2
Partial Differential Equations on Unbounded Domains

In our study of PDEs we noted the differences among the types of equations: parabolic, hyperbolic, and elliptic. Those classifications dictate the types of initial and boundary conditions that should be imposed to obtain a well-posed problem. There is yet another division that makes one *method* of solution preferable over another, namely, the nature and extent of the spatial domain. Spatial domains may be bounded, like a bounded interval, or unbounded, like the entire set of real numbers. It is a matter of preference which type of domain is studied first. It seems that boundaries in a problem, which require boundary conditions, make a problem more difficult. Therefore, we first investigate problems defined on unbounded domains.

2.1 Cauchy Problem for the Heat Equation

We begin with the heat, or diffusion, equation on the real line. That is, we consider the initial value problem

$$u_t = ku_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{2.1}$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Physically, this problem is a model of heat flow in an infinitely long bar where the initial temperature $\phi(x)$ is prescribed. In a chemical or biological con-

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text, the equation governs density variations under a diffusion process. Notice that there are no boundaries in the problem, so we do not prescribe boundary conditions explicitly. However, for problems on infinite domains, conditions at infinity are sometimes either stated explicitly or understood. Such a condition might require boundedness of the solution or some type of decay condition on the solution to zero as $x \to \pm\infty$. In mathematics, a pure initial value problem like (2.1)–(2.2) is often called a **Cauchy problem**.

Deriving the solution of (2.1)–(2.2) can be accomplished in two steps. First we solve the problem for a step function $\phi(x)$, and then we construct the solution to (2.1)–(2.2) using that solution. Therefore, let us consider the problem

$$w_t = kw_{xx}, \quad x \in \mathbb{R}, \ t > 0, \tag{2.3}$$

$$w(x, 0) = 0 \ \text{ for } x < 0; \quad w(x, 0) = u_0 \ \text{ for } x > 0, \tag{2.4}$$

where we have taken the initial condition to be a step function with jump u_0.

We motivate our approach with a simple idea from the subject of dimensional analysis. Dimensional analysis deals with the study of units (seconds, meters, kilograms, and so forth) and dimensions (time, length, mass, and so forth) of the quantities in a problem and how they relate to each other. Equations must be dimensionally consistent (one cannot add apples to oranges), and important conclusions can be drawn from this fact. The cornerstone result in dimensional analysis is called the *pi theorem*. The pi theorem guarantees that whenever there is a physical law relating dimensioned quantities $q_1, \ldots, q_m$, then there is an equivalent physical law relating the independent dimensionless quantities that can be formed from $q_1, \ldots, q_m$. By a dimensionless quantity we mean one in which all the dimensions (length, time, mass, etc.) cancel out. For a simple example take the law

$$h = -\frac{1}{2}gt^2 + vt,$$

which gives the height h of an object at time t when it is thrown upward with initial velocity v; the constant g is the acceleration due to gravity. Here the dimensioned quantities are h, t, v, and g, having dimensions length, time, length per time, and length per time-squared. This law can be rearranged and written equivalently as

$$\frac{h}{vt} = -\frac{1}{2}\left(\frac{gt}{v}\right) + 1$$

in terms of the two dimensionless quantities

$$\pi_1 \equiv \frac{h}{vt} \quad \text{and} \quad \pi_2 \equiv \frac{gt}{v}.$$

For example, h is a length and vt, a velocity times a time, is also a length; so π_1, or h divided by vt, has no dimensions. Similarly, $\pi_2 = gt/v$ is dimensionless.

A law in dimensioned variables can always be reformulated in dimensionless quantities. So the physical law can be written as $\pi_1 = -\frac{1}{2}\pi_2 + 1$.

We use similar reasoning to guess the form of the solution of the initial value problem (2.3)–(2.4). First we list all the variables and constants in the problem: x, t, w, u_0, k. These have dimensions length, time, degrees, degrees, and length-squared per time, respectively. We notice that w/u_0 is a dimensionless quantity (degrees divided by degrees); the only other independent dimensionless quantity in the problem is $x/\sqrt{4kt}$ (the "4" is included for convenience). By the pi theorem we expect that the solution can be written as some combination of these dimensionless variables, or

$$\frac{w}{u_0} = f\left(\frac{x}{\sqrt{4kt}}\right)$$

for some function f yet to be determined. In fact, this is the case. So let us substitute

$$w = f(z), \quad z = \frac{x}{\sqrt{4kt}}$$

into the PDE (2.3). We have taken $u_0 = 1$ for simplicity. The chain rule allows us to compute the partial derivatives as

$$\begin{aligned} w_t &= f'(z)z_t = -\frac{1}{2}\frac{x}{\sqrt{4kt^3}}f'(z), \\ w_x &= f'(z)z_x = \frac{1}{\sqrt{4kt}}f'(z), \\ w_{xx} &= \frac{\partial}{\partial x}w_x = \frac{1}{4kt}f''(z). \end{aligned}$$

Substituting into (2.3) gives, after some cancelation, an ordinary differential equation,

$$f''(z) + 2zf'(z) = 0,$$

for $f(z)$. This equation is easily solved by multiplying through by the integrating factor e^{z^2} and integrating to get

$$f'(z) = c_1 e^{-z^2},$$

where c_1 is a constant of integration. Integrating from 0 to z gives

$$f(z) = c_1 \int_0^z e^{-r^2}dr + c_2,$$

where c_2 is another constant of integration. Therefore we have determined solutions of (2.3) of the form

$$w(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-r^2}dr + c_2.$$

Next we apply the initial condition (2.4) (taking $u_0 = 1$) to determine the constants c_1 and c_2. For a fixed $x < 0$ we take the limit as $t \to 0$ to get

$$0 = w(x,0) = c_1 \int_0^{-\infty} e^{-r^2} dr + c_2.$$

For a fixed $x > 0$ we take the limit as $t \to 0$ to get

$$1 = w(x,0) = c_1 \int_0^{\infty} e^{-r^2} dr + c_2.$$

Recalling that

$$\int_0^{\infty} e^{-r^2} dr = \frac{\sqrt{\pi}}{2},$$

we can solve the last two equations to get $c_1 = 1/\sqrt{\pi},\ c_2 = 1/2$. Therefore, the solution to (2.3)–(2.4) with $u_0 = 1$ is

$$w(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2} dr. \tag{2.5}$$

This solution can be written nicely as

$$w(x,t) = \frac{1}{2}\left(1 + \text{ erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \tag{2.6}$$

in terms of a special function called the "erf" function, which is defined by

$$\text{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^{z} e^{-r^2} dr.$$

Figure 2.1 shows a graph of several time snapshots of the solution (2.6).

Now we will use (2.5) and a physical argument to deduce a solution to the Cauchy problem (2.1)–(2.2). Later, in Section 2.7, we present an analytical argument based on Fourier transforms. We make some observations. First, if a function w satisfies the heat equation, then so does w_x, the partial derivative of that function with respect to x. This is easy to see because

$$0 = (w_t - kw_{xx})_x = (w_x)_t - k(w_x)_{xx}.$$

Therefore, since $w(x,t)$ solves the heat equation, the function

$$G(x,t) \equiv w_x(x,t)$$

solves the heat equation. By direct differentiation of $w(x,t)$ we find that

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)}. \tag{2.7}$$

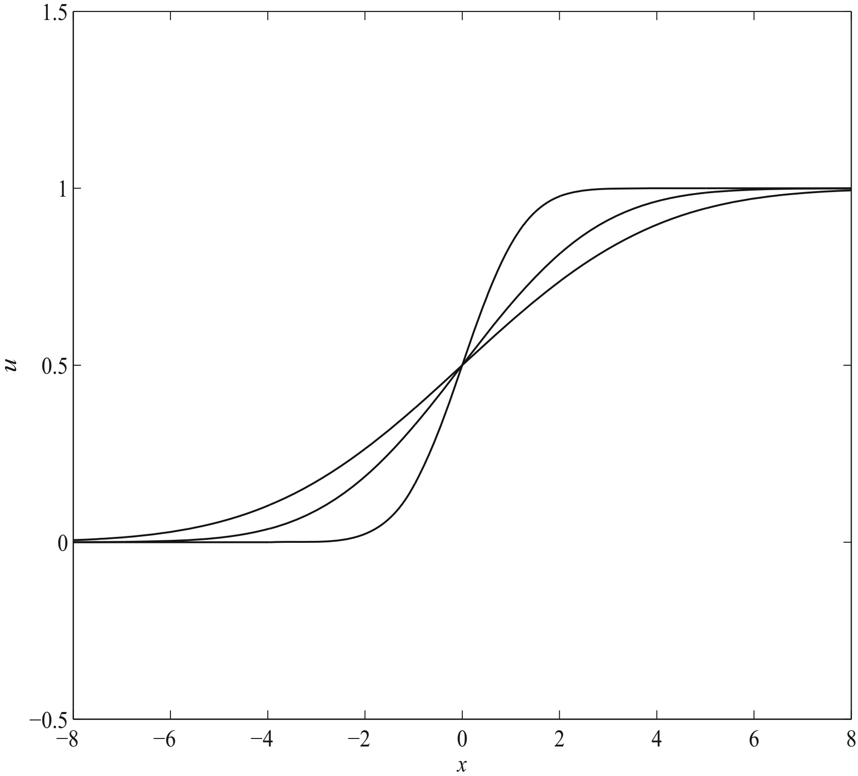

Figure 2.1 Temperature profiles $u = \frac{1}{2}\left(1 + \operatorname{erf}\left(x/\sqrt{4kt}\right)\right)$ at three different times t when the initial temperature is a step function and $k = 1$. As time increases, the profiles are smearing out

The function G is called the **heat kernel** or **fundamental solution** to the heat equation; the reader will note that for each $t > 0$ it graphs as a bell-shaped curve (see Exercise 1, Section 1.1), and the area under the curve for each $t > 0$ is one; that is,

$$\int_{-\infty}^{\infty} G(x,t)dx = 1, \quad t > 0.$$

$G(x,t)$ is the temperature surface that results from an initial unit heat source, i.e., injecting a unit amount of heat at $x = 0$ at time $t = 0$. We further observe that shifting the temperature profile again leads to a solution to the heat equation. Thus, $G(x-y,t)$, which is the temperature surface caused by an initial unit heat source at y, solves the heat equation for any fixed, but arbitrary, y. If $\phi(y)$, rather than unity, is the magnitude of the source at y, then $\phi(y)G(x-y,t)$ gives the resulting temperature surface; the area under a

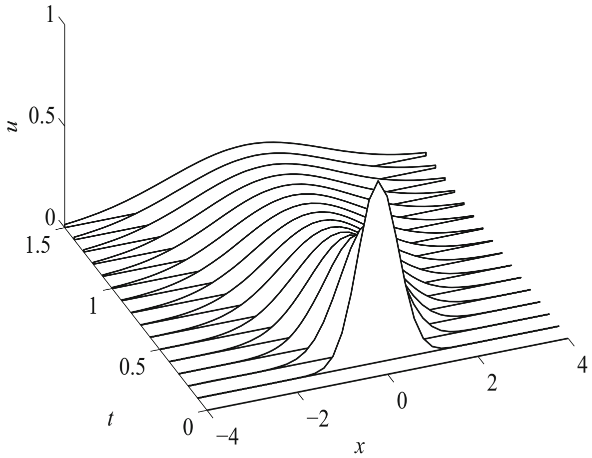

Figure 2.2 Plot of the fundamental solution $u = G(x, t)$ to the diffusion equation (2.7). As $t \to 0^+$ the solution approaches a unit 'point source' at $t = 0$

temperature profile is now $\phi(y)$, where y is the location of the source. Now, let us regard the initial temperature function ϕ in (2.2) as a continuous distribution of sources $\phi(y)$ for each $y \in \mathbb{R}$. Then, superimposing all the effects $\phi(y)G(x-y,t)$ for all y gives the total effect of all these isolated sources; that is,

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \phi(y) G(x-y,t)\,dy \\ &= \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/(4kt)}\,dy \end{aligned}$$

is a solution to the Cauchy problem (2.1)–(2.2) for reasonable assumptions on the initial condition ϕ. More precisely:

Theorem 2.1

Consider the initial value problem for the heat equation,

$$\begin{aligned} u_t &= k u_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x), \quad x \in \mathbb{R}, \end{aligned}$$

where ϕ is a bounded continuous function on $\mathbb{R}$. Then

$$u(x,t) = \int_{-\infty}^{\infty} \phi(y) \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/(4kt)}\,dy \tag{2.8}$$

is a solution to the heat equation for $x \in \mathbb{R}$, $t > 0$, and it has the property that $u(x,t) \to \phi(x)$ as $t \to 0^+$. If ϕ is piecewise continuous, i.e., it has only finitely many jump discontinuities in any bounded interval, then $u(x,t)$ is a solution to the heat equation on $x \in \mathbb{R}$, $t > 0$; and, as $t \to 0^+$, the solution approaches the average value of the left and right limits at a point of discontinuity of ϕ; in symbols,

$$u(x,t) \to \frac{1}{2}\left(\phi(x^-) + \phi(x^+)\right) \text{ as } t \to 0^+. \quad \square$$

This discussion of the Cauchy problem for the heat equation has been intuitive, and it provides a good basis for understanding why the solution has the form it does.

