4
Partial Differential Equations on Bounded Domains

This chapter treats a standard and important method called **separation of variables**, or the method of **eigenfunction expansions**, for solving partial differential equations on bounded spatial domains. This method, due to Fourier in the early 1800s, with contributions by Euler and D'Alembert in the late 1700s, is fundamental. In fact, many textbooks emphasize this method over all others because of its extensive applications in physics and engineering where most problems are on bounded spatial domains. In a nutshell, the essential feature of the method is the replacement of the partial differential by a set of ordinary differential equations which then are solved subject to given initial and boundary conditions. To fix the idea, assume the unknown function in the PDE is a $u = u(x,t)$, where the independent variables are x and t. We make the assumption that u can be written as a *product* of a function of x and a function of t, that is,

$$u(x,t) = y(x)g(t).$$

If the method is to be successful, when this product is substituted into the PDE, the PDE *separates* into two ODEs, one for $y(x)$ and one for $g(t)$. Substitution of the product into the boundary conditions leads to boundary conditions on the function $y(x)$. Therefore, we are faced with a spatial ODE boundary value problem for $y(x)$ and a temporal ODE problem for $g(t)$. When the equations for $y(x)$ and $g(t)$ are solved, we can form a product solution $u(x,t) = y(x)g(t)$ of the PDE that satisfies the boundary conditions. The boundary value problem we obtain for $y(x)$ is a special type of eigenvalue problem called a

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Sturm–Liouville problem and it has infinitely many solutions. Consequently, we will have infinitely many product solutions $u_1(x,t)$, $u_2(x,t)$, $u_3(x,t), \ldots$ that satisfy the boundary conditions. By superimposing these solutions, or adding them up in a special way, we determine a solution of the PDE and boundary conditions that *also* satisfies the initial condition(s). In other words, we form the series

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t) + c_3 u_3(x,t) + \cdots$$

and choose the constants c_n such that the sum satisfies the initial condition(s) as well. The result of the calculation is an infinite series representation of the solution to the the original initial boundary value problem for the PDE.

Section 1 amplifies the preceding remarks to fully motivate the method. Then, Section 2 presents the basics of Sturm–Liouville eigenvalue problems, and the final sections of the chapter treat many important examples that occur in engineering and physics. Many references discuss these methods in detail; particularly we mention Churchill (1963), and subsequent editions, Strauss (1992), Farlow (1993), and Haberman (2013).

4.1 Overview of Separation of Variables

In this first section we work through detailed examples of Fourier's method in the contexts of heat transfer and wave motion.

Example 4.1

We revisit the basic idea of Fourier that we discussed in Section 3.1, which motivated the study of orthogonal expansions and Fourier series. Consider the initial boundary value problem for heat conduction,

$$u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0, \tag{4.1}$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0, \tag{4.2}$$

$$u(x,0) = f(x), \quad 0 < x < \pi. \tag{4.3}$$

The separation of variables method consists of looking for solutions in the form of products, i.e.,

$$u(x,t) = y(x)g(t).$$

Substituting this product into the PDE (4.1) and boundary conditions (4.2), we obtain

$$y(x)g'(t) = y''(x)g(t), \quad y(0)g(t) = 0,\ \ y(\pi)g(t) = 0.$$

Because $g(t)$ is not the zero function, we can write the ODE and boundary conditions as

$$\frac{g'(t)}{g(t)} = \frac{y''(x)}{y(x)} = -\lambda, \quad y(0) = y(\pi) = 0,$$

for some yet to be determined constant λ. This is valid because the only way a function of t can equal a function of x *for all* t and x, is if both are equal to the same constant. The yet unknown constant $-\lambda$ is called the **separation constant**; placing a minus sign on the constant is for convenience and does not mean that it is negative. Therefore, we obtain an ordinary differential equation for g in the time domain, namely,

$$g'(t) = -\lambda g(t),$$

and we obtain a boundary value problem for y in the spatial domain, namely,

$$-y''(x) = \lambda y(x), \quad 0 < x < \pi, \tag{4.4}$$

$$y(0) = 0, \quad y(\pi) = 0. \tag{4.5}$$

Therefore, the PDE problem separated into two ODE problems, one for $g(t)$ and one for $y(x)$. We can easily solve the the DE for g; leaving out the arbitrary multiplicative constant, its solution is

$$g(t) = e^{-\lambda t}.$$

Next we solve the boundary value problem, which, as we later observe, is a special case of a Sturm–Liouville problem: It is obvious that $y(x) = 0$ is a solution to this problem; but we are interested in nontrivial solutions. The question is: *for which values of the separation constant* λ *are there nontrivial solutions*? (For example, if $\lambda = 2$ then the general solution is $y(x) = A\cos\sqrt{2}\,x + B\sin\sqrt{2}\,x$ which is not zero at $x = 0$ and $x = \pi$ unless $A = B = 0$.) One way to determine such values of λ is to write the general solution of (4.4) and go through a case argument by separately considering $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. We examine different cases because the general solution of (4.4) has a different form depending on the sign of λ. (We prove later that λ cannot be a complex number.) If $\lambda = 0$, then the ODE (4.4) has the form $y'' = 0$, whose general solution is the linear function $y(x) = Ax + B$. But the boundary condition $y(0) = 0$ implies $B = 0$, and then $y(\pi) = 0$ implies $A = 0$. Hence, in this case we get only the uninteresting trivial solution and so $\lambda = 0$ is not a possible value. If $\lambda < 0$, say for definiteness $\lambda = -\alpha^2$, then the ODE (4.4) has the form $y'' - \alpha^2 y = 0$, which has a general solution

$$y(x) = Ae^{\alpha x} + Be^{-\alpha x}.$$

The boundary conditions (4.5) force

$$y(0) = A + B = 0,$$
$$y(\pi) = Ae^{\alpha\pi} + Be^{-\alpha\pi} = 0.$$

It is easy to see that these two equations for A and B force $A = B = 0$, and again we obtain only the trivial solution. Thus, we must have $\lambda > 0$, or $\lambda = \alpha^2$. In this case the ODE (4.4) takes the form

$$y'' + \alpha^2 y = 0,$$

which has solutions built from sines and cosines:

$$y(x) = A\cos\alpha x + B\sin\alpha x.$$

First, $y(0) = 0$ forces $A = 0$. Thus $y(x) = B\sin\alpha x$. The right-hand boundary condition yields

$$y(\pi) = B\sin\alpha\pi = 0.$$

But now we are not required to take $B = 0$; rather, we can select α, and hence the unknown λ, to make this equation hold. Clearly, because the sine function vanishes at multiples of π, we have $\alpha = n$, a nonzero positive integer, or

$$\lambda = \lambda_n = n^2, \quad n = 1, 2, \ldots.$$

These are the values of λ that lead to nontrivial solutions

$$y = y_n(x) = \sin nx, \quad n = 1, 2, \ldots.$$

Here we arbitrarily selected the constant $B = 1$; but we can always multiply this solution by any constant to get another (not independent) solution.

These special values of λ for which (4.4–4.5) has a nontrivial solution are called the **eigenvalues** for the problem and the corresponding solutions $y_n(x)$ are called the **eigenfunctions**. For example, $\lambda = 1$ has corresponding eigenfunction $y_1(x) = \sin x$, $\lambda = 4$ has corresponding eigenfunction $y_1(x) = \sin 2x$, and so on. Observe further that the *eigenfunctions form an orthogonal system on the interval* $[0, \pi]$; the eigenfunctions we found are the basis of a Fourier sine series studied in the last chapter.

Next we return to the time equation which has solutions $g(t) = e^{\lambda t}$. Substituting the known values of $\lambda = \lambda_n$ gives

$$g_n(t) = e^{-n^2 t}, \quad n = 1, 2, \ldots.$$

Therefore, product solutions to the PDE and boundary conditions (4.1–4.2) are

$$u_n(x,t) = g_n(t)y_n(x) = e^{-n^2 t}\sin nx, \quad n = 1, 2, \ldots.$$

To find a solution that also satisfies the initial condition (4.3), we use superposition and form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx, \tag{4.6}$$

where the b_n are arbitrary constants. Equation (4.3) implies

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The right side is the Fourier sine series for the initial function $f(x)$ in terms of the orthogonal functions $\sin nx$! From Chapter 3 we conclude that the coefficients b_n are the Fourier coefficients given by

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx. \tag{4.7}$$

In summary, the solution to initial boundary value problem (4.1), (4.2), (4.3) is given by the infinite series (4.6) where the b_n are given by (4.7). □

This is the method of separation of variables, and it applies to most all of the problems on bounded domains in this text. For the method to work:

1. The boundary conditions must be homogeneous.
2. The PDE must be homogeneous.

Later in this chapter we indicate how to deal with nonhomogeneous problems.

