Chapter 7
Determinants of Matrices

The determinant is a map that assigns to every square matrix $A \in R^{n,n}$, where R is a commutative ring with unit, an element of R. This map has very interesting and important properties. For instance it yields a necessary and sufficient condition for the invertibility of $A \in R^{n,n}$. Moreover, it forms the basis for the definition of the characteristic polynomial of a matrix in Chap. 8.

7.1 Definition of the Determinant

There are several different approaches to define the determinant of a matrix. We use the constructive approach via permutations.

Definition 7.1 Let $n \in \mathbb{N}$ be given. A bijective map

$$\sigma \ : \ \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}, \quad j \mapsto \sigma(j),$$

is called a *permutation* of the numbers $\{1, 2, \ldots, n\}$. We denote the set of all these maps by S_n.

A permutation $\sigma \in S_n$ can be written in the form

$$\big[\sigma(1)\ \sigma(2)\ \ldots\ \sigma(n)\big].$$

For example $S_1 = \{[1]\}$, $S_2 = \{[1\,2], [2\,1]\}$, and

$$S_3 = \{\,[1\,2\,3], [1\,3\,2], [2\,1\,3], [2\,3\,1], [3\,1\,2], [3\,2\,1]\,\}.$$

From Lemma 2.17 we know that $|S_n| = n! = 1 \cdot 2 \cdot \ldots \cdot n$.

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The set S_n with the composition of maps "$\circ$" forms a group (cp. Exercise 3.3), which is sometimes called the *symmetric group*. The neutral element in this group is the permutation $[1\,2\,\ldots\,n]$.

While S_1 and S_2 are commutative groups, the group S_n for $n \geq 3$ is non-commutative. As an example consider $n = 3$ and the permutations $\sigma_1 = [2\,3\,1]$, $\sigma_2 = [1\,3\,2]$. Then

$$\sigma_1 \circ \sigma_2 = [\sigma_1(\sigma_2(1))\ \sigma_1(\sigma_2(2))\ \sigma_1(\sigma_2(3))] = [\sigma_1(1)\ \sigma_1(3)\ \sigma_1(2)] = [2\ 1\ 3],$$
$$\sigma_2 \circ \sigma_1 = [\sigma_2(\sigma_1(1))\ \sigma_2(\sigma_1(2))\ \sigma_2(\sigma_1(3))] = [\sigma_2(2)\ \sigma_2(3)\ \sigma_2(1)] = [3\ 2\ 1].$$

Definition 7.2 Let $n \geq 2$ and $\sigma \in S_n$. A pair $(\sigma(i), \sigma(j))$ with $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$ is called an *inversion* of σ. If k is the number of inversions of σ, then $\mathrm{sgn}(\sigma) := (-1)^k$ is called the *sign* of σ. For $n = 1$ we define $\mathrm{sgn}([1]) := 1 = (-1)^0$.

In short, an inversion of a permutation σ is a pair that is "out of order". The term inversion should not be confused with the inverse map σ^{-1} (which exists, since σ is bijective). The sign of a permutation is sometimes also called the *signature*.

Example 7.3 The permutation $[2\,3\,1\,4] \in S_4$ has the inversions $(2, 1)$ and $(3, 1)$, so that $\mathrm{sgn}([2\,3\,1\,4]) = 1$. The permutation $[4\,1\,2\,3] \in S_4$ has the inversions $(4, 1)$, $(4, 2)$, $(4, 3)$, so that $\mathrm{sgn}([4\,1\,2\,3]) = -1$.

We can now define the determinant map.

Definition 7.4 Let R be a commutative ring with unit and let $n \in \mathbb{N}$. The map

$$\det : R^{n,n} \to R, \quad A = [a_{ij}] \mapsto \det(A) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}, \tag{7.1}$$

is called the *determinant*, and the ring element $\det(A)$ is called the *determinant of* A.

The formula (7.1) for $\det(A)$ is called the *signature formula of Leibniz*.[1] The term $\mathrm{sgn}(\sigma)$ in this definition is to be interpreted as an element of the ring R, i.e., either $\mathrm{sgn}(\sigma) = 1 \in R$ or $\mathrm{sgn}(\sigma) = -1 \in R$, where $-1 \in R$ is the unique additive inverse of the unit $1 \in R$.

Example 7.5 For $n = 1$ we have $A = [a_{11}]$ and thus $\det(A) = \mathrm{sgn}([1])a_{11} = a_{11}$. For $n = 2$ we get

$$\begin{aligned}\det(A) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) &= \mathrm{sgn}([1\,2])a_{11}a_{22} + \mathrm{sgn}([2\,1])a_{12}a_{21} \\ &= a_{11}a_{22} - a_{12}a_{21}.\end{aligned}$$

[1] Gottfried Wilhelm Leibniz (1646–1716).

For $n = 3$ we have the *Sarrus rule*[2]:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

In order to compute $\det(A)$ using the signature formula of Leibniz we have to form $n!$ products with n factors each. For large n this is too costly even on modern computers. As we will see in Corollary 7.16, there are more efficient ways for computing $\det(A)$. The signature formula is mostly of theoretical relevance, since it represents the determinant of A explicitly in terms of the entries of A. Considering the n^2 entries as variables, we can interpret $\det(A)$ as a polynomial in these variables. If $R = \mathbb{R}$ or $R = \mathbb{C}$, then standard techniques of Analysis show that $\det(A)$ is a continuous function of the entries of A.

We will now study the group of permutations in more detail. The permutation $\sigma = [3\,2\,1] \in S_3$ has the inversions $(3, 2)$, $(3, 1)$ and $(2, 1)$, so that $\operatorname{sgn}(\sigma) = -1$. Moreover,

$$\prod_{1 \le i < j \le 3} \frac{\sigma(j) - \sigma(i)}{j - i} = \frac{\sigma(2) - \sigma(1)}{2 - 1} \, \frac{\sigma(3) - \sigma(1)}{3 - 1} \, \frac{\sigma(3) - \sigma(2)}{3 - 2} \\ = \frac{2 - 3}{2 - 1} \, \frac{1 - 3}{3 - 1} \, \frac{1 - 2}{3 - 2} = (-1)^3 = -1 = \operatorname{sgn}(\sigma).$$

This observation can be generalized as follows.

