# Chapter 9
# Vector Spaces

In the previous chapters we have focussed on matrices and their properties. We have defined algebraic operations with matrices and derived important concepts associated with them, including their rank, determinant, characteristic polynomial, and eigenvalues. In this chapter we place these concepts in a more abstract framework by introducing the idea of a vector space. Matrices form one of the most important examples of vector spaces, and properties of certain (namely, finite dimensional) vector spaces can be studied in a transparent way using matrices. In the next chapter we will study (linear) maps between vector spaces, and there the connection with matrices will play a central role as well.

## 9.1 Basic Definitions and Properties of Vector Spaces

We begin with the definition of a vector space over a field $K$.

**Definition 9.1** Let $K$ be a field. A *vector space over $K$*, or shortly $K$*-vector space*, is a set $\mathcal{V}$ with two operations,

$$+ \; : \; \mathcal{V} \times \mathcal{V} \to \mathcal{V}, \qquad (v, w) \mapsto v + w, \qquad \text{(addition)}$$
$$\cdot \; : \; K \times \mathcal{V} \to \mathcal{V}, \qquad (\lambda, v) \mapsto \lambda \cdot v, \qquad \text{(scalar multiplication)}$$

that satisfy the following:

(1) $(\mathcal{V}, +)$ is a commutative group.
(2) For all $v, w \in \mathcal{V}$ and $\lambda, \mu \in K$ the following assertions hold:

    (a) $\lambda \cdot (\mu \cdot v) = (\lambda \mu) \cdot v$.
    (b) $1 \cdot v = v$.
    (c) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$.
    (d) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$.

An element $v \in \mathcal{V}$ is called a *vector*,[1] an element $\lambda \in K$ is called a *scalar*.

Again, we usually omit the sign of the scalar multiplication, i.e., we usually write $\lambda v$ instead of $\lambda \cdot v$. If it is clear from the context (or not important) which field we are using, we often omit the explicit reference to $K$ and simply write vector space instead of $K$-vector space.

*Example 9.2*

(1) The set $K^{n,m}$ with the matrix addition and the scalar multiplication forms a $K$-vector space. For obvious reasons, the elements of $K^{n,1}$ and $K^{1,m}$ are sometimes called *column* and *row* vectors, respectively.
(2) The set $K[t]$ forms a $K$-vector space, if the addition is defined as in Example 3.17 (usual addition of polynomials) and the scalar multiplication for $p = \alpha_0 + \alpha_1 t + \ldots + \alpha_n t^n \in K[t]$ is defined by

$$\lambda \cdot p := (\lambda \alpha_0) + (\lambda \alpha_1)t + \ldots + (\lambda \alpha_n)t^n.$$

(3) The continuous and real valued functions defined on a real interval $[\alpha, \beta]$ with the pointwise addition and scalar multiplication, i.e.,

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\lambda \cdot f)(x) := \lambda f(x),$$

form an $\mathbb{R}$-vector space. This can be shown by using that the addition of two continuous functions as well as the multiplication of a continuous function by a real number yield again a continuous function.

Since, by definition, $(\mathcal{V}, +)$ is a commutative group, we already know some vector space properties from the theory of groups (cp. Chap. 3). In particular, every vector space contains a unique neutral element (with respect to addition) $0_{\mathcal{V}}$, which is called the *null vector*. Every vector $v \in \mathcal{V}$ has a unique (additive) inverse $-v \in \mathcal{V}$ with $v + (-v) = v - v = 0_{\mathcal{V}}$. As usual, we will write $v - w$ instead of $v + (-w)$.

**Lemma 9.3** *Let $\mathcal{V}$ be a $K$-vector space. If $0_K$ and $0_{\mathcal{V}}$ are the neutral (null) elements of $K$ and $\mathcal{V}$, respectively, then the following assertions hold:*

*(1)  $0_K \cdot v = 0_{\mathcal{V}}$ for all $v \in \mathcal{V}$.*
*(2)  $\lambda \cdot 0_{\mathcal{V}} = 0_{\mathcal{V}}$ for all $\lambda \in K$.*
*(3)  $-(\lambda \cdot v) = (-\lambda) \cdot v = \lambda \cdot (-v)$ for all $v \in \mathcal{V}$ and $\lambda \in K$.*

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[1]This term was introduced in 1845 by Sir William Rowan Hamilton (1805–1865) in the context of his *quaternions*. It is motivated by the Latin verb "vehi" ("vehor", "vectus sum") which means to ride or drive. Also the term "scalar" was introduced by Hamilton; see the footnote on the scalar multiplication (4.2).

*Proof*

(1) For all $v \in \mathcal{V}$ we have $0_K \cdot v = (0_K + 0_K) \cdot v = 0_K \cdot v + 0_K \cdot v$. Adding $-(0_K \cdot v)$ on both sides of this identity gives $0_\mathcal{V} = 0_K \cdot v$.
(2) For all $\lambda \in K$ we have $\lambda \cdot 0_\mathcal{V} = \lambda \cdot (0_\mathcal{V} + 0_\mathcal{V}) = \lambda \cdot 0_\mathcal{V} + \lambda \cdot 0_\mathcal{V}$. Adding $-(\lambda \cdot 0_\mathcal{V})$ on both sides of this identity gives $0_\mathcal{V} = \lambda \cdot 0_\mathcal{V}$.
(3) For all $\lambda \in K$ and $v \in \mathcal{V}$ we have $\lambda \cdot v + (-\lambda) \cdot v = (\lambda - \lambda) \cdot v = 0_K \cdot v = 0_\mathcal{V}$, as well as $\lambda \cdot v + \lambda \cdot (-v) = \lambda \cdot (v - v) = \lambda \cdot 0_\mathcal{V} = 0_\mathcal{V}$. $\qquad\square$

In the following we will write $0$ instead of $0_K$ and $0_\mathcal{V}$ when it is clear which null element is meant.

As in groups, rings and fields we can identify substructures in vector spaces that are again vector spaces.

**Definition 9.4** Let $(\mathcal{V}, +, \cdot)$ be a $K$-vector space and let $\mathcal{U} \subseteq \mathcal{V}$. If $(\mathcal{U}, +, \cdot)$ is a $K$-vector space, then it is called a *subspace* of $(\mathcal{V}, +, \cdot)$.

A substructure must be closed with respect to the given operations, which here are addition and scalar multiplication.