There is another, standard way to write the solution (2.8) to the Cauchy problem (2.1)–(2.2). If we change variables in the integral using the substitution $r = (x-y)/\sqrt{4kt}$, then $dr = -dy/\sqrt{4kt}$, and (2.8) becomes

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} \phi(x - r\sqrt{4kt})dr. \tag{2.9}$$

This formula is called the **Poisson integral representation**.

We make several observations. First, the solution of the Cauchy problem is an integral representation. Although the formula is not complicated, for most initial conditions $\phi(x)$ the integration cannot be performed analytically. Therefore, numerical or computer evaluation of the integral is ultimately required if temperature profiles are desired. Also, notice that the temperature $u(x,t)$ is nonzero for every real x, even if ϕ is zero outside a small interval about the origin. Thus, a signal propagated by the heat, or diffusion, equation travels infinitely fast; according to this model, if odors diffuse, a bear would instantly smell a newly opened can of tuna ten miles away. Next, although we do not give a proof, the solution given by (2.8) is very smooth; that is, u is infinitely differentiable in both x and t in the domain $t > 0$; this is true even if ϕ is piecewise continuous. Initial signals propagated by the heat equation are immediately smoothed out.

Finally, we note that the heat kernel $G(x,t)$ defined in (2.7) is also called the **Green's function** for the Cauchy problem. In general, the Green's function for a problem is the response of a system, or the effect, caused by a point source. In heat flow on the real line, $G(x,t)$ is the response, i.e., the temperature surface caused by a unit, point heat source given to the system at $x = 0,\ t = 0$. Some of the references discuss the construction of a Green's functions for a variety of problems. Because of the basic role this function plays in diffusion problems, $G(x,t)$ is also called the **fundamental solution** to the heat equation. The reader should review Section 1.4 for a discussion of the origin of the fundamental solution from a probability discussion.

EXERCISES

1. Solve the Cauchy problem (2.1)–(2.2) for the following initial conditions.

   a) $\phi(x) = 1$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| > 1$.

   b) $\phi(x) = e^{-x}$, $x > 0$; $\phi(x) = 0$, $x < 0$.

   In both cases write the solutions in terms of the erf function. Hint: In (b) complete the square with respect to y in the exponent of e.

2. If $|\phi(x)| \le M$ for all x, where M is a positive constant, show that the solution u to the Cauchy problem (2.1)–(2.2) satisfies $|u(x,t)| \le M$ for all x and $t > 0$. Hint: Use the calculus fact that the absolute value of an integral is less than or equal to the integral of the absolute value: $|\int f| \le \int |f|$.

3. Consider the problem (2.3)–(2.4) with $u_0 = 1$. For a fixed $x = x_0$, what is the approximate temperature $w(x_0,t)$ for very large t? Hint: Expand the integrand in the formula for the solution in a power series and integrate term by term.

4. Show that if $u(x,t)$ and $v(x,t)$ are any two solutions to the heat equation (2.1), then $w(x,y,t) = u(x,t)v(y,t)$ solves the two-dimensional heat equation $w_t = k(w_{xx} + w_{yy})$. Can you guess the solution to the two-dimensional Cauchy problem

$$w_t = k(w_{xx} + w_{yy}), \quad (x,y) \in \mathbb{R}^2, \ t > 0,$$
$$w(x,y,0) = \psi(x,y), \quad (x,y) \in \mathbb{R}^2?$$

5. Let the initial temperature in the Cauchy problem (2.1)–(2.2) be given by $\phi(x) = e^{-|x+2|} + e^{-|x-2|}$, with $k = 1$. Use the numerical integration operation in a computer algebra package to draw temperature profiles at several times to illustrate how heat flows in this system. Exhibit the temperature profiles on a single set of coordinate axes.

6. Verify that

$$\int_{-\infty}^{\infty} G(x,t)dx = 1, \quad t > 0.$$

   Hint: Change variables as in the derivation of Poisson's integral representation.

7. Consider the Cauchy problem for the heat equation

$$u_t = ku_{xx}, \ x \in \mathbb{R}, \ t > 0; \ u(x,0) = e^{-x}, \ x \in \mathbb{R}.$$

   Verify that $u(x,t) = e^{-x+kt}$ is an unbounded solution. Is this a contradiction to the theorem?

2.2 Cauchy Problem for the Wave Equation

The one-dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0. \tag{2.10}$$

We observed in Section 1.5 that it models the amplitude of transverse displacement waves on a taut string as well as small amplitude disturbances in acoustics. It also arises in electromagnetic wave propagation, in the mechanical vibrations of elastic media, as well as in other problems. It is a hyperbolic equation and is one of the three fundamental equations in PDEs (along with the diffusion equation and Laplace's equation). Under the transformation of variables (to characteristic coordinates)

$$\xi = x - ct, \quad \tau = x + ct,$$

the wave equation is transformed into the canonical form

$$U_{\tau\xi} = 0, \quad U = U(\xi, \tau),$$

which can be integrated twice to obtain the general solution

$$U(\xi, \tau) = F(\xi) + G(\tau),$$

where F and G are arbitrary functions. Thus, the general solution to (2.10) is

$$u(x,t) = F(x - ct) + G(x + ct). \tag{2.11}$$

Hence, solutions of the wave equation are the superposition (sum) of right- and left-traveling waves moving at speed c.

The Cauchy problem for the wave equation is

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in , t > 0, \tag{2.12}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}. \tag{2.13}$$

Here, f defines the initial displacement of an infinite string, and g defines its initial velocity. The equation is second-order in t, so both the position and velocity must be specified initially.

There is a simple analytical formula for the solution to the Cauchy problem (2.12)–(2.13). It is called **d'Alembert's formula**, and it is given by

$$u(x,t) = \frac{1}{2}(f(x - ct) + f(x + ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \tag{2.14}$$

If f'' and g' are continuous, then it is a straightforward exercise in differential calculus, using Leibniz's formula, to verify that this formula solves (2.12)–(2.13). The formula can be derived (see Exercise 1) by determining the two functions F and G in (2.11) using the initial data (2.13).

Example 2.2

Insight into the behavior of solutions comes from examining the special case where the initial velocity is zero and the initial displacement is a bell-shaped curve. Specifically, we consider the problem (with $c = 2$)

$$u_{tt} - 4u_{xx} = 0, \quad x \in , t > 0,$$
$$u(x,0) = e^{-x^2}, \quad u_t(x,0) = 0, \quad x \in \mathbb{R}.$$

The exact solution is, by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}(e^{-(x-2t)^2} + e^{-(x+2t)^2}).$$

Either the solution surface or wave profiles can be graphed easily using a computer algebra package. Figure 2.3 shows the solution surface; observe how the initial signal at $t = 0$ splits into two smaller signals, and those travel off in opposite directions at speed $c = 2$. In the exercises the reader is asked to examine the case where $f = 0$ and $g \neq 0$; this is the case where the initial displacement is zero and the string is given an initial velocity, or impulse, by, say, striking the string with an object. □

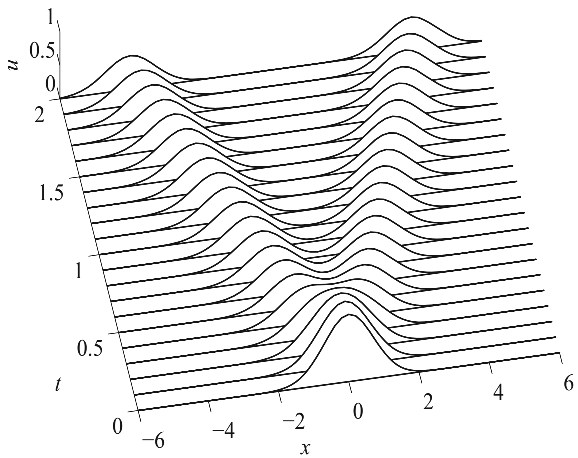

Figure 2.3 Time profiles of the solution surface. The initial signal splits into two signals which move at speeds c and $-c$ along the positive and negative characteristics, $x - ct =$ const., $x + ct =$ const., respectively

Close examination of d'Alembert's formula reveals a fundamental property of the wave equation. If the initial disturbance is *supported* in some interval

$a \le x \le b$ (this means that it is zero outside that interval, so the signal is located only in $a \le x \le b$), then the signal is always zero outside the region bounded by the two straight lines $x + ct = a$ and $x - ct = b$. See Figure 2.4.

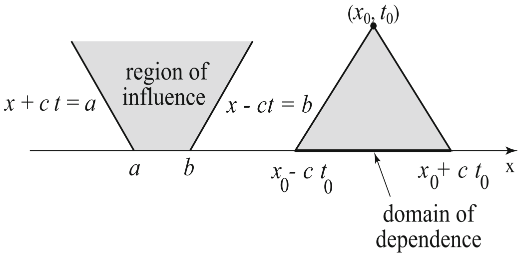

Figure 2.4 Region of influence and domain of dependence

This region is called the **region of influence** of the interval $[a, b]$. An initial signal in $[a, b]$ can never affect the solution outside this region. The lines $x + ct = constant$ are paths in space–time along which signals are propagated at velocity $-c$; the lines $x - ct = constant$ are paths in space–time along which signals are propagated with velocity c. These two families of lines are called the **negative** and **positive characteristics**, respectively. If the interval $[a, b]$ is shrunk to a point, then the region of influence shrinks to a cone, which is called the *light cone*. Looking at the situation in reverse, we can ask what initial data can affect the solution at a point (x_0, t_0). From the d'Alembert formula, only the initial values in the interval $[x_0 - ct_0, x_0 + ct_0]$ will affect the solution at (x_0, t_0). This interval is called the **domain of dependence**, and it is found by tracing the characteristics emanating from the point (x_0, t_0) backward in time to the x-axis.

In summary, there are important points to note regarding the characteristic curves. First, they are curves that carry the signals forward in space–time with velocity c and $-c$. Second, they define a special coordinate system $\xi = x - ct$, $\tau = x + ct$ under which the wave equation $u_{tt} - c^2 u_{xx} = 0$ is reduced to the simple canonical form $u_{\xi\tau} = 0$. In hyperbolic problems there is always a set of characteristic curves that play these roles. Even first order PDEs, which are actually wave-like, have one family of such curves that carry signals and provide a distinguished coordinate system where the problem simplifies (recall the examples in Section 1.2).

