle f(w), v \rangle = \langle w, g(v) \rangle = \langle g(v), w \rangle$$

for all $v, w \in \mathcal{V}$, where we have used the symmetry of the scalar product. Similarly, if $\langle v, f(w) \rangle = \langle g(v), w \rangle$ holds for all $v, w \in \mathcal{V}$, then also $\langle f(v), w \rangle = \langle v, g(w) \rangle$ for all $v, w \in \mathcal{V}$.

Example 13.6 Consider the Euclidean vector space $\mathbb{R}^{3,1}$ with the scalar product

$$\langle v, w\rangle = w^T D v, \quad \text{where} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the linear map

$$f : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}, \quad v \mapsto Fv, \quad \text{where} \quad F = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}.$$

For all $v, w \in \mathbb{R}^{3,1}$ we then have

$$\langle f(v), w\rangle = w^T D F v = w^T D F D^{-1} D v = (D^{-T} F^T D^T w)^T D v = \langle v, f^{ad}(w)\rangle,$$

and thus

$$f^{ad} : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}, \quad v \mapsto D^{-1} F^T D v = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix} v,$$

where we have used that D is symmetric.

We now show that uniquely determined adjoint maps also exist in the unitary case. However, we cannot conclude this directly from Corollary 13.4, since a scalar product on a $\mathbb{C}$-vector space is not a symmetric bilinear form, but a Hermitian sesquilinear form. In order to show the existence of the adjoint map in the unitary case we construct it explicitly. This construction works also in the Euclidean case.

Let $\mathcal{V}$ be a unitary vector space with the scalar product $\langle \cdot, \cdot \rangle$ and let $\{u_1, \dots, u_n\}$ be an orthonormal basis of $\mathcal{V}$. For a given $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ we define the map

$$g : \mathcal{V} \to \mathcal{V}, \quad v \mapsto \sum_{i=1}^{n} \langle v, f(u_i)\rangle u_i.$$

If $v, w \in \mathcal{V}$ and $\lambda, \mu \in \mathbb{C}$, then

$$\begin{aligned} g(\lambda v + \mu w) &= \sum_{i=1}^{n} \langle \lambda v + \mu w, f(u_i)\rangle u_i = \sum_{i=1}^{n} \big(\lambda \langle v, f(u_i)\rangle u_i + \mu \langle v, f(u_i)\rangle u_i\big) \\ &= \lambda g(v) + \mu g(w), \end{aligned}$$

and hence $g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. Let now $v = \sum_{i=1}^{n} \lambda_i u_i \in \mathcal{V}$ and $w \in \mathcal{V}$, then

$$\langle v, g(w) \rangle = \left\langle \sum_{i=1}^{n} \lambda_i u_i, \sum_{j=1}^{n} \langle w, f(u_j) \rangle u_j \right\rangle = \sum_{i=1}^{n} \lambda_i \overline{\langle w, f(u_i) \rangle} = \sum_{i=1}^{n} \lambda_i \langle f(u_i), w \rangle$$
$$= \langle f(v), w \rangle.$$

Furthermore,

$$\langle v, f(w) \rangle = \overline{\langle f(w), v \rangle} = \overline{\langle w, g(v) \rangle} = \langle g(v), w \rangle$$

for all $v, w \in \mathcal{V}$. If $\widetilde{g} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ satisfies $\langle f(v), w \rangle = \langle v, \widetilde{g}(w) \rangle$ for all $v, w \in \mathcal{V}$, then $g = \widetilde{g}$, since the scalar product is positive definite. We can therefore formulate the following result analogously to Corollary 13.5.

Corollary 13.7 *If $\mathcal{V}$ is a finite dimensional unitary vector space with the scalar product $\langle \cdot, \cdot \rangle$, then for every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ there exists a unique $f^{ad} \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ with*

$$\langle f(v), w \rangle = \langle v, f^{ad}(w) \rangle \quad \text{and} \quad \langle v, f(w) \rangle = \langle f^{ad}(v), w \rangle \tag{13.5}$$

for all $v, w \in \mathcal{V}$. The map f^{ad} is called the adjoint *of f (with respect to $\langle \cdot, \cdot \rangle$).*

As in the Euclidean case, again the validity of one of the two equations in (13.5) for all $v, w \in \mathcal{V}$ implies the validity of the other for all $v, w \in \mathcal{V}$.

