Chapter 11
Linear Forms and Bilinear Forms

In this chapter we study different classes of maps between one or two K-vector spaces and the one dimensional K-vector space defined by the field K itself. These maps play an important role in many areas of Mathematics, including Analysis, Functional Analysis and the solution of differential equations. They will also be essential for the further developments in this book: Using bilinear and sesquilinear forms, which are introduced in this chapter, we will define and study Euclidean and unitary vector spaces in Chap. 12. Linear forms and dual spaces will be used in the existence proof of the Jordan canonical form in Chap. 16.

11.1 Linear Forms and Dual Spaces

We start with the set of linear maps from a K-vector space to the vector space K.

Definition 11.1 If $\mathcal{V}$ is a K-vector space, then $f \in \mathcal{L}(\mathcal{V}, K)$ is called a *linear form* on $\mathcal{V}$. The K-vector space $\mathcal{V}^* := \mathcal{L}(\mathcal{V}, K)$ is called the *dual space* of $\mathcal{V}$.

A linear form is sometimes called a *linear functional* or a *one-form*, which stresses that it (linearly) maps into a one dimensional vector space.

Example 11.2 If $\mathcal{V}$ is the $\mathbb{R}$-vector space of the continuous and real valued functions on the real interval $[\alpha, \beta]$ and if $\gamma \in [\alpha, \beta]$, then the two maps

$$f_1 : \mathcal{V} \to \mathbb{R}, \quad g \mapsto g(\gamma),$$

$$f_2 : \mathcal{V} \to \mathbb{R}, \quad g \mapsto \int_\alpha^\beta g(x)dx,$$

are linear forms on $\mathcal{V}$.

If $\dim(\mathcal{V}) = n$, then $\dim(\mathcal{V}^*) = n$ by Theorem 10.16. Let $B_1 = \{v_1, \dots, v_n\}$ be a basis of $\mathcal{V}$ and let $B_2 = \{1\}$ be a basis of the K-vector space K. If $f \in \mathcal{V}^*$, then $f(v_i) = \alpha_i$ for some $\alpha_i \in K$, $i = 1, \dots, n$, and

$$[f]_{B_1, B_2} = [\alpha_1, \dots, \alpha_n] \in K^{1,n}.$$

For an element $v = \sum\limits_{i=1}^{n} \lambda_i v_i \in \mathcal{V}$ we have

$$f(v) = f\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i f(v_i) = \sum_{i=1}^{n} \lambda_i \alpha_i = \underbrace{[\alpha_1, \dots, \alpha_n]}_{\in K^{1,n}} \underbrace{\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}}_{\in K^{n,1}}$$

$$= [f]_{B_1, B_2}\, \Phi_{B_1}(v),$$

where we have identified the isomorphic vector spaces K and $K^{1,1}$ with each other.

For a given basis of a finite dimensional vector space $\mathcal{V}$ we will now construct a special, uniquely determined basis of the dual space $\mathcal{V}^*$.

Theorem 11.3 *If $\mathcal{V}$ is K-vector space with the basis $B = \{v_1, \dots, v_n\}$, then there exists a unique basis $B^* = \{v_1^*, \dots, v_n^*\}$ of $\mathcal{V}^*$ such that*

$$v_i^*(v_j) = \delta_{ij}, \quad i, j = 1, \dots, n,$$

which is called the dual basis *of B.*

Proof By Theorem 10.4, a unique linear map from $\mathcal{V}$ to K can be constructed by prescribing its images at the given basis B. Thus, for each $i = 1, \dots, n$, there exists a unique map $v_i^* \in \mathcal{L}(\mathcal{V}, K)$ with $v_i^*(v_j) = \delta_{ij}$, $j = 1, \dots, n$.

It remains to show that $B^* := \{v_1^*, \dots, v_n^*\}$ is a basis of $\mathcal{V}^*$. If $\lambda_1, \dots, \lambda_n \in K$ are such that

$$\sum_{i=1}^{n} \lambda_i v_i^* = 0_{\mathcal{V}^*} \in \mathcal{V}^*,$$

then

$$0 = 0_{\mathcal{V}^*}(v_j) = \sum_{i=1}^{n} \lambda_i v_i^*(v_j) = \lambda_j, \quad j = 1, \dots, n.$$

Thus, $v_1^*, \dots, v_n^*$ are linearly independent, and $\dim(\mathcal{V}^*) = n$ implies that B^* is a basis of $\mathcal{V}^*$ (cp. Exercise 9.6). $\qquad\square$

Example 11.4 Consider $\mathcal{V} = K^{n,1}$ with the canonical basis $B = \{e_1, \dots, e_n\}$. If $\{e_1^*, \dots, e_n^*\}$ is the dual basis of B, then $e_i^*(e_j) = \delta_{ij}$, which shows that $[e_i^*]_{B,\{1\}} = e_i^T \in K^{1,n}$, $i = 1, \dots, n$.