Lemma 7.6 *For each $\sigma \in S_n$ we have*

$$\operatorname{sgn}(\sigma) = \prod_{1 \le i < j \le n} \frac{\sigma(j) - \sigma(i)}{j - i}. \tag{7.2}$$

Proof If $n = 1$, then the left hand side of (7.2) is an empty product, which is defined to be 1 (cp. Sect. 3.2), so that (7.2) holds for $n = 1$.

Let $n > 1$ and $\sigma \in S_n$ with $\operatorname{sgn}(\sigma) = (-1)^k$, i.e., k is the number of pairs $(\sigma(i), \sigma(j))$ with $i < j$ but $\sigma(i) > \sigma(j)$. Then

$$\prod_{1 \le i < j \le n} (\sigma(j) - \sigma(i)) = (-1)^k \prod_{1 \le i < j \le n} |\sigma(j) - \sigma(i)| = (-1)^k \prod_{1 \le i < j \le n} (j - i).$$

In the last equation we have used the fact that the two products have the same factors (except possibly for their order). □

[2]Pierre Frédéric Sarrus (1798–1861).

Theorem 7.7 *For all* $\sigma_1, \sigma_2 \in S_n$ *we have* $\operatorname{sgn}(\sigma_1 \circ \sigma_2) = \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2)$. *In particular,* $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$ *for all* $\sigma \in S_n$.

Proof By Lemma 7.6 we have

$$
\begin{aligned}
\operatorname{sgn}(\sigma_1 \circ \sigma_2) &= \prod_{1 \le i < j \le n} \frac{\sigma_1(\sigma_2(j)) - \sigma_1(\sigma_2(i))}{j - i} \\
&= \left(\prod_{1 \le i < j \le n} \frac{\sigma_1(\sigma_2(j)) - \sigma_1(\sigma_2(i))}{\sigma_2(j) - \sigma_2(i)} \right) \left(\prod_{1 \le i < j \le n} \frac{\sigma_2(j) - \sigma_2(i)}{j - i} \right) \\
&= \left(\prod_{1 \le \sigma_2(i) < \sigma_2(j) \le n} \frac{\sigma_1(\sigma_2(j)) - \sigma_1(\sigma_2(i))}{\sigma_2(j) - \sigma_2(i)} \right) \operatorname{sgn}(\sigma_2) \\
&= \left(\prod_{1 \le i < j \le n} \frac{\sigma_1(j) - \sigma_1(i)}{j - i} \right) \operatorname{sgn}(\sigma_2) \\
&= \operatorname{sgn}(\sigma_1)\operatorname{sgn}(\sigma_2).
\end{aligned}
$$

For each $\sigma \in S_n$ we have $1 = \operatorname{sgn}([1\,2\,\ldots\,n]) = \operatorname{sgn}(\sigma \circ \sigma^{-1}) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma^{-1})$, so that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$. □

Theorem 7.7 shows that the map sgn is a homomorphism between the groups $(S_n, \circ)$ and $(\{1, -1\}, \cdot)$, where the operation in the second group is the standard multiplication of the integers 1 and -1.

Definition 7.8 A *transposition* is a permutation $\tau \in S_n$, $n \ge 2$, that exchanges exactly two distinct elements $k, \ell \in \{1, 2, \ldots, n\}$, i.e., $\tau(k) = \ell$, $\tau(\ell) = k$ and $\tau(j) = j$ for all $j \in \{1, 2, \ldots, n\} \setminus \{k, \ell\}$.

Obviously $\tau^{-1} = \tau$ for every transposition $\tau \in S_n$.

Lemma 7.9 *Let* $\tau \in S_n$ *be the transposition, that exchanges* k *and* ℓ *for some* $1 \le k < \ell \le n$. *Then* τ *has exactly* $2(\ell - k) - 1$ *inversions and, hence,* $\operatorname{sgn}(\tau) = -1$.

Proof We have $\ell = k + j$ for a $j \ge 1$ and thus τ is given by

$$\tau = [1, \ldots, k-1, \;\; k+j, \;\; k+1, \ldots, k+(j-1), \;\; k, \;\; \ell+1, \ldots, n],$$

where the points denote values of τ in increasing and thus "correct" order. A simple counting argument shows that τ has exactly $2j - 1 = 2(\ell - k) - 1$ inversions. □

7.2 Properties of the Determinant

In this section we prove important properties of the determinant map.

Lemma 7.10 *For $A \in R^{n,n}$ the following assertions hold:*

(1) For $\lambda \in R$,

$$\det\left(\left[\begin{array}{c|c} \lambda & \star \\ \hline 0_{n,1} & A \end{array}\right]\right) = \det\left(\left[\begin{array}{c|c} \lambda & 0_{1,n} \\ \hline \star & A \end{array}\right]\right) = \lambda \det(A).$$

(2) If $A = [a_{ij}]$ is upper or lower triangular, then $\det(A) = \prod_{i=1}^{n} a_{ii}$.
(3) If A has a zero row or column, then $\det(A) = 0$.
(4) If $n \geq 2$ and A has two equal rows or two equal columns, then $\det(A) = 0$.
(5) $\det(A) = \det(A^T)$.

Proof

(1) Exercise.
(2) This follows by an application of (1) to the upper (or lower) triangular matrix A.
(3) If A has a zero row or column, then for every $\sigma \in S_n$ at least one factor in the product $\prod_{i=1}^{n} a_{i,\sigma(i)}$ is equal to zero and thus $\det(A) = 0$.
(4) Let the rows k and ℓ, with $k < \ell$, of $A = [a_{ij}]$ be equal, i.e., $a_{kj} = a_{\ell j}$ for $j = 1, \ldots, n$. Let $\tau \in S_n$ be the transposition that exchanges the elements k and ℓ, and let

$$T_n := \{\sigma \in S_n \mid \sigma(k) < \sigma(\ell)\}.$$

Since the set T_n contains all permutations $\sigma \in S_n$ for which $\sigma(k) < \sigma(\ell)$, we have $|T_n| = |S_n|/2$ and

$$S_n \setminus T_n = \{\sigma \circ \tau \mid \sigma \in T_n\}.$$

Moreover,

$$a_{i,(\sigma\circ\tau)(i)} = \begin{cases} a_{i,\sigma(i)}, & i \neq k, \ell, \\ a_{k,\sigma(\ell)}, & i = k, \\ a_{\ell,\sigma(k)}, & i = \ell. \end{cases}$$