Example 4.2

We apply the procedure, but with less detail, to the initial boundary value problem for the heat equation

$$u_t = k u_{xx}, \;\; 0 < x < l, \; t > 0, \tag{4.8}$$

$$u(0,t) = 0, \;\; u(l,t) = 0, \;\; t > 0, \tag{4.9}$$

$$u(x,0) = f(x), \;\; 0 < x < l, \tag{4.10}$$

where we have included a diffusivity k in the PDE and the spatial interval is $[0, l]$ instead of $[0, \pi]$.

Step one is to assume a solution of the form $u(x,t) = y(x)g(t)$ and substitute it into the PDE (4.8) and boundary conditions (4.9). Substituting into the PDE gives

$$y(x)g'(t) = ky''(x)g(t),$$

or, upon dividing by kyg,

$$\frac{g'(t)}{kg(t)} = \frac{y''(x)}{y(x)}.$$

Notice that the variables in the equations have been separated—the left side is a function of t, and the right side is a function of x. Now, when can a function of t be equal to a function of x for all $x \in (0, l)$ and all $t > 0$? Only if the two functions are equal to the same constant, which again we call $-\lambda$. That is, we must have

$$\frac{g'(t)}{kg(t)} = -\lambda, \quad \frac{y''(x)}{y(x)} = -\lambda.$$

Therefore, we have two ODEs, for g and y:

$$g'(t) = -\lambda k g(t), \quad -y''(x) = \lambda y(x).$$

Also observe we placed the constant k in the g equation; it could be placed in the y equation, but it is convenient to keep the y equation simple. Next we substitute $u(x, t) = y(x)g(t)$ into the boundary conditions (4.9) to obtain

$$y(0)g(t) = 0, \quad y(l)g(t) = 0.$$

Excluding the uninteresting possibility that $g(t) = 0$, we get $y(0) = 0$ and $y(l) = 0$. Therefore, we are led to the boundary value problem

$$-y''(x) = \lambda y(x), \quad 0 < x < l, \tag{4.11}$$

$$y(0) = 0, \quad y(l) = 0. \tag{4.12}$$

This is an ODE boundary value problem for y. Exactly as in Example 4.1, the eigenvalues λ are positive. When $\lambda = \alpha^2$, the problem becomes

$$y'' + \alpha^2 y = 0, \quad y(0) = 0, \;\; y(l) = 0.$$

The general solution is

$$y(x) = A \cos \alpha x + B \sin \alpha x.$$

Now $y(0) = 0$ forces $A = 0$ and then $y(l) = 0$ forces $B \sin \alpha l = 0$. We do not want to take $B = 0$, so we get $\alpha l = n\pi$, or $\alpha = n\pi/l$. Therefore, the eigenvalues are

$$\lambda = \lambda_n = \frac{n^2\pi^2}{l^2}, \;\; n = 1, 2, \ldots, \tag{4.13}$$

and the corresponding solutions, or eigenfunctions, to the BVP are

$$y_n(x) = \sin \frac{n\pi x}{l}, \;\; n = 1, 2, \ldots. \tag{4.14}$$

To reiterate, the eigenvalues are those values of λ for which the problem (4.11–4.12) has a nontrivial solution; the eigenfunctions are the corresponding nontrivial solutions.

The next step is to solve the time equation for $g(t)$. We easily get

$$g'(t) = -\lambda k g(t) \Rightarrow g(t) = e^{-\lambda k t} = e^{-n^2\pi^2 kt/l^2}.$$

Now we put together the preceding results. We have constructed infinitely many product solutions of the PDE (4.8) having the form

$$u_n(x,t) = e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l}, \quad n = 1, 2, \ldots.$$

These product solutions also clearly satisfy the boundary conditions (4.9), but they do not satisfy the given initial condition (4.10). So the next step is to determine the constants c_n such that the linear combination

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l} \tag{4.15}$$

satisfies the initial condition (4.10). Thus we require

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{l}.$$

But the right side is just the Fourier sine series of the function $f(x)$ on the interval $(0, l)$. Therefore, the coefficients c_n are the Fourier coefficients given by (see Section 3.3)

$$c_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \quad n = 1, 2, \ldots. \tag{4.16}$$

Therefore, we have obtained a solution to (4.8–4.10) given by the infinite series (4.15) where the coefficients c_n are given by (4.16). If the function f is complicated, then a simple formula for the Fourier coefficients c_n cannot be found, and we must resort to numerical integration. □

Remark 4.3

We can obtain a particularly insightful formula for the solution in Example 4.2. Substituting the expression for the c_n into the solution formula (4.15) allows us to write the solution in a different way as

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(\xi) \sin \frac{n\pi\xi}{l}\, d\xi \right) e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi x}{l} \\
&= \int_0^l \left(\frac{2}{l} \sum_{n=1}^{\infty} e^{-n^2\pi^2 kt/l^2} \sin \frac{n\pi\xi}{l} \sin \frac{n\pi x}{l} \right) f(\xi)\, d\xi \\
&\equiv \int_0^l g(x,\xi,t) f(\xi)\, d\xi.
\end{aligned}
$$

The function $g(x,\xi,t)$ is the expression in parentheses and is called the **Green's function** for the problem. This formula shows the form of the solution as an integral operator acting on the initial temperature distribution function f. Similar to the discussion in Chapter 2 on the diffusion equation, we can regard $g(x,\xi,t)f(\xi)$ as the temperature response at x, at time t, of the system due to a local source $f(\xi)$ at the point ξ at time $t=0$; then the formula above adds up the contribution of all of the sources in the interval. In this way, the function $g(x,\xi,t)$ can be interpreted as a **source function**, or the response to a 'point source.' This may remind more advanced readers of delta functions. □

Once a series representation for the solution is found, the work is not over if we want to find temperature profiles or the solution surface. Because we cannot sum an infinite series, we often take the first few terms as an approximation. Such approximations are illustrated in an example below and in the exercises. For the heat equation, taking a two or three term approximation is very accurate because of the decaying exponential in the series (4.15).

Example 4.4

Consider the initial boundary value problem for the heat equation:

$$
\begin{aligned}
u_t &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\
u(0,t) &= u(1,t) = 0, \quad t > 0, \\
u(x,0) &= 10x^3(1-x), \quad 0 < x < 1.
\end{aligned}
$$

Here, $k = l = 1$ and the initial temperature is $f(x) = 10x^3(1-x)$. The solution is given by (4.15) with the coefficients (4.16) in Example 4.2. Specifically,

$$
c_n = 20 \int_0^1 x^3(1-x) \sin(n\pi x)\, dx.
$$

Using a computer algebra system or calculator, one finds

$$
c_1 = 0.7331,\ c_2 = -0.4838,\ c_3 = 0.1304.
$$

A 3-term approximate solution is therefore

$$u(x,t) \approx c_1 e^{-\pi^2 t} \sin(\pi x) + c_2 e^{-4\pi^2 t} \sin(2\pi x) + c_3 e^{-9\pi^2 t} \sin(3\pi x).$$

Figure 4.1 plots four time profiles showing how the bar cools down. □

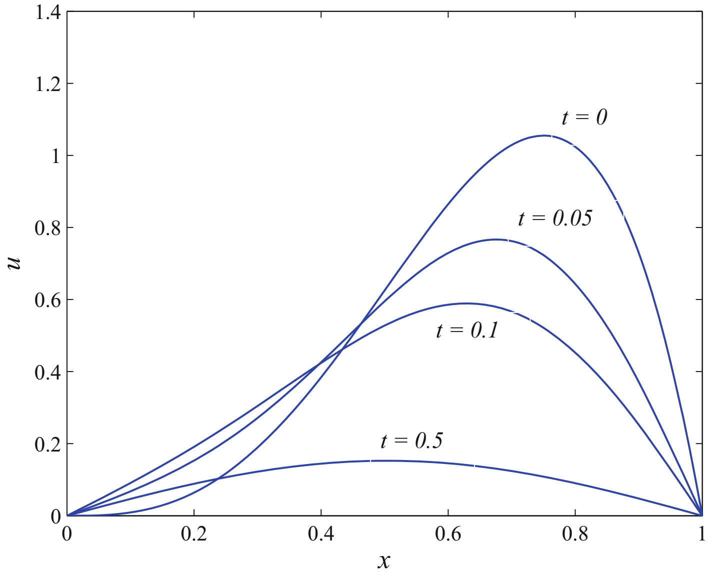

Figure 4.1 Temperature profiles in Example 4.4 at different times. As $t \to 0$ the temperature approaches the steady state solution, $u \equiv 0$

The separation of variables procedure described above can be imitated for a large number of problems. Next we illustrate the method for the wave equation, again with less detail.