Definition 11.5 Let $\mathcal{V}$ and $\mathcal{W}$ be K-vector spaces with their respective dual spaces $\mathcal{V}^*$ and $\mathcal{W}^*$, and let $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. Then

$$f^* : \mathcal{W}^* \to \mathcal{V}^*, \quad h \mapsto f^*(h) := h \circ f,$$

is called the *dual map* of f.

We next derive some properties of the dual map.

Lemma 11.6 *If $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{X}$ are K-vector spaces, then the following assertions hold:*

(1) If $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then the dual map f^* is linear, hence $f^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{V}^*)$.
(2) If $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ and $g \in \mathcal{L}(\mathcal{W}, \mathcal{X})$, then $(g \circ f)^* \in \mathcal{L}(\mathcal{X}^*, \mathcal{V}^*)$ and $(g \circ f)^* = f^* \circ g^*$.
(3) If $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is bijective, then $f^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{V}^*)$ is bijective and $(f^*)^{-1} = (f^{-1})^*$.

Proof (1) If $h_1, h_2 \in \mathcal{W}^*$, $\lambda_1, \lambda_2 \in K$, then

$$f^*(\lambda_1 h_1 + \lambda_2 h_2) = (\lambda_1 h_1 + \lambda_2 h_2) \circ f = (\lambda_1 h_1) \circ f + (\lambda_2 h_2) \circ f$$
$$= \lambda_1 (h_1 \circ f) + \lambda_2 (h_2 \circ f) = \lambda_1 f^*(h_1) + \lambda_2 f^*(h_2).$$

$\square$

(2) and (3) are exercises.

As the following theorem shows, the concepts of the dual map and the transposed matrix are closely related.

Theorem 11.7 *Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional K-vector spaces with bases B_1 and B_2, respectively. Let B_1^* and B_2^* be the corresponding dual bases. If $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then*
$$[f^*]_{B_2^*, B_1^*} = ([f]_{B_1, B_2})^T.$$

Proof Let $B_1 = \{v_1, \ldots, v_m\}$, $B_2 = \{w_1, \ldots, w_n\}$, and let $B_1^* = \{v_1^*, \ldots, v_m^*\}$, $B_2^* = \{w_1^*, \ldots, w_n^*\}$. Let $[f]_{B_1, B_2} = [a_{ij}] \in K^{n,m}$, i.e.,

$$f(v_j) = \sum_{i=1}^{n} a_{ij} w_i, \quad j = 1, \ldots, m,$$

and $[f^*]_{B_2^*, B_1^*} = [b_{ij}] \in K^{m,n}$, i.e.,

$$f^*(w_j^*) = \sum_{i=1}^{m} b_{ij} v_i^*, \quad j = 1, \ldots, n.$$

For every pair (k, ℓ) with $1 \leq k \leq n$ and $1 \leq \ell \leq m$ we then have

$$
\begin{aligned}
a_{k\ell} &= \sum_{i=1}^{n} a_{i\ell} w_k^*(w_i) = w_k^* \left(\sum_{i=1}^{n} a_{i\ell} w_i \right) = w_k^*(f(v_\ell)) = f^* \left(w_k^* \right)(v_\ell) \\
&= \left(\sum_{i=1}^{m} b_{ik} v_i^* \right)(v_\ell) = \sum_{i=1}^{m} b_{ik} v_i^*(v_\ell) \\
&= b_{\ell k},
\end{aligned}
$$

where we have used the definition of the dual map as well as $w_k^*(w_i) = \delta_{ki}$ and $v_i^*(v_\ell) = \delta_{i\ell}$. $\square$

Because of the close relationship between the transposed matrix and the dual map, some authors call the dual map f^* the *transpose* of the linear map f.