**Lemma 9.5** $(\mathcal{U}, +, \cdot)$ *is a subspace of the $K$-vector space $(\mathcal{V}, +, \cdot)$ if and only if* $\emptyset \neq \mathcal{U} \subseteq \mathcal{V}$ *and the following assertions hold:*

*(1)* $v + w \in \mathcal{U}$ *for all $v, w \in \mathcal{U}$,*
*(2)* $\lambda v \in \mathcal{U}$ *for all $\lambda \in K$ and $v \in \mathcal{U}$.*

*Proof* Exercise. $\qquad\square$

*Example 9.6*

(1) Every vector space $\mathcal{V}$ has the trivial subspaces $\mathcal{U} = \mathcal{V}$ and $\mathcal{U} = \{0\}$.
(2) Let $A \in K^{n,m}$ and $\mathcal{U} = \mathcal{L}(A, 0) \subseteq K^{m,1}$, i.e., $\mathcal{U}$ is the solution set of the homogeneous linear system $Ax = 0$. We have $0 \in \mathcal{U}$, so $\mathcal{U}$ is not empty. If $v, w \in \mathcal{U}$, then
$$A(v + w) = Av + Aw = 0 + 0 = 0,$$
i.e., $v + w \in \mathcal{U}$. Furthermore, for all $\lambda \in K$,
$$A(\lambda v) = \lambda (Av) = \lambda 0 = 0,$$
i.e., $\lambda v \in \mathcal{U}$. Hence, $\mathcal{U}$ is a subspace of $K^{m,1}$.
(3) For every $n \in \mathbb{N}_0$ the set $K[t]_{\leq n} := \{p \in K[t] \mid \deg(p) \leq n\}$ is a subspace of $K[t]$.

**Definition 9.7** Let $\mathcal{V}$ be a $K$-vector space, $n \in \mathbb{N}$, and $v_1, \ldots, v_n \in \mathcal{V}$. A vector of the form
$$\lambda_1 v_1 + \ldots + \lambda_n v_n = \sum_{i=1}^{n} \lambda_i v_i \in \mathcal{V}$$

is called a *linear combination* of $v_1, \ldots, v_n$ with the *coefficients* $\lambda_1, \ldots, \lambda_n \in K$. The *(linear) span* of $v_1, \ldots, v_n$ is the set

$$\text{span}\{v_1, \ldots, v_n\} := \Big\{ \sum_{i=1}^{n} \lambda_i v_i \mid \lambda_1, \ldots, \lambda_n \in K \Big\}.$$

Let $M$ be a set and suppose that for every $m \in M$ we have a vector $v_m \in \mathcal{V}$. Let the set of all these vectors, called the *system* of these vectors, be denoted by $\{v_m\}_{m \in M}$. Then the *(linear) span* of the system $\{v_m\}_{m \in M}$, denoted by span $\{v_m\}_{m \in M}$, is defined as the set of all vectors $v \in \mathcal{V}$ that are linear combinations of finitely many vectors of the system.

This definition can be consistently extended to the case $n = 0$. In this case $v_1, \ldots, v_n$ is a list of length zero, or an *empty list*. If we define the empty sum of vectors as $0 \in \mathcal{V}$, then we obtain span$\{v_1, \ldots, v_n\} = $ span $\emptyset = \{0\}$.

If in the following we consider a list of vectors $v_1, \ldots, v_n$ or a set of vectors $\{v_1, \ldots, v_n\}$, we usually mean that $n \geq 1$. The case of empty list and the associated *zero vector space* $\mathcal{V} = \{0\}$ will sometimes be discussed separately.

*Example 9.8* The vector space $K^{1,3} = \{[\alpha_1, \alpha_2, \alpha_3] \mid \alpha_1, \alpha_2, \alpha_3 \in K\}$ is spanned by the vectors $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. The set $\{[\alpha_1, \alpha_2, 0] \mid \alpha_1, \alpha_2 \in K\}$ forms a subspace of $K^{1,3}$ that is spanned by the vectors $[1, 0, 0], [0, 1, 0]$.

**Lemma 9.9** *If $\mathcal{V}$ is a vector space and $v_1, \ldots, v_n \in \mathcal{V}$, then* span$\{v_1, \ldots, v_n\}$ *is a subspace of $\mathcal{V}$.*

*Proof* It is clear that $\emptyset \neq$ span$\{v_1, \ldots, v_n\} \subseteq \mathcal{V}$. Furthermore, span$\{v_1, \ldots, v_n\}$ is by definition closed with respect to addition and scalar multiplication, so that (1) and (2) in Lemma 9.5 are satisfied.                                                                 $\square$

## 9.2   Bases and Dimension of Vector Spaces

We will now discuss the central theory of bases and dimension of vector spaces, and start with the concept of linear independence.

**Definition 9.10** Let $\mathcal{V}$ be a $K$-vector space.

(1)  The vectors $v_1, \ldots, v_n \in \mathcal{V}$ are called *linearly independent* if the equation

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \text{with} \quad \lambda_1, \ldots, \lambda_n \in K$$

always implies that $\lambda_1 = \cdots = \lambda_n = 0$. Otherwise, i.e., when $\sum_{i=1}^{n} \lambda_i v_i = 0$ holds for some scalars $\lambda_1, \ldots, \lambda_n \in K$ that are not all equal to zero, then the vectors $v_1, \ldots, v_n$ are called *linearly dependent*.

(2) The empty list is linear independent.
(3) If $M$ is a set and for every $m \in M$ we have a vector $v_m \in \mathcal{V}$, the corresponding system $\{v_m\}_{m \in M}$ is called *linearly independent* when finitely many vectors of the system are always linearly independent in the sense of (1). Otherwise the system is called *linearly dependent*.

The vectors $v_1, \ldots, v_n$ are linearly independent if and only if the zero vector can be linearly combined only in the trivial way $0 = 0 \cdot v_1 + \ldots + 0 \cdot v_n$. Consequently, if one of these vectors is the zero vector, then $v_1, \ldots, v_n$ are linearly dependent. A single vector $v$ is linearly independent if and only if $v \neq 0$.

The following result gives a useful characterization of the linear independence of finitely many (but at least two) given vectors.

**Lemma 9.11** *The vectors $v_1, \ldots, v_n$, $n \geq 2$, are linearly independent if and only if no vector $v_i$, $i = 1, \ldots, n$, can be written as a linear combination of the others.*

*Proof* We prove the assertion by contraposition. The vectors $v_1, \ldots, v_n$ are linearly dependent if and only if

$$\sum_{i=1}^{n} \lambda_i v_i = 0$$

with at least one scalar $\lambda_j \neq 0$. Equivalently,

$$v_j = -\sum_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_j^{-1} \lambda_i) \, v_i,$$

so that $v_j$ is a linear combination of the other vectors.                                  $\square$

Using the concept of linear independence we can now define the concept of the basis of a vector space.

**Definition 9.12** Let $\mathcal{V}$ be a vector space.

(1) A set $\{v_1, \ldots, v_n\} \subseteq \mathcal{V}$ is called a *basis* of $\mathcal{V}$, when $v_1, \ldots, v_n$ are linearly independent and $\mathrm{span}\{v_1, \ldots, v_n\} = \mathcal{V}$.
(2) The set $\emptyset$ is the basis of the zero vector space $\mathcal{V} = \{0\}$.
(3) Let $M$ be a set and suppose that for every $m \in M$ we have a vector $v_m \in \mathcal{V}$. The set $\{v_m \mid m \in M\}$ is called a *basis* of $\mathcal{V}$ if the corresponding system $\{v_m\}_{m \in M}$ is linearly independent and $\mathrm{span}\{v_m\}_{m \in M} = \mathcal{V}$.