Example 4.5

Recall that the wave equation models the small deflections of an elastic string fixed at both ends with an initial position and velocity given at time $t = 0$. We consider the problem

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \ t > 0, \tag{4.17}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad t > 0, \tag{4.18}$$

$$u(x,0) = F(x), \quad u_t(x,0) = G(x), \quad 0 < x < l, \tag{4.19}$$

where F and G are the initial position and velocity of the string. Assuming $u(x,t) = y(x)g(t)$ and substituting into the PDE (4.17) yields

$$y(x)g''(t) = c^2 y''(x)g(t),$$

or, upon dividing by $c^2 yg$,

$$\frac{g''(t)}{c^2 g(t)} = \frac{y''(x)}{y(x)}.$$

Setting each term equal to a constant $-\lambda$, we obtain the two ODEs

$$g''(t) + c^2\lambda g(t) = 0, \quad -y''(x) = \lambda y(x).$$

Next we substitute $u(x,t) = y(x)g(t)$ into the boundary conditions (4.18) to get

$$y(0)g(t) = 0, \quad y(l)g(t) = 0,$$

which gives $y(0) = 0$ and $y(l) = 0$. Therefore, we are led to the boundary value problem

$$-y''(x) = \lambda y(x), \;\; 0 < x < l; \quad y(0) = 0, \;\; y(l) = 0.$$

This boundary value problem is exactly the same as in the heat flow example above. The eigenvalues and eigenfunctions are (see (4.13–4.14))

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \;\; y_n(x) = \sin\frac{n\pi x}{l}, \;\; n = 1, 2 \ldots . \tag{4.20}$$

Solving the ODE for $g(t)$,

$$g(t) = g_n(t) = C\sin\frac{n\pi ct}{l} + D\cos\frac{n\pi ct}{l}.$$

Therefore, we have constructed infinitely many product solutions of the PDE (4.17) having the form

$$u_n(x,t) = \left(c_n \sin\frac{n\pi ct}{l} + d_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}, \;\; n = 1, 2, \ldots,$$

where we have replaced the arbitrary constants C and D by arbitrary constants c_n and d_n depending on n. These product solutions $u_n(x,t)$ represent modes of vibrations. The temporal part is periodic in time with period $2l/(nc)$; the spatial part has frequency $2l/n$. These product solutions also satisfy the boundary conditions (4.18) but do not satisfy the given initial conditions (4.19). So we form the linear combination

$$u(x,t) = \sum_{n=1}^{\infty}\left(c_n \sin\frac{n\pi ct}{l} + d_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \tag{4.21}$$

and select the constants c_n and d_n such that the initial conditions hold. Thus we require

$$u(x,0) = F(x) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi x}{l}.$$

The right side is the Fourier sine series of the function $F(x)$ on the interval $(0, l)$. Therefore, the coefficients d_n are the Fourier coefficients given by

$$d_n = \frac{2}{l} \int_0^l F(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \ldots. \tag{4.22}$$

To apply the other initial condition $u_t(x, 0) = G(x)$ we need to calculate the time derivative of $u(x, t)$. We obtain

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{nc\pi}{l} \left(c_n \cos \frac{n\pi ct}{l} - d_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}.$$

Thus

$$u_t(x,0) = G(x) = \sum_{n=1}^{\infty} \frac{nc\pi}{l} c_n \sin \frac{n\pi x}{l}.$$

Again, the right side is the Fourier sine series of G on $(0, l)$, so the coefficients of $\sin \frac{n\pi x}{l}$, which are $\frac{nc\pi}{l} c_n$, are the Fourier coefficients; that is,

$$\frac{nc\pi}{l} c_n = \frac{2}{l} \int_0^l G(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \ldots,$$

or

$$c_n = \frac{2}{nc\pi} \int_0^l G(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \ldots. \tag{4.23}$$

Therefore, the solution of the initial boundary value problem is given by the infinite series (4.21) where the coefficients are given by (4.22) and (4.23).

Again, numerical calculations on a computer algebra system or a scientific calculator can be used to determine a few of the coefficients to obtain an approximate solution. However, for the wave equation several terms may be required for accuracy because, unlike the heat equation, there is no decaying exponential in the solution (4.21) to cause rapid decay of the terms in the series. □

What we have described is the separation of variables method, and the procedure can be imitated on a large number of initial boundary value problems. Each problem leads to a boundary value problem for an ODE that has infinitely many solutions that form an orthogonal system. In the next section we focus attention on the ODE boundary value problem, called a Sturm–Liouville problem.

EXERCISES

1. Solve the heat flow problem with both boundaries insulated:

$$u_t = ku_{xx},\ 0 < x < l,\ t > 0,$$
$$u_x(0,t) = u_x(l,t) = 0,\ t > 0,$$
$$u(x,0) = 1 + 2\cos\frac{3\pi x}{l},\quad 0 \le x \le l.$$

2. In the heat flow problem (4.1–4.3) take $k = 1$, $l = \pi$, and $f(x) = 0$ if $0 < x < \pi/2$, $f(x) = 1$ if $\pi/2 < x < \pi$. (a) Find an infinite series representation of the solution. (b) Use the first four terms in the series to obtain an approximate solution, and on the same set of coordinate axes sketch several time snapshots of the approximate temperature distribution in the bar in order to show how the bar cools down. (c) Estimate the error in these approximate distributions by considering the first neglected term. (d) What is $\lim_{t\to 0^+} u(\pi/2,t)$? Comment.

3. In the vibration problem (4.17–4.19) take $c = 1$, $l = \pi$, and $F(x) = x$ if $0 < x < \pi/2$, and $F(x) = \pi - x$ if $\pi/2 < x < \pi$, and take $G(x) \equiv 0$. (a) Find an infinite series representation of the solution. (b) Use the first four terms in the series to obtain an approximate solution, and on the same set of coordinate axes sketch several time snapshots of the wave. (c) Can you estimate the error in these approximate wave forms?

4. The initial boundary value problem for the damped wave equation,

$$u_{tt} + ku_t = c^2u_{xx},\quad 0 < x < 1,\ t > 0,$$
$$u(0,t) = 0,\quad u(1,t) = 0,\quad t > 0,$$
$$u(x,0) = f(x),\quad u_t(x,0) = 0,\quad 0 < x < 1,$$

governs the displacement of a string immersed in a fluid. The string has unit length and is fixed at its ends; its initial displacement is f, and it has no initial velocity. The constant k is the damping constant and c is the wave speed. Use $k = 2$, $c = 1$ and apply the method of separation of variables to find the solution.

5. Consider heat flow in a rod of length l where the heat is lost across the lateral boundary is given by Newton's law of cooling. The model is

$$u_t = ku_{xx} - hu,\quad 0 < x < l,$$
$$u = 0 \text{ at } x = 0,\ x = l,\ \text{for all } t > 0,$$
$$u = f(x) \text{ at } t = 0,\ 0 \le x \le l,$$

where $h > 0$ is the heat loss coefficient.

a) Find the equilibrium temperature.

b) Solve the problem.

4.2 Sturm–Liouville Problems

In this section we examine a broad class of initial boundary value problems of the form

$$u_t = (p(x)u_x)_x - q(x)u, \quad a < x < b,\ t > 0, \tag{4.24}$$

$$\alpha_1 u(a,t) + \alpha_2 u_x(a,t) = 0, \tag{4.25}$$

$$\beta_1 u(b,t) + \beta_2 u_x(b,t) = 0, \tag{4.26}$$

$$u(x,0) = f(x),\ \ a < x < b. \tag{4.27}$$

The standard boundary conditions (i.e., Dirichlet, Neumann, and radiation) are included in (4.25–4.26) as special cases by choosing appropriate values of the constants.

When we assume a separable solution $u(x,t) = y(x)g(t)$, we obtain an ODE for $y = y(x)$ of the form

$$-(p(x)y')' + q(x)y = \lambda y, \quad a < x < b,$$

where λ is the separation constant (see Exercise 1). This type of ODE is called a **Sturm–Liouville differential equation**. The homogenous boundary conditions on the PDE transform into boundary conditions on y, namely,

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \beta_1 y(b) + \beta_2 y'(b) = 0.$$

Solving this important boundary value problem for the function $y(x)$ on a bounded interval $[a,b]$ is the subject of a large portion of this section and the basis for the separation of variables method. We state the problem in a general form.

Definition 4.6

Let p, p', and q be continuous functions on the bounded interval $[a,b]$, and $p(x) \neq 0$ for all $x \in [a,b]$; assume further that $\alpha_1^2 + \alpha_2^2 \neq 0$ and $\beta_1^2 + \beta_2^2 \neq 0$. A **regular Sturm–Liouville problem** (**SLP**) is the problem of determining values of the constant λ for which the boundary value problem

$$-(p(x)y')' + q(x)y = \lambda y, \quad a < x < b, \tag{4.28}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{4.29}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0. \tag{4.30}$$

has a nontrivial solution $y = y(x)$. By a solution we mean a function that is continuous and has a continuous derivative. We refer to these types of problems as (differential) **eigenvalue problems**. □

If the interval $[a, b]$ is infinite, or if $p(x_0) = 0$ for some $x_0 \in [a, b]$ the SLP is called **singular**. In most problems, p is positive. Different types of boundary conditions also lead to singular problems. SLPs are named after J. C. F. Sturm and J. Liouville, who studied them in the mid 1830s. The condition that the constants α_1 and α_2 are not both zero, and the constants $\beta_1$and β_2 are not both zero, guarantees that a boundary condition does not collapse at an endpoint.