Finally, we point out again the important differences between parabolic and hyperbolic problems. Hyperbolic, or wave-like, equations propagate signals at a finite speed along characteristics; there is coherency in the wave form as it propagates, and therefore information is retained. Parabolic, or diffusion, equations propagate signals at infinite speed; because the signals diffuse or smear out, there is a gradual loss of information. A good way to understand how different equations propagate information is to determine how a signal is propagated in a special case. For example, suppose the initial signal is a Gaussian function or bell-shaped curve $\exp(-x^2)$. Think of this signal as being a bit of information. The convection equation $u_t + cu_x = 0$, which is a wave-like equation, propagates this signal via

$$u(x,t) = e^{-(x-ct)^2}.$$

That is, it moves it at speed c without distortion. The wave equation $u_{tt} = c^2 u_{xx}$ moves it via

$$u(x,t) = 0.5(e^{-(x-ct)^2} + e^{-(x+ct)^2}).$$

So the signal breaks into two pieces, and they propagate in opposite directions at speed c. The diffusion equation $u_t = ku_{xx}$ propagates the signal via

$$u(x,t) = \frac{1}{\sqrt{1+4kt}} e^{-x^2/(1+4kt)}.$$

So the signal stays at the same place, but it spreads out and decreases in amplitude. Any information in the signal is eventually lost.

EXERCISES

1. Derive d'Alembert's formula (2.14) by determining the two arbitrary functions F and G in the general solution (2.11) using the initial conditions (2.13).

2. Calculate the exact solution to the Cauchy problem when $c = 2$, the initial displacement is $f(x) = 0$, and the initial velocity is $g(x) = 1/(1 + 0.25x^2)$. Plot the solution surface and discuss the effect of giving a string at rest an initial impulse. Contrast the solution with the case when $f \neq 0$ and $g = 0$.

3. Solve the Cauchy problem

$$u_{tt} = c^2 u_{xx}, \;\; u(x,0) = \phi(x), \; u_t(x,0) = 0,$$

by differentiating the solution to to the Cauchy problem

$$u_{tt} = c^2 u_{xx}, \;\; u(x,0) = 0, \; u_t(x,0) = \phi(x).$$

4. Solve the outgoing signal problem

$$u_{tt} - c^2 u_{xx} = 0, \quad x > 0, \ -\infty < t < \infty; \quad u_x(0,t) = s(t), \quad -\infty < t < \infty,$$

where $s(t)$ is a known signal. Hint: Look for a right-traveling wave solution.

5. The three-dimensional wave equation is

$$u_{tt} - c^2 \Delta u = 0,$$

where $u = u(x, y, z, t)$ and Δ is the Laplacian operator. For waves with spherical symmetry, $u = u(\rho, t)$, where $\rho = \sqrt{x^2 + y^2 + z^2}$. In this special case the Laplacian is given by (Section 1.8) $\Delta u = u_{\rho\rho} + \frac{2}{\rho} u_\rho$. By introducing a change of dependent variable $U = \rho u$, show that the general solution for the spherically symmetric wave equation

$$u_{tt} = c^2 (u_{\rho\rho} + \frac{2}{\rho} u_\rho)$$

is

$$u = \frac{1}{\rho}(F(\rho - ct) + G(\rho + ct)).$$

Why do you think an outward-moving wave $u = F(\rho - ct)/\rho$ decays in amplitude? Give a physical interpretation.

6. Solve the Cauchy problem

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x, 0) = e^{-|x|}, \quad u_t(x, 0) = \cos x, \quad x \in \mathbb{R}.$$

Use a computer algebra program to graph wave profiles at $t = 1, 2, 3$. Take $c = 1$.

7. In Section 1.7 we showed that any solution to Laplace's equation has the property that its value at a point is approximately the average of four nearby values surrounding the point. Can we make a statement about solutions to the wave equation? Consider any characteristic parallelogram (see Figure 2.5) whose sides are positive and negative characteristics, and let A, B, C, D be the vertices as shown. Show that any solution to the wave equation satisfies the relation

$$u(A) + u(C) = u(B) + u(D).$$

8. Let $u = u(x, t)$ solve the wave equation; show that $v = v(x, t)$ defined by

$$v(x,t) = \frac{c}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} u(s, x) e^{-s^2 c^2/(4kt)} ds.$$

solves the heat equation.

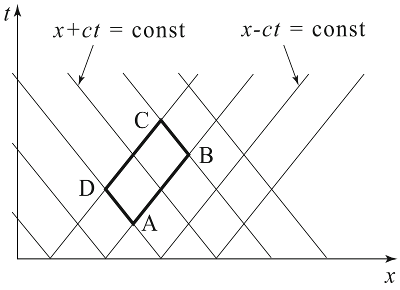

Figure 2.5 Characteristic parallelogram ABCD

9. Using the hint below and verifying the steps, find a particular solution to the nonhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t),\ \ x \in \mathbb{R},\ t > 0;\ \ u(x,0) = u_t(x,0) = 0,\ \ x \in \mathbb{R}.$$

Hint: Integrate the equation over the characteristic triangle T with corners $(\xi - c\tau, 0)$, $(\xi + c\tau, 0)$, (ξ, τ) to get

$$\begin{aligned} \int\int_T f\,dxdt &= \int_T (u_{tt} - c^2 u_{xx})\,dxdt \\ &= \int_C -c^2 u_x\,dt - u_t\,dx \qquad \text{(by Green's Theorem)} \\ &= 2cu(\xi, \tau). \end{aligned}$$

This gives the solution at any point (ξ, τ). Here, C is the boundary of T.

2.3 Well-Posed Problems

In the last two sections we solved the Cauchy problem, or pure initial value problem, for the heat equation and for the wave equation. Now let us set up a similar problem for Laplace's equation. Immediately the reader should be skeptical because Laplace's equation is an elliptic equilibrium equation that does not involve time; we expect boundary conditions rather than initial conditions. Nevertheless, let us consider the two-dimensional Laplace's equation in the upper-half plane $y > 0$ and prescribe u *and* its y derivative u_y along $y = 0$, or the x axis. That is, let us consider the problem

$$u_{xx} + u_{yy} = 0,\ \ x \in \mathbb{R},\ y > 0, \tag{2.15}$$

$$u(x,0) = f(x),\ \ u_y(x,0) = g(x),\quad x \in \mathbb{R}. \tag{2.16}$$

Notice the similarity to the initial value problem for the wave equation (2.12)–(2.13); but here we are using y instead of t, and the equation is elliptic rather than hyperbolic. As an aside, observe that (2.15)–(2.16) is neither a Dirichlet problem nor a Neumann problem because both u and u_y are specified along the boundary $y = 0$.

Let us carefully analyze problem (2.15)–(2.16) in a special case. Suppose first that the boundary conditions are

$$f(x) = g(x) = 0.$$

Then it is clear that a solution (the only solution) to (2.15)–(2.16) is the zero solution $u(x, y) \equiv 0$ for all $x \in \mathbb{R},\ y > 0$. Now let us change the conditions along the boundary by taking

$$f(x) = \frac{1}{n} \cos nx,\ g(x) = 0,\ \ x \in \mathbb{R}.$$

Then, as one can easily check, the solution to (2.15)–(2.16) is

$$u(x, y) = \frac{1}{n} \cos nx \cosh ny.$$

Suppose n is large (say $n = 100$); then we have changed the boundary condition $u(x, 0) = f(x)$ by only a small amount. Yet the solution has changed from zero by a large amount! For example, along the line $x = 0$ (y axis) the solution is

$$u(0, y) = \frac{\cosh ny}{n} \to +\infty \ \text{ as } y \to \infty.$$

To summarize, in this problem a small change of the data on the boundary produced a large change in the solution. This behavior is disturbing, because in a physical problem we expect that the solution should depend continuously on the boundary data—a small change in the boundary data should produce a small change in the solution. After all, we want to be confident in the accuracy of our solution even if we get the boundary data only approximately correct. This latter property of continuous dependence on data is called **stability**, and the problem (2.15)–(2.16) does not have it. So the Cauchy problem for Laplace's equations (2.15)–(2.16) is *not* a well-posed physical problem. In fact, since Laplace's equation models steady heat flow, it seems physically reasonable that we need only specify the temperature $u(x, 0)$ along the boundary to be able to solve the problem, or specify the flux $u_y(x, 0)$, but not both as in (2.16).

The term well–posed in PDEs has a technical meaning. We say that a boundary value problem, initial value problem, or an initial boundary value problem is **well-posed** if:

(1) (**Existence**) it has a solution

(2) (**Uniqueness**) the solution is unique

(3) (**Stability**) the solution depends continuously on the initial and/or boundary data.

If a problem is not well-posed, then it is called **ill-posed**. Resolving these three issues for various PDE models occupies much of the theory of PDEs.

Now that we have shown that the Cauchy problem for Laplace's equation is not stable, the reader may be skeptical about other problems such as the Cauchy problem for the heat equation. We can easily observe that solutions to this problem have the stability property.

Example 2.3

Consider the two problems

$$\begin{aligned} u_t &= ku_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x), \quad x \in \mathbb{R}, \end{aligned}$$

and

$$\begin{aligned} v_t &= kv_{xx}, \quad x \in \mathbb{R},\ t > 0, \\ v(x,0) &= \psi(x), \quad x \in \mathbb{R}, \end{aligned}$$

where ϕ and ψ are continuous, bounded functions and close in the sense that $|\phi(x) - \psi(x)| \leq \delta$ for all x, where δ is a small number. We would like to show that the corresponding solutions $u(x,t)$ and $v(x,t)$ are close. We define $w(x,t) = u(x,t) - v(x,t)$ and note that w satisfies the Cauchy problem

$$\begin{aligned} w_t &= kw_{xx}, \quad x \in \mathbb{R}, t > 0, \\ w(x,0) &= \phi(x) - \psi(x), \quad x \in \mathbb{R}. \end{aligned}$$

The solution formula for the Cauchy problem gives

$$w(x,t) = \int_{-\infty}^{\infty} (\phi(y) - \psi(y))G(x-y,t)dy,$$

where $G(x,t)$ is the fundamental solution. Therefore, for each $t > 0$,

$$\begin{aligned} |u(x,t) - v(x,t)| &\leq \int_{-\infty}^{\infty} \mid \phi(y) - \psi(y) \mid\mid G(x-y,t) \mid dy \\ &\leq \int_{-\infty}^{\infty} \delta G(x-y,t)dy = \delta, \end{aligned}$$

since $\int G(x-y,t)dy = 1$. Therefore, in the sense interpreted above, closeness of the initial data implies closeness of the solution. □

EXERCISES

1. Show that the Cauchy problem for the *backward* diffusion equation,

$$u_t + u_{xx} = 0, \quad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = f(x), \quad x \in \mathbb{R},$$

is unstable by considering the solutions

$$u(x,t) = 1 + \frac{1}{n}e^{n^2 t} \sin nx$$

for large n. Hint: Follow the method used in this section for Laplace's equation.

2. Let $u = u(x,y)$. Is the problem

$$u_{xy} = 0, \quad 0 < x, y < 1,$$

on the unit square, where the value of u is prescribed on the boundary of the square, a well-posed problem? Discuss fully.

3. Consider the two Cauchy problems for the wave equation with different initial data:

$$u^i_{tt} = c^2 u^i_{xx}, \quad x \in \mathbb{R}, \ 0 < t < T,$$
$$u^i(x,0) = f^i(x), \ u^i_t(x,0) = g^i(x), \quad x \in \mathbb{R},$$

for $i = 1, 2$, where f^1, f^2, g^1, and g^2 are given functions (the superscripts are indices and not exponents). If for all $x \in \mathbb{R}$ we have

$$| f^1(x) - f^2(x) | \leq \delta_1, \quad | g^1(x) - g^2(x) | \leq \delta_2,$$

show that $| u^1(x,t) - u^2(x,t) | \leq \delta_1 + \delta_2 T$ for all $x \in \mathbb{R}$, $0 < t < T$. What does this mean with regard to stability? Hint: Follow the method in Example 2.3.