In short, a basis is a *linearly independent spanning set* of a vector space.

*Example 9.13*

(1) Let $E_{ij} \in K^{n,m}$ be the matrix with entry 1 in position $(i, j)$ and all other entries 0 (cp. Sect. 5.1). Then the set

$$\{E_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \tag{9.1}$$

is a basis of the vector space $K^{n,m}$ (cp. (1) in Example 9.2): The matrices $E_{ij} \in K^{n,m}$, $1 \leq i \leq n$ and $1 \leq j \leq m$, are linearly independent, since

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} E_{ij} = [\lambda_{ij}]$$

implies that $\lambda_{ij} = 0$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. For any $A = [a_{ij}] \in K^{n,m}$ we have

$$A = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} E_{ij},$$

and hence

$$\text{span}\{E_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} = K^{n,m}.$$

The basis (9.1) is called the *canonical* or *standard* basis of the vector space $K^{n,m}$. For $m = 1$ we denote the canonical basis vectors of $K^{n,1}$ by

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \ldots, \; e_n := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

These vectors are also called *unit vectors*; they are the $n$ columns of the identity matrix $I_n$.

(2) A basis of the vector space $K[t]$ (cp. (2) in Example 9.2) is given by the set $\{t^m \mid m \in \mathbb{N}_0\}$, since the corresponding system $\{t^m\}_{m \in \mathbb{N}_0}$ is linearly independent, and every polynomial $p \in K[t]$ is a linear combination of finitely many vectors of the system.

The next result is called the *basis extension theorem*.

**Theorem 9.14** *Let $\mathcal{V}$ be a vector space and let $v_1, \ldots, v_r, w_1, \ldots, w_\ell \in \mathcal{V}$, where $r, \ell \in \mathbb{N}_0$. If $v_1, \ldots, v_r$ are linearly independent and $\text{span}\{v_1, \ldots, v_r, w_1, \ldots, w_\ell\} = \mathcal{V}$, then the set $\{v_1, \ldots, v_r\}$ can be extended to a basis of $\mathcal{V}$ using vectors from the set $\{w_1, \ldots, w_\ell\}$.*

*Proof* Note that for $r = 0$ the list $v_1, \ldots, v_r$ is empty and hence linearly independent due to (2) in Definition 9.10.

We prove the assertion by induction on $\ell$. If $\ell = 0$, then $\text{span}\{v_1, \ldots, v_r\} = \mathcal{V}$, and the linear independence of $\{v_1, \ldots, v_r\}$ shows that this set is a basis of $\mathcal{V}$.

Let the assertion hold for some $\ell \geq 0$. Suppose that $v_1, \ldots, v_r, w_1, \ldots, w_{\ell+1} \in \mathcal{V}$ are given, where $v_1, \ldots, v_r$ are linearly independent and span$\{v_1, \ldots, v_r, w_1, \ldots, w_{\ell+1}\} = \mathcal{V}$. If $\{v_1, \ldots, v_r\}$ already is a basis of $\mathcal{V}$, then we are done. Suppose, therefore, that span$\{v_1, \ldots, v_r\} \subset \mathcal{V}$. Then there exists at least one $j$, $1 \leq j \leq \ell+1$, such that $w_j \notin$ span$\{v_1, \ldots, v_r\}$. In particular, we have $w_j \neq 0$. Then

$$\lambda w_j + \sum_{i=1}^{r} \lambda_i v_i = 0$$

implies that $\lambda = 0$ (otherwise we would have $w_j \in$ span$\{v_1, \ldots, v_r\}$) and, therefore, $\lambda_1 = \cdots = \lambda_r = 0$ due to the linear independence of $v_1, \ldots, v_r$. Thus, $v_1, \ldots, v_r, w_j$ are linearly independent. By the induction hypothesis we can extend the set $\{v_1, \ldots, v_r, w_j\}$ to a basis of $\mathcal{V}$ using vectors from the set $\{w_1, \ldots, w_{\ell+1}\} \setminus \{w_j\}$, which contains $\ell$ elements.                    □

*Example 9.15* Consider the vector space $\mathcal{V} = K[t]_{\leq 3}$ (cp. (3) in Example 9.6) and the vectors $v_1 = t$, $v_2 = t^2$, $v_3 = t^3$. These vectors are linearly independent, but $\{v_1, v_2, v_3\}$ is not a basis of $\mathcal{V}$, since span$\{v_1, v_2, v_3\} \neq \mathcal{V}$. For example, the vectors $w_1 = t^2 + 1$ and $w_2 = t^3 - t^2 - 1$ are elements of $\mathcal{V}$, but $w_1, w_2 \notin$ span$\{v_1, v_2, v_3\}$. We have span$\{v_1, v_2, v_3, w_1, w_2\} = \mathcal{V}$. If we extend $\{v_1, v_2, v_3\}$ by $w_1$, then we get the linearly independent vectors $v_1, v_2, v_3, w_1$ which indeed span $\mathcal{V}$. Thus, $\{v_1, v_2, v_3, w_1\}$ is a basis of $\mathcal{V}$.

By the basis extension theorem every vector space that is spanned by finitely many vectors has a basis consisting of finitely many elements. A central result of the theory of vector spaces is that every such basis has the same number of elements. In order to show this result we first prove the following *exchange lemma*.

**Lemma 9.16** *Let $\mathcal{V}$ be a vector space, let $v_1, \ldots, v_m \in \mathcal{V}$ and let $w = \sum_{i=1}^{m} \lambda_i v_i \in \mathcal{V}$ with $\lambda_1 \neq 0$. Then span$\{w, v_2, \ldots, v_m\} =$ span$\{v_1, v_2, \ldots, v_m\}$.*

*Proof* By assumption we have

$$v_1 = \lambda_1^{-1} w - \sum_{i=2}^{m} \left(\lambda_1^{-1} \lambda_i\right) v_i.$$

If $y \in$ span$\{v_1, \ldots, v_m\}$, say $y = \sum_{i=1}^{m} \gamma_i v_i$, then

$$y = \gamma_1 \left(\lambda_1^{-1} w - \sum_{i=2}^{m} \left(\lambda_1^{-1}\lambda_i\right) v_i\right) + \sum_{i=2}^{m} \gamma_i v_i$$

$$= \left(\gamma_1 \lambda_1^{-1}\right) w + \sum_{i=2}^{m} \left(\gamma_i - \gamma_1 \lambda_1^{-1} \lambda_i\right) v_i \in \text{span}\{w, v_2, \ldots, v_m\}.$$

If, on the other hand, $y = \alpha_1 w + \sum_{i=2}^{m} \alpha_i v_i \in$ span$\{w, v_2, \ldots, v_m\}$, then

$$y = \alpha_1 \left( \sum_{i=1}^{m} \lambda_i v_i \right) + \sum_{i=2}^{m} \alpha_i v_i$$

$$= \alpha_1 \lambda_1 v_1 + \sum_{i=2}^{m} (\alpha_1 \lambda_i + \alpha_i) v_i \in \text{span}\{v_1, \ldots, v_m\},$$

and thus $\text{span}\{w, v_2, \ldots, v_m\} = \text{span}\{v_1, v_2, \ldots, v_m\}$. $\qquad\qquad\qquad\qquad \square$

