

Chapter 1

Linear Algebra in Every Day Life

One has to familiarize the student with actual questions from applications, so that he learns to deal with real world problems.¹

Lothar Collatz (1910–1990)

1.1 The PageRank Algorithm

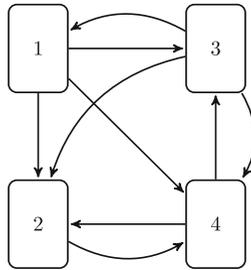
The *PageRank algorithm* is a method to assess the “importance” of documents with mutual *links*, such as web pages, on the basis of the link structure. It was developed by Sergei Brin and Larry Page, the founders of Google Inc., at Stanford University in the late 1990s. The basic idea of the algorithm is the following:

Instead of counting links, PageRank essentially interprets a link of page A to page B as a vote of page A for page B. PageRank then assesses the importance of a page by the number of received votes. PageRank also considers the importance of the page that casts the vote, since votes of some pages have a higher value, and thus also assign a higher value to the page they point to. Important pages will be rated higher and thus lead to a higher position in the search results.²

Let us describe (model) this idea mathematically. Our presentation uses ideas from the article [BryL06]. For a given set of web pages, every page k will be assigned an importance value $x_k \geq 0$. A page k is more important than a page j if $x_k > x_j$. If a page k has a link to a page j , we say that page j has a *backlink* from page k . In the above description these backlinks are the votes. As an example, consider the following link structure:

¹“Man muss den Lernenden mit konkreten Fragestellungen aus den Anwendungen vertraut machen, dass er lernt, konkrete Fragen zu behandeln.”

²Translation of a text found in 2010 on <http://www.google.de/corporate/tech.html>.



Here the page 1 has links to the pages 2, 3 and 4, and a backlink from page 3.

The easiest approach to define importance of web pages is to count its backlinks; the more votes are cast for a page, the more important the page is. In our example this gives the importance values

$$x_1 = 1, \quad x_2 = 3, \quad x_3 = 2, \quad x_4 = 3.$$

The pages 2 and 4 are thus the most important pages, and they are equally important.

However, the intuition and also the above description from Google suggests that backlinks from important pages are more important for the value of a page than those from less important pages. This idea can be modeled by defining x_k as the sum of all importance values of the backlinks of the page k . In our example this results in four equations that have to be satisfied simultaneously,

$$x_1 = x_3, \quad x_2 = x_1 + x_3 + x_4, \quad x_3 = x_1 + x_4, \quad x_4 = x_1 + x_2 + x_3.$$

A disadvantage of this approach is that it does not consider the number of links of the pages. Thus, it would be possible to (significantly) increase the importance of a page just by adding links to that page. In order to avoid this, the importance values of the backlinks in the PageRank algorithm are divided by the number of links of the corresponding page. This creates a kind of “internet democracy”: Every page can vote for other pages, where in total it can cast one vote. In our example this gives the equations

$$x_1 = \frac{x_3}{3}, \quad x_2 = \frac{x_1}{3} + \frac{x_3}{3} + \frac{x_4}{2}, \quad x_3 = \frac{x_1}{3} + \frac{x_4}{2}, \quad x_4 = \frac{x_1}{3} + x_2 + \frac{x_3}{3}. \quad (1.1)$$

These are four equations for the four unknowns, and all equations are *linear*,³ i.e., the unknowns occur only in first power. In Chap. 6 we will see how to write the equations in (1.1) in form of a *linear system of equations*. Analyzing and solving such systems is one of the most important tasks of Linear Algebra. The example of the PageRank algorithm shows that Linear Algebra presents a powerful modeling

³The term “linear” originates from the Latin word “linea”, which means “(straight) line”, and “linearis” means “consisting of (straight) lines”.

tool: We have turned the real world problem of assessing the importance of web pages into a problem of Linear Algebra. This problem will be examined further in Sect. 8.3.

For completeness, we mention that a solution for the four unknowns (computed with MATLAB and rounded to the second significant digit) is given by

$$x_1 = 0.14, \quad x_2 = 0.54, \quad x_3 = 0.41, \quad x_4 = 0.72.$$

Thus, page 4 is the most important one. It is possible to multiply the solution, i.e., the importance values x_k , by a positive constant. Such a multiplication or scaling is often advantageous for computational methods or for the visual display of the results. For example, the scaling could be used to give the most important page the value 1.00. A scaling is allowed, since it does not change the ranking of the pages, which is the essential information provided by the PageRank algorithm.

1.2 No Claim Discounting in Car Insurances

Insurance companies compute the premiums for their customers on the basis of the insured risk: the higher the risk, the higher the premium. It is therefore important to identify the factors that lead to higher risk. In the case of a car insurance these factors include the number of miles driven per year, the distance between home and work, the marital status, the engine power, or the age of the driver. Using such information, the company calculates the initial premium.