As we saw in Example 4.1, a regular SLP need *not* have a nontrivial solution for some values of the constant λ. (Clearly, the zero function $y(x) \equiv 0$ is always a solution, but there is no interest in it). A value of λ for which there is a nontrivial solution of (4.28–4.30) is called an **eigenvalue**, and the corresponding nonzero solution is called an **eigenfunction**. Observe that any constant multiple of an eigenfunction is also an eigenfunction corresponding to the same eigenvalue; but it is not independent. The important fact about regular SLPs is that they have an infinite number of eigenvalues, and the corresponding eigenfunctions form a complete, orthogonal set, which makes orthogonal expansions possible. This is a *key idea* in applied mathematics, and it especially applies to the separation of variables method, which is the origin of the entire subject of Fourier series and orthogonal expansions.

Before proving some general facts about regular SLPs, we give an additional example with *mixed* boundary conditions. In this, and subsequent examples, we write y for $y(x)$, y' for $y'(x)$, etc., because the independent variable x is understood.

Example 4.7

Consider the regular SLP

$$-y'' = \lambda y, \;\; 0 < x < 1,$$
$$y'(0) = 0, \;\; y(1) = 0.$$

We carry out a case argument to determine the possible eigenvalues. Exactly as in previous examples, we find that λ cannot be negative or zero. If $\lambda = \alpha^2 > 0$ then the general solution of the ODE is $y(x) = A\cos\alpha x + B\sin\alpha x$. Now, $y'(0) = \alpha B = 0$, giving $B = 0$; hence $y(x) = A\cos\alpha x$ and $y(1) = A\cos\alpha = 0$. There is no need to take $A = 0$, so we take $\cos\alpha = 0$, and thus α must be $\pi/2 + n\pi$, $n = 0, 1, 2, \ldots$. Therefore the eigenvalues are

$$\lambda_n = \left(\frac{\pi}{2} + n\pi\right)^2 = \left(\frac{\pi}{2}(2n+1)\right)^2, \;\; n = 0, 1, 2, \ldots.$$

The corresponding eigenfunctions are

$$y_n(x) = \cos\left(\frac{\pi}{2}(2n+1)x\right), \quad n = 0, 1, 2, \ldots.$$

One can verify that the eigenfunctions form an orthogonal system on the interval $[0, 1]$. □

We can always carry out a case argument as in the preceding example to determine the eigenvalues, their signs, and corresponding eigenfunctions. For many equations this can be tedious, and so it is advantageous to have some general results that allow us to reject or affirm certain cases immediately, without detailed calculation. This is discussed in the next section. For the present we state the central result regarding regular SLPs.

Theorem 4.8

For a regular Sturm–Liouville problem (4.28–4.30) the following hold:

1. There are infinitely many discrete eigenvalues λ_n, $n = 1, 2, \ldots$, and the eigenvalues are real with $\lim_{n\to\infty} |\lambda_n| = +\infty$.
2. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.
3. An eigenvalue can have only a single independent eigenfunction.
4. The set of all eigenfunctions $y_n(x)$ is complete in the sense that every square-integrable function f on $[a, b]$ can be expanded in a generalized Fourier series
$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad c_n = \frac{(y_n, f)}{\|y_n\|^2},$$
which converges to f in the mean-square sense on $[a, b]$. □

It is beyond our scope to prove the existence of the eigenvalues (see, for example, Birkhoff and Rota (1989)). But it is straightforward to demonstrate (2) and (3). First we introduce some notation that streamlines writing the formulas.

Notation. Let L denote the second-order differential operator defined by the left side of the Sturm–Liouville differential equation:

$$Ly = -(p(x)y')' + q(x)y, \quad a < x < b.$$

(It is common to write Ly and not $L(y)$.) So the SLP can be written simply as

$$Ly = \lambda y, \quad a < x < b, \tag{4.31}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{4.32}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0. \tag{4.33}$$

We make liberal use of the inner product of two functions:

$$(y_1, y_2) = \int_a^b y_1(x) y_2(x) dx.$$

Finally, we introduce the **Wronskian** of two functions y_1, y_2, defined by

$$W\big[y_1, y_2\big](x) = y_1(x) y_2'(x) - y_1'(x) y_2(x).$$

Notice the Wronskian is a function of x, and we usually write it simply as $W(x)$ if the two functions are understood. The reader may be familiar with this quantity from an elementary differential equations course.

The following lemma contains two extremely important identities that are used in the calculations.

Lemma 4.9

Let y_1 and y_2 be two C^1 functions on the interval $[a, b]$. Then,

$$y_2 L y_1 - y_1 L y_2 = \frac{d}{dx}\big[p(x) W(x)\big].$$

This identity is called **Lagrange's identity**. It follows that

$$(y_2, L y_1) - (y_1, L y_2) = \big[p(x) W(x)\big] \big|_a^b = p(b) W(b) - p(a) W(a).$$

This is **Green's identity**. □

The proof of Lagrange's identity is straightforward. Just take the derivative of $p(x)W(x)$ and verify, using the product rule, that it coincides with the left side:

$$\begin{aligned}
\frac{d}{dx}[p(x)W(x)] &= \frac{d}{dx}\big[p y_1 y_2'\big] - \frac{d}{dx}\big[p y_1' y_2\big] \\
&= \big(p y_2'\big)' y_1 + p y_2' y_1' - \big(p y_1'\big)' y_2 - p y_1' y_2' \\
&= \big(p y_2'\big)' y_1 - \big(p y_1'\big)' y_2 \\
&= -\big(p y_1'\big)' y_2 + q y_1 y_2 + \big(p y_2'\big)' y_1 - q y_1 y_2 \\
&= \big[-\big(p y_1'\big)' + q y_1\big] y_2 - \big[-\big(p y_2'\big)' + q y_2\big] y_1 = y_2 L y_1 - y_1 L y_2.
\end{aligned}$$

To prove Green's identity integrate both sides of Lagrange's identity over $[a, b]$ and use the inner product notation. One can also prove Green's identity directly using integration by parts. □

Now we demonstrate the proofs of various parts of Theorem 4.8.

Orthogonality of eigenfunctions. Let λ_1 and λ_2 be distinct eigenvalues of the SLP (4.31–4.33) with corresponding eigenfunctions $y_1(x)$ and $y_2(x)$. By Green's identity,

$$\begin{aligned}\left(y_2, Ly_1\right) - \left(y_1, Ly_2\right) &= \left(y_2, \lambda y_1\right) - \left(y_1, \lambda y_2\right) \\ &= \left(\lambda_1 - \lambda_2\right)\left(y_1, y_2\right) \\ &= p(b)W(b) - p(a)W(a).\end{aligned}$$

Now we calculate $W(a)$ and $W(b)$ using the boundary conditions. Because both y_1 and y_2 satisfy the first boundary condition (4.32), we have

$$\begin{aligned}\alpha_1 y_1(a) + \alpha_2 y_1'(a) = 0, \\ \alpha_1 y_2(a) + \alpha_2 y_2'(a) = 0.\end{aligned}$$

These expressions can be regarded as a homogenous linear system in two unknowns α_1 and α_2:

$$\begin{pmatrix} y_1(a) & y_1'(a) \\ y_2(a) & y_2'(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By assumption, α_1 and α_2 are not both zero. Hence, the determinant of the coefficient matrix must be zero. That is, $y_1(a)y_2'(a) - y_1'(a)y_1(a) = W(a) = 0$. Similarly $W(b) = 0$. Therefore,

$$\left(\lambda_1 - \lambda_2\right)\left(y_1, y_2\right) = 0.$$

Because $\lambda_1 \neq \lambda_2$, we have $\left(y_1, y_2\right) = 0$, which means y_1 and y_2 are orthogonal. □

The eigenvalues are real. Let λ be an eigenvalue with corresponding eigenfunction y. Then $Ly = \lambda y$ and y satisfies the boundary conditions (4.32–4.33). Taking the complex conjugate of the differential equation we get

$$L\overline{y} = \overline{\lambda}\overline{y}$$

and the comlex conjugate $\overline{y}$ satisfies the boundary conditions as well (with y replaced by $\overline{y}$). This means $\overline{\lambda}$, $\overline{y}$ is also an eigenpair. As above, the Wronskian is zero at $x = a$ and $x = b$. Applying Green's identity then gives

$$\begin{aligned}0 = (y, L\overline{y}) - (\overline{y}, Ly) &= (y, \overline{\lambda}\overline{y}) - (\overline{y}, \lambda y) = (\overline{\lambda} - \lambda)(\overline{y}, y) \\ &= \int_a^b \overline{y}(x)y(x)dx = \int_a^b \mid y(x) \mid^2 dx.\end{aligned}$$