We have $a_{k,\sigma(\ell)} = a_{\ell,\sigma(\ell)}$ and $a_{\ell,\sigma(k)} = a_{k,\sigma(k)}$. Thus, using Theorem 7.7 and Lemma 7.9, we obtain

$$\begin{aligned}
\sum_{\sigma \in S_n \setminus T_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} &= \sum_{\sigma \in T_n} \operatorname{sgn}(\sigma \circ \tau) \prod_{i=1}^{n} a_{i,(\sigma\circ\tau)(i)} \\
&= \sum_{\sigma \in T_n} (-\operatorname{sgn}(\sigma)) \prod_{i=1}^{n} a_{i,(\sigma\circ\tau)(i)} \\
&= -\sum_{\sigma \in T_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.
\end{aligned}$$

This implies

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$
$$= \sum_{\sigma \in T_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} + \sum_{\sigma \in S_n \setminus T_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = 0.$$

The proof for the case of two equal columns is analogous.

(5) We observe first that

$$\{ (\sigma(i), i) \mid 1 \leq i \leq n \} = \{ (i, \sigma^{-1}(i)) \mid 1 \leq i \leq n \}$$

for every $\sigma \in S_n$. To see this, let i with $1 \leq i \leq n$ be fixed. Then $\sigma(i) = j$ if and only if $i = \sigma^{-1}(j)$. Thus, $(\sigma(i), i) = (j, i)$ is an element of the first set if and only if $(j, \sigma^{-1}(j)) = (j, i)$ is an element of the second set. Since σ is bijective, the two sets are equal.
Let $A = [a_{ij}]$ and $A^T = [b_{ij}]$ with $b_{ij} = a_{ji}$. Then

$$\det(A^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_{i=1}^{n} a_{\sigma(i),i} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma^{-1}) \prod_{i=1}^{n} a_{i,\sigma^{-1}(i)}$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} = \det(A).$$

Here we have used that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$ (cp. Theorem 7.7) and the fact that the two products $\prod_{i=1}^{n} a_{\sigma(i),i}$ and $\prod_{i=1}^{n} a_{i,\sigma^{-1}(i)}$ have the same factors. □

Example 7.11 For the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 1 & 4 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

from $\mathbb{Z}^{3,3}$ we obtain $\det(A) = 1 \cdot 4 \cdot 6 = 24$ by (2) in Lemma 7.10, and $\det(B) = \det(C) = 0$ by (3) and (4) in Lemma 7.10. We may also compute these determinants using the Sarrus rule from Example 7.5.

Item (2) in Lemma 7.10 shows in particular that $\det(I_n) = 1$ for the identity matrix $I_n = [e_1, e_2, \ldots, e_n] \in R^{n,n}$. For this reason the determinant map is called *normalized*.

For $\sigma \in S_n$ the matrix

$$P_\sigma := [e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)}]$$

is called the permutation matrix associated with σ. This map from the group S_n to the group of permutation matrices in $R^{n,n}$ is bijective. The inverse of a permutation matrix is its transpose (cp. Theorem 4.16) and we can easily check that

$$P_\sigma^{-1} = P_\sigma^T = P_{\sigma^{-1}}.$$

If $A = [a_1, a_2, \dots, a_n] \in R^{n,n}$, i.e., $a_j \in R^{n,1}$ is the jth column of A, then

$$A P_\sigma = [a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}],$$

i.e., the right-multiplication of A with P_σ exchanges the columns of A according to the permutation σ. If, on the other hand, $a_i \in R^{1,n}$ is the ith row of A, then

$$P_\sigma^T A = \begin{bmatrix} a_{\sigma(1)} \\ a_{\sigma(2)} \\ \vdots \\ a_{\sigma(n)} \end{bmatrix},$$

i.e., the left-multiplication of A by P_σ^T exchanges the rows of A according to the permutation σ.

We next study the determinants of the elementary matrices.

Lemma 7.12 *(1) For $\sigma \in S_n$ and the associated permutation matrix $P_\sigma \in R^{n,n}$ we have* $\operatorname{sgn}(\sigma) = \det(P_\sigma)$. *If $n \geq 2$ and P_{ij} is defined as in (5.1), then* $\det(P_{ij}) = -1$.

(2) If $M_i(\lambda)$ and $G_{ij}(\lambda)$ are defined as in (5.2) and (5.3), respectively, then $\det(M_i(\lambda)) = \lambda$ *and* $\det(G_{ij}(\lambda)) = 1$.

Proof

(1) If $\widetilde{\sigma} \in S_n$ and $P_{\widetilde{\sigma}} = [a_{ij}] \in R^{n,n}$, then $a_{\widetilde{\sigma}(j),j} = 1$ for $j = 1, 2, \dots, n$, and all other entries of $P_{\widetilde{\sigma}}$ are zero. Hence

$$\det(P_{\widetilde{\sigma}}) = \det(P_{\widetilde{\sigma}}^T) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \underbrace{\prod_{j=1}^{n} a_{\sigma(j),j}}_{=0 \text{ for } \sigma \neq \widetilde{\sigma}} = \operatorname{sgn}(\widetilde{\sigma}) \prod_{j=1}^{n} \underbrace{a_{\widetilde{\sigma}(j),j}}_{=1} = \operatorname{sgn}(\widetilde{\sigma}).$$

The permutation matrix P_{ij} is associated with the transposition that exchanges i and j. Hence, $\det(P_{ij}) = -1$ follows from Lemma 7.9.

(2) Since $M_i(\lambda)$ and $G_{ij}(\lambda)$ are lower triangular matrices, the assertion follows from (2) in Lemma 7.10. □

These results lead to some important computational rules for determinants.