Usually the best indicator for future accidents, and hence future insurance claims, is the number of accidents of the individual customer in the past, i.e., the claims history. In order to incorporate this information into the premium rates, insurers establish a system of *risk classes*, which divide the customers into homogeneous risk groups with respect to their previous claims history. Customers with fewer accidents in the past get a discount on their premium. This approach is called a *no claims discounting scheme*.

For a mathematical model of this scheme we need a set of risk classes and a *transition rule* for moving between the classes. At the end of a policy year, the customer may move to a different class depending on the claims made during the year. The discount is given in percent of the premium in the initial class. As a simple example we consider four risk classes,

	C_1	C_2	C_3	C_4
% discount	0	10	20	40

and the following transition rules:

- No accident: Step up one class (or stay in C_4).

- One accident: Step back one class (or stay in C_1).
- More than one accident: Step back to class C_1 (or stay in C_1).

Next, the insurance company has to estimate the probability that a customer who is in the class C_i in this year will move to the class C_j . This probability is denoted by p_{ij} . Let us assume, for simplicity, that the probability of exactly one accident for every customer is 0.1, i.e., 10%, and the probability of two or more accidents for every customer is 0.05, i.e., 5%. (Of course, in practice the insurance companies determine these probabilities in dependence of the classes.)

For example, a customer in the class C_1 will stay in C_1 in case of at least one accident. This happens with the probability 0.15, so that $p_{11} = 0.15$. A customer in C_1 has no accident with the probability 0.85, so that $p_{12} = 0.85$. There is no chance to move from C_1 to C_3 or C_4 in the next year, so that $p_{13} = p_{14} = 0.00$. In this way we obtain 16 values p_{ij} , $i, j = 1, 2, 3, 4$, which we can arrange in a 4×4 matrix as follows:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} 0.15 & 0.85 & 0.00 & 0.00 \\ 0.15 & 0.00 & 0.85 & 0.00 \\ 0.05 & 0.10 & 0.00 & 0.85 \\ 0.05 & 0.00 & 0.10 & 0.85 \end{bmatrix}. \quad (1.2)$$

All entries of this matrix are nonnegative real numbers, and the sum of all entries in each row is equal to 1.00, i.e.,

$$p_{i1} + p_{i2} + p_{i3} + p_{i4} = 1.00 \quad \text{for each } i = 1, 2, 3, 4.$$

Such a matrix is called *row-stochastic*.

The analysis of matrix properties is a central topic of Linear Algebra that is developed throughout this book. As in the example with the PageRank algorithm, we have translated a practical problem into the language of Linear Algebra, and we can now study it using Linear Algebra techniques. This example of premium rates will be discussed further in Example 4.7.

1.3 Production Planning in a Plant

The production planning in a plant has to consider many different factors, in particular commodity prices, labor costs, and available capital, in order to determine a production plan. We consider a simple example:

A company produces the products P_1 and P_2 . If x_i units of the product P_i are produced, where $i = 1, 2$, then the pair (x_1, x_2) is called a *production plan*. Suppose that the raw materials and labor for the production of one unit of the product P_i cost a_{1i} and a_{2i} Euros, respectively. If b_1 Euros are available for the purchase of raw materials and b_2 Euros for the payment of labor costs, then a production plan must

satisfy the *constraint inequalities*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\leq b_1, \\ a_{21}x_1 + a_{22}x_2 &\leq b_2. \end{aligned}$$

If a production plan satisfies these constraints, it is called *feasible*. Let p_i be the profit from selling one unit of product P_i . Then the goal is to determine a production plan that maximizes the *profit function*

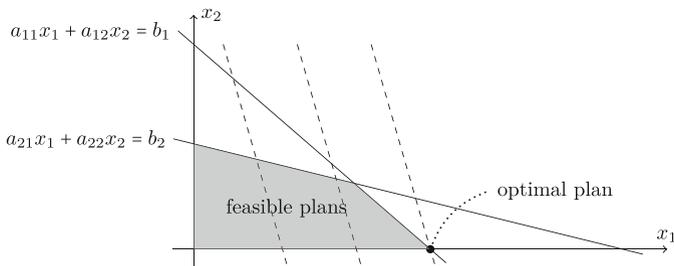
$$\Phi(x_1, x_2) = p_1x_1 + p_2x_2.$$

How can we find this maximum?