2.4 Semi-Infinite Domains

Heat Equation

In Sections 2.1 and 2.2 we solved the heat equation and the wave equation, respectively, on the domain $-\infty < x < \infty$. Now we study these equations when the domain is semi-infinite, i.e., on the interval $0 < x < \infty$. This means that there is a boundary in the problem, at $x = 0$, and one expects that it is necessary to impose a boundary condition there. For example, to determine how the temperature distribution evolves in a semi-infinite bar, one should know the temperature in the bar initially, as well as the temperature at $x = 0$ for all time.

To begin, we consider the initial boundary value problem for the heat equation

$$u_t = ku_{xx}, \quad x > 0, t > 0, \tag{2.17}$$

$$u(0,t) = 0, \quad t > 0, \tag{2.18}$$

$$u(x,0) = \phi(x), \quad x > 0, \tag{2.19}$$

where we specify the temperature to be zero at $x = 0$ for all time. To solve this problem we use the method of *reflection* through the boundary. The idea is to extend the problem (2.17)–(2.19) to the entire real axis by extending the initial data ϕ to an *odd function* ψ defined by

$$\psi(x) = \begin{cases} \phi(x), & x > 0, \\ -\phi(-x), & x < 0, \end{cases}$$

with $\psi(0) = 0$. We then solve the extended problem by formula (2.8) and then restrict that solution to the positive real axis, which will then be the solution to (2.17)–(2.19). Figure 2.6 shows the initial data for the *extended* problem and a resulting odd solution profile $v(x,t)$. Physically, we are attaching a bar occupying the space $-\infty < x < 0$ and giving it an initial temperature equal to the negative of that in the original bar. Thus, let us consider the Cauchy problem for $v(x,t)$:

$$v_t = kv_{xx}, \quad x \in \mathbb{R}, t > 0, \tag{2.20}$$

$$v(x,0) = \psi(x), \quad x \in \mathbb{R}, \tag{2.21}$$

where ψ is the odd extension of the function ϕ as described above. By the formula for the solution to the heat equation over the real line, the solution to (2.20)–(2.21) is given by

$$v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\psi(y)dy,$$

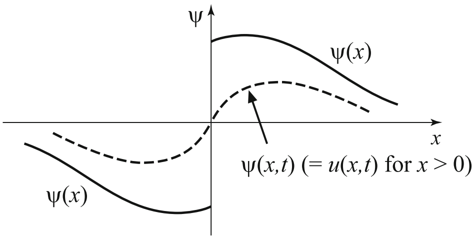

Figure 2.6 The initial data $\psi(x)$ and solution profile $v(x,t)$ for the odd, extended problem

where $G(x,y)$ is the fundamental solution. Breaking this integral into two parts, $y<0$ and $y>0$, we obtain

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{0} G(x-y,t)\psi(y)dy + \int_{0}^{\infty} G(x-y,t)\psi(y)dy \\ &= -\int_{-\infty}^{0} G(x-y,t)\phi(-y)dy + \int_{0}^{\infty} G(x-y,t)\phi(y)dy \\ &= -\int_{0}^{\infty} G(x+y,t)\phi(y)dy + \int_{0}^{\infty} G(x-y,t)\phi(y)dy \\ &= \int_{0}^{\infty} [G(x-y,t) - G(x+y,t)]\phi(y)dy. \end{aligned}$$

We restrict this solution to $x>0$, and therefore the solution to the heat equations (2.17)–(2.19), on the domain $x \ge 0$, is

$$u(x,t) = \int_{0}^{\infty} [G(x-y,t) - G(x+y,t)]\phi(y)dy, \quad x \ge 0.$$

The Wave Equation

The wave equation on a semi-infinite domain can be solved in the same manner. Consider the problem of the transverse vibrations of a string occupying $x>0$ when the end at $x=0$ is held fixed. The initial boundary value problem is

$$u_{tt} = c^2 u_{xx}, \quad x>0, t>0, \tag{2.22}$$

$$u(0,t) = 0, \quad t>0, \tag{2.23}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x>0. \tag{2.24}$$

For $x > ct$ (i.e., ahead of the leading signal $x = ct$ coming from the origin) the interval of dependence lies in $(0, \infty)$, where the initial data are given; therefore, in this domain, the solution is given by d'Alembert's formula:

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \quad x > ct. \tag{2.25}$$

Next, the data given along the $x = 0$ boundary cannot affect the solution in the region $x > ct$, since signals travel outward from the boundary at speed c. See Figure 2.7 To solve the problem in the region $0 < x < ct$ we proceed as we

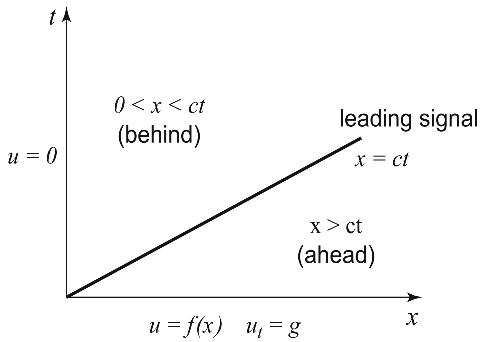

Figure 2.7 Space–time domain $x, t > 0$ where the problem (2.22)–(2.24) is defined. The region $x > ct$ is affected only by the initial data f and g and can be found by d'Alembert's formula

did for the heat equation and extend the initial data f and g to *odd* functions on the entire real axis. Therefore, we consider the problem

$$v_{tt} = c^2 v_{xx}, \quad x \in \mathbb{R}, t > 0, \tag{2.26}$$

$$v(0,t) = 0, \quad t > 0, \tag{2.27}$$

$$v(x,0) = F(x), \quad v_t(x,0) = G(x), \quad x \in \mathbb{R}, \tag{2.28}$$

where

$$F(x) = f(x), \quad x > 0; \quad F(0) = 0; \quad F(x) = -f(-x), \quad x < 0,$$

and

$$G(x) = g(x), \quad x > 0; \quad G(0) = 0; \quad G(x) = -g(-x), \quad x < 0.$$

By d'Alembert's formula the solution to (2.26)–(2.28) is

$$v(x,t) = \frac{1}{2}(F(x-ct) + F(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds.$$

In the region $x < ct$ this becomes (since $x - ct < 0$)

$$v(x,t) = \frac{1}{2}(-f(-x+ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{0} -g(-s)ds$$
$$+ \frac{1}{2c}\int_{0}^{x+ct} g(s)ds.$$

If the variable of integration s in the first integral is replaced by $-s$, then the two integrals can be combined, and we may write

$$u(x,t) = v(x,t) = \frac{1}{2}(f(x+ct) - f(ct-x)) + \frac{1}{2c}\int_{ct-x}^{ct+x} g(s)ds, \qquad (2.29)$$
$$0 < x < ct.$$

In summary, the solution to the initial boundary value problem (2.22)–(2.24) for the wave equation on a half-line is given by the two formulas (2.25) and (2.29), depending on $x > ct$ or $x < ct$.

Why does this reflection method work for the heat equation and wave equation? To reiterate, the solutions to the Cauchy problems for these two equations are odd functions if the initial data is odd. And the restriction of an odd solution to the positive real axis is the solution to the given initial boundary value problem. If this intuitive reasoning leaves the reader perplexed, then one can always verify analytically that the solutions we have obtained by this reflection method are, in fact, solutions to the given problems.

Finally, if the boundary condition (2.18) along $x = 0$ in the heat flow problem (2.17)–(2.19) is replaced by a Neumann condition

$$u_x(0,t) = 0, \quad t > 0,$$

then the problem can be solved by extending the initial data to an *even* function. The same is true for the wave equation. We leave these calculations as exercises.

EXERCISES

1. Solve the initial boundary value problem for the heat equation,

$$u_t = ku_{xx}, \quad x > 0, t > 0,$$
$$u_x(0,t) = 0, \quad t > 0,$$
$$u(x,0) = \phi(x), \quad x > 0,$$

with an insulated boundary condition, by extending the initial temperature ϕ to the entire real axis as an *even* function. The solution is

$$u(x,y) = \int_0^\infty [G(x-y,t) + G(x+y,t)]\phi(y)dy.$$

2. Find a formula for the solution to the problem

$$u_t = ku_{xx}, \quad x > 0, \, t > 0,$$
$$u(0,t) = 0, \quad t > 0,$$
$$u(x,0) = 1, \quad x > 0.$$

Plot several solution profiles when $k = 0.5$.

3. Find the solution to the problem

$$u_{tt} = c^2 u_{xx}, \quad x > 0, \, t > 0,$$
$$u(0,t) = 0, \quad t > 0,$$
$$u(x,0) = xe^{-x}, \quad u_t(x,0) = 0, \quad x > 0.$$

Pick $c = 0.5$ and sketch several time snapshots of the solution surface to observe the reflection of the wave from the boundary.

4. Solve the problem

$$u_t = ku_{xx}, \quad x > 0, \, t > 0,$$
$$u(0,t) = 1, \quad t > 0,$$
$$u(x,0) = 0, \quad x > 0.$$

Hint: Transform the problem to one of the form (2.17)–(2.19) and use Exercise 2.

5. (Age of the earth) In this exercise we use Lord Kelvin's argument, given in the mid 1860s, to estimate the age of the earth using a measurement of the geothermal gradient at the surface. The geothermal gradient is the temperature gradient u_x measured at surface of the earth. To obtain the estimate, treat the earth as flat at the surface with $x > 0$ measuring the depth from the surface $x = 0$. Take the diffusivity of the earth to be $k = 0.007$ cm^2 per second, the assume that initial temperature was 7000 degrees Fahrenheit (molten rock) at the beginning. Finally, assume the temperature of the surrounding atmosphere has always been 0 degrees. Use a current geothermal gradient value of 3.7×10^{-4} degrees per cm. After determining the approximate age of the earth, estimate the percentage of the original heat that has been lost until the present day? Comment on the accuracy of the Kelvin's argument by searching on line for the approximate age of the earth. Hint: Treat this as a linear, one-dimensional, heat flow problem.

6. (Subsurface temperatures) A clutch of insect eggs lies at a depth of x_1 cm below the ground surface.

a) If the surface is subjected to periodic temperature variations of $T_0 + A\cos\omega t$ over a long time, what is the temperature variation experienced by the egg clutch? Hint: find a complex, plane wave solution of the diffusion equation of the form $u = T_0 + Ae^{i(\gamma x - \omega t)}$ and determine γ in terms of ω and the diffusivity k of the soil; take the real part.

b) At depth x_1 find the phase shift of the temperature and the amplitude attenuation factor.

c) Plot of the amplitude of the temperature variation versus depth. Take $T_0 = 30$, $A = 15$ degrees Celsius, $k = 0.004$ cm^2 per second, and $\omega = 2\pi$ per day, and plot the temperature variations at the surface and at the depth 3 cm.

7. The heat equation governs the flow of heat downward ($x \geq 0$) through the soil due to changes in ambient air temperature $S(t)$ at the surface ($x = 0$) and the absorption of heat caused by solar radiation $W(t)$ falling on the surface, measured in Watts per unit area A, per time. Let $u = u(x,t)$ denote the temperature at a depth x at time, and let C, ρ, and K be the specific heat, density, and thermal conductivity of the soil. Also, let H denote the heat transfer coefficient at the surface and α be the fraction of solar energy that is absorbed at the surface. Derive the boundary condition at the surface,

$$-AKu_x(0,t) = -AH(u(0,t) - S(t)) + A\alpha W(t),$$

being specific about the origin of each term. Give a correct set of units for each quantity appearing. (For an example see A. Parrott & J. D. Logan, 2010, *Ecological Modelling* 221, 1378–1393.)