Lemma 7.13 *For $A \in R^{n,n}$, $n \geq 2$, and $\lambda \in R$ the following assertions hold:*

(1) The multiplication of a row of A by λ leads to the multiplication of $\det(A)$ *by λ:*
$\det(M_i(\lambda)A) = \lambda \det(A) = \det(M_i(\lambda)) \det(A)$.
(2) The addition of the λ–multiple of a row of A to another row of A does not change $\det(A)$*:*
$\det(G_{ij}(\lambda)A) = \det(A) = \det(G_{ij}(\lambda)) \det(A)$*, and*
$\det(G_{ij}(\lambda)^T A) = \det(A) = \det(G_{ij}(\lambda)^T) \det(A)$.
(3) Exchanging two rows of A changes the sign of $\det(A)$*:*
$\det(P_{ij}A) = -\det(A) = \det(P_{ij}) \det A$.

Proof

(1) If $A = [a_{mk}]$ and $\widetilde{A} = M_i(\lambda)A = [\widetilde{a}_{mk}]$, then

$$\widetilde{a}_{mk} = \begin{cases} a_{mk}, & m \neq i, \\ \lambda a_{mk}, & m = i, \end{cases}$$

and hence

$$\det(\widetilde{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{m=1}^{n} \widetilde{a}_{m,\sigma(m)} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \underbrace{\widetilde{a}_{i,\sigma(i)}}_{=\lambda a_{i,\sigma(i)}} \prod_{\substack{m=1 \\ m \neq i}}^{n} \underbrace{\widetilde{a}_{m,\sigma(m)}}_{=a_{m,\sigma(m)}}$$
$$= \lambda \det(A).$$

(2) If $A = [a_{mk}]$ and $\widetilde{A} = G_{ij}(\lambda)A = [\widetilde{a}_{mk}]$, then

$$\widetilde{a}_{mk} = \begin{cases} a_{mk}, & m \neq j, \\ a_{jk} + \lambda a_{ik}, & m = j, \end{cases}$$

and hence

$$\det(\widetilde{A}) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \, (a_{j,\sigma(j)} + \lambda a_{i,\sigma(j)}) \prod_{\substack{m=1 \\ m \neq j}}^{n} a_{m,\sigma(m)}$$
$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{m=1}^{n} a_{m,\sigma(m)} \; + \; \lambda \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{i,\sigma(j)} \prod_{\substack{m=1 \\ m \neq j}}^{n} a_{m,\sigma(m)}.$$

The first term is equal to $\det(A)$, and the second is equal to the determinant of a matrix with two equal columns, and thus equal to zero. The proof for the matrix $G_{ij}(\lambda)^T A$ is analogous.

(3) The permutation matrix P_{ij} exchanges rows i and j of A, where $i < j$. This exchange can be expressed by the following four elementary row operations: Multiply row j by -1; add row i to row j; add the (-1)–multiple of row j to row i; add row i to row j. Therefore,

$$P_{ij} = G_{ij}(1)(G_{ij}(-1))^T G_{ij}(1) M_j(-1).$$

(One may verify this also by carrying out the matrix multiplications.) Using (1) and (2) we obtain

$$\begin{aligned}\det(P_{ij}A) &= \det\left(G_{ij}(1)(G_{ij}(-1))^T G_{ij}(1) M_j(-1)A\right)\\ &= \det(G_{ij}(1))\ \det((G_{ij}(-1))^T)\ \det(G_{ij}(1))\ \det(M_j(-1))\ \det(A)\\ &= (-1)\ \det(A).\end{aligned}$$ □

Since $\det(A) = \det(A^T)$ (cp. (5) in Lemma 7.10), the results in Lemma 7.13 for the rows of A can be formulated analogously for the columns of A.

Example 7.14 Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix} \in \mathbb{Z}^{3,3}.$$

A simple calculation shows that $\det(A) = -4$. Since B is obtained from A by exchanging the first two columns we have $\det(B) = -\det(A) = 4$.

The determinant map can be interpreted as a map of $(R^{n,1})^n$ to R, i.e., as a map of the n columns of the matrix $A \in R^{n,n}$ to the ring R. If $a_i, a_j \in R^{n,1}$ are two columns of A,

$$A = [\ldots a_i \ldots a_j \ldots],$$

then

$$\det(A) = -\det([\ldots a_j \ldots a_i \ldots])$$

by (3) in Lemma 7.13. Due to this property the determinant map is called an *alternating* map of the columns of A. Analogously, the determinant map is an alternating map of the rows of A.

If the kth row of A has the form $\lambda a^{(1)} + \mu a^{(2)}$ for some $\lambda, \mu \in R$ and $a^{(j)} = \left[a_{k1}^{(j)}, \ldots, a_{kn}^{(j)}\right] \in R^{1,n}$, $j = 1, 2$, then

$$\det(A) = \det\left(\begin{bmatrix} \vdots \\ \lambda a^{(1)} + \mu a^{(2)} \\ \vdots \end{bmatrix}\right) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \left(\lambda a^{(1)}_{k,\sigma(k)} + \mu a^{(2)}_{k,\sigma(k)}\right) \prod_{\substack{i=1\\ i\neq k}}^{n} a_{i,\sigma(i)}$$

$$= \lambda \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a^{(1)}_{k,\sigma(k)} \prod_{\substack{i=1\\ i\neq k}}^{n} a_{k,\sigma(k)} \; + \; \mu \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, a^{(2)}_{k,\sigma(k)} \prod_{\substack{i=1\\ i\neq k}}^{n} a_{i,\sigma(i)}$$

$$= \lambda \det\left(\begin{bmatrix} \vdots \\ a^{(1)} \\ \vdots \end{bmatrix}\right) + \mu \det\left(\begin{bmatrix} \vdots \\ a^{(2)} \\ \vdots \end{bmatrix}\right).$$

This property is called the *linearity* of the determinant map with respect to the rows of A. Analogously we have the linearity with respect to the columns of A. Linear maps will be studied in detail in later chapters.

The next result is called the *multiplication theorem for determinants*.

Theorem 7.15 *If K is a field and $A, B \in K^{n,n}$, then* $\det(AB) = \det(A)\det(B)$. *Moreover, if A is invertible, then* $\det(A^{-1}) = (\det(A))^{-1}$.

