

# 3

## *Orthogonal Expansions*

In this chapter we study topics that form the traditional core of applied mathematics—*boundary value problems* and *orthogonal expansions*. These subjects are interrelated in one of the most aesthetic theories in all of mathematics.

In an abstract sense, mathematics can be thought of as imposing structures on sets of objects and studying their properties. In linear algebra, for example, an algebraic structure (addition and scalar multiplication) is defined on a finite-dimensional vector space, typically  $\mathbb{R}^n$ , along with geometric structures that characterize concepts like distance and orthogonality (perpendicularity). Finally, linear transformations, represented by matrices, between vector spaces can be introduced and we can study how those interrelate with the algebraic and geometric structures. This all leads to the solution of linear systems, the algebraic eigenvalue problem, and the decomposition of the vector space and the transformation into fundamental, or canonical, forms. What is exciting is that all of these ideas extend in a straightforward way to infinite-dimensional *function spaces* and linear differential operators defined boundary value problems on those spaces. The theory is not only insightful and unifying, but it has far-reaching applications to science, engineering, and mathematics.

### 3.1 The Fourier Method

Without a lot of detail, we begin with an example that indicates how these ideas apply to initial boundary value problems in PDEs. The key question, as

we observe, is whether a function  $f(x)$  can be expanded in a linear combinations of ‘basis’ functions, analogous to expanding a vector in terms of orthogonal basis vectors. Next we see why this question arises.

The basic technique for solving PDEs on a bounded spatial domain is the Fourier method, named after Joseph Fourier (1768–1830). In this first section we take a nineteenth-century perspective and make some comments about the origin of the method. Our discussion will motivate one of the most fundamental topics in analysis and in PDEs, namely orthogonal expansions.

To fix the notion let us consider heat flow in a finite bar of length  $l = \pi$  and unit diffusivity ( $k = 1$ ), where the ends are held at zero degrees. We choose  $\pi$  as the length to make the constants come out simpler at the end. From Chapter 1 we know that the temperature  $u = u(x, t)$  in a laterally insulated bar must satisfy the model equations

$$u_t = u_{xx}, \quad 0 < x < \pi; \quad t > 0, \quad (3.1)$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0, \quad (3.2)$$

consisting of the heat equation and the boundary conditions. For the moment, we ignore the initial condition. It is easy to check that for any positive integer  $n$ , a solution to (3.1) is given by

$$u_n(x, t) = e^{-n^2 t} \sin nx. \quad (3.3)$$

Each of these solutions clearly satisfies the boundary conditions (3.2) as well. For the moment, put a hold on your curiosity about the origin of the set of solutions in (3.3).

As we know, for a well-posed problem, equations (3.1) and (3.2) are augmented by an initial condition of the form

$$u(x, 0) = f(x), \quad 0 < x < \pi. \quad (3.4)$$

We now ask how a solution to the *initial* BVP can be constructed from only the solutions (3.3) of the boundary value problem? Fourier argued that the solution  $u(x, t)$  to (3.1), (3.2), (3.4) can be taken as a linear combination of the infinite number of solutions  $u_n(x, t)$  in (3.3); that is,

$$u(x, t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t} \sin nx \quad (3.5)$$

for appropriately chosen constants  $a_n$ . To satisfy the initial condition, (3.4) implies that

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin nx. \quad (3.6)$$

This means the initial temperature function  $f(x)$  must be an infinite series representation in terms of the periodic ‘basis’ functions  $\sin x, \sin 2x, \sin 3x, \dots$

On a historical note, the calculation was troublesome in Fourier’s time, in the early 1800s. Concepts like convergence of series were not well understood. L. Euler, D. Bernoulli, and d’Alembert, all of whom had addressed similar problems in the mid-1700s regarding the wave equation and vibrating strings, had wondered about the possibility of expanding a nonperiodic function  $f(x)$  in terms of periodic functions like sines and cosines. It was L. Dirichlet who in 1829 finally established conditions under which such series representations like (3.6) are valid. In fact, these ideas about convergence of Fourier series gave tremendous impetus to the development of mathematical analysis in the 19th century.

To determine the coefficients  $a_n$  that make (3.6) valid, one could proceed formally and multiply both sides of (3.6) by  $\sin mx$ , for some fixed but arbitrary index  $m$ , and then integrate from  $x = 0$  to  $x = \pi$  to obtain

$$\int_0^\pi f(x) \sin mx \, dx = \int_0^\pi \sum_{n=1}^{\infty} a_n \sin nx \sin mx \, dx.$$

If we interchange the integral and the summation sign we obtain

$$\int_0^\pi f(x) \sin mx \, dx = \sum_{n=1}^{\infty} a_n \int_0^\pi \sin nx \sin mx \, dx.$$

The integrals in the right side summand can be calculated. We find that

$$\int_0^\pi \sin nx \sin mx \, dx = 0, \quad n \neq m. \quad (3.7)$$

Amazingly, this fact forces the infinite series to collapse to a single term. So the only term that survives in the infinite series is the one where the dummy summation index  $n$  hits  $m$ , the fixed index. Therefore we get

$$\int_0^\pi f(x) \sin mx \, dx = a_m \int_0^\pi \sin^2 mx \, dx = \frac{\pi}{2} a_m.$$

Because  $m$  is an arbitrary fixed index, we can replace it by  $n$  and so we have shown that the coefficients in (3.6) are given by

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad n = 1, 2, 3, \dots \quad (3.8)$$

Therefore, in summary, in a formal sense<sup>6</sup> the solution to the heat flow problem (3.1)–(3.2)–(3.4) is given by (3.5), where the coefficients  $a_n$  are given

<sup>6</sup> A formal calculation in mathematics is one done without complete rigor, but can be verified under special assumptions.

by (3.8). Indeed, one can verify that this is a solution for reasonable initial temperature distributions  $f(x)$ .

What we described is Fourier's method; namely, we solve a general initial boundary value problem by superimposing a set of solutions to the pure boundary value problem and then choosing the constants such that the initial condition is satisfied as well. The key to the calculation is that the functions  $\sin nx$  satisfy a relation like (3.7), which is called an **orthogonality condition**. It is analogous to the expansion of a vector in  $\mathbb{R}^n$  in terms of orthogonal vectors, or a basis. Orthogonality enables easy calculation of coefficients in the infinite series (3.6). The infinite series (3.6) for  $f(x)$  in terms of the orthogonal functions  $\sin nx$  is called an **orthogonal expansion**. We have glossed over several key steps, but in the next section we state the definitions and concepts in much greater detail and show their validity.

Fourier's method is the basic technique for solving boundary value problems for PDEs on bounded domains. One obvious, remaining, question is how we obtained the set of solutions

$$u_n(x, t) = e^{-n^2 t} \sin nx, \quad n = 0, 1, 2, \dots$$

that forms a basis for the expansion. The answer is: from a differential *eigenvalue problem*. We fully discuss this eigenvalue problem in Chapter 4. But for the present, we focus attention on the representation of functions by infinite series like the orthogonal expansion (3.6).

### EXERCISES

1. Consider the initial boundary value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < \pi, \quad t > 0, \quad (3.9)$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0, \quad (3.10)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 < x < \pi, \quad (3.11)$$

on a bounded spatial domain. Recall that this system models the small, transverse deflections of a taut string fixed at the two endpoints.