The two equations

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{and} \quad a_{21}x_1 + a_{22}x_2 = b_2$$

describe straight lines in the coordinate system that has the variables x_1 and x_2 on its axes. These two lines form boundary lines of the feasible production plans, which are “below” the lines; see the figure below. Note that we also must have $x_i \geq 0$, since we cannot produce negative units of a product. For planned profits $y_i, i = 1, 2, 3, \dots$, the equations $p_1x_1 + p_2x_2 = y_i$ describe parallel straight lines in the coordinate system; see the dashed lines in the figure. If x_1 and x_2 satisfy $p_1x_1 + p_2x_2 = y_i$, then $\Phi(x_1, x_2) = y_i$. The profit maximization problem can now be solved by moving the dashed lines until one of them reaches the corner with the maximal y :



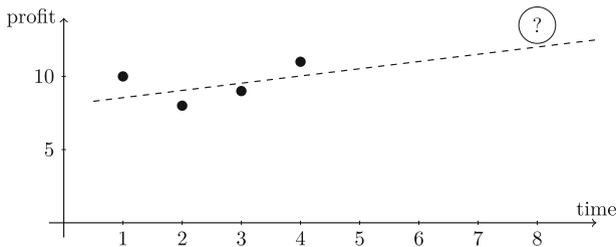
In case of more variables we cannot draw such a simple figure and obtain the solution “graphically”. But the general idea of finding a corner with the maximum profit is still the same. This is an example of a *linear optimization problem*. As before, we have formulated a real world problem in the language of Linear Algebra, and we can use mathematical methods for its solution.

1.4 Predicting Future Profits

The prediction of profits or losses of a company is a central planning instrument of economics. Analogous problems arise in many areas of political decision making,

for example in budget planning, tax estimates or the planning of new infrastructures. We consider a specific example:

In the four quarters of a year a company has profits of 10, 8, 9, 11 million Euros. The board now wants to predict the future profits development on the basis of these values. Evidence suggests, that the profits behave *linearly*. If this was true, then the profits would form a straight line $y(t) = \alpha t + \beta$ that connects the points (1, 10), (2, 8), (3, 9), (4, 11) in the coordinate system having “time” and “profit” as its axes. This, however, does neither hold in this example nor in practice. Therefore one tries to find a straight line that deviates “as little as possible” from the given points. One possible approach is to choose the parameters α and β in order to minimize the sum of the squared distances between the given points and the straight line. Once the parameters α and β have been determined, the resulting line $y(t)$ can be used for estimating or predicting the future profits, as illustrated in the following figure:

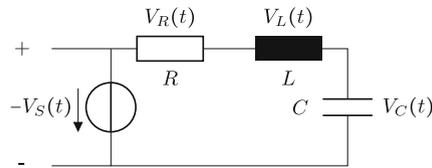


The determination of the parameters α and β that minimize a sum of squares is called a *least squares problem*. We will solve least squares problems using methods of Linear Algebra in Example 12.16. The approach itself is sometimes called a *parameter identification*. In Statistics, the modeling of given data (here the company profits) using a linear predictor function (here $y(t) = \alpha t + \beta$) is known as *linear regression*.

1.5 Circuit Simulation

The current development of electronic devices is very rapid. In short intervals, nowadays often less than a year, new models of laptops or mobile phones have to be issued to the market. To achieve this, continuously new generations of computer chips have to be developed. These typically become smaller and more powerful, and naturally should use as little energy as possible. An important factor in this development is to plan and simulate the chips *virtually*, i.e., in the computer and without producing a physical prototype. This model-based planning and optimization of products is a central method in many high technology areas, and it is based on modern mathematics.

Usually, the switching behavior of a chip is modeled by a mathematical system consisting of differential and algebraic equations that describe the relation between currents and voltages. Without going into details, consider the following circuit:



In this circuit description, $V_S(t)$ is the given input current at time t , and the characteristic values of the components are R for the resistor, L for the inductor, and C for the capacitor. The functions for the potential differences at the three components are denoted by $V_R(t)$, $V_L(t)$, and $V_C(t)$; $I(t)$ is the current.

Applying the Kirchhoff laws⁴ of electrical engineering leads to the following system of linear equations and differential equations that model the dynamic behavior of the circuit:

$$\begin{aligned} L \frac{d}{dt} I &= V_L, \\ C \frac{d}{dt} V_C &= I, \\ R I &= V_R, \\ V_L + V_C + V_R &= V_S. \end{aligned}$$

In this example it is easy to solve the last two equations for V_L and V_R , and hence to obtain a system of differential equations

$$\begin{aligned} \frac{d}{dt} I &= -\frac{R}{L} I - \frac{1}{L} V_C + \frac{1}{L} V_S, \\ \frac{d}{dt} V_C &= -\frac{1}{C} I, \end{aligned}$$

for the functions I and V_C . We will discuss and solve this system in Example 17.13.

This simple example demonstrates that for the simulation of a circuit a system of linear differential equations and algebraic equations has to be solved. Modern computer chips in industrial practice require solving such systems with millions of *differential-algebraic equations*. Linear Algebra is one of central tools for the theoretical analysis of such systems as well as the development of efficient solution methods.

⁴Gustav Robert Kirchhoff (1824–1887).