Proof By Theorem 5.2 we know that for $A \in K^{n,n}$ there exist invertible elementary matrices $S_1, \ldots, S_t$ such that $\widetilde{A} = S_t \ldots S_1 A$ is in echelon form. By Lemma 7.13 we have

$$\det(A) = \det(S_1^{-1}) \cdots \det(S_t^{-1}) \det(\widetilde{A}),$$

as well as

$$\begin{aligned}\det(AB) &= \det\left(S_1^{-1} \cdots S_t^{-1} \widetilde{A} B\right) \\ &= \det(S_1^{-1}) \cdots \det(S_t^{-1}) \det(\widetilde{A} B).\end{aligned}$$

There are two cases: If A is not invertible, then $\widetilde{A}$ and thus also $\widetilde{A}B$ have a zero row. Then $\det(\widetilde{A}) = \det(\widetilde{A}B) = 0$, which implies that $\det(A) = 0$, and hence $\det(AB) = 0 = \det(A)\det(B)$. On the other hand, if A is invertible, then $\widetilde{A} = I_n$, since $\widetilde{A}$ is in echelon form. Now $\det(I_n) = 1$ again gives $\det(AB) = \det(A)\det(B)$.

Finally, if A is invertible, then $1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1})$, and hence $\det(A^{-1}) = (\det(A))^{-1}$. □

Since our proof relies on Theorem 5.2, which is valid for matrices over a field K, we have formulated Theorem 7.15 for $A, B \in K^{n,n}$. However, the multiplication theorem for determinants also holds for matrices over a commutative ring R with unit. A direct proof based on the signature formula of Leibniz can be found, for example, in the book "Advanced Linear Algebra" by Loehr [Loe14, Sect. 5.13]. That book also contains a proof of the *Cauchy-Binet formula* for $\det(AB)$ with $A \in R^{n,m}$ and $B \in R^{m,n}$ for $n \leq m$. Below we will sometimes use that $\det(AB) = \det(A)\det(B)$

holds for all $A, B \in R^{n,n}$, although we have shown the result in Theorem 7.15 only for $A, B \in K^{n,n}$.

The proof of Theorem 7.15 suggests that $\det(A)$ can be easily computed while transforming $A \in K^{n,n}$ into its echelon form using elementary row operations.

Corollary 7.16 *For $A \in K^{n,n}$ let $S_1, \ldots, S_t \in K^{n,n}$ be elementary matrices, such that $\widetilde{A} = S_t \ldots S_1 A$ is in echelon form. Then either $\widetilde{A}$ has a zero row and hence $\det(A) = 0$, or $\widetilde{A} = I_n$ and hence $\det(A) = (\det(S_1))^{-1} \cdots (\det(S_t))^{-1}$.*

As shown in Theorem 5.4, every matrix $A \in K^{n,n}$ can be factorized as $A = PLU$, and hence $\det(A) = \det(P)\det(L)\det(U)$. The determinants of the matrices on the right hand side are easily computed, since these are permutation and triangular matrices. An LU-decomposition of a matrix A therefore yields an efficient way to compute $\det(A)$.

MATLAB-Minute.
Look at the matrices `wilkinson(n)` for `n=2,3,...,10` in MATLAB. Can you find a general formula for their entries? For `n=2,3,...,10` compute
`A=wilkinson(n)`
`[L,U,P]=lu(A)` (LU-decomposition; cp. the MATLAB-Minute above Definition 5.6)
`det(L), det(U), det(P), det(P)*det(L)*det(U), det(A)`
Which permutation is associated with the computed matrix `P`? Why is `det(A)` an integer for odd `n`?

7.3 Minors and the Laplace Expansion

We now show that the determinant can be used for deriving formulas for the inverse of an invertible matrix and for the solution of linear systems of equations. These formulas are, however, more of theoretical than practical relevance.

Definition 7.17 Let R be a commutative ring with unit and let $A \in R^{n,n}$, $n \geq 2$. Then the matrix $A(j,i) \in R^{n-1,n-1}$ that is obtained by deleting the jth row and ith column of A is called a *minor*[3] of A. The matrix

$$\mathrm{adj}(A) = [b_{ij}] \in R^{n,n} \quad \text{with} \quad b_{ij} := (-1)^{i+j} \det(A(j,i)),$$

is called the *adjunct* of A.

The adjunct is also called *adjungate* or *classical adjoint* of A.

[3]This term was introduced in 1850 by James Joseph Sylvester (1814–1897).

Theorem 7.18 *For $A \in R^{n,n}$, $n \geq 2$, we have*

$$A\,\mathrm{adj}(A) = \mathrm{adj}(A)\,A = \det(A)I_n.$$

In particular A is invertible if and only if $\det(A) \in R$ *is invertible. In this case* $(\det(A))^{-1} = \det(A^{-1})$ *and* $A^{-1} = (\det(A))^{-1}\mathrm{adj}(A)$.

Proof Let $B = [b_{ij}]$ have the entries $b_{ij} = (-1)^{i+j}\det(A(j,i))$. Then $C = [c_{ij}] = \mathrm{adj}(A)A$ satisfies

$$c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n}(-1)^{i+k}\det(A(k,i))a_{kj}.$$

Let a_ℓ be the ℓth column of A and let

$$\widetilde{A}(k,i) := [a_1, \ldots, a_{i-1}, e_k, a_{i+1}, \ldots, a_n] \in R^{n,n},$$

where e_k is the kth column of the identity matrix I_n. Then there exist permutation matrices P and Q that perform $k-1$ row and $i-1$ column exchanges, respectively, such that

$$P\widetilde{A}(k,i)Q = \left[\begin{array}{c|c} 1 & \star \\ \hline 0 & A(k,i) \end{array}\right].$$

Using (1) in Lemma 7.10 we obtain

$$\begin{aligned}
\det(A(k,i)) &= \det\left(\left[\begin{array}{c|c} 1 & \star \\ \hline 0 & A(k,i) \end{array}\right]\right) = \det(P\widetilde{A}(k,i)Q) \\
&= \det(P)\,\det(\widetilde{A}(k,i))\,\det(Q) \\
&= (-1)^{(k-1)+(i-1)}\det(\widetilde{A}(k,i)) \\
&= (-1)^{k+i}\det(\widetilde{A}(k,i)).
\end{aligned}$$

The linearity of the determinant with respect to the columns now gives

$$\begin{aligned}
c_{ij} &= \sum_{k=1}^{n}(-1)^{i+k}(-1)^{k+i}a_{kj}\det(\widetilde{A}(k,i)) \\
&= \det([a_1, \ldots, a_{i-1}, a_j, a_{i+1}, \ldots, a_n]) \\
&= \begin{cases} 0, & i \neq j \\ \det(A), & i = j \end{cases} \\
&= \delta_{ij}\,\det(A),
\end{aligned}$$

and thus $\mathrm{adj}(A)A = \det(A)I_n$. Analogously we can show that $A\,\mathrm{adj}(A) = \det(A)I_n$.