- a) Verify that the set of functions,

$$u_n(x, t) = \cos nct \sin nx, \quad n = 1, 2, \dots,$$

satisfies the PDE (3.9) and boundary conditions (3.10).

- b) These solutions are called **fundamental modes** and they represent **standing waves**. Plot several time profiles of the fundamental mode corresponding to  $n = 2$  and  $c = 1$ .

c) Superimpose the fundamental modes by defining

$$u(x, t) = \sum_n a_n \cos nct \sin nx$$

and determine the coefficients  $a_n$  for which  $u(x, t)$  satisfies the initial conditions (3.11).

d) Show formally that  $u(x, t)$  solves the initial BVP (3.9)–(3.11). (You may differentiate term-by-term.) These calculations amount to resolving an arbitrary wave into its fundamental modes.

e) Find a formal solution as above if the initial conditions are changed to

$$u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad 0 < x < \pi.$$

## 3.2 Orthogonal Expansions

From elementary calculus the reader is familiar with the notion of expanding a given function  $f(x)$  in an infinite series of simpler functions. For example, if  $f$  has infinitely many derivatives at a point  $x = a$ , then its *Taylor series* about  $x = a$  is

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,$$

where the  $c_n$ , the Taylor coefficients, are given by

$$c_n = f^{(n)}(a)/n!.$$

For example, the Taylor series for the exponential function about  $x = a = 0$  is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots.$$

Another type of series is a Fourier series. If  $f$  is defined and integrable on the interval  $[-\pi, \pi]$ , then its **Fourier series** is an infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad (3.12)$$

where the coefficients are calculated by the formulas

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nxdx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nxdx, \quad n = 1, 2, \dots$$

For Taylor series we use the set of polynomial functions  $1, x - a, (x - a)^2, \dots$  as the basis for the expansion, and for Fourier series we use the trigonometric functions  $1, \cos x, \cos 2x, \dots, \sin x, \sin 2x, \dots$  as the basis for the expansion. Each expansion has its advantages. From calculus we know that a few terms of the Taylor series can provide a good approximation to  $f$  locally near  $x = a$ . On the other hand, the first few terms of a Fourier series can provide a good global approximation to periodic signals.

### Example 3.1

We illustrate how to calculate the Fourier series (3.12) of the step function

$$f(x) = 0, \quad -\pi \leq x < 0, \quad f(x) = 1, \quad 0 \leq x \leq \pi$$

on the interval  $[-\pi, \pi]$ . The coefficients, given above, are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} 1 dx = 1,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = \frac{1}{n\pi} \sin nx \Big|_0^{\pi} = 0,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = -\frac{1}{n\pi} \cos nx \Big|_0^{\pi},$$

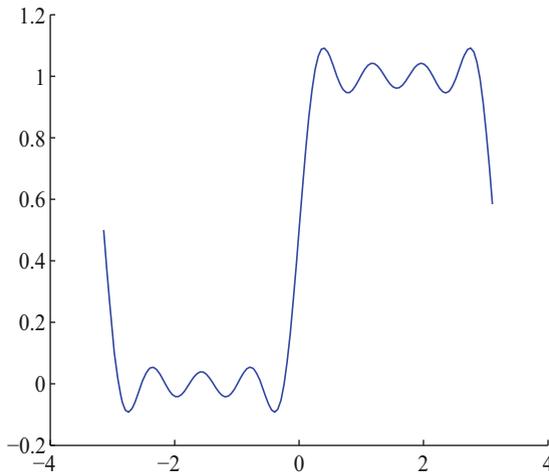
$$= -\frac{1}{n\pi} (\cos n\pi - 1) = \frac{1}{n\pi} (1 - (-1)^{n+1}).$$

Here we used the fact that  $\cos n\pi = (-1)^n$ . Thus, the Fourier series (3.12) is

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \dots \right).$$

A plot of the first five terms is shown in Figure 3.1. Note that the value of the Fourier series at the discontinuity  $x = 0$  seems to be  $1/2$ , the midpoint value at the jump.  $\square$

In this section we want to understand the underlying idea of a Fourier expansion by allowing the basis functions to be any set of functions with an *orthogonality* property. The result will be a general theory that incorporates Fourier series as a special case. Orthogonal is another word for perpendicular, but it has meaning in more general settings, such as with respect to sets of functions.



**Figure 3.1** A five-term Fourier ‘wavy’ series approximation of the function  $f(x) = 0$ ,  $-\pi \leq x < 0$ ,  $f(x) = 1$ ,  $0 \leq x \leq \pi$ . More terms give a better approximation in the open intervals  $(-\pi, 0)$  and  $(0, \pi)$

To fix the context, let us define a space of functions  $L^2 \equiv L^2[a, b]$  consisting of all the real-valued functions  $f$  defined on the interval  $[a, b]$  for which

$$\int_a^b |f(x)|^2 dx < \infty.$$

When this property holds, we say that  $f$  is **square integrable**<sup>7</sup> and write  $f \in L^2[a, b]$ . In this set of functions we define an inner product (or scalar product), which is analogous to the dot product for vectors. If  $f, g \in L^2$ , then the **inner product** of  $f$  and  $g$  is a real number, denoted by  $(f, g)$ , and is defined by

$$(f, g) = \int_a^b f(x)g(x)dx.$$

By the properties of integrals, the inner product of functions satisfies the same properties of dot products for vectors, namely, *symmetry*, *additivity*, and

<sup>7</sup> The integral here is, appropriately, the Lebesgue integral studied in advanced analysis courses rather than the Riemann integral, which is studied in elementary calculus; however, there will be no harm in interpreting the integrals in this text as a Riemann integrals since all of the functions that we examine are Riemann integrable (and therefore Lebesgue integrable).

*homogeneity:*

$$\begin{aligned}(f, g) &= (g, f), \\ (f, g + h) &= (f, g) + (f, h), \\ (f, \alpha g) &= \alpha(f, g), \quad \alpha \in \mathbb{R}.\end{aligned}$$

We also introduce the notion of length, or size, in the set  $L^2[a, b]$ . If  $f \in L^2[a, b]$ , then we define the **norm** of  $f$  to be the nonnegative real number

$$\|f\| \equiv \sqrt{(f, f)} = \sqrt{\int_a^b |f(x)|^2 dx}.$$

Observe that this definition is exactly analogous to the definition of the length of a vector  $\mathbf{v}$  in Euclidean space  $\mathbb{R}^n$ :  $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ . Introducing an inner product puts a geometric structure on  $L^2$  that allows us to define lengths and perpendicularity, just as in  $\mathbb{R}^n$ .

The set  $L^2$  is an example of a **function space** (a vector space of functions) analogous to  $\mathbb{R}^n$ , which is simply called a vector space. One can add functions in  $L^2$  in the usual way and multiply functions in  $L^2$  by scalars (real numbers). Those operations satisfy the same rules as addition of vectors in  $\mathbb{R}^n$  and multiplication of vectors by scalars. Thus  $L^2$  and  $\mathbb{R}^n$  are similar in their algebraic structures and their geometric structures. However, a big difference is that  $\mathbb{R}^n$  is finite dimensional, whereas the function space  $L^2$  is infinite dimensional. This means there is not a *finite* number of functions that form a basis for  $L^2$ .