Applied to matrices, Lemma 11.6 and Theorem 11.7 yield the following rules known from Chap. 4:

$$
\begin{aligned}
(AB)^T &= B^T A^T \quad \text{for } A \in K^{n,m} \text{ and } B \in K^{m,\ell}, \text{ and} \\
(A^{-1})^T &= (A^T)^{-1} \quad \text{for } A \in GL_n(K).
\end{aligned}
$$

Example 11.8 For the two bases of $\mathbb{R}^{2,1}$,

$$
B_1 = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}, \quad B_2 = \left\{ w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},
$$

the elements of the corresponding dual bases are given by

$$
v_1^* : \mathbb{R}^{2,1} \to \mathbb{R}, \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mapsto \alpha_1 + 0, \qquad v_2^* : \mathbb{R}^{2,1} \to \mathbb{R}, \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mapsto 0 + \frac{1}{2} \alpha_2,
$$

$$
w_1^* : \mathbb{R}^{2,1} \to \mathbb{R}, \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mapsto \alpha_1 - \alpha_2, \quad w_2^* : \mathbb{R}^{2,1} \to \mathbb{R}, \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mapsto 0 + \alpha_2.
$$

The matrix representations of these maps are

$$
\begin{aligned}
\left[v_1^* \right]_{B_1, \{1\}} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & \left[v_2^* \right]_{B_1, \{1\}} &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \\
\left[w_1^* \right]_{B_2, \{1\}} &= \begin{bmatrix} 1 & 0 \end{bmatrix}, & \left[w_2^* \right]_{B_2, \{1\}} &= \begin{bmatrix} 0 & 1 \end{bmatrix}.
\end{aligned}
$$

For the linear map

$$
f : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}, \quad \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \mapsto \begin{bmatrix} \alpha_1 + \alpha_2 \\ 3\alpha_2 \end{bmatrix},
$$

we have

$$[f]_{B_1,B_2} = \begin{bmatrix} 1 & -4 \\ 0 & 6 \end{bmatrix}, \quad [f^*]_{B_2^*,B_1^*} = \begin{bmatrix} 1 & 0 \\ -4 & 6 \end{bmatrix}.$$

11.2 Bilinear Forms

We now consider special maps from a pair of K-vector spaces to the K-vector space K.

Definition 11.9 Let $\mathcal{V}$ and $\mathcal{W}$ be K-vector spaces. A map $\beta : \mathcal{V} \times \mathcal{W} \to K$ is called a *bilinear form on $\mathcal{V} \times \mathcal{W}$*, when

(1) $\beta(v_1 + v_2, w) = \beta(v_1, w) + \beta(v_2, w)$,
(2) $\beta(v, w_1 + w_2) = \beta(v, w_1) + \beta(v, w_2)$,
(3) $\beta(\lambda v, w) = \beta(v, \lambda w) = \lambda\beta(v, w)$,

hold for all $v, v_1, v_2 \in \mathcal{V}$, $w, w_1, w_2 \in \mathcal{W}$, and $\lambda \in K$.

A bilinear form β is called *non-degenerate in the first variable*, if $\beta(v, w) = 0$ for all $w \in \mathcal{W}$ implies that $v = 0$. Analogously, it is called *non-degenerate in the second variable*, if $\beta(v, w) = 0$ for all $v \in \mathcal{V}$ implies that $w = 0$. If β is non-degenerate in both variables, then β is called *non-degenerate* and the spaces $\mathcal{V}, \mathcal{W}$ are called a *dual pair* with respect to β.

If $\mathcal{V} = \mathcal{W}$, then β is called a *bilinear form on $\mathcal{V}$*. If additionally $\beta(v, w) = \beta(w, v)$ holds for all $v, w \in \mathcal{V}$, then β is called *symmetric*. Otherwise, β is called *nonsymmetric*.

Example 11.10

(1) If $A \in K^{n,m}$, then

$$\beta : K^{m,1} \times K^{n,1} \to K, \quad (v, w) \mapsto w^T A v,$$

is a bilinear form on $K^{m,1} \times K^{n,1}$ that is non-degenerate if and only if $n = m$ and $A \in GL_n(K)$, (cp. Exercise 11.10).
(2) The bilinear form

$$\beta : \mathbb{R}^{2,1} \times \mathbb{R}^{2,1} \to \mathbb{R}, \quad (x, y) \mapsto y^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} x,$$

is degenerate in both variables: For $\widehat{x} = [1, -1]^T$, we have $\beta(\widehat{x}, y) = 0$ for all $y \in \mathbb{R}^{2,1}$; for $\widehat{y} = [1, -1]^T$ we have $\beta(x, \widehat{y}) = 0$ for all $x \in \mathbb{R}^{2,1}$. The set of all $x = [x_1, x_2]^T \in \mathbb{R}^{2,1}$ with $\beta(x, x) = 1$ is equal to the solution set of the quadratic equation in two variables $x_1^2 + 2x_1x_2 + x_2^2 = 1$, or $(x_1 + x_2)^2 = 1$, for $x_1, x_2 \in \mathbb{R}$. Geometrically, this set is given by the two straight lines $x_1 + x_2 = 1$ and $x_1 + x_2 = -1$ in the cartesian coordinate system of $\mathbb{R}^2$.