2.5 Sources and Duhamel's Principle

How do we proceed if the PDE contains a source term? For example, consider the heat-flow problem

$$u_t = ku_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0, \tag{2.30}$$

$$u(x,0) = 0, \quad x \in \mathbb{R}, \tag{2.31}$$

where f is a given heat source. The key to the analysis of this problem can be discovered by examining an ordinary differential equation with a source term. For example, consider the initial value problem

$$y'(t) + ay = F(t), \quad t > 0; \quad y(0) = 0. \tag{2.32}$$

Multiplying by the integrating factor e^{at} makes the left side a total derivative, and we obtain

$$\frac{d}{dt}(e^{at}y) = e^{at}F(t).$$

Integrating from 0 to t then gives

$$e^{at}y(t) = \int_0^t e^{a\tau}F(\tau)d\tau,$$

which can be rewritten as

$$y(t) = \int_0^t e^{-a(t-\tau)}F(\tau)d\tau. \tag{2.33}$$

Now let us consider another problem, where we put the source term in as the initial condition. Let $w = w(t;\tau)$ be the solution to the problem

$$w'(t;\tau) + aw(t;\tau) = 0, \quad t > 0; \quad w(0;\tau) = F(\tau),$$

where a new parameter τ has been introduced. It is straightforward to see that the solution to this problem is

$$w(t;\tau) = F(\tau)e^{-at}.$$

So the solution to (2.32), the problem with a source, is the integral of the solution $w(t,\tau)$ (with t replaced by $t-\tau$) of the associated homogeneous problem where the source is included as an initial condition. That is,

$$y(t) = \int_0^t w(t-\tau;\tau)F(\tau)\,\tau = \int_0^t F(\tau)e^{-a(t-\tau)}\,d\tau,$$

which is (2.33).

The fact that a particular solution of a linear equation can be deduced from the solution of the homogeneous equation is called Duhamel's principle. For ODEs we state the principle as follows:

Theorem 2.4

(**Duhamel's principle**) The solution of the problem

$$y'(t) + ay = F(t), \quad t > 0; \quad y(0) = 0$$

is given by

$$y(t) = \int_0^t w(t-\tau,\tau)d\tau,$$

where $w = w(t,\tau)$ solves the homogeneous problem

$$w'(t;\tau) + aw(t;\tau) = 0, \quad t > 0; \quad w(0;\tau) = F(\tau). \quad \Box$$

The same type of result is true for second-order ODEs as well; the reader may recall that the variation of parameters method uses the homogeneous solutions to construct a particular solution. See the Appendix for the formula.

Now let us extrapolate this idea and apply it to the heat flow problem (2.30)–(2.31). If Duhamel's principle is valid in this case, then the solution of (2.30)–(2.31) should be

$$u(x,t) = \int_0^t w(x, t-\tau, \tau)d\tau,$$

where $w(x,t;\tau)$ solves the homogeneous problem

$$w_t = kw_{xx}, \quad x \in ,t > 0, \tag{2.34}$$

$$w(x,0;\tau) = f(x,\tau), \quad x \in \mathbb{R}. \tag{2.35}$$

In fact, we can write down the explicit formula; by (2.8) the solution to (2.34)–(2.35) is

$$w(x,t;\tau) = \int_{-\infty}^{\infty} G(x-y,t)f(y,\tau)dy,$$

where G is the heat kernel. Therefore, the solution to (2.30)–(2.31) should be given by

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)f(y,\tau)dyd\tau.$$

Indeed, one can verify that this is the case.

It is not surprising that the solution turned out to be an integral. The PDE (2.30) has the form $Hu = f$, where $H = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial t^2}$ is a differential operator. If we formally write $u = H^{-1}f$ (as we might do in matrix theory if H were a matrix and u and f vectors), then we would expect H^{-1}, the inverse of H, to be an integral operator, since integration and differentiation are inverse processes.

We may now write down the formula for the solution to the problem

$$u_t = ku_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0, \tag{2.36}$$

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}, \tag{2.37}$$

where the initial condition is no longer zero. By linearity, the solution to (2.36)–(2.37) is the sum of the solutions to the two problems

$$u_t = ku_{xx}, \quad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = \phi(x), \quad x \in \mathbb{R},$$

and

$$u_t = ku_{xx} + f(x,t), \quad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = 0, \quad x \in \mathbb{R}.$$

Thus, the solution to (2.36)–(2.37) is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-\tau)f(y,\tau)dyd\tau. \quad (2.38)$$

This is the **variation of parameters formula** for the problem (2.36)–(2.37).

Example 2.5

(**Wave equation**) Duhamel's principle can also be applied to the wave equation. The solution to the problem

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in \mathbb{R}, t > 0, \quad (2.39)$$

$$u(x,0) = u_t(x,0) = 0, \quad x \in \mathbb{R}, \quad (2.40)$$

is

$$u(x,t) = \int_0^t w(x,t-\tau,\tau)d\tau,$$

where $w = w(x,t;\tau)$ is the solution to

$$w_{tt} - c^2 w_{xx} = 0, \quad x \in \mathbb{R}, t > 0,$$

$$w(x,0;\tau) = 0, \quad w_t(x,0;\tau) = f(x,\tau), \quad x \in \mathbb{R}.$$

We put the source term f into the initial condition for w_t rather than for w because f, like w_t, is an acceleration; note that the displacement u is a time integral of w, so w must be a velocity, making w_t an acceleration. From d'Alembert's formula,

$$w(x,t;\tau) = \frac{1}{2c}\int_{x-ct}^{x+ct} f(s,\tau)ds,$$

and therefore by Duhamel's principle the solution to (2.39)–(2.40) is given by the formula

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(s,\tau)dsd\tau. \quad \square \quad (2.41)$$

Source terms also arise in PDEs when problems are transformed in order to homogenize the boundary conditions.

Example 2.6

Consider the diffusion problem

$$u_t = ku_{xx}, \quad x > 0, t > 0, \quad (2.42)$$

$$u(x,0) = \phi(x), \quad x > 0, \quad (2.43)$$

$$u(0,t) = g(t), \quad t > 0. \quad (2.44)$$

We solved this problem in the last section when $g(t) = 0$. So let us attempt to transform the problem into one where the boundary condition is zero. To this end, let $v(x,t) = u(x,t) - g(t)$, or $u(x,t) = v(x,t) + g(t)$; then substituting into (2.42)–(2.44) gives

$$v_t = kv_{xx} - g'(t), \quad x > 0, t > 0, \tag{2.45}$$

$$v(x,0) = \psi(x) \equiv \phi(x) - g(0), \quad x > 0, \tag{2.46}$$

$$v(0,t) = 0, \quad t > 0. \tag{2.47}$$

Therefore, transformation of the dependent variable has changed the problem into one with a homogeneous boundary condition, but a price was paid—an inhomogeneity, or source term $-g'(t)$, was introduced into the PDE. In general, we can always homogenize the boundary conditions in a linear problem, but the result is an inhomogeneous PDE; so inhomogeneous boundary conditions can be traded for inhomogeneous PDEs. We can solve (2.45)–(2.47) for $v(x,t)$ by formulating a Duhamel's principle. In Section 2.6 we show that Laplace transform methods can also be applied to find the solution. □

EXERCISES

1. Write a formula for the solution to the problem

$$u_{tt} - c^2 u_{xx} = \sin x, \quad x \in \mathbb{R}, t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \quad x \in \mathbb{R}.$$

   Graph the solution surface when $c = 1$.

2. Write a formula for the solution to the problem

$$u_t - ku_{xx} = \sin x, \quad x \in \mathbb{R}, \ t > 0,$$
$$u(x,0) = 0, \quad x \in \mathbb{R}.$$

3. Using Duhamel's principle, find a formula for the solution to the initial value problem for the convection equation

$$u_t + cu_x = f(x,t), \quad x \in \mathbb{R}, \ t > 0; \qquad u(x,0) = 0, \quad x \in \mathbb{R}.$$

   Hint: Look at the problem

$$w_t(x,t;\tau) + cw_x(x,t;\tau) = 0, \quad x \in ,t > 0; \qquad w(x,0;\tau) = f(x,\tau), \quad x \in \mathbb{R}.$$

4. Solve the problem

$$u_t + 2u_x = xe^{-t}, \quad x \in \mathbb{R}, t > 0; \quad u(x,0) = 0, \quad x \in \mathbb{R}.$$

5. Formulate Duhamel's principle and solve the initial boundary value problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad x > 0, t > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,t) &= 0, \quad t > 0. \end{aligned}$$

Solution:

$$u(x,t) = \int_0^t \int_0^\infty (G(x-y, t-\tau) - G(x+y, t-\tau)) f(y,\tau) dy d\tau.$$

2.6 Laplace Transforms

Laplace transforms are first encountered in elementary differential equations courses as a technique for solving linear, constant-coefficient, ordinary differential equations; Laplace transforms convert an ODE into an algebra problem. They are particularly useful for nonhomogeneous differential equations with impulse or discontinuous sources. The ideas easily extend to PDEs, where the the operation of Laplace transformation converts PDEs to ODEs.

Let $u = u(t)$ be a piecewise continuous function on $t \geq 0$ that does not grow too fast; for example, assume that u is of *exponential order*, which means that $|u(t)| \leq c\exp(at)$ for t sufficiently large, where $a, c > 0$. Then the **Laplace transform** of u is defined by

$$(\mathcal{L}u)(s) \equiv U(s) = \int_0^\infty u(t)e^{-st}dt. \tag{2.48}$$

The Laplace transform is an example of an integral transform; it takes a given function $u(t)$ in the time domain and converts it to a new function $U(s)$, in the so-called transform domain. U and s are called the transform variables. The Laplace transform is linear in that

$$\mathcal{L}(c_1u + c_2v) = c_1\mathcal{L}u + c_2\mathcal{L}\, v$$

where c_1 and c_2 are constants. If the transform $U(s)$ is known, then $u(t)$ is called the inverse transform of $U(s)$ and we write $\mathcal{L}^{-1}U = u$. Pairs of Laplace transforms and their inverses are tabulated in books of tables, and many software packages, such as MATLAB, have simple commands that yield transform pairs. A short table is given at the end of this section (Table 2.1).

The importance of the Laplace transform, like other transforms, is that it changes derivative operations to multiplication operations in the transform domain. In fact, we have

$$(\mathcal{L}u')(s) = sU(s) - u(0), \tag{2.49}$$

$$(\mathcal{L}u'')(s) = s^2U(s) - su(0) - u'(0). \tag{2.50}$$

Formulas (2.49) and (2.50) are readily derived using integration by parts, and they are the basic operational formulas for solving differential equations.

Example 2.7

Solve the initial value problem

$$u'' + u = 0, \quad t > 0; \quad u(0) = 0, \ u'(0) = 1.$$

Taking Laplace transforms of both sides of the differential equation and using (2.50) gives

$$s^2U(s) - 1 + U(s) = 0,$$

or

$$U(s) = \frac{1}{1+s^2}.$$

So, the ordinary differential equation has been transformed into an algebraic equation, and we solved the problem in the transform domain. To recover $u(t)$ from its transform $U(s)$ we look up the inverse transform in Table 2.1 to find

$$u(t) = \mathcal{L}^{-1}\left(\frac{1}{1+s^2}\right) = \sin t,$$

which is the solution. □

In the preceding example we were recovered a function from its Laplace transform by looking in a table. One may ask, in general, how to determine $u(t)$ from knowledge of its transform $U(s)$. The answer to this question requires knowledge of complex variables; it would take us far afield to give a thorough discussion. However, we do indicate the general formula to compute $u(t)$ from its transform $U(s)$. The **inversion formula** is

$$u(t) = (\mathcal{L}^{-1}U)(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} U(s)e^{st}ds.$$

The integral in this formula is a complex contour integral taken over the vertical straight line in the complex plane from $a - i\infty$ to $a + i\infty$. The number a is any real number for which the resulting path lies to the right of any singularities (poles, essential singular points, or branch points and cuts) of the function $U(s)$. A thorough discussion of the inversion formula can be found in the references .