If $\det(A) \in R$ is invertible, then

$$I_n = (\det(A))^{-1}\mathrm{adj}(A)A = A(\det(A))^{-1}\mathrm{adj}(A),$$

i.e., A is invertible with $A^{-1} = (\det(A))^{-1}\mathrm{adj}(A)$. If, on the other hand, A is invertible, then

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1})\det(A),$$

where we have used the multiplication theorem for determinants over R (cp. our comment following the proof of Theorem 7.15). Thus, $\det(A)$ is invertible with $(\det(A))^{-1} = \det(A^{-1})$, and again $A^{-1} = (\det(A))^{-1}\mathrm{adj}(A)$. □

Example 7.19

(1) For

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \in \mathbb{Z}^{2,2}$$

we have $\det(A) = 2$ and thus A is not invertible. But A is invertible when considered as an element of $\mathbb{Q}^{2,2}$, since in this case $\det(A^{-1}) = (\det(A))^{-1} = \frac{1}{2}$.

(2) For

$$A = \begin{bmatrix} t-1 & t-2 \\ t & t-1 \end{bmatrix} \in (\mathbb{Z}[t])^{2,2}$$

we have $\det(A) = 1$. The matrix A is invertible, since $1 \in \mathbb{Z}[t]$ is invertible.

Note that if $A \in R^{n,n}$ is invertible, then Theorem 7.18 shows that A^{-1} can be obtained by inverting only one ring element, $\det(A)$.

We now use Theorem 7.18 and the multiplication theorem for matrices over a commutative ring with unit to prove a result already announced in Sect. 4.2: In order to show that $\widetilde{A} \in R^{n,n}$ is the (unique) inverse of $A \in R^{n,n}$, only one of the two equations $\widetilde{A}A = I_n$ or $A\widetilde{A} = I_n$ needs to be checked.

Corollary 7.20 *Let $A \in R^{n,n}$. If a matrix $\widetilde{A} \in R^{n,n}$ exists with $\widetilde{A}A = I_n$ or $A\widetilde{A} = I_n$, then A is invertible and $\widetilde{A} = A^{-1}$.*

Proof If $\widetilde{A}A = I_n$, then the multiplication theorem for determinants yields

$$1 = \det(I_n) = \det(\widetilde{A}A) = \det(\widetilde{A})\det(A) = \det(A)\det(\widetilde{A}),$$

i.e., $\det(A) \in R$ is invertible with $(\det(A))^{-1} = \det(\widetilde{A})$. Thus also A is invertible and has a unique inverse A^{-1}. For $n = 1$ this is obvious and for $n \geq 2$ it was shown in Theorem 7.18. If we multiply the equation $\widetilde{A}A = I_n$ from the right with A^{-1} we get $\widetilde{A} = A^{-1}$.

The proof starting from $A\widetilde{A} = I_n$ is analogous. □

Let us summarize the invertibility criteria for a square matrix over a field that we have shown so far:

$$\begin{aligned} A \in GL_n(K) &\overset{\text{Theorem 5.2}}{\Longleftrightarrow} \text{The echelon form of } A \text{ is the identity matrix } I_n \\ &\overset{\text{Definition 5.10}}{\Longleftrightarrow} \operatorname{rank}(A) = n \\ &\overset{\text{clear}}{\Longleftrightarrow} \operatorname{rank}(A) = \operatorname{rank}([A, b]) = n \quad \text{for all } b \in K^{n,1} \\ &\overset{\text{Algorithm 6.6}}{\Longleftrightarrow} |\mathscr{L}(A, b)| = 1 \quad \text{for all } b \in K^{n,1} \\ &\overset{\text{Theorem 7.18}}{\Longleftrightarrow} \det(A) \neq 0. \end{aligned} \tag{7.3}$$

Alternatively we obtain:

$$\begin{aligned} A \notin GL_n(K) &\overset{\text{Theorem 5.2}}{\Longleftrightarrow} \text{The echelon form of } A \text{ has at least one zero row} \\ &\overset{\text{Definition 5.10}}{\Longleftrightarrow} \operatorname{rank}(A) < n \\ &\overset{\text{clear}}{\Longleftrightarrow} \operatorname{rank}([A, 0]) < n \\ &\overset{\text{Algorithm 6.6}}{\Longleftrightarrow} \mathscr{L}(A, 0) \neq \{0\} \\ &\overset{\text{Theorem 7.18}}{\Longleftrightarrow} \det(A) = 0. \end{aligned} \tag{7.4}$$

In the fields $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we have the (usual) absolute value $|\cdot|$ of numbers and can formulate the following useful invertibility criterion for matrices.

Theorem 7.21 *If $A \in K^{n,n}$ with $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ is* diagonally dominant, *i.e., if*

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \quad \textit{for all} \quad i = 1, \ldots, n,$$

then $\det(A) \neq 0$.

Proof We prove the assertion by contraposition, i.e., by showing that $\det(A) = 0$ implies that A is not diagonally dominant.

If $\det(A) = 0$, then $\mathscr{L}(A, 0) \neq \{0\}$, i.e., the homogeneous linear system of equations $Ax = 0$ has at least one solution $\widehat{x} = [\widehat{x}_1, \ldots, \widehat{x}_n]^T \neq 0$. Let $\widehat{x}_m$ be an entry of $\widehat{x}$ with maximal absolute value, i.e., $|\widehat{x}_m| \geq |\widehat{x}_j|$ for all $j = 1, \ldots, n$. In particular, we then have $|\widehat{x}_m| > 0$. The mth row of $A\widehat{x} = 0$ is given by

$$a_{m1}\widehat{x}_1 + a_{m2}\widehat{x}_2 + \ldots + a_{mn}\widehat{x}_n = 0 \quad \Leftrightarrow \quad a_{mm}\widehat{x}_m = -\sum_{\substack{j=1 \\ j \neq m}}^{n} a_{mj}\widehat{x}_j.$$

We now take absolute values on both sides and use the triangle inequality, which yields

$$|a_{mm}|\,|\widehat{x}_m| \leq \sum_{\substack{j=1\\ j\neq m}}^{n} |a_{mj}|\,|\widehat{x}_j| \leq \sum_{\substack{j=1\\ j\neq m}}^{n} |a_{mj}||\widehat{x}_m|, \quad \text{hence} \quad |a_{mm}| \leq \sum_{\substack{j=1\\ j\neq m}}^{n} |a_{mj}|,$$

so that A not diagonally dominant. □

The converse of this theorem does not hold: For example, the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \in \mathbb{Q}^{2,2},$$

has $\det(A) = -2 \neq 0$, but A is not diagonally dominant.