(3) If $\mathcal{V}$ is a K-vector space, then

$$\beta : \mathcal{V} \times \mathcal{V}^* \to K, \quad (v, f) \mapsto f(v),$$

is a bilinear form on $\mathcal{V} \times \mathcal{V}^*$, since

$$\beta(v_1 + v_2, f) = f(v_1 + v_2) = f(v_1) + f(v_2) = \beta(v_1, f) + \beta(v_2, f),$$
$$\beta(v, f_1 + f_2) = (f_1 + f_2)(v) = f_1(v) + f_2(v) = \beta(v, f_1) + \beta(v, f_2),$$
$$\beta(\lambda v, f) = f(\lambda v) = \lambda f(v) = \lambda \beta(v, f) = (\lambda f)(v) = \beta(v, \lambda f),$$

hold for all $v, v_1, v_2 \in \mathcal{V}$, $f, f_1, f_2 \in \mathcal{V}^*$ and $\lambda \in K$. This bilinear form is non-degenerate and thus $\mathcal{V}, \mathcal{V}^*$ are a dual pair with respect to β (cp. Exercise 11.11 for the case $\dim(\mathcal{V}) \in \mathbb{N}$).

Definition 11.11 Let $\mathcal{V}$ and $\mathcal{W}$ be K-vector spaces with bases $B_1 = \{v_1, \dots, v_m\}$ and $B_2 = \{w_1, \dots, w_n\}$, respectively. If β is a bilinear form on $\mathcal{V} \times \mathcal{W}$, then

$$[\beta]_{B_1 \times B_2} = [b_{ij}] \in K^{n,m}, \quad b_{ij} := \beta(v_j, w_i),$$

is called the *matrix representation* of β with respect to the bases B_1 and B_2.

If $v = \sum_{j=1}^{m} \lambda_j v_j \in \mathcal{V}$ and $w = \sum_{i=1}^{n} \mu_i w_i \in \mathcal{W}$, then

$$\beta(v, w) = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \mu_i \beta(v_j, w_i) = \sum_{i=1}^{n} \mu_i \sum_{j=1}^{m} b_{ij} \lambda_j = \left(\Phi_{B_2}(w)\right)^T [\beta]_{B_1 \times B_2} \, \Phi_{B_1}(v),$$

where we have used the coordinate map from Lemma 10.17.

Example 11.12 If $B_1 = \left\{e_1^{(m)}, \dots, e_m^{(m)}\right\}$ and $B_2 = \left\{e_1^{(n)}, \dots, e_n^{(n)}\right\}$ are the canonical bases of $K^{m,1}$ and $K^{n,1}$, respectively, and if β is the bilinear form from (1) in Example 11.10 with $A = [a_{ij}] \in K^{n,m}$, then $[\beta]_{B_1 \times B_2} = [b_{ij}]$, where

$$b_{ij} = \beta\left(e_j^{(m)}, e_i^{(n)}\right) = \left(e_i^{(n)}\right)^T A e_j^{(m)} = a_{ij},$$

and hence $[\beta]_{B_1 \times B_2} = A$.

The following result shows that symmetric bilinear forms have symmetric matrix representations.

Lemma 11.13 *For a bilinear form β on a finite dimensional vector space $\mathcal{V}$ the following statements are equivalent:*

(1) *β is symmetric.*
(2) *For every basis B of $\mathcal{V}$ the matrix $[\beta]_{B \times B}$ is symmetric.*
(3) *There exists a basis B of $\mathcal{V}$ such that $[\beta]_{B \times B}$ is symmetric.*

Proof Exercise.   □

We will now analyze the effect of a basis change on the matrix representation of a bilinear form.

Theorem 11.14 *Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional K-vector spaces with bases B_1, $\widetilde{B}_1$ of $\mathcal{V}$ and B_2, $\widetilde{B}_2$ of $\mathcal{W}$. If β is a bilinear form on $\mathcal{V} \times \mathcal{W}$, then*

$$[\beta]_{B_1 \times B_2} = \big([\mathrm{Id}_\mathcal{W}]_{B_2, \widetilde{B}_2}\big)^T [\beta]_{\widetilde{B}_1 \times \widetilde{B}_2} [\mathrm{Id}_\mathcal{V}]_{B_1, \widetilde{B}_1}.$$