Another important result that is extremely useful in calculations is the convolution theorem.

Theorem 2.8

(**Convolution Theorem**) Let u and v be piecewise continuous on $t \geq 0$ and of exponential order. Then

$$\mathcal{L}(u * v)(s) = U(s)V(s),$$

where

$$(u * v)(t) \equiv \int_0^t u(t-\tau)v(\tau)d\tau$$

is the **convolution** of u and v, and $U = \mathcal{L}u$, $V = \mathcal{L}v$. Therefore,

$$\mathcal{L}^{-1}(U(s)V(s)) = (u * v)(t). \quad \square$$

The convolution theorem tells what to take the transform of in order to get a product of Laplace transforms, namely, the convolution. It also states that the inverse transform of a product of two transforms is the convolution of the two. It is easy to verify that $(u * v)(t) = (v * u)(t)$, so the order of the convolution does not make a difference. We know that the Laplace transform is additive, but this shows it is not multiplicative. That is, the Laplace transform of a product is not the product of the Laplace transforms.

Example 2.9

Find the solution of the IVP

$$u' + 3u = f(t), \quad u(0) = 1.$$

Taking the transform of the equation gives

$$sU(s) - 1 + 3U(s) = F(s), \quad F(s) = \mathcal{L}f(t),$$

or

$$U(s) = \frac{1}{s+3} + \frac{1}{s+3}F(s).$$

Taking the inverse transform yields

$$u(t) = e^{-3t} + \mathcal{L}^{-1}\left(\frac{1}{s+3}F(s)\right).$$

The convolution theorem implies that the second term on the right is

$$e^{-3t} * f(t).$$

Therefore

$$u(t) = e^{-3t} + \int_0^t e^{-3(t-\tau)} f(\tau)d\tau. \qquad \square$$

To review the preceding method, taking the Laplace transform of an ordinary differential equation for $u(t)$ results in an algebraic equation for $U(s)$ in the transform domain. We solve the algebraic equation for $U(s)$ and then recover $u(t)$ by inversion.

Partial Differential Equations

The same strategy applies to partial differential equations where the unknown is a function of two variables, for example $u = u(x,t)$. Now we transform on t, as before, with the variable x being a parameter unaffected by the transform. In particular, we define the **Laplace transform** of $u(x,t)$ by

$$(\mathcal{L}u)(x,s) \equiv U(x,s) = \int_0^\infty u(x,t)e^{-st}dt. \tag{2.51}$$

Then time derivatives transform as in (2.49) and (2.50); for example,

$$(\mathcal{L}u_t)(x,s) = sU(x,s) - u(x,0), \quad (\mathcal{L}u_{tt})(x,s) = s^2U(x,s) - su(x,0) - u_t(x,0).$$

On the other hand, spatial derivatives are left unaffected; for example,

$$(\mathcal{L}u_x)(x,s) = \int_0^\infty \frac{\partial}{\partial x}u(x,t)e^{-st}dt = \frac{\partial}{\partial x}\int_0^\infty u(x,t)e^{-st}dt = U_x(x,s).$$

Therefore, upon taking Laplace transforms, a PDE in x and t is reduced to an ordinary differential equation in x; all the t-derivatives are turned into multiplication in the transform domain. We point out that in some problems the transform may be take on x, with t as a parameter. The choice depends on the type of boundary conditions.

Example 2.10

(**Contaminant transport**) Let $u = u(x,t)$ denote the concentration of a chemical contaminant dissolved in a fluid in the semi-infinite domain $x > 0$ at time t. Initially, assume that the domain is free from contamination. For times $t > 0$ we impose a constant unit concentration of a contaminant on the boundary $x = 0$, and we ask how this contaminant diffuses into the region. Assuming a unit diffusion constant, the mathematical model is

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x > 0,\ t > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,t) &= 1, \quad t > 0; \quad u(x,t) \ \text{bounded}. \end{aligned}$$

Taking Laplace transforms of both sides of the PDE gives

$$sU(x,s) - U_{xx}(x,s) = 0.$$

This is an ordinary differential equation with x as the independent variable, and the solution is

$$U(x,s) = a(s)e^{-\sqrt{s}x} + b(s)e^{\sqrt{s}x}.$$

Because we want bounded solutions, we set $b(s) = 0$. Then

$$U(x,s) = a(s)e^{-\sqrt{s}x}.$$

Now we take the Laplace transform of the boundary condition to get $U(0,s) = 1/s$, where we have used $\mathcal{L}(1) = 1/s$. Therefore, $a(s) = 1/s$, and the solution in the transform domain is

$$U(x,s) = \frac{1}{s}e^{-\sqrt{s}x}.$$

Now we must invert the transform. Consulting Table 2.1 or a computer algebra program, we find that the solution is

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{\sqrt{4t}}\right),$$

where erfc is the **complimentary error function** defined by the formula

$$\operatorname{erfc}(y) = 1 - \frac{2}{\sqrt{\pi}}\int_0^y e^{-r^2}\,dr.$$

Observe that

$$\operatorname{erfc}(y) = 1 - \operatorname{erf}(y).$$

The complimentary error function is plotted in Figure 2.8. □

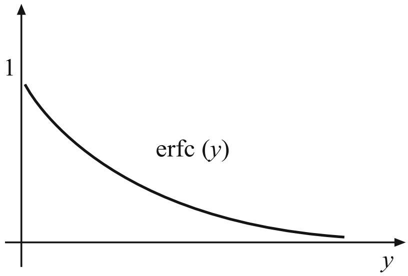

Figure 2.8 Plot of the complimentary error function erfc(y). In the solution $y = x/\sqrt{4t}$

.

Example 2.11

In the contaminant transport model in the last example let us change the boundary condition to a function of time and consider

$$\begin{aligned} u_t - u_{xx} &= 0, \quad x > 0, \ t > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,t) &= f(t), \quad t > 0; \quad u(x,t) \ \text{bounded.} \end{aligned}$$

Taking Laplace transforms of both sides of the PDE gives, as in the example,

$$sU(x,s) - U_{xx}(x,s) = 0,$$

which has solution

$$U(x,s) = a(s)e^{-\sqrt{s}x}.$$

Now we take a Laplace transform of the boundary condition to get $U(0,s) = F(s)$. Therefore, $a(s) = F(s)$ and the solution in the transform domain is

$$U(x,s) = F(s)e^{-\sqrt{s}x}.$$

Consulting Table 2.1, we find that

$$\mathcal{L}^{-1}\left(e^{-\sqrt{s}x}\right) = \frac{x}{\sqrt{4\pi t^3}}e^{-x^2/4t}.$$

Therefore, we can use the convolution theorem to write the solution as

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi(t-\tau)^3}}e^{-x^2/(4(t-\tau))}f(\tau)d\tau. \quad \square$$

A very complete reference for the Laplace transform, which includes an extensive table, theory, and applications, is Churchill (1970).

Parameter Identification Problems

In some physical problems all of the input parameters may not be known a priori. In a **parameter identification problem** we ask whether it is possible to take certain measurements and thereby determine an unknown parameter (a constant) or distributed parameter (a function) in a PDE. For example, suppose the diffusivity of a metal rod is unknown. Will holding the temperature constant at one end and measuring the heat flux out the other end determine the diffusivity? Parameter identification problems are a subclass of **inverse problems**; these can be described qualitatively as problems where the input is to be recovered from the output, rather than conversely, which is the case for so-called *direct problems.* All of the problems we encountered so far are direct problems (given all the data, find the solution), but one can argue that inverse problems play an equally important role in applied science. In many empirical problems we want to measure the output of experiments and use that information to determine properties of the system. In tomography, for example, we can we learn about density variations in tissue by observing the reflection and transmission properties of sound waves. Here we will not be too ambitious, but rather only introduce some of the issues.

Example 2.12

Consider a long ($x \geq 0$) metal slab of unknown thermal conductivity K but known density ρ and specific heat c. For simplicity, take $\rho = c = 1$. Suppose further that measurements can be made only at the face $x = 0$. If a known temperature $f(t)$ at $x = 0$ is applied, can we measure the heat flux at $x = 0$ at a single instant of time t_0 and thereby determine K? We assume $f(0) = 0$. The PDE model is

$$u_t = Ku_{xx}, \quad x, t > 0, \tag{2.52}$$

$$u(0,t) = f(t), \quad t > 0, \tag{2.53}$$

$$u(x,0) = 0, \quad x > 0, \tag{2.54}$$

which is a well-posed problem. We are asking whether we can determine K from a single flux measurement

$$-Ku_x(0,t_0) = a, \tag{2.55}$$

where a is known. Using Laplace transforms we have already solved the direct problem. The solution is

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4K\pi(t-\tau)^3}} e^{-x^2/(4K(t-\tau))} f(\tau)d\tau. \tag{2.56}$$

It appears that the strategy should be to calculate the flux at $(0, t_0)$ from the solution formula (2.56). Indeed this is the case, but calculating the x derivative of u is not a simple matter. The straightforward approach of pulling a partial derivative $\partial/\partial x$ under the integral sign fails because one of the resulting improper integrals cannot be evaluated at $x = 0$; it does not exist. The reader should verify this statement. Therefore, we must be more clever in calculating u_x. To this end we note that (2.56) can be written as

$$u(x,t) = -2K \int_0^t \frac{\partial}{\partial x} G(x, t-\tau) f(\tau)d\tau,$$

where $G(x,t)$ is the heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi K t}} e^{-x^2/(4Kt)}.$$

Therefore,

$$u_x(x,t) = -2K \int_0^t \frac{\partial^2}{\partial x^2} G(x, t-\tau) f(\tau)d\tau \tag{2.57}$$

$$= 2 \int_0^t \frac{\partial}{\partial \tau} G(x, t-\tau) f(\tau)d\tau, \tag{2.58}$$

since $-G_\tau = G_t = KG_{xx}$ (the heat kernel satisfies the heat equation). Now we integrate by parts to obtain

$$u_x(x,t) = -2 \int_0^t G(x, t-\tau) f'(\tau)d\tau.$$

The boundary terms generated by the integration by parts are zero. Consequently, we have

$$-Ku_x(0, t_0) = \sqrt{K} \int_0^{t_0} \frac{f'(\tau)}{\sqrt{\pi(t_0-\tau)}} d\tau = a.$$

This equation uniquely determines K and solves the parameter identification problem.

For example, if $f(t) = \beta t$, i.e., the temperature is increased linearly, then the integral can be calculated exactly to obtain

$$K = \frac{\pi a^2}{4\beta^2 t_0}. \quad \square \tag{2.59}$$

Sections throughout the text provide exercises showing additional examples—typically, inverse problems, and in particular distributed parameter identification problems, lead to an integral equation for the unknown function. These equations are difficult to resolve, and stability is often a problem. That is, a small change in measurement can cause a large change in the calculated parameter value.