By the way, for notational convenience we often drop the explicit dependence of functions on  $x$  and write just  $f$  in place of  $f(x)$ . In some instances, when  $f$  represents a signal, we think of  $x$  as time.

**Remark.** It should be clear that other important spaces of functions on the closed interval  $[a, b]$ , e.g., continuous functions, piecewise continuous functions, differentiable functions, and so forth, are subsets of  $L^2$ . So the set of functions  $L^2$  contains all the functions we normally work with in engineering and science.

### Example 3.2

Let  $f(x) = x^3$  and  $g(x) = 1 + x$  on the interval  $[0, 1]$ . These functions are clearly in  $L^2$ . The inner product is

$$(f, g) = \int_0^1 x^3(1 + x)dx = \frac{9}{20}.$$

The norm  $f$  is measured by

$$\|f\| = \sqrt{\int_0^1 (x^3)^2 dx} = \sqrt{\frac{1}{7}}.$$

The norm is the square root of the area under the graph of the *square* of the given function.  $\square$

We say that a set of functions  $f_1, f_2, f_3, \dots$  in  $L^2[a, b]$  is **orthogonal** on the interval  $[a, b]$  if  $(f_i, f_j) = 0$  for  $i \neq j$  and  $(f_j, f_j) = \|f_j\|^2 \neq 0$ .

### Example 3.3

The set  $1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots$  is orthogonal on  $[-\pi, \pi]$  because if we integrate the product of any two of these over the interval  $[-\pi, \pi]$  we get zero. The reader should verify this statement using formulas from a calculus book.

For another example, we observed in the last section that

$$(\sin mx, \sin nx) = \int_0^\pi \sin mx \sin nx \, dx = 0, \quad n \neq m.$$

Thus the set of functions  $\sin x, \sin 2x, \dots$  is orthogonal on the interval  $[0, \pi]$ . The reader should check that the power functions  $1, x, x^2, \dots$  that appear in a Taylor series do *not* form an orthogonal set on any interval  $[a, b]$ .  $\square$

Finally, we remark that functions can be **normalized** in the same way that a vector can be turned into a unit vector, namely by dividing by its norm, or size. If  $f \in L^2[a, b]$ , then the function  $g$  defined by

$$g(x) = \frac{f(x)}{\|f\|}$$

has unit norm. If each function in an orthogonal set of functions has unit norm, then the set is called **orthonormal** system. Thus, orthogonal systems functions can always be turned into orthonormal systems.

### Example 3.4

The set of functions

$$\sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots,$$

is orthogonal on the interval  $[0, l]$ , and  $\|\sin \frac{n\pi x}{l}\|^2 = l/2$ . Therefore, the functions

$$\sqrt{\frac{2}{l}} \sin \frac{n\pi x}{l}$$

form an orthonormal set of functions on  $[0, l]$ . An expansion of a function  $f(x)$  defined on  $[0, l]$  in terms of this set of orthonormal functions is called a **Fourier sine series**.  $\square$

### Convergence

The next question is a key one. Can an arbitrary function  $f \in L^2$  be expanded in an infinite series of a given orthogonal system of functions  $f_1, f_2, f_3, \dots$ ? That is, can we write

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \quad (3.13)$$

for some set of constants  $c_n$ , and in what sense can we interpret the convergence of the series to  $f(x)$ ? Physically, we think of (3.13) as decomposing a signal  $f$  into fundamental modes  $f_1, f_2, f_3, \dots$ . This is the same question that we asked in Section 3.1 regarding the feasibility of expanding a given initial temperature  $f(x)$  into a sine series.

This question of convergence is of course not an issue in finite dimensional spaces. But in function spaces this is a delicate question and there are several possible answers, depending on how we measure error in a partial sum. Precisely, we formulate three types the errors that can occur in an  $N$ -term approximation

$$S_N(x) \equiv \sum_{n=1}^N c_n f_n(x)$$

to a function  $f(x)$ . Each of these error expressions leads to a different definition of convergence of the infinite series.

1. The **pointwise error**  $E_N(x)$  is the function

$$E_N(x) \equiv f(x) - S_N(x) \equiv f(x) - \sum_{n=1}^N c_n f_n(x).$$

Thus, for a fixed number of terms  $N$ ,  $E_N(x)$  is a function whose values give the error at each point  $x$ . If for each *fixed*  $x$  in  $[a, b]$  we have  $\lim_{N \rightarrow \infty} E_N(x) = 0$ , then we say that the infinite series (3.13) **converges pointwise** to  $f$  on the interval  $[a, b]$ ; in this case, for each fixed  $x$  the series is a numerical series that converges to the numerical value  $f(x)$ , at that  $x$ . Pointwise convergence means that the error goes to zero at each point. This is the type of convergence that a calculus student first encounters.

2. The **mean-square error** in the approximation is the number  $e_N$  defined by

$$e_N \equiv \int_a^b |f(x) - S_N(x)|^2 dx \equiv \int_a^b |f(x) - \sum_{n=1}^N c_n f_n(x)|^2 dx.$$

Note that  $e_N$  is just the square of the pointwise error, integrated over the interval  $[a, b]$ . Thus, the mean-square error is an integrated average over all

the pointwise errors. We say that the infinite series (3.13) converges to  $f$  in the **mean-square** sense if

$$\lim_{N \rightarrow \infty} e_N = 0.$$

Mean-square convergence is also called **convergence in  $L^2[a, b]$** . Thus, we can write the last line in terms of the norm as

$$\lim_{N \rightarrow \infty} \|f - S_N\|^2 = 0.$$

Note that we can remove the square on the norm, which is common. It requires that the integrated pointwise error-squared go to zero as more and more terms are taken. In advanced courses it is shown that pointwise convergence does not imply mean-square convergence, and conversely.

3. The **uniform error**  $\varepsilon_N$  is the maximum pointwise error over the interval  $[a, b]$ , or

$$\varepsilon_N = \max_{a \leq x \leq b} |f(x) - S_N(x)| = \max_{a \leq x \leq b} \left| f(x) - \sum_{n=1}^N c_n f_n(x) \right|.$$

Uniform convergence is a stronger type of convergence; it implies both mean-square and pointwise convergence. We say that the series (3.13) **converges uniformly** to  $f(x)$  on  $[a, b]$  if

$$\lim_{N \rightarrow \infty} \varepsilon_N = \lim_{N \rightarrow \infty} \left\{ \max_{a \leq x \leq b} |f(x) - S_N(x)| \right\} = 0.$$

Thus, for any given tolerance  $\epsilon > 0$ , we can find a number  $N$  (representing the number of terms) such that for all  $n > N$  we have

$$\max_{a \leq x \leq b} |f(x) - S_n(x)| < \epsilon.$$

That is, the error can be made uniformly small over the entire interval by choosing the number of terms large enough. Uniform convergence depends only on the tolerance  $\epsilon$  and not on the position in the interval where the error is calculated. [In some cases “max” must be replaced “sup” (for *supremum*); see an advanced calculus text, e.g., Spivak (2008).]

By the way, all of the definitions above are valid for series of functions, whether or not the functions  $f_n(x)$  are mutually orthogonal.