Using this lemma we now prove the *exchange theorem*.[2]

**Theorem 9.17** *Let $W = \{w_1, \ldots, w_n\}$ and $U = \{u_1, \ldots, u_m\}$ be finite subsets of a vector space, and let $w_1, \ldots, w_n$ be linearly independent. If $W \subseteq \text{span}\{u_1, \ldots, u_m\}$, then $n \leq m$, and $n$ elements of $U$, if numbered appropriately the elements $u_1, \ldots, u_n$, can be exchanged against $n$ elements of $W$ in such a way that*

$$\text{span}\{w_1, \ldots, w_n, u_{n+1}, \ldots, u_m\} = \text{span}\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_m\}.$$

*Proof* By assumption we have $w_1 = \sum_{i=1}^{m} \lambda_i u_i$ for some scalars $\lambda_1, \ldots, \lambda_m$ that are not all zero (otherwise $w_1 = 0$, which contradicts the linear independence of $w_1, \ldots, w_n$). After an appropriate renumbering we have $\lambda_1 \neq 0$, and Lemma 9.16 yields

$$\text{span}\{w_1, u_2, \ldots, u_m\} = \text{span}\{u_1, u_2, \ldots, u_m\}.$$

Suppose that for some $r$, $1 \leq r \leq n-1$, we have exchanged the vectors $u_1, \ldots, u_r$ against $w_1, \ldots, w_r$ so that

$$\text{span}\{w_1, \ldots, w_r, u_{r+1}, \ldots, u_m\} = \text{span}\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}.$$

It is then clear that $r \leq m$.

By assumption we have $w_{r+1} \in \text{span}\{u_1, \ldots, u_m\}$, and thus

$$w_{r+1} = \sum_{i=1}^{r} \lambda_i w_i + \sum_{i=r+1}^{m} \lambda_i u_i$$

for some scalars $\lambda_1, \ldots, \lambda_m$. One of the scalars $\lambda_{r+1}, \ldots, \lambda_m$ must be nonzero (otherwise $w_{r+1} \in \text{span}\{w_1, \ldots, w_r\}$, which contradicts the linear independence of $w_1, \ldots, w_m$). After an appropriate renumbering we have $\lambda_{r+1} \neq 0$, and Lemma 9.16 yields

$$\text{span}\{w_1, \ldots, w_{r+1}, u_{r+2}, \ldots, u_m\} = \text{span}\{w_1, \ldots, w_r, u_{r+1}, \ldots, u_m\}.$$

If we continue this construction until $r = n - 1$, then we obtain

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[2]In the literature, his theorem is sometimes called the *Steinitz exchange theorem* after Ernst Steinitz (1871–1928). The result was first proved in 1862 by Hermann Günther Graßmann (1809–1877).

$$\text{span}\{w_1, \ldots, w_n, u_{n+1}, \ldots, u_m\} = \text{span}\{u_1, \ldots, u_n, u_{n+1}, \ldots u_m\},$$

where in particular $n \leq m$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Using this fundamental theorem, the following result about the unique number of basis elements is a simple corollary.

**Corollary 9.18** *If a vector space $\mathcal{V}$ is spanned by finitely many vectors, then $\mathcal{V}$ has a basis consisting of finitely many elements, and any two bases of $\mathcal{V}$ have the same number of elements.*

*Proof* The assertion is clear for $\mathcal{V} = \{0\}$ (cp. (2) in Definition 9.12). Let $\mathcal{V} = \text{span}\{v_1, \ldots, v_m\}$ with $v_1 \neq 0$. By Theorem 9.14, we can extend $\text{span}\{v_1\}$ using elements of $\{v_2, \ldots, v_m\}$ to a basis of $\mathcal{V}$. Thus, $\mathcal{V}$ has a basis with finitely many elements. Let $U := \{u_1, \ldots, u_\ell\}$ and $W := \{w_1, \ldots, w_k\}$ be two such bases. Then

$$W \subseteq \mathcal{V} = \text{span}\{u_1, \ldots, u_\ell\} \overset{\text{Theorem 9.18}}{\Longrightarrow} k \leq \ell,$$
$$U \subseteq \mathcal{V} = \text{span}\{w_1, \ldots, w_k\} \overset{\text{Theorem 9.18}}{\Longrightarrow} \ell \leq k,$$

and thus $\ell = k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We can now define the dimension of a vector space.

**Definition 9.19** If there exists a basis of a $K$-vector space $\mathcal{V}$ that consists of finitely many elements, then $\mathcal{V}$ is called *finite dimensional*, and the unique number of basis elements is called the *dimension* of $\mathcal{V}$. We denote the dimension by $\dim_K(\mathcal{V})$ or $\dim(\mathcal{V})$, if it is clear which field is meant.

If $\mathcal{V}$ is not spanned by finitely many vectors, then $\mathcal{V}$ is called *infinite dimensional*, and we write $\dim_K(\mathcal{V}) = \infty$.

Note that the zero vector space $\mathcal{V} = \{0\}$ has the basis $\emptyset$ and thus it has dimension zero (cp. (2) in Definition 9.12).

If $\mathcal{V}$ is a finite dimensional vector space and if $v_1, \ldots, v_m \in \mathcal{V}$ with $m > \dim(\mathcal{V})$, then the vectors $v_1, \ldots, v_m$ must be linearly dependent. (If these vectors were linearly independent, then we could extend them via Theorem 9.14 to a basis of $\mathcal{V}$ that would contain more than $\dim(\mathcal{V})$ elements.)

*Example 9.20* The set in (9.1) forms a basis of the vector space $K^{n,m}$. This basis has $n \cdot m$ elements, and hence $\dim(K^{n,m}) = n \cdot m$. On the other hand, the vector space $K[t]$ is not spanned by finitely many vectors (cp. (2) in Example 9.13) and hence it is infinite dimensional.

*Example 9.21* Let $\mathcal{V}$ be the vector space of continuous and real valued functions on the real interval $[0, 1]$ (cp. (3) in Example 9.2). Define for $n = 1, 2, \ldots$ the function $f_n \in \mathcal{V}$ by

$$f_n(x) = \begin{cases} 0, & x < \frac{1}{n+1}, \\[2mm] 0, & \frac{1}{n} < x, \\[2mm] 2n(n+1)x - 2n, & \frac{1}{n+1} \le x \le \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right), \\[2mm] -2n(n+1)x + 2n + 2, & \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < x \le \frac{1}{n}. \end{cases}$$
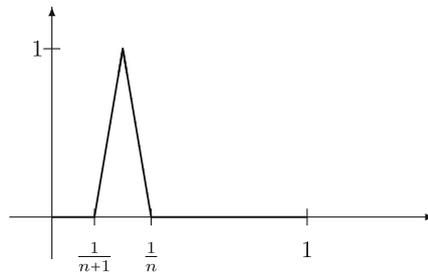


Every linear combination $\sum_{j=1}^{k} \lambda_j f_j$ is a continuous function that has the value $\lambda_j$ at $\frac{1}{2}\left(\frac{1}{j} + \frac{1}{j+1}\right)$. Thus, the equation $\sum_{j=1}^{k} \lambda_j f_j = 0 \in \mathcal{V}$ implies that all $\lambda_j$ must be zero, so that $f_1, \ldots, f_k \in \mathcal{V}$ are linearly independent for all $k \in \mathbb{N}$. Consequently, $\dim(\mathcal{V}) = \infty$.