Because y is not identically zero, the last integral is positive. So $\overline{\lambda} = \lambda$. Because λ equals its complex conjugate, it must be real. □

An eigenvalue λ has a unique eigenfunction, up to a constant multiple. Let y_1 and y_2 be two eigenfunctions associated with the eigenvalue λ. Then $Ly_1 = \lambda y_1$ and $Ly_2 = \lambda y_2$. By Lagrange's identity,

$$0 = y_2 L y_1 - y_1 L y_2 = \frac{d}{dx}\Big[p(x)W(x)\Big].$$

This means $p(x)W(x) = c =$ constant. We can evaluate the constant by either of the boundary conditions. For example,

$$p(a)W(a) = c = 0$$

because, as we showed, the boundary condition forces $W(a) = 0$. Hence, $W(x) = y_1 y_2' - y_1' y_2 = 0$ for all x in $[a, b]$. This is the same as

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = 0, \tag{4.34}$$

or $y_1 = \text{constant} \cdot y_2$. Hence y_1 and y_2 are dependent functions. □

The eigenfunctions are real. Let y be an eigenfunction corresponding to the real eigenvalue λ. If y is complex-valued, then $y = \phi(x) + i\psi(x)$, where ϕ and ψ are real-valued functions. By the linearity of L and the boundary conditions, it follows that both ϕ and ψ are eigenfunctions corresponding to λ. Then ϕ and ψ are linear dependent. So we can replace y with one of these real-valued eigenfunctions. □

Example 4.10

(**Periodic boundary conditions**) Consider the simple boundary value problem

$$-y'' = \lambda y, \quad -\pi < x < \pi,$$
$$y(-\pi) = y(\pi), \;\; y'(-\pi) = y'(\pi).$$

This is not a regular SLP because the boundary conditions, called *periodic boundary conditions*, are of a different type. Nevertheless, it is easy to check that the eigenvalues are $\lambda_n = n^2, \;\; n = 0, 1, 2, \ldots$. Corresponding to $\lambda = 0$ is a *single* eigenfunction $y_0 = 1$, and corresponding to any $n > 0$ there are *two* linearly independent eigenfunctions, $\cos nx$ and $\sin nx$. This does not contradict the theorem. One can note that (4.34), required in the proof of uniqueness of an eigenfunction, cannot hold in this case. However, it remains true that $p(x)W(x) \mid_{-\pi}^{\pi} = 0$. Hence the eigenvalues are real and the corresponding set of eigenfunctions form a complete orthogonal set. This set of eigenfunctions form the basis of the classical Fourier series

$$\frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos nx + b_n \sin nx. \;\; \square$$

Another important question concerns the *sign* of the eigenvalues. We can always carry out a case argument as in previous examples, but often an *energy argument* gives an answer more efficiently, as in the following example.

Example 4.11

(**Energy argument**) Consider the Sturm–Liouville differential equation

$$-y'' + q(x)y = \lambda y, \quad 0 < x < l,$$

with $q(x) > 0$ and mixed boundary conditions

$$y(0) = 0, \quad y'(l) = 0.$$

Let λ be an eigenvalue with corresponding eigenfunction $y = y(x)$. Multiply the differential equation by y and integrate to get

$$\int_0^l -yy''\,dx + \int_0^l q(x)y^2 dx = \lambda \int_0^l y^2(x)dx.$$

This method gets its name from the expression

$$\int_0^l y^2(x)dx = \|y\|^2,$$

which is the *energy* of the function y. (See Section 3.3.) Now integrate the first integral by parts, pulling one derivative off y'' and putting it on y:

$$\int_0^l -yy''\,dx = -y(x)y'(x)\mid_0^l + \int_0^l (y')^2\,dx.$$

Then

$$-y(x)y'(x)\mid_0^l + \int_0^l (y')^2\,dx + \int_0^l q(x)y^2 dx = \lambda \int_0^l y^2(x)dx.$$

The boundary conditions force the boundary term to be zero, and therefore

$$\int_0^l (y')^2\,dx + \int_0^l q(x)y^2 dx = \lambda \int_0^l y^2(x)dx.$$

The left side is strictly positive, and the integral on the right is positive. Hence, $\lambda > 0$. This problem has only positive eigenvalues. □

A similar argument using integration by parts can be made for the general Sturm–Liouville problem (4.28–4.30). The reader can show, as above, that if λ and y are an eigenvalue–eigenfunction pair, then

$$\lambda = \frac{\int_a^b \left(p(x)y'^2 + q(x)y^2\right)dx - p(x)y(x)y'(x)\mid_a^b}{\|y\|^2}. \tag{4.35}$$

This expression for the eigenvalue λ is called the **Rayleigh quotient**. The boundary term in the numerator involving the Wronskian may be simplified by the SLP boundary conditions (4.29–4.30).

Example 4.12

Consider the SLP

$$-y'' = \lambda y, \quad 0 < x < 1,$$
$$y(0) = 0, \;\; y(1) + \beta y_1'(1) = 0, \;\; \beta > 0.$$

Here $p(x) = 1$ and $q(x) = 0$. Therefore, if λ and y are an eigenpair, then the Rayleigh quotient gives

$$\lambda = \frac{\int_0^1 y'^2 dx - y(x)y'(x) \mid_0^1}{\|y\|^2}.$$

But

$$-y(1)y'(1) + y(0)y'(0) = \beta y'(1)^2 > 0.$$

Hence, the eigenvalues are positive. □

Next we solve a problem with a radiation-type boundary condition.

Example 4.13

Consider the diffusion problem

$$u_t = ku_{xx}, \quad 0 < x < 1, \; t > 0, \tag{4.36}$$
$$u(0,t) = 0, \;\; u(1,t) + u_x(1,t) = 0, \quad t > 0, \tag{4.37}$$
$$u(x,0) = f(x), \quad 0 < x < 1, \tag{4.38}$$

where the left end is fixed and the right end satisfies a radiation-type condition; note that if $u > 0$ at $x = 1$ then $u_x < 0$ at $x = 1$, so heat is radiating out of the bar. We assume $u(x,t) = y(x)g(t)$ and substitute into the PDE to obtain

$$\frac{g'(t)}{kg(t)} = \frac{y''(x)}{y(x)} = -\lambda,$$

where $-\lambda$ is the separation constant. This gives the two ODEs

$$g'(t) = -\lambda k g(t), \quad -y''(x) = \lambda y(x).$$

Substituting the product $u = yg$ into the boundary conditions gives

$$y(0)g(t) = 0, \quad y(1)g(t) + y'(1)g(t) = 0.$$

Since $g(t) \neq 0$, we are forced to take $y(0) = 0$ and $y(1) + y'(1) = 0$, and we are left with the Sturm–Liouville problem

$$-y'' = \lambda y, \quad 0 < x < 1; \;\; y(0) = 0, \;\; y(1) + y'(1) = 0.$$

We know from Example 4.12 that the eigenvalues are positive. For $\lambda > 0$ the solution to the ODE is $y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. The boundary condition $y(0) = 0$ forces $A = 0$, and the boundary condition $y(1) + y'(1) = 0$ forces the condition

$$\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda} = 0,$$

or

$$\sqrt{\lambda} = -\tan\sqrt{\lambda}.$$

This equation, where the variable λ is tied up in a trigonometric function, cannot be solved analytically. But plots of the functions $\sqrt{\lambda}$ and $-\tan\sqrt{\lambda}$ versus λ (see Figure 4.2) show that the equation has infinitely many positive solutions $\lambda_1, \lambda_2, \ldots$, represented by the intersection points of the two graphs. Numerically, we find $\lambda_1 = 4.115858$, $\lambda_2 = 24.13934$, $\lambda_3 = 63.65911$.