### Example 3.5

Investigate the convergence the series

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad -1 < x < 1.$$

Here we can find the partial sums explicitly. We have

$$S_N(x) = \sum_{n=0}^N (-1)^n x^{2n} = \sum_{n=0}^N (-x^2)^n = \frac{1 - (-x^2)^{N+1}}{1 + x^2},$$

where we have used the geometric sum

$$1 + z + z^2 + z^3 + \cdots + z^N = \frac{1 - z^{N+1}}{1 - z}.$$

Simplifying,

$$S_N(x) = \frac{1 + (-1)^{N+2} x^{2(N+1)}}{1 + x^2} \rightarrow \frac{1}{1 + x^2}, \quad -1 < x < 1.$$

Therefore the series converges pointwise to  $1/(1 + x^2)$  on  $-1 < x < 1$ .

To check uniform convergence we note that the uniform error is

$$\varepsilon_N = \max_{-1 < x < 1} \left| S_N(x) - \frac{1}{1 + x^2} \right| = \max_{-1 < x < 1} \frac{x^{2(N+1)}}{1 + x^2} = \frac{1}{2}.$$

This does not go to zero as  $N \rightarrow \infty$ , so the convergence is *not* uniform.

For mean-square convergence,

$$E_N = \int_{-1}^1 \left| S_N(x) - \frac{1}{1 + x^2} \right|^2 dx = \int_{-1}^1 \frac{x^{4N+4}}{(1 + x^2)^2} dx.$$

But, on the interval  $-1 < x < 1$  we have  $\frac{1}{(1+x^2)^2} \leq 1$ . Therefore

$$E_N = \int_{-1}^1 \frac{x^{4N+4}}{(1 + x^2)^2} dx < \int_{-1}^1 x^{4N+4} dx = \frac{2}{4N + 5} \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Therefore the series converges in the mean-square sense.  $\square$

A novice may not appreciate subtle differences in convergence and wonder why we have defined three types (actually, there are others). Suffice it to say that the error may not go to zero in a pointwise or uniform manner, but that does not mean the approximation is not useful; a weaker form of convergence, like mean-square convergence, may be all that is required.

Now suppose (3.13) holds in the sense of mean-square convergence where  $f$  is a given function in  $L^2$  and  $f_n$  is a *given orthogonal system* in  $L^2$ . How can we

compute the coefficients  $c_n$ ? The demonstration here is exactly that which is given in Section 3.1 where we showed how to compute the coefficients in a sine series. Orthogonality is crucial. We assume in this argument that the series in (3.13) can be integrated term by term (it can be proved). If we multiply both sides of (3.13) by  $f_m$ , where  $m$  is a fixed, but arbitrary index, and integrate over the interval  $[a, b]$ , we get, using the inner product notation,

$$\begin{aligned}(f, f_m) &= \int_a^b \sum_{n=1}^{\infty} c_n f_n(x) f_m(x) dx \\ &= \sum_{n=1}^{\infty} c_n (f_n, f_m) \\ &= c_m (f_m, f_m) = c_m \|f_m\|^2.\end{aligned}$$

Note that the infinite sum collapsed to a single term because of the orthogonality of the  $f_n$ . So the  $c_n$  are given by

$$c_n = \frac{1}{\|f_n\|^2} (f, f_n), \quad n = 1, 2, \dots \quad (3.14)$$

We summarize in the following theorem.

### Theorem 3.6

Let  $f \in L^2[a, b]$  and  $\{f_n\}$  be an orthogonal system of  $L^2$  functions on  $[a, b]$ . If

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x) \quad (3.15)$$

in the mean square sense, then the coefficients  $c_n$  are given by (3.14).  $\square$

The  $c_n$  are called the **Fourier coefficients**, and when the coefficients are computed in this way, the orthogonal series (3.15) is called the **generalized Fourier series** for  $f$ . In analogy with finite dimensional vector spaces, like  $\mathbb{R}^n$ , the Fourier coefficient  $c_n$  is interpreted as the projection of  $f$  onto the subspace defined by  $f_n$ .

### Remark 3.7

If the series expansion (3.15) is valid in the mean-square sense and the  $f_n$  are **orthonormal**, then the Fourier coefficients (3.14) are given simply by

$$c_n = (f, f_n), \quad (f_n \text{ orthonormal}). \quad \square \quad (3.16)$$

There is another useful property of Fourier coefficients.

### Theorem 3.8

**(Best approximation)** Let  $\{c_n\}$  be the set of Fourier coefficients in the finite expansion of  $f \in L^2[a, b]$  in the orthonormal system  $f_1, f_2, \dots, f_N$ . If  $\{a_n\}$  is any other sequence of coefficients, then

$$\|f - \sum_{n=1}^N c_n f_n\|^2 \leq \|f - \sum_{n=1}^N a_n f_n\|^2. \quad (3.17)$$

That is, the mean-square error is minimized for the approximation with the Fourier coefficients  $c_n$  given by (3.16).  $\square$

This result is called **best approximation**. It answers the question how we can best approximate  $f$  by a finite linear combination of the  $f_n$ . Just use the Fourier coefficients.

The demonstration of (3.17) is straightforward using the definition of norm and properties of the inner product. We have

$$\begin{aligned} \|f - \sum_{n=1}^N a_n f_n\|^2 &= \left( f - \sum_{n=1}^N a_n f_n, f - \sum_{n=1}^N a_n f_n \right) \\ &= (f, f) - 2 \sum_{n=1}^N a_n (f, f_n) + \sum_{n=1}^N a_n^2 \\ &= (f, f) - 2 \sum_{n=1}^N a_n c_n + \sum_{n=1}^N a_n^2 + \sum_{n=1}^N c_n^2 - \sum_{n=1}^N c_n^2 \\ &= (f, f) - \sum_{n=1}^N c_n^2 + \sum_{n=1}^N (a_n - c_n)^2 \\ &= \|f - \sum_{n=1}^N c_n f_n\|^2 + \sum_{n=1}^N (a_n - c_n)^2. \end{aligned}$$

Because the second term on the right is nonnegative, (3.17) is valid. If the  $f_n$  are not orthonormal, and just orthogonal, then the same result is true with the  $c_n$  given by (3.14).  $\square$

In the last line of the proof we used the fact that

$$\|f - \sum_{n=1}^N c_n f_n\|^2 = \|f\|^2 - \sum_{n=1}^N c_n^2,$$

which the reader should verify. But this equality leads to another interesting inequality. Because the left side is nonnegative, we have

$$\sum_{n=1}^N c_n^2 \leq \|f\|^2.$$

This inequality is valid for each  $N$ , and so

$$\sum_{n=1}^{\infty} c_n^2 \leq \|f\|^2, \quad (\text{Bessel's Inequality})$$

which is **Bessel's inequality**. It shows that the series of squared Fourier coefficients converges, and thus  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ . So the Fourier coefficients get smaller and smaller.