Table 2.1 Laplace Transforms

$u(t)$	$U(s)$
1	$s^{-1}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin at$ and $\cos at$	$\frac{a}{s^2+a^2}$ and $\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$ and $\cosh at$	$\frac{a}{s^2-a^2}$ and $\frac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
$H(t-a)$	$s^{-1}e^{-as}, \quad s > 0$
$\delta(t-a)$	e^{-as}
$H(t-a)f(t-a)$	$F(s)e^{-as}$
$f(t)e^{-at}$	$F(s+a)$
$H(t-a)f(t)$	$e^{-as}\mathcal{L}(f(t+a))$
$\text{erf}\sqrt{t}$	$s^{-1}(1+s)^{-1/2}, \quad s > 0$
$\frac{1}{\sqrt{t}}\exp\left(\frac{-a^2}{4t}\right)$	$\sqrt{\pi/s}\,e^{-a\sqrt{s}}, \quad (s > 0)$
$1-\text{erf}\left(\frac{a}{2\sqrt{t}}\right)$	$s^{-1}e^{-a\sqrt{s}}, \quad s > 0$
$\frac{a}{2t^{3/2}}\exp\left(\frac{-a^2}{4t}\right)$	$\sqrt{\pi}\,e^{-a\sqrt{s}}, \quad s > 0$

EXERCISES

1. Consider the nonhomogeneous initial value problem

$$u'' + 4u = te^{-t}, \quad u(0) = 0, \;\; u'(0) = 0.$$

a) Use Duhamel's principle to solve this problem. That is, show

$$u(t) = \int_0^t w(t-\tau,\tau)d\tau$$

where (w,τ) is the solution to

$$w'' + 4w = 0, \quad w(0,\tau) = 0, \;\; w'(0,\tau) = \tau e^{-\tau}.$$

b) Use Laplace transforms to verify your solution.

2. Find the inverse transform of

$$U(s) = \frac{1}{s(s^2+1)}$$

using the convolution theorem.

3. Using the integral definition of Laplace transform, find the transform of $u(t) = \sqrt{t}$. Hint: Make the substitution $r^2 = st$.

4. Show that $\mathcal{L}\left(\int_0^t f(\tau)d\tau\right) = \frac{F(s)}{s}$.

5. Show that $\mathcal{L}\left(H(t-a)f(t-a)\right) = e^{-as}F(s)$, where H is the unit step function (the Heaviside function) defined by $H(x) = 0$ for $x < 0$, and $H(x) = 1$ for $x \geq 0$.

6. Solve $u_{tt} = u_{xx}$ for $x, t > 0$ with $u(0,t) = \sin t, \;\; t > 0$, and $u(x,0) = 0, \;\; u_t(x,0) = 1, \;\; x > 0$.

7. Solve $u_t = u_{xx}$ on $x, t > 0$ with $u(0,t) = a, \;\; t > 0$ and $u(x,0) = b$, where a and b are constants.

8. Use Laplace transforms to solve the initial boundary value problem

$$u_t = u_{xx}, \;\; x > 0, \; t > 0,$$
$$u_x(0,t) - u(0,t) = 0, \;\; t > 0,$$
$$u(x,0) = u_0, \;\; x > 0.$$

Interpret this model physically in the context of heat flow.

9. Find a bounded solution to

$$u_{tt} = c^2 u_{xx} - f(t), \;\; x > 0, \; t > 0,$$
$$u(0,t) = 0, \;\; t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \;\; x > 0.$$

10. Solve the following problem using Laplace transforms.

$$u_{tt} = c^2 u_{xx} - g, \;\; x > 0, \; t > 0,$$
$$u(0,t) = 0, \;\; t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \;\; x > 0.$$

The solution shows what happens to a falling cable lying on a table that is suddenly removed. Sketch some time snapshots of the solution.

11. In the quarter plane $x, y > 0$, where the temperature is initially zero, heat flows only in the y-direction; along the edge $y = 0$ heat is convected along the x-axis, and the temperature is constantly 1 at the point $x = y = 0$. The boundary value problem for the temperature $u(x, y, t)$ is

$$u_t = u_{yy}, \quad x, t, y > 0,$$
$$u(x, y, 0) = 0, \quad x, y > 0,$$
$$u(0, 0, t) = 1, \quad t > 0,$$
$$u_t(x, 0, t) + u_x(x, 0, t) = 0, \quad x, t > 0.$$

Find a bounded solution using Laplace transforms.

12. A very deep container of liquid is insulated on its sides. Initially, its temperature is a constant u_0 degrees, and for $t > 0$ heat radiates from its exposed top surface according to Newton's law of cooling (see Section 1.3). The air temperature is zero degrees. (a) Formulate an initial boundary value problem for the temperature of the liquid, and find a formula for the temperature at various depths at various times. (b) Take $\rho = c = K = \beta = 1$, and use a computer algebra system to plot some temperature profiles.

13. Derive the solution $u(x, t) = H(t - x/c)g(t - x/c)$ to the problem

$$u_{tt} = c^2 u_{xx}, \quad x, t > 0,$$
$$u(x, 0) = u_t(x, 0) = 0, \quad x > 0,$$
$$u(0, t) = g(t), \quad t > 0.$$

14. (Parameter identification) Suppose a chemical reactor occupying the space $x < 0$ operates at some unknown temperature $F(t)$. To determine $F(t)$ we insert a long, laterally insulated metal probe of unit diffusivity, and we measure the temperature $U(t)$ of the probe at $x = 1$. Assume that the probe occupies $0 \leq x < \infty$. Formulate the model equations and show that $F(t)$ satisfies the integral equation

$$U(t) = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{F(\tau)}{(t - \tau)^{3/2}} e^{-1/4(t-\tau)} d\tau.$$

Hint: Solve the direct problem by Laplace transforms and use the table entry

$$\mathcal{L}^{-1}(e^{-\sqrt{s}x}) = \frac{x}{2\sqrt{\pi}\, t^{3/2}} e^{-x^2/(4t)}.$$

Suppose the temperature of the reactor $F(t)$ is a constant F_0; find F_0 if the temperature at $x = 1$ is found to be 10 degrees at $t = 5$.

15. Verify (2.59).

2.7 Fourier Transforms

The Fourier transform is another integral operator with properties similar to the Laplace transform in that derivatives are turned into multiplication operations in the transform domain. Thus the Fourier transform, like the Laplace transform, is useful as a computational tool in solving differential equations. In PDEs the Laplace transform is usually applied to the time variable, while the Fourier transform is often applied to the spatial variable when it varies over $(-\infty, \infty)$.

We begin with functions of one variable. The **Fourier transform** of a function $u = u(x)$, $x \in \mathbb{R}$, is defined by the equation

$$(\mathcal{F}u)(\xi) \equiv \hat{u}(\xi) = \int_{-\infty}^{\infty} u(x)e^{i\xi x}\,dx. \tag{2.60}$$

If u is absolutely integrable, i.e., $\int_{-\infty}^{\infty} |u|dx < \infty$, then $\hat{u} = \hat{u}(\xi)$ can be shown to exist.

The transform of a function $u = u(x)$ may, or may not, turn out to be a complex-valued function. The variable ξ is real, but the value $\hat{u}(\xi)$ can be complex. Further, there are a lot of common, simple functions that do not have a classical Fourier transform because the improper integral does not exist. For example,

$$u(x) = e^{x} \quad \text{and} \quad u(x) = C$$

do not have Fourier transforms. Only functions that decay at $\pm\infty$ sufficiently fast have Fourier transforms.

In the theory of Fourier transforms, it is common to work with the set $\mathcal{S}$ of **rapidly decreasing functions** on $\mathbb{R}$: these are the functions with continuous derivatives of all orders, and for which each function and all its derivatives decay to zero as $x \to \pm\infty$ *faster than any power function* (functions like $1/x^2$ and $1/x^6$). The function $\exp(-x^2)$ defining the bell-shaped curve is a such a rapidly decreasing function. More technically, if the set of functions that have continuous derivatives of all orders on $\mathbb{R}$ is denoted by C^∞, then

$$\mathcal{S} = \{u \in C^\infty : \ |u^{(k)}(x)| \le M|x|^{-N} \text{ as } |x| \to \infty, \\ k = 0, 1, 2, \ldots; \text{for all integers}\, N\}.$$

The set $\mathcal{S}$ is called the **Schwartz class** of functions, and one can show that if $u \in \mathcal{S}$, then $\hat{u} \in \mathcal{S}$, and conversely. So $\mathcal{S}$ is a closed set under both Fourier transformation and inversion, which makes it a good set to work with. In this text, however, a purely formal approach is not taken; rather, the goal is to understand how Fourier transforms provide a useful tool for problem-solving.

There is one important remark about notation. There is no standard convention on how to define the Fourier transform (2.60); some put a factor of $1/(2\pi)$ or $1/\sqrt{2\pi}$ in front of the integral, and some have a negative power in the exponential, or even include a factor of 2π in the exponential. One should be aware of these variations when consulting other sources, especially tables of transforms.

A basic property of the Fourier transform is that the kth derivative $u^{(k)}$ $(k = 1, 2, \ldots)$ transforms to an algebraic expression. That is,

$$(\mathcal{F}u^{(k)})(\xi) = (-i\xi)^k \hat{u}(\xi), \quad u \in \mathcal{S}, \tag{2.61}$$

confirming our comment that derivatives are transformed to multiplication (by a factor of $(-i\xi)^k$). This formula is easily proved using integration by parts (as for the Laplace transform); all the boundary terms generated in the integration by parts are zero, since u and all its spatial derivatives vanish at $\pm\infty$.

For functions of two variables, say $u = u(x,t)$, the variable t acts as a parameter, and we define on the variable x as

$$(\mathcal{F}u)(\xi,t) \equiv \hat{u}(\xi,t) = \int_{-\infty}^{\infty} u(x,t)e^{i\xi x}\,dx.$$

Then, under Fourier transformation, x-derivatives turn into multiplication, and t derivatives remain unaffected; for example,

$$\begin{aligned}(\mathcal{F}u_x)(\xi,t) &= (-i\xi)\hat{u}(\xi,t),\\ (\mathcal{F}u_{xx})(\xi,t) &= (-i\xi)^2\hat{u}(\xi,t),\\ (\mathcal{F}u_t)(\xi,t) &= \hat{u}_t(\xi,t).\end{aligned}$$

Solving a differential equation for u involves first transforming the problem into the transform domain ($\hat{u}$ and ξ) and then solving the resulting differential equation for $\hat{u}$. One is then faced with the inversion problem, or the problem of determining the u for which $\mathcal{F}u = \hat{u}$. Another nice property of the Fourier transform is the simple form of the **inversion formula**, or inverse transform. It is

$$(\mathcal{F}^{-1}\hat{u})(x) \equiv u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)e^{-i\xi x}\,d\xi. \tag{2.62}$$

This result is called the **Fourier integral theorem**; it dictates how to get back from the transform domain.

Some Fourier transforms can be calculated directly, but many require complex contour integration. In the next example we calculate the transform using a differential equation technique.