## 9.3   Coordinates and Changes of the Basis

We will now study the linear combinations of basis vectors of a finite dimensional vector space. In particular, we will study what happens with a linear combination if we change to another basis of the vector space.

**Lemma 9.22** *If $\{v_1, \ldots, v_n\}$ is a basis of a $K$-vector space $\mathcal{V}$, then for every $v \in \mathcal{V}$ there exist uniquely determined scalars $\lambda_1, \ldots, \lambda_n \in K$ with $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$. These scalars are called the* coordinates *of $v$ with respect to the basis $\{v_1, \ldots, v_n\}$.*

*Proof* Let $v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \mu_i v_i$ for some scalars $\lambda_i, \mu_i \in K, i = 1, \ldots, n$, then

$$0 = v - v = \sum_{i=1}^{n} (\lambda_i - \mu_i) v_i.$$

The linear independence of $v_1, \ldots, v_n$ implies that $\lambda_i = \mu_i$ for $i = 1, \ldots, n$. $\qquad\square$

By definition, the coordinates of a vector depend on the given basis. In particular, they depend on the ordering (or numbering) of the basis vectors. Because of this, some authors distinguish between the basis as "set", i.e., a collection of elements without a particular ordering, and an "ordered basis". In this book we will keep the set notation for a basis $\{v_1, \ldots, v_n\}$, where the indices indicate the ordering of the basis vectors.

Let $\mathcal{V}$ be a $K$-vector space, $v_1, \ldots, v_n \in \mathcal{V}$ (they need not be linearly independent) and
$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n$$
for some coefficients $\lambda_1, \ldots, \lambda_n \in K$. Let us write

$$(v_1, \ldots, v_n) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} := \lambda_1 v_1 + \ldots + \lambda_n v_n. \tag{9.2}$$

Here $(v_1, \ldots, v_n)$ is an $n$-tuple over $\mathcal{V}$, i.e.,

$$(v_1, \ldots, v_n) \in \mathcal{V}^n = \underbrace{\mathcal{V} \times \ldots \times \mathcal{V}}_{n \text{ times}}.$$

For $n = 1$ we have $\mathcal{V}^1 = \mathcal{V}$. We then skip the parentheses and write $v$ instead of $(v)$ for a 1-tuple. The notation (9.2) formally defines a "multiplication" as map from $\mathcal{V}^n \times K^{n,1}$ to $\mathcal{V}$.

For all $\alpha \in K$ we have

$$\alpha \cdot v = (\alpha \cdot \lambda_1)v_1 + \ldots + (\alpha \cdot \lambda_n)v_n = (v_1, \ldots, v_n) \begin{bmatrix} \alpha\lambda_1 \\ \vdots \\ \alpha\lambda_n \end{bmatrix}.$$

If $\mu_1, \ldots, \mu_n \in K$ and

$$u = \mu_1 v_1 + \ldots + \mu_n v_n = (v_1, \ldots, v_n) \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

then

$$v + u = (\lambda_1 + \mu_1)v_1 + \ldots + (\lambda_n + \mu_n)v_n = (v_1, \ldots, v_n) \begin{bmatrix} \lambda_1 + \mu_1 \\ \vdots \\ \lambda_n + \mu_n \end{bmatrix}.$$

This shows that if vectors are given by linear combinations, then the operations scalar multiplication and addition correspond to operations with the coefficients of the vectors with respect to the linear combinations.

We can further extend this notation. Let $A = [a_{ij}] \in K^{n,m}$ and let

$$u_j = (v_1, \ldots, v_n) \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix}, \quad j = 1, \ldots, m.$$

Then we write the $m$ linear combinations for $u_1, \ldots, u_m$ as the system

$$(u_1, \ldots, u_m) =: (v_1, \ldots, v_n)A. \tag{9.3}$$

On both sides of this equation we have elements of $\mathcal{V}^m$. The right-multiplication of an arbitrary $n$-tuple $(v_1, \ldots, v_n) \in \mathcal{V}^n$ with a matrix $A \in K^{n,m}$ thus corresponds to forming $m$ linear combinations of the vectors $v_1, \ldots, v_n$, with the corresponding coefficients given by the entries of $A$. Formally, this defines a "multiplication" as a map from $\mathcal{V}^n \times K^{n,m}$ to $\mathcal{V}^m$.

**Lemma 9.23** *Let $\mathcal{V}$ be a $K$-vector space, let $v_1, \ldots, v_n \in \mathcal{V}$ be linearly independent, let $A \in K^{n,m}$, and let $(u_1, \ldots, u_m) = (v_1, \ldots, v_n)A$. Then the vectors $u_1, \ldots, u_m$ are linearly independent if and only if $\operatorname{rank}(A) = m$.*

*Proof* Exercise.                                                                                          □

Now consider also a matrix $B = [b_{ij}] \in K^{m,\ell}$. Using (9.3) we obtain

$$(u_1, \ldots, u_m)B = ((v_1, \ldots, v_n)A)B.$$

**Lemma 9.24** *In the previous notation,*

$$((v_1, \ldots, v_n)A)B = (v_1, \ldots, v_n)(AB).$$

*Proof* Exercise.                                                                                          □

Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be bases of $\mathcal{V}$ and let $v \in \mathcal{V}$. By Lemma 9.22 there exist (unique) coordinates $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$, respectively, with

$$v = (v_1, \ldots, v_n) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = (w_1, \ldots, w_n) \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}.$$

We will now describe a method for transforming the coordinates $\lambda_1, \ldots, \lambda_n$ with respect to the basis $\{v_1, \ldots, v_n\}$ into the coordinates $\mu_1, \ldots, \mu_n$ with respect to the basis $\{w_1, \ldots, w_n\}$, and vice versa.