Example 13.8 Consider the unitary vector space $\mathbb{C}^{3,1}$ with the scalar product

$$\langle v, w \rangle = w^H D v, \quad \text{where} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the linear map

$$f : \mathbb{C}^{3,1} \to \mathbb{C}^{3,1}, \quad v \mapsto Fv, \quad \text{where} \quad F = \begin{bmatrix} 1 & 2\mathbf{i} & 2 \\ \mathbf{i} & 0 & -\mathbf{i} \\ 2 & 0 & 3\mathbf{i} \end{bmatrix}.$$

For all $v, w \in \mathbb{C}^{3,1}$ we then have

$$\langle f(v), w \rangle = w^H D F v = w^H D F D^{-1} D v = (D^{-H} F^H D^H w)^H D v$$
$$= \langle v, f^{ad}(w) \rangle,$$

and thus

$$f^{ad} : \mathbb{C}^{3,1} \to \mathbb{C}^{3,1}, \quad v \mapsto D^{-1} F^H D v = \begin{bmatrix} 1 & -2\mathbf{i} & 2 \\ -\mathbf{i} & 0 & 0 \\ 2 & 2\mathbf{i} & -3\mathbf{i} \end{bmatrix} v,$$

where we have used that D is real and symmetric.

We next investigate the properties of the adjoint map.

Lemma 13.9 *Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space.*

(1) If $f_1, f_2 \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ and $\lambda_1, \lambda_2 \in K$ (where $K = \mathbb{R}$ in the Euclidean and $K = \mathbb{C}$ in the unitary case), then

$$(\lambda_1 f_1 + \lambda_2 f_2)^{ad} = \overline{\lambda}_1 f_1^{ad} + \overline{\lambda}_2 f_2^{ad}.$$

In the Euclidean case the map $f \mapsto f^{ad}$ is therefore linear, and in the unity case semilinear.

(2) We have $(\mathrm{Id}_{\mathcal{V}})^{ad} = \mathrm{Id}_{\mathcal{V}}$.

(3) For every $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ we have $(f^{ad})^{ad} = f$.

(4) If $f_1, f_2 \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, then $(f_2 \circ f_1)^{ad} = f_1^{ad} \circ f_2^{ad}$.

Proof

(1) If $v, w \in \mathcal{V}$ and $\lambda_1, \lambda_2 \in K$, then

$$\begin{aligned}\langle(\lambda_1 f_1 + \lambda_2 f_2)(v), w\rangle &= \lambda_1\langle f_1(v), w\rangle + \lambda_2\langle f_2(v), w\rangle \\ &= \lambda_1 \left\langle v, f_1^{ad}(w)\right\rangle + \lambda_2 \left\langle v, f_2^{ad}(w)\right\rangle \\ &= \left\langle v, \overline{\lambda}_1 f_1^{ad}(w) + \overline{\lambda}_2 f_2^{ad}(w)\right\rangle \\ &= \left\langle v, \left(\overline{\lambda}_1 f_1^{ad} + \overline{\lambda}_2 f_2^{ad}\right)(w)\right\rangle,\end{aligned}$$

and thus $(\lambda_1 f_1 + \lambda_2 f_2)^{ad} = \overline{\lambda}_1 f_1^{ad} + \overline{\lambda}_2 f_2^{ad}$.

(2) For all $v, w \in \mathcal{V}$ we have $\langle \mathrm{Id}_{\mathcal{V}}(v), w\rangle = \langle v, w\rangle = \langle v, \mathrm{Id}_{\mathcal{V}}(w)\rangle$, and thus $(\mathrm{Id}_{\mathcal{V}})^{ad} = \mathrm{Id}_{\mathcal{V}}$.

(3) For all $v, w \in \mathcal{V}$ we have $\langle f^{ad}(v), w\rangle = \langle v, f(w)\rangle$, and thus $(f^{ad})^{ad} = f$.