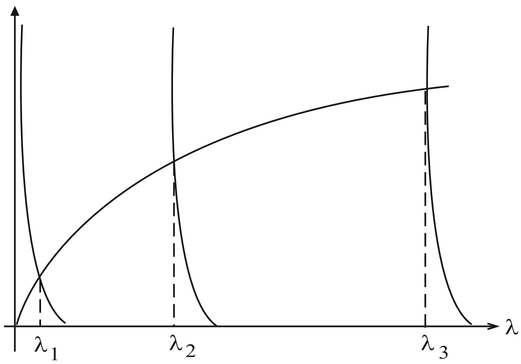

Figure 4.2 Graphical representation of the first three eigenvalues of the SLP at the intersection of $y = -\tan(\sqrt{\lambda})$ and $y = \sqrt{(\lambda)}$

These values λ_n are the eigenvalues for the Sturm–Liouville problem, and the corresponding eigenfunctions are

$$y_n = \sin\sqrt{\lambda_n}x, \quad n = 1, 2, \ldots.$$

By the Sturm–Liouville theorem we know that the eigenfunctions $y_n(x)$ are orthogonal on $0 < x < 1$, which means that

$$\int_0^1 \sin\sqrt{\lambda_n}x\sin\sqrt{\lambda_m}x dx = 0 \;\text{ for }\; m \neq n.$$

The solution to the time equation is

$$g(t) = ce^{-\lambda kt},$$

and since there are infinitely many values of λ, we have infinitely many such solutions,

$$g_n(t) = c_n e^{-\lambda_n kt}.$$

Summarizing, we have obtained infinitely many product solutions of the form

$$c_n e^{-\lambda_n kt} \sin \sqrt{\lambda_n} x, \quad n = 1, 2 \ldots .$$

These functions will solve the PDE (4.36) and the boundary conditions (4.37). We superimpose these to form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n kt} \sin \sqrt{\lambda_n} x, \tag{4.39}$$

and we select the coefficients c_n such that u will satisfy the initial condition (4.38). We have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin \sqrt{\lambda_n} x.$$

This equation is an expansion of the function $f(x)$ in terms of the eigenfunctions $y_n(x)$. We know from Chapter 3 that the coefficients c_n are given by the formula

$$c_n = \frac{1}{\|y_n\|^2} \int_0^1 f(x) \sin \sqrt{\lambda_n} x \, dx, \quad n = 1, 2 \ldots ,$$

or

$$c_n = \frac{1}{\int_0^1 \sin^2 \sqrt{\lambda_n} x dx} \int_0^1 f(x) \sin \sqrt{\lambda_n} x \, dx, \quad n = 1, 2, \ldots . \tag{4.40}$$

Consequently, the solution of (4.36–4.38) is given by (4.39), where the coefficients c_n are given by (4.40).

Using MATLAB to calculate a three-term approximation to the solution in the case $k = 0.1$ with initial temperature $f(x) = 10x(1-x)$, we find the coefficients c_1, c_2, c_3 are given by

$$c_1 = 2.13285, \; c_2 = 1.040488, \; c_3 = -0.219788.$$

Therefore, a three-term approximate solution is

$$\begin{aligned} u(x,t) &\approx 2.13285 e^{-4.115858t} \sin \sqrt{4.115858} x \\ &+ 1.040488 e^{-24.13934t} \sin \sqrt{24.13934} x \\ &- 0.219788 e^{-63.65911t} \sin \sqrt{63.65911} x. \quad \square \end{aligned}$$

We end this section with some summarizing comments and terminology. For the regular Sturm–Liouville problem (4.31–4.33), Green's identity

$$\Big(y_1, Ly_2\Big) - \Big(y_2, Ly_1\Big) = p\Big[y_1 y_2' - y_1' y_2\Big] \,|_a^b$$

was an essential tool in obtaining results concerning eigenvalues, orthogonality of eigenfunctions, and so on. As we showed, the boundary conditions have the property that for any two functions y_1, y_2 satisfying the same set of boundary conditions (4.32–4.33),

$$p\left[y_1 y_2' - y_1' y_2\right] \,|_a^b = 0. \tag{4.41}$$

In this case we say the boundary conditions are **symmetric**. Whenever we have symmetric boundary conditions for a regular SLP, it is clear that

$$(y_1, Ly_2) = (y_2, Ly_1).$$

When this property holds with corresponding symmetric boundary conditions, we say that the differential operator L is symmetric, or **self-adjoint**.

Example 4.14

We showed that the differential operator $Ly = -y''$, $a < x < b$, with periodic boundary conditions is self-adjoint. However, one can check (an exercise) that the operator $L = -(p(x)y')'$, $a < x < b$ is not self-adjoint with periodic boundary conditions unless $p(a) = p(b)$. □

EXERCISES

1. Show that substitution of $u(x,t) = g(t)y(x)$ into the PDE

   $$u_t = (p(x)u_x)_x - q(x)u, \quad a < x < b,\ t > 0,$$

   leads to the pair of differential equations

   $$g'(t) = -\lambda g(t), \;\; -(p(x)y')' + q(x)y = \lambda y,$$

   where λ is some constant.

2. Show that the SLP

   $$-y''(x) = \lambda y(x), \quad 0 < x < l,$$
   $$y'(0) = 0, \;\; y(l) = 0,$$

with mixed Dirichlet and Neumann boundary conditions has eigenvalues

$$\lambda_n = \left(\frac{(1+2n)\pi}{2l}\right)^2$$

and corresponding eigenfunctions

$$y_n(x) = \cos\frac{(1+2n)\pi x}{2l}$$

for $n = 0, 1, 2, \ldots$.

3. Consider the SLP

$$-y'' = \lambda y, \quad 0 < x < 1; \quad y(0) + y'(0) = 0, \quad y(1) = 0.$$

Is $\lambda = 0$ an eigenvalue? Are there any negative eigenvalues? Show that there are infinitely many positive eigenvalues by finding an equation whose roots are those eigenvalues, and show graphically that there are infinitely many roots.

4. Show that the SLP

$$-y'' = \lambda y, \quad 0 < x < 2; \quad y(0) + 2y'(0) = 0, \quad 3y(2) + 2y'(2) = 0,$$

has exactly one negative eigenvalue. Is zero an eigenvalue? How many positive eigenvalues are there?

5. For the SLP

$$-y'' = \lambda y, \quad 0 < x < l; \quad y(0) - ay'(0) = 0, \quad y(l) + by'(l) = 0,$$

show that $\lambda = 0$ is an eigenvalue if and only if $a + b = -l$.

6. What can you say about the sign of the eigenvalues for the SLP

$$-y'' + xy = -\lambda y, \quad 0 < x < 1, \quad y(0) = y(1) = 0.$$

Use a computer algebra package to find the eigenvalues and eigenfunctions. Hint: Look up *Airy's differential equation.*

7. Consider the regular SLP

$$-y'' + q(x)y = \lambda y, \quad 0 < x < l,$$
$$y(0) = y(l) = 0,$$

where $q(x) > 0$ on $[0, l]$. Show that if λ and y are an eigenvalue and eigenfunction, respectively, then

$$\lambda = \frac{\int_0^l \left(y'^2 + qy^2\right)dx}{||y||^2}.$$

Is $\lambda > 0$? Can $y(x) = \text{constant}$ be an eigenfunction?

8. Does the boundary value problem

$$-y'' = \lambda y, \quad a < x < b,$$
$$y(a) = y(b), \;\; y'(b) = 2y'(b),$$

have symmetric boundary conditions? Is it self-adjoint?

9. Find the eigenvalues and eigenfunctions for the following problem with *periodic* boundary conditions:

$$-y''(x) = \lambda y(x), \quad 0 < x < l,$$
$$y(0) = y(l), \;\; y'(0) = y'(l).$$

10. Consider a large, circular, tubular ring of circumference $2l$ that contains a chemical of concentration $c(x,t)$ dissolved in water. Let x be the arc-length parameter with $0 < x < 2l$. See Figure 4.3. If the concentration

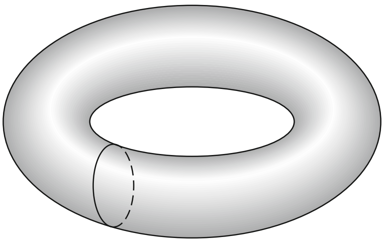

Figure 4.3 Circular ring

of the chemical is initially given by $f(x)$, then $c(x,t)$ satisfies the initial boundary value problem

$$c_t = Dc_{xx}, \quad 0 < x < 2l, \; t > 0,$$
$$c(0,t) = c(2l,t), \;\; c_x(0,t) = c_x(2l,t), \quad t > 0,$$
$$c(x,0) = f(x), \quad 0 < x < 2l.$$

These boundary conditions are called **periodic boundary conditions**, and D is the diffusion constant. Apply the separation of variables method and show that the associated Sturm–Liouville problem has eigenvalues $\lambda_n = (n\pi/l)^2$ for $n = 0,1,2,\ldots$ and eigenfunctions $y_0(x) = 1, y_n(x) = A_n\cos(n\pi x/l) + B_n\sin(n\pi x/l)$ for $n = 1,2,\ldots$. Show that the concentration is given by

$$c(x,t) = \frac{A_0}{2} + \sum_{n=0}^{\infty}(A_n\cos(n\pi x/l) + B_n\sin(n\pi x/l))e^{-n^2\pi^2 Dt/l^2}$$

and find formulas for the A_n and B_n.

11. Find eigenvalues and eigenfunctions for the singular problem

$$-y'' + x^2 y = \lambda y, \quad x \in \mathbb{R}; \quad y \in L^2(\mathbb{R}).$$

Use a computer algebra system if needed.