For every basis vector $v_j$, $j = 1, \ldots, n$, there exist (unique) coordinates $p_{ij}$, $i = 1, \ldots, n$, such that

$$v_j = (w_1, \ldots, w_n) \begin{bmatrix} p_{1j} \\ \vdots \\ p_{nj} \end{bmatrix}, \quad j = 1, \ldots, n.$$

Defining $P = [p_{ij}] \in K^{n,n}$ we can write these $n$ equations for the vectors $v_j$ analogous to (9.3) as

$$(v_1, \ldots, v_n) = (w_1, \ldots, w_n) P. \tag{9.4}$$

In the same way, for every basis vector $w_j$, $j = 1, \ldots, n$, there exist (unique) coordinates $q_{ij}$, $i = 1, \ldots, n$, such that

$$w_j = (v_1, \ldots, v_n) \begin{bmatrix} q_{1j} \\ \vdots \\ q_{nj} \end{bmatrix}, \quad j = 1, \ldots, n.$$

If we set $Q = [q_{ij}] \in K^{n,n}$, then analogously to (9.4) we get

$$(w_1, \ldots, w_n) = (v_1, \ldots, v_n) Q.$$

Thus,

$$(w_1, \ldots, w_n) = (v_1, \ldots, v_n) Q = ((w_1, \ldots, w_n) P) Q = (w_1, \ldots, w_n)(PQ),$$

which implies that

$$(w_1, \ldots, w_n)(I_n - PQ) = (0, \ldots, 0).$$

This means that the $n$ linear combinations of the basis vectors $w_1, \ldots, w_n$, with their corresponding coordinates given by the entries of the $n$ columns of $I_n - PQ$, are all equal to the zero vector. Since the basis vectors are linearly independent, all coordinates must be zero, and hence $I_n - PQ = 0 \in K^{n,n}$, or $PQ = I_n$. Analogously we obtain the equation $QP = I_n$. Therefore the matrix $P \in K^{n,n}$ is invertible with $P^{-1} = Q$. Furthermore, we have

$$v = (v_1, \ldots, v_n) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = ((w_1, \ldots, w_n) P) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = (w_1, \ldots, w_n) \left( P \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \right).$$

Due to the uniqueness of the coordinates of $v$ with respect to the basis $\{w_1, \ldots, w_n\}$ we obtain

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = P \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = P^{-1} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}.$$

Hence a multiplication with the matrix $P$ transforms the coordinates of $v$ with respect to the basis $\{v_1, \ldots, v_n\}$ into those with respect to the basis $\{w_1, \ldots, w_n\}$; a multiplication with $P^{-1}$ yields the inverse transformation. Therefore, $P$ and $P^{-1}$ are called *coordinate transformation matrices*.

We can summarize the results obtained above as follows.

**Theorem 9.25** *Let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ be bases of a $K$-vector space $\mathcal{V}$. Then the uniquely determined matrix $P \in K^{n,n}$ is (9.4) is invertible and yields the coordinate transformation from $\{v_1, \ldots, v_n\}$ to $\{w_1, \ldots, w_n\}$: If*

$$v = (v_1, \ldots, v_n) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = (v_1, \ldots, v_n) \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix},$$

*then*

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = P \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

*Example 9.26* Consider the vector space $\mathcal{V} = \mathbb{R}^2 = \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in \mathbb{R}\}$ with the entrywise addition and scalar multiplication. A basis of $\mathcal{V}$ is given by the set $\{e_1 = (1, 0), e_2 = (0, 1)\}$, and we have $(\alpha_1, \alpha_2) = \alpha_1 e_1 + \alpha_2 e_2$ for all $(\alpha_1, \alpha_2) \in \mathcal{V}$. Another basis of $\mathcal{V}$ is the set $\{v_1 = (1, 1), v_2 = (1, 2)\}$. The corresponding coordinate transformation matrices can be obtained from the defining equations $(v_1, v_2) = (e_1, e_2)P$ and $(e_1, e_2) = (v_1, v_2)Q$ as

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad Q = P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

## 9.4   Relations Between Vector Spaces and Their Dimensions

Our first result describes the relation between a vector space and a subspace.

**Lemma 9.27** *If $\mathcal{V}$ is a finite dimensional vector space and $\mathcal{U} \subseteq \mathcal{V}$ is a subspace, then $\dim(\mathcal{U}) \leq \dim(\mathcal{V})$ with equality if and only if $\mathcal{U} = \mathcal{V}$.*

*Proof* Let $\mathcal{U} \subseteq \mathcal{V}$ and let $\{u_1, \ldots, u_m\}$ be a basis of $\mathcal{U}$, where $\{u_1, \ldots, u_m\} = \emptyset$ for $\mathcal{U} = \{0\}$. Using Theorem 9.14 we can extend this set to a basis of $\mathcal{V}$. If $\mathcal{U}$ is a proper subset of $\mathcal{V}$, then at least one basis vector needs to be added and hence $\dim(\mathcal{U}) < \dim(\mathcal{V})$. If $\mathcal{U} = \mathcal{V}$, then every basis of $\mathcal{V}$ is also a basis of $\mathcal{U}$, and thus $\dim(\mathcal{U}) = \dim(\mathcal{V})$. $\qquad\square$

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are subspaces of a vector space $\mathcal{V}$, then their *intersection* is given by

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \{u \in \mathcal{V} \mid u \in \mathcal{U}_1 \ \wedge \ u \in \mathcal{U}_2\}$$

(cp. Definition 2.6). The *sum* of the two subspaces is defined as

$$\mathcal{U}_1 + \mathcal{U}_2 := \{u_1 + u_2 \in \mathcal{V} \mid u_1 \in \mathcal{U}_1 \ \wedge \ u_2 \in \mathcal{U}_2\}.$$

**Lemma 9.28** *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are subspaces of a vector space $\mathcal{V}$, then the following assertions hold:*

*(1) $\mathcal{U}_1 \cap \mathcal{U}_2$ and $\mathcal{U}_1 + \mathcal{U}_2$ are subspaces of $\mathcal{V}$.*
*(2) $\mathcal{U}_1 + \mathcal{U}_1 = \mathcal{U}_1$.*
*(3) $\mathcal{U}_1 + \{0\} = \mathcal{U}_1$.*
*(4) $\mathcal{U}_1 \subseteq \mathcal{U}_1 + \mathcal{U}_2$, with equality if and only if $\mathcal{U}_2 \subseteq \mathcal{U}_1$.*

*Proof* Exercise. $\qquad\square$

An important result is the following *dimension formula for subspaces*.

**Theorem 9.29** *If $\mathcal{U}_1$ and $\mathcal{U}_2$ are finite dimensional subspaces of a vector space $\mathcal{V}$, then*

$$\dim(\mathcal{U}_1 \cap \mathcal{U}_2) + \dim(\mathcal{U}_1 + \mathcal{U}_2) = \dim(\mathcal{U}_1) + \dim(\mathcal{U}_2).$$

*Proof* Let $\{v_1, \ldots, v_r\}$ be a basis of $\mathcal{U}_1 \cap \mathcal{U}_2$. We extend this set to a basis $\{v_1, \ldots, v_r, w_1, \ldots, w_\ell\}$ of $\mathcal{U}_1$ and to a basis $\{v_1, \ldots, v_r, x_1, \ldots, x_k\}$ of $\mathcal{U}_2$, where we assume that $r, \ell, k \geq 1$. (If one of the lists is empty, then the following argument is easily modified.)