Proof Let $B_1 = \{v_1, \ldots, v_m\}$, $\widetilde{B}_1 = \{\widetilde{v}_1, \ldots, \widetilde{v}_m\}$, $B_2 = \{w_1, \ldots, w_n\}$, $\widetilde{B}_2 = \{\widetilde{w}_1, \ldots, \widetilde{w}_n\}$, and

$$(v_1, \ldots, v_m) = (\widetilde{v}_1, \ldots, \widetilde{v}_m)P, \quad \text{where} \quad P = [p_{ij}] = [\mathrm{Id}_\mathcal{V}]_{B_1, \widetilde{B}_1},$$
$$(w_1, \ldots, w_n) = (\widetilde{w}_1, \ldots, \widetilde{w}_n)Q, \quad \text{where} \quad Q = [q_{ij}] = [\mathrm{Id}_\mathcal{W}]_{B_2, \widetilde{B}_2}.$$

With $[\beta]_{\widetilde{B}_1 \times \widetilde{B}_2} = [\widetilde{b}_{ij}]$, where $\widetilde{b}_{ij} = \beta(\widetilde{v}_j, \widetilde{w}_i)$, we then have

$$\beta(v_j, w_i) = \beta\Big(\sum_{k=1}^{m} p_{kj}\widetilde{v}_k, \sum_{\ell=1}^{n} q_{\ell i}\widetilde{w}_\ell\Big) = \sum_{\ell=1}^{n} q_{\ell i} \sum_{k=1}^{m} \beta(\widetilde{v}_k, \widetilde{w}_\ell)p_{kj}$$

$$= \sum_{\ell=1}^{n} q_{\ell i} \sum_{k=1}^{m} \widetilde{b}_{\ell k} p_{kj}$$

$$= \begin{bmatrix} q_{1i} \\ \vdots \\ q_{ni} \end{bmatrix}^T [\beta]_{\widetilde{B}_1 \times \widetilde{B}_2} \begin{bmatrix} p_{1j} \\ \vdots \\ p_{mj} \end{bmatrix},$$

which implies that $[\beta]_{B_1 \times B_2} = Q^T [\beta]_{\widetilde{B}_1 \times \widetilde{B}_2} P$, and hence the assertion follows.   □

If $\mathcal{V} = \mathcal{W}$ and B_1, B_2 are two bases of $\mathcal{V}$, then we obtain the following special case of Theorem 11.14:

$$[\beta]_{B_1 \times B_1} = \big([\mathrm{Id}_\mathcal{V}]_{B_1, B_2}\big)^T [\beta]_{B_2 \times B_2} [\mathrm{Id}_\mathcal{V}]_{B_1, B_2}.$$

The two matrix representations $[\beta]_{B_1 \times B_1}$ and $[\beta]_{B_2 \times B_2}$ of β in this case are *congruent*, which we formally define as follows.

Definition 11.15 If for two matrices $A, B \in K^{n,n}$ there exists a matrix $Z \in GL_n(K)$ with $B = Z^T A Z$, then A and B are called *congruent*.

Lemma 11.16 *Congruence is an equivalence relation on the set $K^{n,n}$.*

Proof Exercise.   □

11.3   Sesquilinear Forms

For *complex* vector spaces we introduce another special class of forms.

Definition 11.17   Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{C}$-vector spaces. A map $s : \mathcal{V} \times \mathcal{W} \to \mathbb{C}$ is called a *sesquilinear form* on $\mathcal{V} \times \mathcal{W}$, when

(1) $s(v_1 + v_2, w) = s(v_1, w) + s(v_2, w)$,
(2) $s(\lambda v, w) = \lambda s(v, w)$,
(3) $s(v, w_1 + w_2) = s(v, w_1) + s(v, w_2)$,
(4) $s(v, \lambda w) = \overline{\lambda} s(v, w)$,

hold for all $v, v_1, v_2 \in \mathcal{V}$, $w, w_1, w_2 \in \mathcal{W}$ and $\lambda \in \mathbb{C}$.
    If $\mathcal{V} = \mathcal{W}$, then s is called a *sesquilinear form on* $\mathcal{V}$. If additionally $s(v, w) = \overline{s(w, v)}$ holds for all $v, w \in \mathcal{V}$, then s is called *Hermitian*.[1]

    The prefix *sesqui* is Latin and means "one and a half". Note that a sesquilinear form is linear in the first variable and *semilinear* ("half linear") in the second variable.
    The following result characterizes Hermitian sesquilinear forms.