12. Find an infinite series representation for the solution to the wave problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) &= u_x(l,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \ \ u_t(x,0) = 0, \quad 0 < x < l. \end{aligned}$$

Interpret this problem in the context of waves on a string.

13. Find an infinite-series representation for the solution to the heat absorption–radiation problem

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) - a_0 u(0,t) = 0,& \ \ u_x(1,t) + a_1 u(1,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < 1, \end{aligned}$$

where $a_0 < 0$, $a_1 > 0$, and $a_0 + a_1 > -a_0 a_1$. State why there is radiation at $x = 1$, absorption at $x = 0$, and the radiation greatly exceeds the absorption. Choose $a_0 = -0.25$ and $a_1 = 4$ and find the first four eigenvalues; if $f(x) = x(1-x)$, find an approximate solution to the problem and plot either the approximate solution surface or time snapshots.

4.3 Generalization and Singular Problems

A straightforward generalization of a regular Sturm–Liouville problem is the SLP

$$L_w y = -(p(x)y')' + q(x)y = \lambda r(x) y, \quad a < x < b, \tag{4.42}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{4.43}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0, \tag{4.44}$$

which now includes a positive, continuous **weight function** $r = r(x)$ on the right side of the differential equation. Here, the same conditions on p, q, and the boundary conditions hold as before. Many physical processes lead to this type of equation after separating variables.

The Sturm–Liouville theorem holds exactly as before, but with a different definition of orthogonality. We define the **weighted inner product** of two functions by

$$\Big(y_1, y_2\Big)_w = \int_a^b y_1(x)y_2(x)r(x)dx.$$

This leads to the **weighted norm**

$$\|y\|_w = \sqrt{\Big(y_1, y_2\Big)_w}.$$

The conclusions of Theorem 4.8 are the same but with the fact that the eigenfunctions are orthogonal with repect to the weighted inner product. Moreover, the eigenfunctions $y_n(x)$, $n = 1, 2, 3, \ldots$ form a complete orthogonal set of functions on $[a, b]$; that is, if f is square integrable in the weighted norm, then

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x),$$

in the mean square sense, where

$$c_n = \frac{\Big(f, y_n\Big)_w}{\|y\|_w^2}, \quad n = 1, 2, 3, \ldots.$$

Example 4.15

Consider the eigenvalue problem

$$-y'' + y' = \lambda y, \quad 0 < x < 1,$$
$$y(0) = y(1) = 0.$$

As it stands, this problem is not in self-adjoint form, so we cannot apply Theorem 4.8 directly. However, if we multiply by the integrating factor e^{-x}, the equation becomes

$$-\Big(e^{-x}y'\Big)' = \lambda e^{-x}y,$$

which is of regular Sturm–Liouville type, with weight function $r(x) = e^{-x}$. One can use an energy argument, or case argument, to easily show that the eigenvalues are positive and are given by

$$\lambda_n = \frac{1}{4} + n^2\pi^2, \quad n = 1, 2, \ldots$$

with eigenfunctions

$$y_n(x) = e^{x/2}\sin n\pi x, \quad n = 1, 2, \ldots.$$

The eigenfunctions are orthogonal with respect to weight function e^{-x}:

$$\Big(y_n, y_m\Big)_w = \int_0^1 e^{x/2} \sin n\pi x \cdot e^{x/2} \sin n\pi x \, e^{-x} dx$$
$$= \int_0^1 \sin n\pi x \sin n\pi x \, dx = 0.$$

Therefore, if f is square integrable, then we can write

$$f(x) = \sum_{n=1}^{\infty} c_n e^{-x} \sin n\pi x,$$

in the mean square sense, where c_n

$$c_n = \frac{\Big(f, y_n\Big)_w}{\Big(y_n, y_n\Big)_w^2}, \quad n = 1, 2, 3, \ldots . \quad \square$$

Remark 4.16

A differential equation of the form

$$y'' + a(x)y' + b(x)y = \lambda y$$

can be put into self-adjoint form my multiplying by the integrating factor

$$\mu(x) = e^{\int_a^x a(\xi)d\xi}.$$

Then, note $\mu'(x) = \mu(x)a(x)$, and the differential equation becomes

$$(\mu(x)y')' + \mu(x)b(x)y = \lambda\mu(x)y,$$

or

$$(\mu(x)y')' + q(x)y = \lambda\mu(x)y, \quad q(x) = \mu(x)b(x). \quad \square$$

Singular Problems

Singular problems occur for the SLP

$$L_w y = -(p(x)y')' + q(x)y = \lambda r(x)y, \quad a < x < b, \tag{4.45}$$
$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \tag{4.46}$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0, \tag{4.47}$$

when $p(x_0) = 0$ at some point x_0 in the interval $[a, b]$. Here we assume either $p(a) = 0$ or $p(b) = 0$.[8] In the case $p(a) = 0$ we generally replace the boundary condition at $x = a$ by the condition that $y(a)$ is bounded.

[8] If $p(x) = 0$ at an interior point in the interval, then that point is called a *turning point*. Turning points are examined in more advanced treatments.

Example 4.17

The singular SLP

$$-(xy')' = \lambda xy, \quad 0 < x < R,$$
$$y(0) \text{ bounded}, \quad y(R) = 0$$

occurs in the initial BVP for the heat equation in a circular disk of radius R. This is discussed in detail in Section 4.6. The differential equation is called **Bessel's equation**. We show here that this problem is self-adjoint. Denoting $Ly = -(xy')'$ and letting y_1 and y_2 be two functions that satisfy the boundary conditions, by Green's identity we get

$$\begin{aligned}\Big(y_1, Ly_2\Big) - \Big(y_2, Ly_1\Big) &= x\Big[y_1(x)y_2'(x) - y_1'(x)y_2(x)\ \big|_0^R \\ &= -(0)\Big[y_1(0)y_2'(0) - y_1'(0)y_2(0)\Big] = 0.\end{aligned}$$

So, L is self-adjoint. Therefore the results of the Sturm–Liouville theorem hold: there are infinitely many discrete eigenvalues and the eigenfunctions are orthogonal with weight function $r(x) = x$. Refer to Section 4.6. □

Example 4.18

Consider the singular SLP

$$-y'' = \lambda y, \quad x > 0,$$
$$y(0) = 0, \quad y(x) \text{ bounded}.$$

The problem is singular because the interval is infinite. It is clear that $\lambda > 0$ because negative λ gives exponential solutions which cannot satisfy the boundary conditions; zero cannot be an eigenvalue for the same reason. The general solution is $y(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$. The left boundary condition $y(0) = 0$ forces $B = 0$. We are left with $y(x) = A\sin\sqrt{\lambda}\,x$. Therefore, the bounded solutions, or eigenfunctions, are

$$y_\lambda(x) = \sin\sqrt{\lambda}, \quad \lambda > 0,$$

and the eigenvalues are *all* positive values of λ; this is a *continuous* set of eigenvalues, rather than discrete. In this case there is no classical orthogonality property on $(0, \infty)$. However, there is a continuous version of a Fourier expansion. If f is piecewise continuous on $x \geq 0$, and if f is absolutely integrable, or

$$\int_0^\infty |\ f(x)\ |\ dx < \infty,$$

then we can represent f by a continuous version of a Fourier sine series, namely,

$$f(x) = \frac{2}{\pi}\int_0^\infty F(\alpha)\sin\alpha x\,d\alpha, \tag{4.48}$$

where we have let $\alpha = \sqrt{\lambda}$; it can be shown that the coefficient function $F(\alpha)$ is given by

$$F(\alpha) = \int_0^\infty f(x)\sin\alpha x\,dx. \tag{4.49}$$

The function $F(\alpha)$ in (4.49) is called the **Fourier sine transform** of f and is commonly denoted by $\mathcal{S}(f)$. It acts in a similar way to other transforms where derivative operations are turned into multiplication operations. Fourier sine transforms are convenient for solving differential equations on $0 \le x < \infty$. Other sources contain tables of sine (and cosine) transforms and their inverses, similar to those for Laplace transforms. We refer readers to Churchill (1969, 1972) or Farlow (1993), for example. □

EXERCISES

1. Is the partial differential equation

$$\left(x^2u_x\right)_x + x^2u_{tt} = 0$$

separable? Is

$$u_{xx} + (x+y)^2u_{yy} = 0$$

separable?

2. Use an energy argument to prove that the SLP

$$-\left(e^{-x}y'\right)' = \lambda e^{-x}y,\ \ 0 < x < 1,\ \ y(0) = y(1) = 0$$

has only positive eigenvalues.

3. Consider the SLP

$$-\left(xy'\right)' = \lambda\frac{1}{x}y,\ \ 1 < x < b,\ \ y(1) = y(b) = 0.$$

Find the eigenvalues and eigenfunctions, and write down the Fourier expansion for a given function $f(x)$ on the interval.