(4) For all $v, w \in \mathcal{V}$ we have

$$\begin{aligned}\langle(f_2 \circ f_1)(v), w\rangle = \langle f_2(f_1(v)), w\rangle &= \left\langle f_1(v), f_2^{ad}(w)\right\rangle = \left\langle v, f_1^{ad}\left(f_2^{ad}(w)\right)\right\rangle \\ &= \left\langle v, \left(f_1^{ad} \circ f_2^{ad}\right)(w)\right\rangle,\end{aligned}$$

and thus $(f_2 \circ f_1)^{ad} = f_1^{ad} \circ f_2^{ad}$. □

The following result shows relations between the image and kernel of an endomorphism and of its adjoint.

Theorem 13.10 *If $\mathcal{V}$ is a finite dimensional Euclidean or unitary vector space and $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, then the following assertions hold:*

(1) $\ker(f^{ad}) = \operatorname{im}(f)^{\perp}$.
(2) $\ker(f) = \operatorname{im}(f^{ad})^{\perp}$.

Proof

(1) If $w \in \ker(f^{ad})$, then $f^{ad}(w) = 0$ and

$$0 = \langle v, f^{ad}(w)\rangle = \langle f(v), w\rangle$$

for all $v \in \mathcal{V}$, hence $w \in \operatorname{im}(f)^{\perp}$. If, on the other hand, $w \in \operatorname{im}(f)^{\perp}$, then

$$0 = \langle f(v), w\rangle = \langle v, f^{ad}(w)\rangle$$

for all $v \in \mathcal{V}$. Since $\langle \cdot, \cdot\rangle$ is non-degenerate, we have $f^{ad}(w) = 0$ and, hence, $w \in \ker(f^{ad})$.

(2) Using $(f^{ad})^{ad} = f$ and (1) we get $\ker(f) = \ker((f^{ad})^{ad}) = \operatorname{im}(f^{ad})^{\perp}$. □

Example 13.11 Consider the unitary vector space $\mathbb{C}^{3,1}$ with the standard scalar product and the linear map

$$f : \mathbb{C}^{3,1} \to \mathbb{C}^{3,1}, \quad v \mapsto Fv, \quad \text{with} \quad F = \begin{bmatrix} 1 & \mathbf{i} & \mathbf{i} \\ \mathbf{i} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then

$$f^{ad} : \mathbb{C}^{3,1} \to \mathbb{C}^{3,1}, \quad v \mapsto F^H v, \quad \text{with} \quad F^H = \begin{bmatrix} 1 & -\mathbf{i} & 1 \\ -\mathbf{i} & 0 & 0 \\ -\mathbf{i} & 0 & 0 \end{bmatrix}.$$

The matrices F and F^H have rank 2. Therefore, $\dim(\ker(f)) = \dim(\ker(f^{ad})) = 1$. A simple calculation shows that

$$\ker(f) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\} \quad \text{and} \quad \ker(f^{ad}) = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \\ \mathbf{i} \end{bmatrix}\right\}.$$

The dimension formula for linear maps implies that $\dim(\operatorname{im}(f)) = \dim(\operatorname{im}(f^{ad})) = 2$. From the matrices F and F^H we can see that

$$\operatorname{im}(f) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ \mathbf{i} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \text{and} \quad \operatorname{im}(f^{ad}) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -\mathbf{i} \\ -\mathbf{i} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

The equations $\ker(f^{ad}) = \operatorname{im}(f)^{\perp}$ and $\ker(f) = \operatorname{im}(f^{ad})^{\perp}$ can be verified by direct computation.

13.2 Adjoint Endomorphisms and Matrices

We now study the relation between the matrix representations of an endomorphism and its adjoint. Let $\mathcal{V}$ be a finite dimensional unitary vector space with the scalar product $\langle \cdot, \cdot \rangle$ and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. For an orthonormal basis $B = \{u_1, \dots, u_n\}$ of $\mathcal{V}$ let $[f]_{B,B} = [a_{ij}] \in \mathbb{C}^{n,n}$, i.e.,

$$f(u_j) = \sum_{k=1}^{n} a_{kj} u_k, \quad j = 1, \dots, n,$$

and hence

$$\langle f(u_j), u_i \rangle = \left\langle \sum_{k=1}^{n} a_{kj} u_k,\, u_i \right\rangle = a_{ij}, \quad i, j = 1, \dots, n.$$

If $[f^{ad}]_{B,B} = [b_{ij}] \in \mathbb{C}^{n,n}$, i.e.,

$$f^{ad}(u_j) = \sum_{k=1}^{n} b_{kj} u_k, \quad j = 1, \dots, n,$$

then

$$b_{ij} = \langle f^{ad}(u_j), u_i \rangle = \langle u_j, f(u_i) \rangle = \overline{\langle f(u_i), u_j \rangle} = \overline{a}_{ji}.$$

Thus, $[f^{ad}]_{B,B} = ([f]_{B,B})^H$. The same holds for a finite dimensional Euclidean vector space, but then we can omit the complex conjugation. Therefore, we have shown the following result.