From Theorem 7.18 we obtain the *Laplace expansion*[4] of the determinant, which is particularly useful when A contains many zero entries (cp. Example 7.24 below).

Corollary 7.22 *For $A \in R^{n,n}$, $n \geq 2$, the following assertions hold:*

(1) For each $i = 1, 2, \ldots, n$ we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i, j)).$$

(Laplace expansion of $\det(A)$ *with respect to the ith row A.)*

(2) For each $j = 1, 2, \ldots, n$ we have

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A(i, j)).$$

(Laplace expansion of $\det(A)$ *with respect to the jth column of A.)*

Proof The two expansions for $\det(A)$ follow immediately by comparison of the diagonal entries in the matrix equations $\det(A)\, I_n = A\, \mathrm{adj}(A)$ and $\det(A)\, I_n = \mathrm{adj}(A)\, A$. □

The Laplace expansions allows a recursive definition of the determinant: For $A \in R^{n,n}$ with $n \geq 2$, let $\det(A)$ be defined as in (1) or (2) in Corollary 7.22. We can choose an arbitrary row or column of A. The formula for $\det(A)$ then contains only matrices of size $(n-1)\times(n-1)$. For each of these we can use the Laplace expansion again, now expressing each determinant in terms of determinants of $(n-2) \times (n-2)$ matrices. We can do this recursively until only 1×1 matrices remain. For $A = [a_{11}] \in R^{1,1}$ we define $\det(A) := a_{11}$.

Finally we state *Cramer's rule*,[5] which gives an explicit formula for the solution of a linear system in form of determinants. This rule is only of theoretical value, because in order to compute the n components of the solution it requires the evaluation of $n + 1$ determinants of $n \times n$ matrices.

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[4] Pierre-Simon Laplace (1749–1827) published this expansion in 1772.

[5] Gabriel Cramer (1704–1752).

Corollary 7.23 *Let K be a field, $A \in GL_n(K)$ and $b \in K^{n,1}$. Then the unique solution of the linear system of equations $Ax = b$ is given by*

$$\widehat{x} = [\widehat{x}_1, \ldots, \widehat{x}_n]^T = A^{-1}b = (\det(A))^{-1} \operatorname{adj}(A)b,$$

with

$$\widehat{x}_i = \frac{\det[a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n]}{\det(A)}, \quad i = 1, \ldots, n.$$

Example 7.24 Consider

$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \in \mathbb{Q}^{4,4}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \in \mathbb{Q}^{4,1}.$$

The Laplace expansion with respect to the last column yields

$$\det(A) = 1 \cdot \det\left(\begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}\right) = 1 \cdot 1 \cdot \det\left(\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\right) = 1 \cdot 1 \cdot (-1) = -1.$$

Thus, A is invertible and $Ax = b$ has a unique solution $\widehat{x} = A^{-1}b \in \mathbb{Q}^{4,1}$, which by Cramer's rule has the following entries:

$$\widehat{x}_1 = \det\left(\begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 \end{bmatrix}\right) / \det(A) = -4/(-1) = 4,$$

$$\widehat{x}_2 = \det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}\right) / \det(A) = 1/(-1) = -1,$$

$$\widehat{x}_3 = \det\left(\begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}\right) / \det(A) = 1/(-1) = -1,$$

$$\widehat{x}_4 = \det\left(\begin{bmatrix} 1 & 3 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}\right) / \det(A) = -1/(-1) = 1.$$

Exercises

7.1 A permutation $\sigma \in S_n$ is called an *r-cycle* if there exists a subset $\{i_1, \dots, i_r\} \subseteq \{1, 2, \dots, n\}$ with $r \geq 1$ elements and

$$\sigma(i_k) = i_{k+1} \text{ for } k = 1, 2, \dots, r-1, \quad \sigma(i_r) = i_1, \quad \sigma(i) = i \text{ for } i \notin \{i_1, \dots, i_r\}.$$

We write an r-cycle as $\sigma = (i_1, i_2, \dots, i_r)$. In particular, a transposition $\tau \in S_n$ is a 2-cycle.

(a) Let $n = 4$ and the 2-cycles $\tau_{1,2} = (1, 2)$, $\tau_{2,3} = (2, 3)$ and $\tau_{3,4} = (3, 4)$ be given. Compute $\tau_{1,2} \circ \tau_{2,3}$, $\tau_{1,2} \circ \tau_{2,3} \circ \tau_{1,2}^{-1}$, and $\tau_{1,2} \circ \tau_{2,3} \circ \tau_{3,4}$.
(b) Let $n \geq 4$ and $\sigma = (1, 2, 3, 4)$. Determine σ^j for $j = 2, 3, 4, 5$.
(c) Show that the inverse of the cycle $(i_1, \dots, i_r)$ is given by $(i_r, \dots, i_1)$.
(d) Show that two cycles with disjoint elements, i.e. $(i_1, \dots, i_r)$ and $(j_1, \dots, j_s)$ with $\{i_1, \dots, i_r\} \cap \{j_1, \dots, j_s\} = \emptyset$, commute.
(e) Show that every permutation $\sigma \in S_n$ can be written as product of disjoint cycles that are, except for the order, uniquely determined by σ.

7.2 Prove Lemma 7.10 (1) using (7.1).

7.3 Show that the group homomorphism sgn : $(S_n, \circ) \to (\{1, -1\}, \cdot)$ satisfies the following assertions:

(a) The set $A_n = \{\sigma \in S_n \mid \text{sgn}(\sigma) = 1\}$ is a subgroup of S_n (cp. Exercise 3.8).
(b) For all $\sigma \in A_n$ and $\pi \in S_n$ we have $\pi \circ \sigma \circ \pi^{-1} \in A_n$.