If suffices to show that $\{v_1, \ldots, v_r, w_1, \ldots, w_\ell, x_1, \ldots, x_k\}$ is a basis of $\mathcal{U}_1 + \mathcal{U}_2$. Obviously,
$$\text{span}\{v_1, \ldots, v_r, w_1, \ldots, w_\ell, x_1, \ldots, x_k\} = \mathcal{U}_1 + \mathcal{U}_2,$$

and hence it suffices to show that $v_1, \ldots, v_r, w_1, \ldots, w_\ell, x_1, \ldots, x_k$ are linearly independent. Let

$$\sum_{i=1}^{r} \lambda_i v_i + \sum_{i=1}^{\ell} \mu_i w_i + \sum_{i=1}^{k} \gamma_i x_i = 0,$$

then

$$\sum_{i=1}^{k} \gamma_i x_i = -\left( \sum_{i=1}^{r} \lambda_i v_i + \sum_{i=1}^{\ell} \mu_i w_i \right).$$

On the left hand side of this equation we have, by definition, a vector in $\mathcal{U}_2$; on the right hand side a vector in $\mathcal{U}_1$. Therefore, $\sum_{i=1}^{k} \gamma_i x_i \in \mathcal{U}_1 \cap \mathcal{U}_2$. By construction, however, $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{U}_1 \cap \mathcal{U}_2$ and the vectors $v_1, \ldots, v_r, w_1, \ldots, w_\ell$ are linearly independent. Therefore, $\sum_{i=1}^{\ell} \mu_i w_i = 0$ implies that $\mu_1 = \cdots = \mu_\ell = 0$. But then also

$$\sum_{i=1}^{r} \lambda_i v_i + \sum_{i=1}^{k} \gamma_i x_i = 0,$$

and hence $\lambda_1 = \cdots = \lambda_r = \gamma_1 = \cdots = \gamma_k = 0$ due to the linear independence of $v_1, \ldots, v_r, x_1, \ldots, x_k$.                                                                                                 □

If at least one of the subspaces in Theorem 9.29 is infinite dimensional, then the assertion is still formally correct, since in this case $\dim(\mathcal{U}_1 + \mathcal{U}_2) = \infty$ and $\dim(\mathcal{U}_1) + \dim(\mathcal{U}_2) = \infty$.

*Example 9.30*  For the subspaces

$$\mathcal{U}_1 = \{[\alpha_1, \alpha_2, 0] \mid \alpha_1, \alpha_2 \in K\}, \quad \mathcal{U}_2 = \{[0, \alpha_2, \alpha_3] \mid \alpha_2, \alpha_3 \in K\} \subset K^{1,3}$$

we have $\dim(\mathcal{U}_1) = \dim(\mathcal{U}_2) = 2$,

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \{[0, \alpha_2, 0] \mid \alpha_2 \in K\}, \quad \dim(\mathcal{U}_1 \cap \mathcal{U}_2) = 1,$$
$$\mathcal{U}_1 + \mathcal{U}_2 = K^{1,3}, \qquad\qquad\qquad \dim(\mathcal{U}_1 + \mathcal{U}_2) = 3.$$

The above definition of the sum can be extended to an arbitrary (but finite) number of subspaces: If $\mathcal{U}_1, \ldots, \mathcal{U}_k$, $k \geq 2$, are subspaces of the vector space $\mathcal{V}$, then we define

$$\mathcal{U}_1 + \ldots + \mathcal{U}_k = \sum_{j=1}^{k} \mathcal{U}_j := \left\{ \sum_{j=1}^{k} u_j \mid u_j \in \mathcal{U}_j, \ j = 1, \ldots, k \right\}.$$

This sum is called *direct*, if

$$\mathcal{U}_i \cap \sum_{\substack{j=1 \\ j \neq i}}^{k} \mathcal{U}_j = \{0\} \quad \text{for } i = 1, \ldots, k,$$

and in this case we write the (direct) sum as

$$\mathcal{U}_1 \oplus \ldots \oplus \mathcal{U}_k = \bigoplus_{j=1}^{k} \mathcal{U}_j.$$

In particular, a sum $\mathcal{U}_1 + \mathcal{U}_2$ of two subspaces $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{V}$ is direct if $\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$.

The following theorem presents two equivalent characterizations of the direct sum of subspaces.

**Theorem 9.31** *If $\mathcal{U} = \mathcal{U}_1 + \ldots + \mathcal{U}_k$ is a sum of $k \geq 2$ subspaces of a vector space $\mathcal{V}$, then the following assertions are equivalent:*

(1) *The sum $\mathcal{U}$ is direct, i.e., $\mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j = \{0\}$ for $i = 1, \ldots, k$.*
(2) *Every vector $u \in \mathcal{U}$ has a representation of the form $u = \sum_{j=1}^{k} u_j$ with uniquely determined $u_j \in \mathcal{U}_j$ for $j = 1, \ldots, k$.*
(3) *$\sum_{j=1}^{k} u_j = 0$ with $u_j \in \mathcal{U}_j$ for $j = 1, \ldots, k$ implies that $u_j = 0$ for $j = 1, \ldots, k$.*

*Proof*

(1) $\Rightarrow$ (2): Let $u = \sum_{j=1}^{k} u_j = \sum_{j=1}^{k} \widetilde{u}_j$ with $u_j, \widetilde{u}_j \in \mathcal{U}_j$, $j = 1, \ldots, k$. For every $i = 1, \ldots, k$ we then have

$$u_i - \widetilde{u}_i = - \sum_{j \neq i} (u_j - \widetilde{u}_j) \in \mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j.$$

Now $\mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j = \{0\}$ implies that $u_i - \widetilde{u}_i = 0$, and hence $u_i = \widetilde{u}_i$ for $i = 1, \ldots, k$.
(2) $\Rightarrow$ (3): This is obvious.
(3) $\Rightarrow$ (1): For a given $i$, let $u \in \mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j$. Then $u = \sum_{j \neq i} u_j$ for some $u_j \in \mathcal{U}_j$, $j \neq i$, and hence $-u + \sum_{j \neq i} u_j = 0$. In particular, this implies that $u = 0$, and thus $\mathcal{U}_i \cap \sum_{j \neq i} \mathcal{U}_j = \{0\}$.                          $\square$

**Exercises**

(In the following exercises $K$ is an arbitrary field.)

9.1.   Which of the following sets (with the usual addition and scalar multiplication) are $\mathbb{R}$-vector spaces?

$$\left\{ [\alpha_1, \alpha_2] \in \mathbb{R}^{1,2} \,|\, \alpha_1 = \alpha_2 \right\}, \quad \left\{ [\alpha_1, \alpha_2] \in \mathbb{R}^{1,2} \,|\, \alpha_1^2 + \alpha_2^2 = 1 \right\},$$

$$\left\{ [\alpha_1, \alpha_2] \in \mathbb{R}^{1,2} \,|\, \alpha_1 \geq \alpha_2 \right\}, \quad \left\{ [\alpha_1, \alpha_2] \in \mathbb{R}^{1,2} \,|\, \alpha_1 - \alpha_2 = 0 \text{ and } 2\alpha_1 + \alpha_2 = 0 \right\}.$$

Determine, if possible, a basis and the dimension.
9.2.   Determine a basis of the $\mathbb{R}$-vector space $\mathbb{C}$ and $\dim_{\mathbb{R}}(\mathbb{C})$. Determine a basis of the $\mathbb{C}$-vector space $\mathbb{C}$ and $\dim_{\mathbb{C}}(\mathbb{C})$.
9.3.   Show that $a_1, \ldots, a_n \in K^{n,1}$ are linearly independent if and only if $\det([a_1, \ldots, a_n]) \neq 0$.