Lemma 11.18   *A sesquilinear form on the $\mathbb{C}$-vector space $\mathcal{V}$ is Hermitian if and only if $s(v, v) \in \mathbb{R}$ for all $v \in \mathcal{V}$.*

Proof If s is Hermitian then, in particular, $s(v, v) = \overline{s(v, v)}$ for all $v \in \mathcal{V}$, and thus $s(v, v) \in \mathbb{R}$.
    If, on the other hand, $v, w \in \mathcal{V}$, then by definition

$$s(v + w, v + w) = s(v, v) + s(v, w) + s(w, v) + s(w, w), \qquad (11.1)$$
$$s(v + \mathbf{i}w, v + \mathbf{i}w) = s(v, v) + \mathbf{i}s(w, v) - \mathbf{i}s(v, w) + s(w, w). \qquad (11.2)$$

The first equation implies that $s(v, w) + s(w, v) \in \mathbb{R}$, since $s(v + w, v + w), s(v, v), s(w, w) \in \mathbb{R}$ by assumption. The second equation implies analogously that $\mathbf{i}s(w, v) - \mathbf{i}s(v, w) \in \mathbb{R}$. Therefore,

$$s(v, w) + s(w, v) = \overline{s(v, w)} + \overline{s(w, v)},$$
$$-\mathbf{i}s(v, w) + \mathbf{i}s(w, v) = \overline{\mathbf{i}s(v, w)} - \overline{\mathbf{i}s(w, v)}.$$

Multiplying the second equation with $\mathbf{i}$ and adding the resulting equation to the first we obtain $s(v, w) = \overline{s(w, v)}$ $\square$

Corollary 11.19   *For a sesquilinear form s on the $\mathbb{C}$-vector space $\mathcal{V}$ we have*

$$2 s(v, w) = s(v + w, v + w) + \mathbf{i}s(v + \mathbf{i}w, v + \mathbf{i}w) - (\mathbf{i} + 1)(s(v, v) + s(w, w)).$$

for all $v, w \in \mathcal{V}$.

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[1]Charles Hermite (1822–1901).

Proof The result follows from multiplication of (11.2) with **i** and adding the result to (11.1). □

Corollary 11.19 shows that a sesquilinear form on a $\mathbb{C}$-vector space $\mathcal{V}$ is uniquely determined by the values of $s(v, v)$ for all $v \in \mathcal{V}$.

Definition 11.20 The *Hermitian transpose* of $A = [a_{ij}] \in \mathbb{C}^{n,m}$ is the matrix

$$A^H := [\overline{a}_{ij}]^T \in \mathbb{C}^{m,n}.$$

If $A = A^H$, then A is called *Hermitian*.

If a matrix A has real entries, then obviously $A^H = A^T$. Thus, a real symmetric matrix is also Hermitian. If $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is Hermitian, then in particular $a_{ii} = \overline{a}_{ii}$ for $i = 1, \ldots, n$, i.e., Hermitian matrices have real diagonal entries.

The Hermitian transposition satisfies similar rules as the (usual) transposition (cp. Lemma 4.6).

Lemma 11.21 *For $A, \widehat{A} \in \mathbb{C}^{n,m}$, $B \in \mathbb{C}^{m,\ell}$ and $\lambda \in \mathbb{C}$ the following assertions hold:*

(1) $(A^H)^H = A$.
(2) $(A + \widehat{A})^H = A^H + \widehat{A}^H$.
(3) $(\lambda A)^H = \overline{\lambda} A^H$.
(4) $(AB)^H = B^H A^H$.

Proof Exercise. □

Example 11.22 For $A \in \mathbb{C}^{n,m}$ the map

$$s : \mathbb{C}^{m,1} \times \mathbb{C}^{n,1} \to \mathbb{C}, \quad (v, w) \mapsto w^H A v,$$

is a sesquilinear form.

The matrix representation of a sesquilinear form is defined analogously to the matrix representation of bilinear forms (cp. Definition 11.11).

Definition 11.23 Let $\mathcal{V}$ and $\mathcal{W}$ be $\mathbb{C}$-vector spaces with bases $B_1 = \{v_1, \ldots, v_m\}$ and $B_2 = \{w_1, \ldots, w_n\}$, respectively. If s is a sesquilinear form on $\mathcal{V} \times \mathcal{W}$, then

$$[s]_{B_1 \times B_2} = [b_{ij}] \in \mathbb{C}^{n,m}, \quad b_{ij} := s(v_j, w_i),$$

is called the *matrix representation* of s with respect to the bases B_1 and B_2.

Example 11.24 If $B_1 = \left\{e_1^{(m)}, \ldots, e_m^{(m)}\right\}$ and $B_2 = \left\{e_1^{(n)}, \ldots, e_n^{(n)}\right\}$ are the canonical bases of $\mathbb{C}^{m,1}$ and $\mathbb{C}^{n,1}$, respectively, and s is the sesquilinear form of Example 11.22 with $A = [a_{ij}] \in \mathbb{C}^{n,m}$, then $[s]_{B_1 \times B_2} = [b_{ij}]$ with

$$b_{ij} = s\left(e_j^{(m)}, e_i^{(n)}\right) = \left(e_i^{(n)}\right)^H A e_j^{(m)} = \left(e_i^{(n)}\right)^T A e_j^{(m)} = a_{ij}$$

and, hence, $[s]_{B_1 \times B_2} = A$.