### Example 3.9

**(Best approximation)** Note that the set of functions  $f_n(x) = \sin nx$  forms an orthogonal set of functions on  $[0, \pi]$ . Find the best approximation of the function  $f(x) = 8x(\pi - x)e^{-x^2}$  of the form

$$c_1 \sin x + c_2 \sin 2x + c_3 \sin 3x + c_4 \sin 4x + c_5 \sin 5x.$$

We know that the  $c_n$  are the Fourier coefficients given by

$$c_n = \frac{(f, \sin nx)}{\|\sin nx\|^2} = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Using a calculator we obtain

$$c_1 = 3.708, \quad c_2 = 4.040, \quad c_3 = 2.230, \quad c_4 = 0.798, \quad c_5 = 0.295.$$

The reader should plot  $f(x)$  and the Fourier approximation on the interval  $[0, \pi]$  to observe how much the two differ.  $\square$

Note that all the previous results assume that  $\sum c_n f_n$  converges in the mean square sense to  $f$ . So we need another result to round out our discussion: namely, if the  $\{f_n\}$  form an orthonormal set and we form the generalized Fourier series (3.13) with  $c_n$  the Fourier coefficients given by (3.16), does the series automatically converge to  $f$  in one sense or another? The answer to the question is yes, in the mean-square sense, provided that the orthonormal system  $\{f_n\}$  is complete.

### Definition 3.10

An orthonormal system  $\{f_n\}$  in  $L^2$  is said to be **complete** if, and only if, there is *no* nontrivial  $f \in L^2$  which is orthogonal to all the  $f_n$ . That is, if  $(f, f_n) = 0$  for  $n = 1, 2, 3, \dots$ , then  $f \equiv 0$ .  $\square$

Thus, an orthonormal sequence  $\{f_n\}$  is complete if the only function having all its Fourier coefficients vanish is the zero function. Sometimes it is difficult to show completeness, and we shall just state whether a given orthonormal system is complete. It is shown in advanced treatments that completeness is equivalent to the property that *any* function  $f \in L^2$  can be expanded in a generalized Fourier series (3.13), where the  $c_n$  are the Fourier coefficients, and that the series converges to  $f$  in the mean-square sense. Furthermore, it can be proved that completeness is equivalent to strict equality holding in the Bessel inequality; i.e.,  $\{f_n\}$  is complete if, and only if, for each  $f \in L^2$ ,

$$\sum_{n=1}^{\infty} c_n^2 = \|f\|^2.$$

This equation is called **Parseval's equality**.

Pointwise convergence results, or even stronger uniform convergence results, are more difficult to obtain. Typically, for these these types of results continuity and differentiability conditions on the function  $f$  are required. We state these results in the next section for orthogonal, trigonometric systems.

We can extend the modal interpretation of the generalized Fourier series still further. We can think of  $f$  as a signal and the orthonormal set  $\{f_n\}$  as fundamental modes. The Fourier coefficient  $c_n$  determines the contribution of the  $n$ th mode, and the generalized Fourier series is the decomposition of the signal into fundamental modes. The sequence of squared coefficients,  $c_1^2, c_2^2, c_3^2, \dots$ , is called the **energy spectrum**, and  $c_n^2$  is called the energy of the  $n$ th mode; by Parseval's equality, the **total energy** in the signal is the norm-squared,  $\|f\|^2$ .

### EXERCISES

The trigonometric identities

$$\begin{aligned}\sin a \sin b &= \frac{1}{2}(\cos(a - b) - \cos(a + b)) \\ \sin a \cos b &= \frac{1}{2}(\sin(a + b) + \sin(a - b)),\end{aligned}$$

along with others, are useful in the calculations.

1. Derive a trigonometric identity similar to the ones above for  $\cos a \cos b$ .

2. a) Verify that the set of functions  $1, \cos x, \cos 2x, \dots$  forms an orthogonal system on the interval  $[0, \pi]$ .

- b) Verify that the set of functions

$$\cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \dots,$$

form an orthogonal system on the interval  $[0, l]$ .

- c) If

$$f(x) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi x}{l}\right)$$

in the mean-square sense on  $[0, l]$ , what is the formula for the  $c_n$ ? This series is called the **Fourier cosine series** for  $f$  on  $[0, l]$ .

3. Let  $f(x) = 0$  for  $0 < x < 1$  and  $f(x) = 1$  for  $1 < x < 3$ .

- a) Find the first 4 nonzero terms of the Fourier cosine series of  $f$ .  
 b) What is the sum of this series on  $0 \leq x \leq 3$ ?  
 c) Why does the series converge to  $f(x)$  in the mean-square sense?  
 d) Find the value of the infinite sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \dots$$

4. Consider the infinite series

$$\sum_{n=1}^{\infty} (1-x)x^{n-1}, \quad 0 < x < 1.$$

- a) Show that the series converges pointwise. Hint: Find the partial sums.  
 b) Show that the convergence is not uniform.  
 c) Show that the series converges in the mean-square sense.
5. Consider the infinite series

$$\sum_{n=1}^{\infty} \left( \frac{n}{1+n^2x^2} - \frac{n-1}{1+(n-1)^2x^2} \right), \quad 0 < x < 1.$$

- a) Show that the series converges pointwise. Hint: The series ‘telescopes.’  
 b) Show that the convergence is not uniform.  
 c) Show that the series does not converge in the mean-square sense.

6. If  $c_n$  are Fourier coefficients of  $f$  and  $f_n$  is an orthonormal set, show that

$$\left( \sum_{n=1}^N c_n f_n, f - \sum_{n=1}^N c_n f_n \right) = 0.$$

7. (**Gram–Schmidt orthogonalization**) We know from elementary linear algebra that any set of linearly independent vectors may be turned into a set of orthogonal vectors by the Gram–Schmidt orthogonalization process. The same process works for functions in  $L^2[a, b]$ . Let  $f_1, f_2, f_3, \dots$  be an independent set of functions in  $L^2$ . Define the set of functions  $g_n$  by

$$g_1 = f_1, \quad g_2 = f_2 - \frac{(f_2, g_1)}{\|g_1\|^2} g_1, \quad g_3 = f_3 - \frac{(f_3, g_2)}{\|g_2\|^2} g_2 - \frac{(f_3, g_1)}{\|g_1\|^2} g_1, \dots$$

Show that  $g_n$  is an orthogonal set.

8. The functions  $1, x, x^2, x^3$  are independent functions on the interval  $[-1, 1]$ .
- Use the preceding exercise to generate a set of four orthogonal polynomials  $P_0(x), \dots, P_3(x)$  on  $[-1, 1]$ . These are called **Legendre polynomials**.

- b) Find the best approximation of  $e^x$  on  $[-1, 1]$  of the form

$$e^x \approx c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x).$$

- c) Plot  $e^x$  and the approximation on a set of coordinate axes.

- d) What is the pointwise error? What is the maximum pointwise error over  $[-1, 1]$ . What is the mean-square error?

9. For  $f, g \in L^2[a, b]$ , prove the **Cauchy–Schwarz inequality**

$$|(f, g)| \leq \|f\| \|g\|.$$

Hint: Define  $Q(t) = (f + tg, f + tg)$  for any real number  $t$ . Use the rules of inner product to expand this expression and obtain a quadratic polynomial in  $t$ ; because  $Q(t) \geq 0$  (why?), the quadratic polynomial can have at most one real root. Examine the discriminant of the polynomial.