Example 2.13

We calculate the transform of the Gaussian function $u(x) = e^{-ax^2}, \quad a > 0$, or the classical bell-shaped curve. By definition,

$$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{-ax^2} e^{i\xi x} dx.$$

Differentiating with respect to ξ and then integrating by parts gives

$$\begin{aligned} \hat{u}'(\xi) &= i \int_{-\infty}^{\infty} x e^{-ax^2} e^{i\xi x} dx \\ &= \frac{-i}{2a} \int_{-\infty}^{\infty} \frac{d}{dx} e^{-ax^2} e^{i\xi x} dx \\ &= \frac{-\xi}{2a} \hat{u}(\xi). \end{aligned}$$

Therefore, we have a differential equation $\hat{u}' = \frac{-\xi}{2a}\hat{u}$ for $\hat{u}$. Separating variables and integrating gives the general solution

$$\hat{u}(\xi) = Ce^{-\xi^2/(4a)}.$$

The constant C can be determined by noticing that

$$\hat{u}(0) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$

Consequently, we have

$$\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}} e^{-\xi^2/(4a)}. \tag{2.63}$$

So, the Fourier transform of a Gaussian function is itself a Gaussian; likewise, the inverse transform of a Gaussian is a Gaussian. Equation (2.63) has an important interpretation. Let us regard $u(x) = e^{-ax^2}, \quad a > 0$ as a signal representing information. If a is large, then the signal, i.e., the bell-shaped curve, has a very high,narrow peak; information in the signal is localized and coherent. In the transform domain, however, the signal $\hat{u}(\xi)$ is broad with a low peak. The information has spread and coherence is lost. The opposite is true when a is small: broad signals transform to narrow ones. □

Remark. We can think of a Fourier transform as resolving a function into all of its frequencies. A measurement of the **frequency spectrum** is the modulus $|\hat{u}(\xi)|$. □

Example 2.14

If $u(x) = H(x+1) - H(x-1)$ (a square pulse), then it is easily shown that $\hat{u}(\xi) = \frac{2\sin\xi}{\xi}$. The frequency spectrum is

$$\mid \hat{u}(\xi) \mid = \frac{2\sin \mid \xi \mid}{\mid \xi \mid}.$$

The plot of the frequency spectrum shows the frequency composition of the square wave pulse. $\square$

Similar to the case of Laplace transforms, a convolution relation holds for Fourier transforms. If $u, v \in \mathcal{S}$, then we define their **convolution**, which is in $\mathcal{S}$, by

$$(u * v)(x) = \int_{-\infty}^{\infty} u(x-y)v(y)dy.$$

Then we have the following theorem.

Theorem 2.15

(**Convolution theorem**) If $u, v \in \mathcal{S}$, then

$$\mathcal{F}(u * v)(\xi) = \hat{u}(\xi)\hat{v}(\xi). \quad \square$$

By the Fourier integral theorem it follows immediately that

$$(u * v)(x) = \mathcal{F}^{-1}(\hat{u}(\xi)\hat{v}(\xi)).$$

This formula states that the inverse transform of a product of Fourier transforms is a convolution; this is a useful relationship in solving differential equations.

Example 2.16

Let $f \in \mathcal{S}$ and determine u for which

$$u'' - u = f(x), \quad x \in \mathbb{R}.$$

Taking the transform of both sides yields

$$(-i\xi)^2\hat{u} - \hat{u} = \hat{f},$$

or

$$\hat{u}(\xi) = -\frac{1}{1+\xi^2}\hat{f}(\xi).$$

In the transform domain the solution is a product of transforms, and so we apply the convolution theorem. From the Exercises we have

$$\mathcal{F}\left(\frac{1}{2}e^{-|x|}\right) = \frac{1}{1+\xi^2}.$$

Therefore,

$$u(x) = -\frac{1}{2}e^{-|x|} * f(x) = -\frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)dy. \quad \square$$

The strategy in applying transform methods to solve partial differential equations is to proceed formally, making any assumptions that are required to obtain an answer; for example, assume that all the data is in $\mathcal{S}$. When a solution is obtained one can then attempt to verify that it is indeed a solution to the problem. Often one can prove that the solution obtained holds under less severe conditions than required in the application of the transform method.

Now we apply the Fourier transform method to the Cauchy problem for the heat equation. We will derive the same solution formula (2.8) that we obtained in Section 2.1 by a different method.

Example 2.17

(**Heat equation**) Use Fourier transforms to solve the pure initial value problem for the heat equation:

$$u_t - ku_{xx} = 0, \quad x \in , t > 0; \quad u(x,0) = f(x), \ x \in \mathbb{R}. \tag{2.64}$$

Again we assume that $f \in \mathcal{S}$. Taking Fourier transforms of the PDE gives

$$\hat{u}_t = -\xi^2 k\hat{u},$$

which is an ordinary differential equation in t for $\hat{u}(\xi, t)$, with ξ as a parameter. Its solution is

$$\hat{u}(\xi, t) = Ce^{-\xi^2 kt}.$$

But the initial condition gives $\hat{u}(\xi, 0) = \hat{f}(\xi)$, and so $C = \hat{f}(\xi)$. Therefore,

$$\hat{u}(\xi, t) = e^{-\xi^2 kt}\hat{f}(\xi).$$

Replacing a by $1/(4kt)$ in formula (2.63) gives

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi kt}}e^{-x^2/(4kt)}\right) = e^{-\xi^2 kt}.$$

Thus, by the convolution theorem we have

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}e^{-(x-y)^2/(4kt)} f(y)dy. \tag{2.65}$$

This solution was derived under the assumption that f is in the Schwartz class. But now that we have it, we can attempt to show that it is a solution under milder restrictions on f. For example, one can prove that (2.65) is a solution to (2.64) if f is a continuous, bounded function on $\mathbb{R}$. $\square$

Example 2.18

(**Laplace's equation in a half plane**) Now we solve a basic problem involving Laplace's equation in the upper half plane. Consider

$$u_{xx} + u_{yy} = 0, \quad x \in , y > 0; \quad u(x,0) = f(x), \quad x \in \mathbb{R}.$$

We also append the condition that the solution u remains bounded as $y \to \infty$. This example is similar to the last example. Taking the transform (on x with y as a parameter) of the PDE, we obtain

$$\hat{u}_{yy} - \xi^2 \hat{u} = 0,$$

which has general solution

$$\hat{u}(\xi, y) = a(\xi)e^{-\xi y} + b(\xi)e^{\xi y}.$$

The boundedness condition on u forces $b(\xi) = 0$ if $\xi > 0$ and $a(\xi) = 0$ if $\xi < 0$. So we take

$$\hat{u}(\xi, y) = c(\xi)e^{-|\xi| y}.$$

Upon applying Fourier transforms to the boundary condition, we get $c(\xi) = \hat{f}(\xi)$. Therefore, the solution in the transform domain is

$$\hat{u}(\xi, y) = e^{-|\xi| y}\hat{f}(\xi).$$

Then, using the convolution theorem, we obtain the solution

$$u(x,y) = \frac{y}{\pi}\frac{1}{x^2+y^2} * f = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\tau)d\tau}{(x-\tau)^2+y^2}. \quad \square$$

An excellent, brief introduction to Fourier transforms can be found in Vretblad (2006).

EXERCISES

1. Find the convolution of the functions $f(x) = x$ and $g(x) = e^{-x^2}$.
2. Show that the inverse Fourier transform of $e^{-a|\xi|}$, $a > 0$, is

$$\frac{a}{\pi}\frac{1}{x^2+a^2}.$$

3. Verify the following properties of the Fourier transform:

   a) $(\mathcal{F}u)(\xi) = 2\pi(\mathcal{F}^{-1}u)(-\xi)$.

   b) $\mathcal{F}(e^{iax}u)(\xi) = \hat{u}(\xi + a)$.

   c) $\mathcal{F}(u(x + a)) = e^{-ia\xi}\hat{u}(\xi)$.

   Formula (a) states that if a transform is known, so is its inverse, and conversely.

4. If $\mathcal{F}\left(xe^{-|x|}\right) = \frac{4i\xi}{(1+\xi^2)^2}$, find $\mathcal{F}\left(\frac{x}{(1+x^2)^2}\right)$.

5. If $u(x) = e^{-|x|}$, compute $u * u$ and then find the inverse Fourier transform of $\frac{1}{(1+\xi^2)^2}$.

6. Find the Fourier transform of the function u defined by $u(x) = e^{-ax}$ if $x > 0$, and $u(x) = 0$ if $x \le 0$. Plot the frequency spectrum.

7. Compute $\mathcal{F}(xe^{-ax^2})$. Hint: Use (2.61).

8. Compute the Fourier transform of $u(x) = \cos ax$ if $| x |< \pi/2a$, and $u(x) = 0$ if $| x |> \pi/2a$.

9. Solve the following initial value problem for the inhomogeneous heat equation:
$$u_t = u_{xx} + F(x,t), \quad x \in , t > 0 \ \ u(x,0) = 0, \quad x \in \mathbb{R}.$$

10. Find a formula for the solution to the following initial value problem for the free Schrödinger equation:
$$u_t = iu_{xx}, \quad x \in , t > 0; \quad u(x,0) = e^{-x^2}, \ x \in \mathbb{R}.$$

11. Find a bounded solution to the Neumann problem
$$u_{xx} + u_{yy} = 0, \quad x \in , y > 0,$$
$$u_y(x,0) = g(x), \quad x \in \mathbb{R}.$$

    Hint: Let $v = u_y$ and reduce the problem to a Dirichlet problem. The solution is
$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x-\xi)\ln(y^2+\xi^2)d\xi + C.$$

12. Solve the boundary value problem
$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \ y > 0,$$
$$u(x,0) = 1, \ |x| \le l; \quad u(x,0) = 0, \quad |x| > l.$$

13. Use integration by parts to verify (assume $u \in \mathcal{S}$)

$$(\mathcal{F}u_x)(\xi, t) = (-i\xi)\hat{u}(\xi, t),$$
$$(\mathcal{F}u_{xx})(\xi, t) = (-i\xi)^2\hat{u}(\xi, t).$$

14. Let $u(x)$ be a square wave, i.e, $u(x) = 1$ if $|x| \leq a$ and $u(x) = 0$ if $|x| > a$. Show that

$$(\mathcal{F}u)(\xi) = \frac{2\sin a\xi}{\xi}.$$

Plot the frequency spectrum.

15. Solve the Cauchy problem for the advection–diffusion equation using Fourier transforms:

$$u_t = Du_{xx} - cu_x, \quad x \in \mathbb{R},\ t > 0; \quad u(x, 0) = \phi(x), \quad x \in \mathbb{R}.$$

16. This exercise explores the role of a term u_{xxx} (called a **dispersion** term) in a PDE by examining the equation

$$u_t + u_{xxx} = 0.$$

This equation is sometimes called the linearized Korteweg–deVries (KdV) equation.

a) What relation between ω and k would have to hold if a solution of the form

$$u(x, t) = e^{i(kx - \omega t)}$$

exists? What do these solutions look like, and how does their speed depend on k? What does your conclusion mean, qualitatively? (Recall that the real and imaginary parts of a complex solution to a linear equation are both real solutions.)

b) Use Fourier transforms to solve the Cauchy problem for the linearized KdV equation, and write your answer in terms of the **Airy function** defined by

$$\text{Ai}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\left(\frac{z^3}{3} + xz\right) dz.$$

17. Take a Gaussian function $\exp(-x^2)$ as the initial condition for the Cauchy problem for the heat equation and find the solution profile when $t = 1$.

18. Use Fourier transforms to derive d'Alembert's solution (2.14) to the Cauchy problem for the wave equations (2.12)–(2.13) when the initial velocity is zero, i.e., $g(x) \equiv 0$.

19. (Parameter identification) It is believed that nerve impulses are transmitted along axons by both diffusion and convection. In fundamental experiments the biologists Hodgkin and Keyes used radioactive potassium ^{42}K to measure how ions convect and diffuse in squid axons. If $u = u(x,t)$ is the concentration of potassium in a long axon, then the convection–diffusion model is

$$u_t = Du_{xx} - vu_x, \quad x \in \mathbb{R}, \quad t > 0.$$

The velocity v can be measured directly, but the diffusion constant D is difficult to measure. To determine D take a known initial concentration $u(x,0) = e^{-x^2/a}$ and solve for the concentration $u(x,t)$. Obtain

$$u(x,t) = \frac{\sqrt{a}}{\sqrt{a+4Dt}} e^{-(x-vt)^2/(a+4Dt)}.$$

Show how D can be recovered from a continuous measurement $U(t) = u(x_0,t)$ of the potassium concentration at some fixed location $x = x_0$.

20. Solve the integral equation for f:

$$\int_{-\infty}^{\infty} f(x-y)e^{-y^2}\,dy = e^{-x^2/4}.$$