Exercises

(In the following exercises K is an arbitrary field.)

11.1. Let $\mathcal{V}$ be a finite dimensional K-vector space and $v \in \mathcal{V}$. Show that $f(v) = 0$ for all $f \in \mathcal{V}^*$ if and only if $v = 0$.

11.2. Consider the basis $B = \{10, t-1, t^2 - t\}$ of the 3-dimensional vector space $\mathbb{R}[t]_{\leq 2}$. Compute the dual basis B^* to B.

11.3. Let $\mathcal{V}$ be an n-dimensional K-vector space and let $\{v_1^*, \ldots, v_n^*\}$ be a basis of $\mathcal{V}^*$. Prove or disprove: There exists a unique basis $\{v_1, \ldots, v_n\}$ of $\mathcal{V}$ with $v_i^*(v_j) = \delta_{ij}$.

11.4. Let $\mathcal{V}$ be a finite dimensional K-vector space and let $f, g \in \mathcal{V}^*$ with $f \neq 0$. Show that $g = \lambda f$ for a $\lambda \in K \setminus \{0\}$ holds if and only if $\ker(f) = \ker(g)$. Is it possible to omit the assumption $f \neq 0$?

11.5. Let $\mathcal{V}$ be a K-vector space and let $\mathcal{U}$ be a subspace of $\mathcal{V}$. The set

$$\mathcal{U}^0 := \{f \in \mathcal{V}^* \mid f(u) = 0 \text{ for all } u \in \mathcal{U}\}$$

is called the *annihilator* of $\mathcal{U}$. Show the following assertions:

(a) $\mathcal{U}^0$ is a subspace of $\mathcal{V}^*$.
(b) For subspaces $\mathcal{U}_1, \mathcal{U}_2$ of $\mathcal{V}$ we have

$$(\mathcal{U}_1 + \mathcal{U}_2)^0 = \mathcal{U}_1^0 \cap \mathcal{U}_2^0, \quad (\mathcal{U}_1 \cap \mathcal{U}_2)^0 = \mathcal{U}_1^0 + \mathcal{U}_2^0,$$

and if $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then $\mathcal{U}_2^0 \subseteq \mathcal{U}_1^0$.
(c) If $\mathcal{W}$ is a K-vector space and $f \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, then $\ker(f^*) = (\operatorname{im}(f))^0$.

11.6. Prove Lemma 11.6 (2) and (3).

11.7. Let $\mathcal{V}$ and $\mathcal{W}$ be K-vector spaces. Show that the set of all bilinear forms on $\mathcal{V} \times \mathcal{W}$ with the operations

$$+ \; : \; (\beta_1 + \beta_2)(v, w) := \beta_1(v, w) + \beta_2(v, w),$$
$$\cdot \; : \; (\lambda \cdot \beta)(v, w) := \lambda \cdot \beta(v, w),$$

is a K-vector space.

11.8. Let $\mathcal{V}$ and $\mathcal{W}$ be K-vector spaces with bases $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ and corresponding dual bases $\{v_1^*, \ldots, v_m^*\}$ and $\{w_1^*, \ldots, w_n^*\}$, respectively. For $i = 1, \ldots, m$ and $j = 1, \ldots, n$ let

$$\beta_{ij} : \mathcal{V} \times \mathcal{W} \to K, \quad (v, w) \mapsto v_i^*(v) w_j^*(w).$$

(a) Show that β_{ij} is a bilinear form on $\mathcal{V} \times \mathcal{W}$.

(b) Show that the set $\{\beta_{ij} \mid i = 1, \ldots, m, \ j = 1, \ldots, n\}$ is a basis of the K-vector space of bilinear forms on $\mathcal{V} \times \mathcal{W}$ (cp. Exercise 11.7) and determine the dimension of this space.

11.9. Let $\mathcal{V}$ be the $\mathbb{R}$-vector space of the continuous and real valued functions on the real interval $[\alpha, \beta]$. Show that

$$\beta : \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \quad (f, g) \mapsto \int_{\alpha}^{\beta} f(x)g(x)dx,$$

is a symmetric bilinear form on $\mathcal{V}$. Is β degenerate?

11.10. Show that the map β from (1) in Example 11.10 is a bilinear form, and show that it is non-degenerate if and only if $n = m$ and $A \in GL_n(K)$.