10. Verify the Cauchy–Schwarz inequality for  $f(x) = x$  and  $g(x) = e^{-x}$  on  $0 \leq x \leq 1$ .
11. Let  $f, g \in L^2[a, b]$ . Show that the triangle inequality  $\|f + g\| \leq \|f\| + \|g\|$  holds. Hint: Consider  $\|f + g\|^2$ .
12. For which powers  $r$  is the function  $f(x) = x^r$  in  $L^2[0, 1]$ ? In  $L^2[1, \infty)$ ?

13. Let  $f$  be defined and integrable on  $[0, l]$ . The orthogonal expansion

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx,$$

is called the **Fourier sine series** for  $f$  on  $[0, l]$ . Find the Fourier sine series for  $f(x) = \cos x$  on  $[0, \pi/2]$  and plot a 6-term approximation. What is the Fourier sine series of  $f(x) = \sin x$  on  $[0, \pi]$ ?

14. Find the numbers  $b_1, b_2, b_3$  such that

$$y(x) = b_1 \sin \frac{\pi x}{2} + b_2 \sin \frac{2\pi x}{2} + b_3 \sin \frac{2\pi x}{2}$$

is the best approximation to  $f(x) = 1$  on  $(0, 2)$ . Draw a plot of the approximation.

15. Expand the function  $f(x) = 1$  on  $[0, 1]$  in a Fourier sine series and derive the interesting formula

$$\frac{\pi}{4} = \sin 1 + \frac{1}{3} \sin 3 + \frac{1}{5} \sin 5 + \dots$$

### 3.3 Classical Fourier Series

In the last section we introduced the concept of representing a given function  $f(x)$  in terms of an infinite series of orthogonal functions  $f_n(x)$ . Now we focus those concepts on a special set of orthogonal functions where the  $f_n$  are given by sines and cosines. The resulting series is called a (classical) **Fourier series**, or **trigonometric series**. In this section we drop the adjective “classical.” To contrast it with Fourier cosine and Fourier sine series, sometimes we refer to it as the full Fourier series.

We work on an arbitrary symmetric interval  $[-l, l]$  about the origin. If  $f$  is an integrable function on  $[-l, l]$ , then its **Fourier series** is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \quad (3.18)$$

where the **Fourier coefficients**  $a_n$  and  $b_n$  are given by the formulas

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots, \quad (3.19)$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots \quad (3.20)$$

Here, the set of functions

$$\frac{1}{2}, \cos \frac{n\pi x}{l}, \sin \frac{n\pi x}{l}, \quad n = 1, 2, \dots, \quad (3.21)$$

forms an orthogonal system on the interval  $[-l, l]$ . They are playing the role of the  $f_n(x)$  in formula (3.13), and the  $a_n$  and  $b_n$  are playing the role of the coefficients  $c_n$ . It is shown in more advanced texts that the set of functions (3.21) is complete, and therefore the Fourier series (3.18) converges in the mean-square sense to  $f$  when  $f \in L^2[-l, l]$ .

### Example 3.11

Let  $f(x) = x$  for  $-l \leq x \leq l$ . Then the Fourier coefficients are easily computed using integration by parts to get

$$a_n = \frac{1}{l} \int_{-l}^l x \cos \frac{n\pi x}{l} dx = 0, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{l} \int_{-l}^l x \sin \frac{n\pi x}{l} dx = \frac{2l}{n\pi} (-1)^{n+1}, \quad n = 1, 2, \dots$$

Therefore the Fourier series for  $f$  is

$$\frac{2l}{\pi} \left( \sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \dots \right).$$

We make two observations. First, at  $x = \pm l$  the series clearly converges to zero and thus does not converge to  $f(x)$  at these two points. So the series does not converge pointwise to  $f(x)$  on the interval  $[-l, l]$ . It does, however, converge pointwise to  $f(x) = x$  on  $(-l, l)$ . Second, the derived series obtained by differentiating term-by-term,

$$2 \sum_{n=1}^{\infty} (-1)^{n+1} \cos \frac{n\pi x}{l},$$

does not converge at all, much less to the derivative  $f'(x) = 1$ . (Notice the  $n$ th term does not go to zero.) So the series cannot be differentiated term by term. We do know from Section 3.2 that the series converges to  $f$  in the mean-square sense.  $\square$

For the Fourier series (3.18) we define the  $n$ th **harmonic** to be

$$a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}.$$

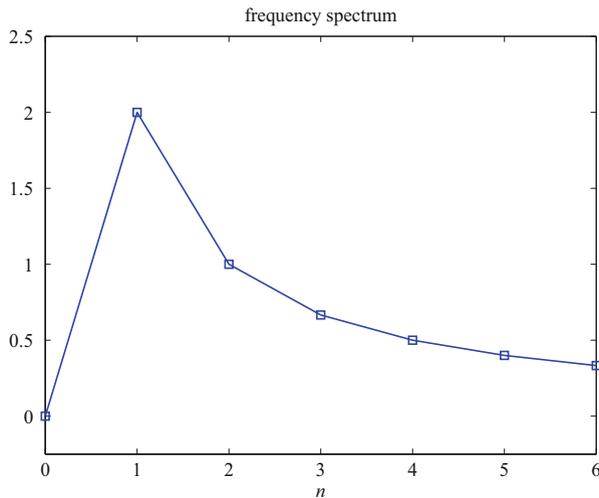
Each harmonic in the series has a higher frequency (and thus more oscillations) than the preceding harmonic. The **frequency spectrum** of the series is the sequence of numbers  $\gamma_n$  defined by

$$\gamma_0 = \frac{|a_0|}{\sqrt{2}}, \quad \gamma_n = \sqrt{a_n^2 + b_n^2} \quad (n \geq 1).$$

The frequency spectrum is a measure of the contribution of the various harmonics in the decomposition of  $f$ . The numbers  $\gamma_n^2$ ,  $n \geq 0$ , form the **energy spectrum**. The reader is invited to show that **Parseval's equality** takes the form

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{l} \|f\|^2.$$

The frequency spectrum for  $f(x) = x$  in the last example is  $\gamma_0 = 0$ ,  $\gamma_n = 2l/(n\pi)$ ,  $n \geq 1$ . It is often graphed, as in Figure 3.2, to show visually the contribution of each harmonic.



**Figure 3.2** Frequency spectrum for the function  $f(x) = x$  on  $[-\pi, \pi]$

A Fourier series simplifies considerably if  $f$  is an even or an odd function. First observe that  $\sin \frac{n\pi x}{l}$  is an odd function and  $\cos \frac{n\pi x}{l}$  is an even function; moreover, an even function times an odd function is odd. Therefore, if  $f$  is an even function, then the product  $f(x) \sin \frac{n\pi x}{l}$ , which is the integrand in the expression for  $b_n$  in (3.18), is an odd function; so all of the coefficients  $b_n$  are

zero because an odd function integrated over a symmetric interval about the origin is zero. Hence:

If  $f$  is an even function, then its Fourier series reduces to a cosine series having the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l},$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots$$

Similarly,

If  $f$  is an odd function, then the coefficients  $a_n$  in (3.18) vanish, and the Fourier series reduces to a sine series

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

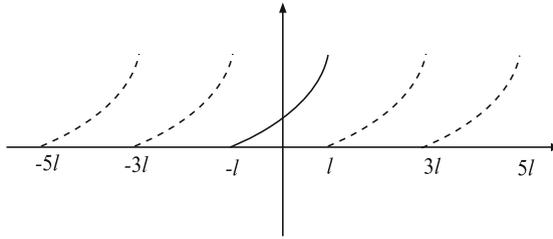
$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots$$

This last fact is illustrated in the previous example, where we found the Fourier sine series for  $f(x) = x$ , an odd function.