7.4 Compute the determinants of the following matrices:

(a) $A = [e_n, e_{n-1}, \dots, e_1] \in \mathbb{Z}^{n,n}$, where e_i is the ith column of the identity matrix.
(b) $B = \left[b_{ij}\right] \in \mathbb{Z}^{n,n}$ with

$$b_{ij} = \begin{cases} 2 & \text{for } |i-j| = 0, \\ -1 & \text{for } |i-j| = 1, \\ 0 & \text{for } |i-j| \geq 2. \end{cases}$$

(c)

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & e^{\pi} & 4 & 5 & 1 & \sqrt{\pi} \\ e^2 & 1 & \frac{17}{31} & \sqrt{6} & \sqrt{7} & \sqrt{8} & \sqrt{10} \\ e^3 & 0 & -e & \pi & e & 0 & \pi^e \\ e^4 & 0 & 10001 & 0 & \pi^{-1} & 0 & e^2\pi \\ e^6 & 0 & \sqrt{2} & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{7,7}.$$

(d) The 4×4 *Wilkinson matrix*[6] (cp. the MATLAB-Minute at the end of Sect. 7.2).

7.5 Construct matrices $A, B \in \mathbb{R}^{n,n}$ for some $n \geq 2$ and with $\det(A + B) \neq \det(A) + \det(B)$.

7.6 Let R be a commutative ring with unit, $n \geq 2$ and $A \in R^{n,n}$. Show that the following assertions hold:

(a) $\mathrm{adj}(I_n) = I_n$.
(b) $\mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)$, if A and $B \in R^{n,n}$ are invertible.
(c) $\mathrm{adj}(\lambda A) = \lambda^{n-1}\mathrm{adj}(A)$ for all $\lambda \in R$.
(d) $\mathrm{adj}(A^T) = \mathrm{adj}(A)^T$.
(e) $\det(\mathrm{adj}(A)) = (\det(A))^{n-1}$, if A is invertible.
(f) $\mathrm{adj}(\mathrm{adj}(A)) = \det(A)^{n-2}A$.
(g) $\mathrm{adj}(A^{-1}) = \mathrm{adj}(A)^{-1}$, if A is invertible.

Can one drop the requirement of invertibility in (b) or (e)?

7.7 Let $n \geq 2$ and $A = [a_{ij}] \in \mathbb{R}^{n,n}$ with $a_{ij} = \frac{1}{x_i + y_j}$ for some $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. Hence, in particular, $x_i + y_j \neq 0$ for all i, j. (Such a matrix A is called a *Cauchy matrix*.[7])

(a) Show that

$$\det(A) = \frac{\prod_{1 \leq i < j \leq n} (x_j - x_i)(y_j - y_i)}{\prod_{i,j=1}^{n} (x_i + y_j)}.$$

(b) Use (a) to derive a formula for the determinant of the $n \times n$ *Hilbert matrix* (cp. the MATLAB-Minute above Definition 5.6).

7.8 Let R be a commutative ring with unit. If $\alpha_1, \ldots, \alpha_n \in R$, $n \geq 2$, then

$$V_n := \left[\alpha_i^{j-1}\right] = \begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{bmatrix} \in R^{n,n}$$

is called a *Vandermonde matrix*.[8]

(a) Show that

$$\det(V_n) = \prod_{1 \leq i < j \leq n} (\alpha_j - \alpha_i).$$

[6] James Hardy Wilkinson (1919–1986).
[7] Augustin Louis Cauchy (1789–1857).
[8] Alexandre-Théophile Vandermonde (1735–1796).

(b) Let K be a field and let $K[t]_{\leq n-1}$ be the set of polynomials in the variable t of degree at most $n-1$. Show that two polynomials $p, q \in K[t]_{\leq n-1}$ are equal if there exist pairwise distinct $\beta_1, \ldots, \beta_n \in K$ with $p(\beta_j) = q(\beta_j)$.

7.9 Show the following assertions:

(a) Let K be a field with $1 + 1 \neq 0$ and let $A \in K^{n,n}$ with $A^T = -A$. If n is odd, then $\det(A) = 0$.
(b) If $A \in GL_n(\mathbb{R})$ with $A^T = A^{-1}$, then $\det(A) \in \{1, -1\}$.

7.10 Let K be a field and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

for some $A_{11} \in K^{n_1,n_1}$, $A_{12} \in K^{n_1,n_2}$, $A_{21} \in K^{n_2,n_1}$, $A_{22} \in K^{n_2,n_2}$. Show the following assertions:

(a) If $A_{11} \in GL_{n_1}(K)$, then $\det(A) = \det(A_{11}) \det\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)$.
(b) If $A_{22} \in GL_{n_2}(K)$, then $\det(A) = \det(A_{22}) \det\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)$.
(c) If $A_{21} = 0$, then $\det(A) = \det(A_{11}) \det(A_{22})$.

Can you show this also when the matrices are defined over a commutative ring with unit?

7.11 Construct matrices $A_{11}, A_{12}, A_{21}, A_{22} \in \mathbb{R}^{n,n}$ for $n \geq 2$ with

$$\det\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \neq \det(A_{11})\det(A_{22}) - \det(A_{12})\det(A_{21}).$$

7.12 Let $A = [a_{ij}] \in GL_n(\mathbb{R})$ with $a_{ij} \in \mathbb{Z}$ for $i, j = 1, \ldots, n$. Show that the following assertions hold:

(a) $A^{-1} \in \mathbb{Q}^{n,n}$.
(b) $A^{-1} \in \mathbb{Z}^{n,n}$ if and only if $\det(A) \in \{-1, 1\}$.
(c) The linear system of equations $Ax = b$ has a unique solution $\widehat{x} \in \mathbb{Z}^{n,1}$ for every $b \in \mathbb{Z}^{n,1}$ if and only if $\det(A) \in \{-1, 1\}$.

7.13 Show that $G = \{A \in \mathbb{Z}^{n,n} \mid \det(A) \in \{-1, 1\}\}$ is a subgroup of $GL_n(\mathbb{Q})$.