11.11. Let $\mathcal{V}$ be a finite dimensional K-vector space. Show that $\mathcal{V}, \mathcal{V}^*$ is a dual pair with respect to the bilinear form β from (3) in Example 11.10, i.e., that the bilinear form β is non-degenerate.

11.12. Let $\mathcal{V}$ be a finite dimensional K-vector space and let $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{W} \subseteq \mathcal{V}^*$ be subspaces with $\dim(\mathcal{U}) = \dim(\mathcal{W}) \geq 1$. Prove or disprove: The spaces $\mathcal{U}, \mathcal{W}$ form a dual pair with respect to the bilinear form $\beta : \mathcal{U} \times \mathcal{W} \to K$, $(v, h) \mapsto h(v)$.

11.13. Let $\mathcal{V}$ and $\mathcal{W}$ be finite dimensional K-vector spaces with the bases B_1 and B_2, respectively, and let β be a bilinear form on $\mathcal{V} \times \mathcal{W}$.

(a) Show that the following statements are equivalent:
(1) $[\beta]_{B_1 \times B_2}$ is not invertible.
(2) β is degenerate in the second variable.
(3) β is degenerate in the first variable.
(b) Conclude from (a): β is non-degenerate if and only if $[\beta]_{B_1 \times B_2}$ is invertible.

11.14. Prove Lemma 11.16.

11.15. Prove Lemma 11.13.

11.16. For a bilinear form β on a K-vector space $\mathcal{V}$, the map $q_\beta : \mathcal{V} \to K$, $v \mapsto \beta(v, v)$, is called the *quadratic form* induced by β. Show the following assertion:
If $1+1 \neq 0$ in K and β is symmetric, then $\beta(v, w) = \frac{1}{2}(q_\beta(v+w) - q_\beta(v) - q_\beta(w))$ holds for all $v, w \in \mathcal{V}$.

11.17. Show that a sesquilinear form s on a $\mathbb{C}$-vector space $\mathcal{V}$ satisfies the *polarization identity*

$$s(v, w) = \frac{1}{4}\big(s(v+w, v+w) - s(v-w, v-w) + \mathbf{i}s(v+\mathbf{i}w, v+\mathbf{i}w) - \mathbf{i}s(v-\mathbf{i}w, v-\mathbf{i}w)\big)$$

for all $v, w \in \mathcal{V}$.

11.18. Consider the following maps from $\mathbb{C}^{3,1} \times \mathbb{C}^{3,1}$ to $\mathbb{C}$:

(a) $\beta_1(x, y) = 3x_1\overline{x}_1 + 3y_1\overline{y}_1 + x_2\overline{y}_3 - x_3\overline{y}_2,$
(b) $\beta_2(x, y) = x_1\overline{y}_2 + x_2\overline{y}_3 + x_3\overline{y}_1,$
(c) $\beta_3(x, y) = x_1 y_2 + x_2 y_3 + x_3 y_1,$

(d) $\beta_4(x, y) = 3x_1\bar{y}_1 + x_1\bar{y}_2 + x_2\bar{y}_1 + 2\mathbf{i}x_2\bar{y}_3 - 2\mathbf{i}x_3\bar{y}_2 + x_3\bar{y}_3$.

Which of these are bilinear forms or sesquilinear forms on $\mathbb{C}^{3,1}$? Test whether the bilinear form is symmetric or the sesquilinear form is Hermitian, and derive the corresponding matrix representations with respect to the canonical basis $B_1 = \{e_1, e_2, e_3\}$ and the basis $B_2 = \{e_1, e_1 + \mathbf{i}e_2, e_2 + \mathbf{i}e_3\}$.

11.19. Prove Lemma 11.21.

11.20. Let $A \in \mathbb{C}^{n,n}$ be Hermitian. Show that

$$ s \ : \ \mathbb{C}^{n,1} \times \mathbb{C}^{n,1}, \quad (v, w) \mapsto w^H A v, $$

is a Hermitian sesquilinear form on $\mathbb{C}^{n,1}$.

11.21. Let $\mathcal{V}$ be a finite dimensional $\mathbb{C}$-vector space with the basis B, and let s be a sesquilinear form on $\mathcal{V}$. Show that s is Hermitian if and only if $[s]_{B \times B}$ is Hermitian.

11.22. Show the following assertions for $A, B \in \mathbb{C}^{n,n}$:

(a) If $A^H = -A$, then the eigenvalues of A are purely imaginary.
(b) If $A^H = -A$, then $\text{trace}(A^2) \leq 0$ and $(\text{trace}(A))^2 \leq 0$.
(c) If $A^H = A$ and $B^H = B$, then $\text{trace}((AB)^2) \leq \text{trace}(A^2B^2)$.