9.4. Let $\mathcal{V}$ be a $K$-vector space, $\Omega$ a nonempty set and $\mathrm{Map}(\Omega, \mathcal{V})$ the set of maps from $\Omega$ to $\mathcal{V}$. Show that $\mathrm{Map}(\Omega, \mathcal{V})$ with the operations

$$+ : \mathrm{Map}(\Omega, \mathcal{V}) \times \mathrm{Map}(\Omega, \mathcal{V}) \to \mathrm{Map}(\Omega, \mathcal{V}), \quad (f, g) \mapsto f + g,$$
$$\text{with } (f + g)(x) := f(x) + g(x) \text{ for all } x \in \Omega,$$
$$\cdot : K \times \mathrm{Map}(\Omega, \mathcal{V}) \to \mathrm{Map}(\Omega, \mathcal{V}), \quad (\lambda, f) \mapsto \lambda \cdot f,$$
$$\text{with}(\lambda \cdot f)(x) := \lambda f(x) \text{ for all } x \in \Omega,$$

is a $K$-vector space.

9.5. Show that the functions sin and cos in $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ are linearly independent.

9.6. Let $\mathcal{V}$ be a vector space with $n = \dim(\mathcal{V}) \in \mathbb{N}$ and let $v_1, \ldots, v_n \in \mathcal{V}$. Show that the following statements are equivalent:

(1) $v_1, \ldots, v_n$ are linearly independent.
(2) $\mathrm{span}\{v_1, \ldots, v_n\} = \mathcal{V}$.
(3) $\{v_1, \ldots, v_n\}$ is a basis of $\mathcal{V}$.

9.7. Show that $(K^{n,m}, +, \cdot)$ is a $K$-vector space (cp. (1) in Example 9.2). Find a subspace of this $K$-vector space.

9.8. Show that $(K[t], +, \cdot)$ is a $K$-vector space (cp. (2) in Example 9.2). Show further that $K[t]_{\leq n}$ is a subspace of $K[t]$ (cp. (3) in Example 9.6) and determine $\dim(K[t]_{\leq n})$.

9.9. Show that the polynomials $p_1 = t^5 + t^4$, $p_2 = t^5 - 7t^3$, $p_3 = t^5 - 1$, $p_4 = t^5 + 3t$ are linearly independent in $\mathbb{Q}[t]_{\leq 5}$ and extend $\{p_1, p_2, p_3, p_4\}$ to a basis of $\mathbb{Q}[t]_{\leq 5}$.

9.10. Let $n \in \mathbb{N}$ and

$$K[t_1, t_2] := \left\{ \sum_{i,j=0}^{n} \alpha_{ij} t_1^i t_2^j \mid \alpha_{ij} \in K \right\}.$$

An element of $K[t_1, t_2]$ is called *bivariate polynomial* over $K$ in the unknowns $t_1$ and $t_2$. Define a scalar multiplication and an addition so that $K[t_1, t_2]$ becomes a vector space. Determine a basis of $K[t_1, t_2]$.

9.11. Show Lemma 9.5.

9.12. Let $A \in K^{n,m}$ and $b \in K^{n,1}$. Is the solution set $\mathscr{L}(A, b)$ of $Ax = b$ a subspace of $K^{m,1}$?

9.13. Let $A \in K^{n,n}$ and let $\lambda \in K$ be an eigenvalue of $A$. Show that the set $\{v \in K^{n,1} \mid Av = \lambda v\}$ is a subspace of $K^{n,1}$.

9.14. Let $A \in K^{n,n}$ and let $\lambda_1 \neq \lambda_2$ be two eigenvalues of $A$. Show that any two associated eigenvectors $v_1$ and $v_2$ are linearly independent.

9.15. Show that $B = \{B_1, B_2, B_3, B_4\}$ and $C = \{C_1, C_2, C_3, C_4\}$ with

$$B_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

are bases of the vector space $K^{2,2}$, and determine corresponding coordinate transformation matrices.

9.16. Examine the elements of the following sets for linear independence in the vector space $K[t]_{\leq 3}$:

$$U_1 = \{t, t^2 + 2t, t^2 + 3t + 1, t^3\}, \quad U_2 = \{1, t, t + t^2, t^2 + t^3\},$$
$$U_3 = \{1, t^2 - t, t^2 + t, t^3\}.$$

Determine the dimensions of the subspaces spanned by the elements of $U_1$, $U_2$, $U_3$. Is one of these sets a basis of $K[t]_{\leq 3}$?

9.17. Show that the set of sequences $\{(\alpha_1, \alpha_2, \alpha_3, \ldots) \mid \alpha_i \in \mathbb{Q}, \ i \in \mathbb{N}\}$ with entry-wise addition and scalar multiplication forms an infinite dimensional vector space, and determine a basis system.

9.18. Prove Lemma 9.23.

9.19. Prove Lemma 9.24.

9.20. Prove Lemma 9.28.

9.21. Let $\mathcal{U}_1, \mathcal{U}_2$ be finite dimensional subspaces of a vector space $\mathcal{V}$. Show that the sum $\mathcal{U}_1 + \mathcal{U}_2$ is direct if $\dim(\mathcal{U}_1 + \mathcal{U}_2) = \dim(\mathcal{U}_1) + \dim(\mathcal{U}_2)$.

9.22. Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$, $k \geq 3$, be finite dimensional subspaces of a vector space $\mathcal{V}$. Suppose that $\mathcal{U}_i \cap \mathcal{U}_j = \{0\}$ for all $i \neq j$. Is the sum $\mathcal{U}_1 + \ldots + \mathcal{U}_k$ direct?

9.23. Let $\mathcal{U}$ be a subspace of a finite dimensional vector space $\mathcal{V}$. Show that there exists another subspace $\widetilde{\mathcal{U}}$ with $\mathcal{U} \oplus \widetilde{\mathcal{U}} = \mathcal{V}$. (The subspace $\widetilde{\mathcal{U}}$ is called a *complement* of $\mathcal{U}$.)

9.24. Determine three subspaces $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ of $\mathcal{V} = \mathbb{R}^{3,1}$ with $\mathcal{U}_2 \neq \mathcal{U}_3$ and $\mathcal{V} = \mathcal{U}_1 \oplus \mathcal{U}_2 = \mathcal{U}_1 \oplus \mathcal{U}_3$. Is there a subspace $\mathcal{U}_1$ of $\mathcal{V}$ with a uniquely determined complement?