Theorem 13.12 *If $\mathcal{V}$ is a finite dimensional Euclidean or unitary vector space with the orthonormal basis B and $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, then*

$$[f^{ad}]_{B,B} = ([f]_{B,B})^H.$$

(In the Euclidean case $([f]_{B,B})^H = ([f]_{B,B})^T$.)

An important special class are the selfadjoint endomorphisms.

Definition 13.13 Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space. An endomorphism $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is called *selfadjoint* when $f = f^{ad}$.

Trivial examples of selfadjoint endomorphism in $\mathcal{L}(\mathcal{V}, \mathcal{V})$ are $f = 0$ and $\mathrm{Id}_{\mathcal{V}}$.

Corollary 13.14

(1) If $\mathcal{V}$ is a finite dimensional Euclidean vector space, $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is selfadjoint and B is an orthonormal basis of $\mathcal{V}$, then $[f]_{B,B}$ is a symmetric matrix.

(2) If $\mathcal{V}$ is a finite dimensional unitary vector space, $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ is selfadjoint and B is an orthonormal basis of $\mathcal{V}$, then $[f]_{B,B}$ is an Hermitian matrix.

The selfadjoint endomorphisms again form a vector space. However, one has to be careful to use the appropriate field over which this vector space is defined. In particular, the set of selfadjoint endomorphisms on a unitary vector space $\mathcal{V}$ does *not* form a $\mathbb{C}$-vector space. If $f = f^{ad} \in \mathcal{L}(\mathcal{V}, \mathcal{V}) \setminus \{0\}$, then $(\mathbf{i}f)^{ad} = -\mathbf{i}f^{ad} = -\mathbf{i}f \neq \mathbf{i}f$ (cp. (1) in Lemma 13.9). Similarly, the Hermitian matrices in $\mathbb{C}^{n,n}$ do *not* form a $\mathbb{C}$-vector space. If $A = A^H \in \mathbb{C}^{n,n} \setminus \{0\}$ is Hermitian, then $(\mathbf{i}A)^H = -\mathbf{i}A^H = -\mathbf{i}A \neq \mathbf{i}A$.

Lemma 13.15

(1) If $\mathcal{V}$ is an n-dimensional Euclidean vector space, then the set of selfadjoint endomorphisms $\{f \in \mathcal{L}(\mathcal{V}, \mathcal{V}) \mid f = f^{ad}\}$ forms an $\mathbb{R}$-vector space of dimension $n(n+1)/2$.

(2) If $\mathcal{V}$ is an n-dimensional unitary vector space, then the set of selfadjoint endomorphisms $\{f \in \mathcal{L}(\mathcal{V}, \mathcal{V}) \mid f = f^{ad}\}$ forms an $\mathbb{R}$-vector space of dimension n^2.

Proof Exercise. □

A matrix $A \in \mathbb{C}^{n,n}$ with $A = A^T$ is called *complex symmetric*. Unlike the Hermitian matrices, the complex symmetric matrices form a $\mathbb{C}$-vector space.

Lemma 13.16 *The set of complex symmetric matrices in $\mathbb{C}^{n,n}$ forms a $\mathbb{C}$-vector space of dimension $n(n+1)/2$.*

Proof Exercise. □

Lemmas 13.15 and 13.16 will be used in Chap. 15 in our proof of the Fundamental Theorem of Algebra.

Exercises

13.1. Let $\beta(v, w) = w^T B v$ with $B = \operatorname{diag}(1, -1)$ be defined for $v, w \in \mathbb{R}^{2,1}$. Consider the linear maps $f : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$, $v \mapsto Fv$, and $h : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$, $w \mapsto Hw$, where

$$F = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2,2}, \quad H = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{2,2}.$$

Determine β_v, $\beta^{(1)}$ and $\beta^{(2)}$ as in (13.1)–(13.2) as well as the right adjoint of f and the left adjoint of h with respect to β.