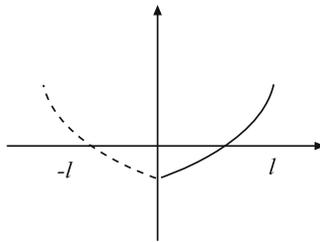
### **Periodic functions on $\mathbb{R}$**

Given any integrable function on  $(-l, l)$ , we can always form its full Fourier series (3.18). However, the Fourier series (3.18) itself is a  $2l$ -periodic function, meaning that it repeats its values every interval of length  $2l$ . (Recall that a function  $\phi$  is  $P$ -periodic, or periodic of period  $P$ , if it repeats itself in every interval of length  $P$ ; that is,  $f(x + P) = f(x)$  for all  $x$ .) This means that the Fourier series represents not only the function  $f$  on  $(-l, l)$ , but also represents its  $2l$  **periodic extension** to the entire real line. The  $2l$  periodic extension of  $f$  is found by plotting the graph of  $f$  on  $(-l, l)$  and then translating its graph to the left and to the right to every interval of length  $2l$ ; that is, to the intervals  $(l, 3l)$ ,  $(3l, 5l)$ ,  $\dots$ ,  $(-3l, -l)$ ,  $(-5l, -3l)$ ,  $\dots$ . See Figure 3.3.

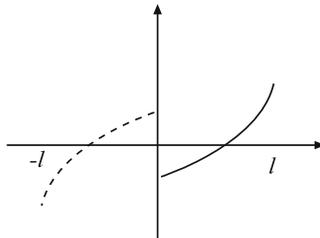
If a function  $f$  is defined only on  $(0, l)$ , then we can extend it to an even function on  $(-l, l)$ . See Figure 3.4. This is called its **even extension**. In this case, the full Fourier series reduces to a cosine series. If we extend  $f$  to an odd function on  $(-l, l)$ , the so-called odd extension (see Figure 3.5), then its full Fourier series reduces to a sine series.



**Figure 3.3** A function  $f$  defined on  $(-l, l)$  (*solid*) and its  $2l$ -periodic extension (*dashed*)



**Figure 3.4** A function  $f$  defined on  $(0, l)$  (*solid*) and its even extension (*dashed*) to  $(-l, l)$



**Figure 3.5** A function  $f$  defined on  $(0, l)$  (*solid*) and its odd extension (*dashed*) to  $(-l, l)$

### Example 3.12

Consider the 2-periodic signal shown in Figure 3.6, which is called a triangular wave. Analytically it is given by  $f(x) = x + 1$  if  $-1 < x \leq 0$ ;  $f(x) = 1 - x$  if  $0 < x \leq 1$ ; and otherwise 2-periodic. We compute its Fourier series. Here,  $f$  is an even function, and so  $b_n = 0$  for all  $n$ . The coefficients  $a_n$  are given by

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x dx \\ &= \int_{-1}^0 (x + 1) \cos n\pi x dx + \int_0^1 (1 - x) \cos n\pi x dx. \end{aligned}$$

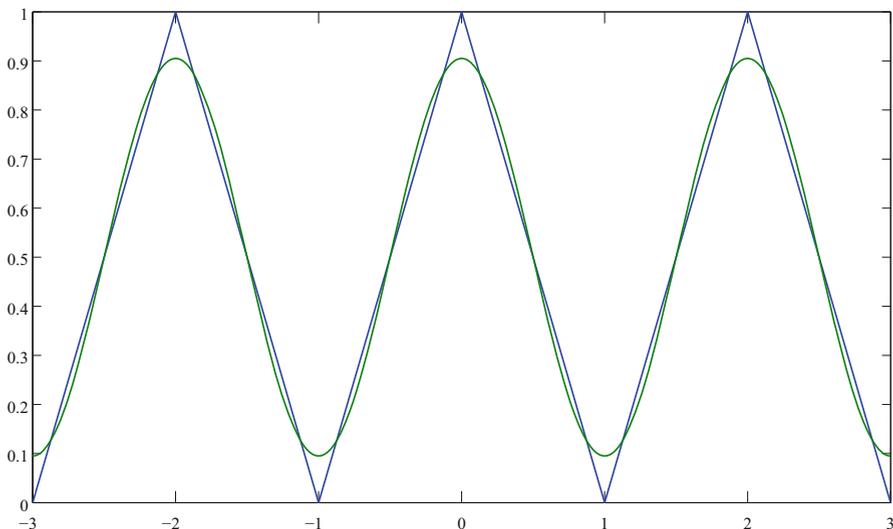
When  $n = 0$ , we easily get  $a_0 = 1$ . For  $n \geq 1$  we can calculate  $a_n$  by hand (using integration by parts), use a calculator, or use a computer algebra package. The first five coefficients are

$$a_1 = 0.405, \quad a_2 = 0, \quad a_3 = 0.045, \quad a_4 = 0, \quad a_5 = 0.016.$$

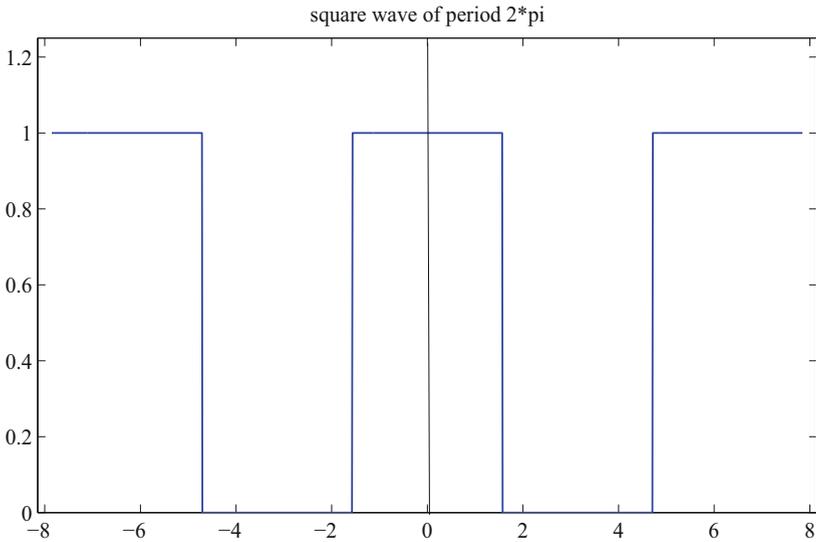
Therefore, a four-term Fourier approximation to the triangular wave is

$$f(x) \approx 0.5 + 0.405 \cos \pi x + 0.045 \cos 3\pi x + 0.016 \cos 5\pi x.$$

Figure 3.6 shows a plot of  $f$  and its approximation. From the last section we know that this approximation is the best, in the mean-square sense.  $\square$



**Figure 3.6** A triangular wave and its 4-term Fourier expansion



**Figure 3.7** Graph of a  $2\pi$ -periodic square wave. See Exercise 1

### Remark 3.13

**(Exponential form of Fourier Series)** Using the fact that sine and cosine functions may be written in terms of the complex exponential function, that is,

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}), \quad \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}),$$

it is evident that the full Fourier series for a function  $f$  on  $(-l, l)$  can be written in exponential form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.$$

It is not hard to see that

$$\int_{-l}^l e^{in\pi x/l} e^{-im\pi x/l} dx = \begin{cases} 0, & m \neq n \\ 2l, & m = n \end{cases}.$$

(Note the minus sign on the second factor in the integral.) This entails a modification of the definition of the inner product for complex-valued functions to

$$(f, g) = \int_{-l}^l f(x) \overline{g(x)} dx,$$

where the *overbar* means complex conjugation. By previous methods we find that the complex Fourier coefficients are given by

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-in\pi x/l} dx.$$

The complex form is useful in many calculation involving exponential and hyperbolic functions. We can always return to the real Fourier series by isolating the real and imaginary parts of the series.  $\square$

Finally we make some comments about convergence of Fourier series. We already noted that if  $f$  is square integrable, then mean-square convergence is automatic. To obtain pointwise convergence results some additional assumptions must be made regarding the smoothness of  $f$ . Our assumption is that the graph of  $f$  is made up of finitely many smooth pieces; the function is not necessarily continuous. This will cover most of the interesting functions encountered in science and engineering. A function  $f$  is **piecewise continuous** on  $[a, b]$  if it is continuous except possibly at a finite number of points in  $[a, b]$  where it has simple jump discontinuities;  $f$  has a simple jump discontinuity at  $x = c$  if both one-sided limits  $f(c^+)$  and  $f(c^-)$  exist and are finite at  $c$ . The function may or may not be defined at a jump discontinuity. We say  $f$  is **piecewise smooth** on  $[a, b]$  if both  $f$  and  $f'$  are piecewise continuous on  $[a, b]$ . We say  $f$  is piecewise smooth on  $(-\infty, \infty)$  if it is piecewise smooth on each bounded subinterval  $[a, b]$  of the real line. Then the basic pointwise convergence theorem is the following:

### Theorem 3.14

**(Pointwise convergence)** If  $f$  is piecewise smooth on  $[-l, l]$  and otherwise  $2l$ -periodic, then its Fourier series (3.18) converges pointwise for all  $x \in \mathbb{R}$  to the value  $f(x)$  if  $f$  is continuous at  $x$ , and to the average value of its left and right limits at  $x$ , namely  $\frac{1}{2}(f(x^-) + f(x^+))$ , if  $f$  is discontinuous at  $x$ .  $\square$

To get stronger convergence results, like uniform convergence, additional smoothness conditions on  $f$  are required. Also, continuity of  $f$  is not enough to guarantee pointwise convergence; incredible as it may seem, there are continuous functions whose Fourier series diverge at every point! Next is a basic result for uniform convergence of the Fourier series for a given function.

### Theorem 3.15

**(Uniform convergence)** Let  $f$  be a continuous function on the interval  $[-l, l]$  with  $f(-l) = f(l)$ , and suppose its derivative is piecewise continuous on that interval. Then the Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

converges absolutely and uniformly to  $f(x)$  on that interval.  $\square$

### EXERCISES

1. Find the Fourier series for the  $2\pi$ -periodic square wave shown in Figure 3.7. Sketch a two-term, a four-term, and a six-term approximation.
2. Show that the Fourier cosine series for  $\sin x$  on  $0 < x < \pi$  is

$$\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

Show graphically the function represented by the series on the interval  $(-\pi, \pi)$ . On the entire real line?

3. Find the Fourier series for  $f(x) = x^2$  on  $[-\pi, \pi]$ . Sketch a graph of the function defined on all of  $\mathbb{R}$  to which the Fourier series converges. Is the convergence pointwise? Use the Fourier series to show

$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Graph the frequency spectrum.

4. Derive the series representation

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

using the function  $f(x) = |x|$ ,  $-\pi \leq x \leq \pi$ .

5. Find the Fourier sine series of  $f(x) = \pi - x$  on  $(0, \pi)$ .
6. Let  $f(x) = 1$ ,  $0 < x < \pi/2$  and  $f(x) = 0$ ,  $\pi/2 < x < \pi$ . Find the Fourier cosine series and the Fourier sine series. What is the full Fourier series? Explicitly characterize the values of  $x \in \mathbb{R}$  where each converges pointwise.
7. Consider the function (signal) defined by  $f(x) = x+1$  for  $-2\pi < x \leq 0$  and  $f(x) = x$  for  $0 < x \leq 2\pi$ , and otherwise  $4\pi$  periodic. Sketch a graph of the signal and find its Fourier series. Find and graph the frequency spectrum. To what value does the Fourier series converge at  $x = 0$ ? At  $x = 2\pi$ ? At  $x = \pi$ ? Graph the sum of the first four harmonics and observe how well it approximates  $f$ .
8. Show that for all  $x$  in  $(0, 1)$ ,

$$\cos \pi x = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin 2n\pi x.$$

9. Show that

$$e^x = \frac{\sinh \pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 - in} e^{inx}$$

for  $\pi < x < \pi$ . Write the series in real quantities.

10. Show that the Fourier series for  $f(x) = \cos ax$  on  $[-\pi, \pi]$ , where  $a$  is not an integer, is

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left( \frac{1}{2a^2} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{a^2 - n^2} \right).$$

Show

$$\csc a\pi = \frac{1}{a\pi} + \frac{2a}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2 - a^2}.$$

11. Let  $f(x) = -\frac{1}{2}$  on  $-\pi < x \leq 0$  and  $f(x) = \frac{1}{2}$  on  $0 \leq x \leq \pi$ . Show that the Fourier series for  $f$  is

$$\sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)x.$$

If  $s_N(x)$  denotes the sum of the first  $N$  terms, sketch a graph of  $s_1(x)$ ,  $s_3(x)$ ,  $s_7(x)$ , and  $s_{10}(x)$  and compare to  $f(x)$ . Observe that the approximations overshoot  $f(x)$  in a neighborhood of  $x = 0$ , and the overshoot is not improved regardless of how many terms are taken in the approximation. This overshoot behavior of Fourier series near a discontinuity is called **Gibbs phenomenon**. The series does converge pointwise to  $f(x)$  on  $(-\pi, 0)$  and on  $(0, \pi)$ . At  $x = 0$  it converges pointwise to the average of the left and right limits, or zero. It does not converge uniformly on any interval containing  $x = 0$ .

12. Show that the Fourier sine series of  $f(x) = 100$  on  $[0, l]$  is

$$\frac{4}{100} \left( \sin(\pi x/l) + \frac{1}{3} \sin(3\pi x/l) + \frac{1}{5} \sin(5\pi x/l) + \dots \right).$$

Using software, plot the first 50 terms of the series on the scaled interval  $0 \leq x/l \leq 1$ . Observe Gibbs phenomena at the points  $x/l = 0$  and  $x/l = 1$ , and calculate the percentage of the maximum overshoot at those two points. (Answer: about 9%.)

13. Find the complex form of the full Fourier series for  $\cosh x$  on  $(-1, 1)$ . Find the real form.