13.2. Let $(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}})$ and $(\mathcal{W}, \langle\cdot,\cdot\rangle_{\mathcal{W}})$ be two finite dimensional Euclidean vector spaces and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Show that there exists a unique $g \in \mathcal{L}(\mathcal{W}, \mathcal{V})$ with $\langle f(v), w\rangle_{\mathcal{W}} = \langle v, g(w)\rangle_{\mathcal{V}}$ for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$.

13.3. Let $\langle v, w\rangle = w^T B v$ for all $v, w \in \mathbb{R}^{2,1}$ with

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{2,2}.$$

(a) Show that $\langle v, w\rangle = w^T Bv$ is a scalar product on $\mathbb{R}^{2,1}$.
(b) Using this scalar product, determine the adjoint map f^{ad} of $f : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$, $v \mapsto Fv$, with $F \in \mathbb{R}^{2,2}$.
(c) Investigate which properties F needs to satisfy so that f is selfadjoint.

13.4. Let $n \geq 2$ and

$$f : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}, \quad [x_1, \dots, x_n]^T \mapsto [0, x_1, \dots, x_{n-1}]^T.$$

Determine the adjoint f^{ad} of f with respect to the standard scalar product of $\mathbb{R}^{n,1}$.

13.5. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. Show that $\ker(f^{ad} \circ f) = \ker(f)$ and $\operatorname{im}(f^{ad} \circ f) = \operatorname{im}(f^{ad})$.

13.6. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space, let $\mathcal{U} \subseteq \mathcal{V}$ be a subspace and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ with $f(\mathcal{U}) \subseteq \mathcal{U}$. Show that then $f^{ad}(\mathcal{U}^{\perp}) \subseteq \mathcal{U}^{\perp}$.

13.7. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space, let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ and $v \in \mathcal{V}$. Show that $v \in \operatorname{im}(f)$ if and only if $v \in \ker(f^{ad})^{\perp}$.
"Matrix version": For $A \in \mathbb{C}^{n,n}$ and $b \in \mathbb{C}^{n,1}$ the linear system of equations $Ax = b$ has a solution if and only if $b \in \mathscr{L}(A^H, 0)^{\perp}$.

13.8. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space and let $f, g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ be selfadjoint. Show that $f \circ g$ is selfadjoint if and only if f and g commute, i.e., $f \circ g = g \circ f$.

13.9. Let $\mathcal{V}$ be a finite dimensional unitary vector space and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. Show that f is selfadjoint if and only if $\langle f(v), v\rangle \in \mathbb{R}$ holds for all $v \in \mathcal{V}$.

13.10. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{V})$ be a *projection*, i.e., f satisfies $f^2 = f$. Show that f is selfadjoint if and only if $\ker(f) \perp \operatorname{im}(f)$, i.e., $\langle v, w\rangle = 0$ holds for all $v \in \ker(f)$ and $w \in \operatorname{im}(f)$.

13.11. Let $\mathcal{V}$ be a finite dimensional Euclidean or unitary vector space and let $f, g \in \mathcal{L}(\mathcal{V}, \mathcal{V})$. Show that if $g^{ad} \circ f = 0 \in \mathcal{L}(\mathcal{V}, \mathcal{V})$, then $\langle v, w\rangle = 0$ holds for all $v \in \operatorname{im}(f)$ and $w \in \operatorname{im}(g)$.

13.12. For two polynomials $p, q \in \mathbb{R}[t]_{\leq n}$ let

$$\langle p, q\rangle := \int_{-1}^{1} p(t)q(t)\, dt.$$

(a) Show that this defines a scalar product on $\mathbb{R}[t]_{\leq n}$.
(b) Consider the map

$$f : \mathbb{R}[t]_{\leq n} \to \mathbb{R}[t]_{\leq n}, \quad p = \sum_{i=0}^{n} \alpha_i t^i \mapsto \sum_{i=1}^{n} i\alpha_i t^{i-1},$$

and determine f^{ad}, $\ker(f^{ad})$, $\operatorname{im}(f)$, $\ker(f^{ad})^{\perp}$ and $\operatorname{im}(f)^{\perp}$.

13.13. Prove Lemma 13.15.

13.14. Prove Lemma 13.16.