4. Consider the SLP

$$-\left(x^2y'\right)' = \lambda y,\ \ 1 < x < \pi,\quad y(1) = y(\pi) = 0.$$

Use an energy argument to show that any eigenvalue must be nonnegative. Find the eigenvalues and eigenfunctions.

5. Show that the differential operator

$$Ly = a(x)y'' + b(x)y' + c(x)y$$

can be written

$$\frac{a(x)}{p(x)}\left[\Big(p(x)y'\Big)' + \frac{c(x)p(x)}{a(x)}y\right],$$

and therefore the eigenvalue problem $Ly = \lambda y$ can be written

$$-\Big(p(x)y'\Big)' + q(x)y = \lambda r(x)y$$

for appropriately defined functions functions p, q, and r.

6. Let L be the differential operator defined by $Ly = a(x)y'' + b(x)y' + c(x)y,\ 0 < x < 1$. Find an operator L^* for which

$$\Big(y_1, Ly_2\Big) = \Big(y_2, L^*y_1\Big) + B(x)\ \Big|_0^1,$$

where $B(x)\ |_0^1$ represents boundary terms. The operator L^* is called the *formal adjoint* of L. Hint: Write out the inner product $\big(y_1, Ly_2\big)$ and integrate by parts twice to remove the derivatives on y_2 and put them on y_1.

7. Consider the initial BVP for the wave equation on $0 < x < l$ with variable sound speed $c(x)$:

$$\begin{aligned} u_{tt} &= c(x)^2 u_{xx}, \quad 0 < x < l,\ t > 0, \\ u(0,t) &= u(l,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \quad 0 \le x \le l. \end{aligned}$$

Find the Sturm–Liouville problem associated this BVP. What is the weight function in the orthogonality relation?

8. (Parameter identification) This problem deals with determining the unknown density of a nonhomogeneous vibrating string by observing one fundamental frequency and the corresponding mode of vibration. Consider a stretched string of unit length and unit tension that is fastened at both ends. The displacement is governed by (see Section 1.5)

$$\rho(x)u_{tt} = u_{xx},\ \ 0 < x < 1,\ t > 0, \tag{4.50}$$

$$u(0,t) = u(1,t) = 0,\ \ t > 0, \tag{4.51}$$

where $\rho(x)$ is the unknown density. If we assume a solution of the form $u(x,t) = y(x)g(t)$, then we obtain a Sturm–Liouville problem

$$-y'' = \lambda\rho(x)y,\ \ 0 < x < 1; \quad y(0) = y(1) = 0.$$

Suppose we observe (say, using a strobe light) a fundamental frequency $\lambda = \lambda_{\mathrm{f}}$ and the associated mode (eigenfunction) $y = y_{\mathrm{f}}(x)$. Show that the unknown density must satisfy the integral equation

$$\int_0^1 (1-x) y_{\mathrm{f}}(x) \rho(x) dx = \frac{y_{\mathrm{f}}'(0)}{\lambda_{\mathrm{f}}}.$$

Hint: Integrate the ODE from $x = 0$ to $x = s$ and then from $s = 0$ to $s = 1$. Determine the density if it is a constant.

9. (Parameter identification) Consider heat flow in a rod of length π and unit diffusivity whose ends are held at constant zero temperature and whose initial temperature is zero degrees. Suppose further that there is an external heat source $f(x)$ supplying heat to the bar. Address the question of recovering the heat source $f(x)$ from a temporal measurement of the temperature $U(t)$ made at the midpoint of the rod. Proceed by formulating the appropriate equations and show that $f(x)$ must be the solution to the integral equation

$$U(t) = \int_0^t \int_0^\pi g(\xi, t-\tau) f(\xi)\, d\xi\, d\tau,$$

where

$$g(\xi, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin \frac{n\pi}{2} \sin n\xi.$$

Show that for any positive integer m, the pair

$$u(x,t) = m^{-3/2}\left(1 - e^{-m^2 t}\right) \sin mx, \quad f(x) = \sqrt{m} \sin mx$$

satisfies the model equations you proposed. Show that for m large, $u(x,t)$ is (uniformly) small, yet $\max |f(x)|$ is large, and thus state why small errors in the measurement $U(t)$ may lead to large differences in $f(x)$. (This means that recovery of f by solving the integral equation above may be unstable and therefore difficult.).

4.4 Laplace's Equation

In Section 1.7 we introduced Laplace's equation, which is given in two and three dimensions by

$$u_{xx} + u_{yy} = 0, \quad u_{xx} + u_{yy} + u_{zz} = 0,$$

respectively. Applications of Laplace's equation are far-reaching, from steady-state heat flow, electrostatics, potential flow of a fluid, deflections of a membrane, to complex analysis. Solutions of Laplace's equation are called **harmonic**, or **potential functions**, referring, for example, to the potential of an electric field. Laplace's equation itself is sometimes called the potential equation.

The separation of variables method can easily be applied to solve Laplace's equation on rectangles in two dimensions, just exactly as the heat equation and wave equation were solved. Exercises at the end of this section provide examples. In Chapter 2, using Fourier transforms, we solved Laplace's equation in a half space $y > 0$. Presently, we extend the eigenfunctions method to circular domains, wedges, and annuli. Later, we develop some general properties of harmonic functions in both two and three dimensions.

Temperatures in a Disk

Suppose we know the temperature on the boundary of a circular, laminar disk of radius R. The goal is to find the equilibrium temperatures inside the disk. Separation of variables works in a straightforward way and leads to a particularly nice integral formula for the solution.

Because the region of interest is a disk, we guess that polar coordinates are more appropriate than rectangular coordinates, so we formulate the steady-state heat flow problem in polar coordinates r and θ, where

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Then a circular disk of radius R can be represented simply as $0 \le r \le R$, $0 \le \theta \le 2\pi$. The unknown temperature inside the disk is $u = u(r,\theta)$, and the prescribed temperature on the boundary of the plate is $u(R,\theta) = f(\theta)$, where f is a given function. We know from prior discussions (Section 1.8) that u must satisfy Laplace's equation $\Delta u = 0$ inside the disk. Therefore, upon representing the Laplacian Δ in polar coordinates (see Section 1.8), we are faced with the following boundary value problem for $u(r,\theta)$:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < R,\ 0 \le \theta \le 2\pi, \tag{4.52}$$

$$u(R,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi. \tag{4.53}$$

Implicit are periodic boundary conditions

$$u(r,0) = u(r,2\pi), \quad u_\theta(r,0) = u_\theta(r,2\pi). \tag{4.54}$$

The separation of variables method works the same way as in rectangular coordinates. We assume a product solution $u(r,\theta) = y(r)g(\theta)$ and substitute into the PDE to obtain

$$y''(r)g(\theta) + \frac{1}{r}y'(r)g(\theta) + \frac{1}{r^2}y(r)g''(\theta) = 0.$$

This can be written

$$-\frac{r^2y''(r) + ry'(r)}{y(r)} = \frac{g''(\theta)}{g(\theta)} = -\lambda,$$

where $-\lambda$ is the separation constant. Therefore, we obtain two ordinary differential equations for y and g, namely,

$$r^2y''(r) + ry'(r) = \lambda y(r), \quad g''(\theta) = -\lambda g(\theta).$$

The *periodic boundary conditions* force $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$. Therefore, we have the following self-adjoint SLP problem for $g(\theta)$:

$$g''(\theta) = -\lambda g(\theta); \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi). \tag{4.55}$$

First, it is clear that $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $g_0(\theta) = 1$. Moreover, there are no negative eigenvalues; if λ is negative, then the ODE has exponential solutions, and exponential solutions cannot satisfy periodicity conditions. (Or, one could apply an energy argument to show nonnegativity.) Therefore, let us assume $\lambda = p^2 > 0$. The differential equation for g has general solution

$$g(\theta) = a\cos p\theta + b\sin p\theta.$$

The periodic boundary conditions force

$$(\cos 2\pi p - 1)a + (\sin 2\pi p)b = 0,$$
$$(\sin 2\pi p)a + (1 - \cos 2\pi p)b = 0.$$

This is a system of two linear homogeneous algebraic equations for the constants a and b. From matrix algebra, we know that it has a nontrivial solution if the determinant of the coefficients is zero, i.e.,

$$(\cos 2\pi p - 1)(1 - \cos 2\pi p) - \sin^2 2\pi p = 0,$$

or, simplifying,

$$\cos 2\pi p = 1.$$

This means that $p = \sqrt{\lambda}$ must be a positive integer, that is,

$$\lambda = \lambda_n = n^2, \quad n = 1, 2, \ldots.$$

Along with $\lambda_0 = 0$, these are the eigenvalues of the problem (4.55). The eigenfunctions are

$$g_0(\theta) = 1, \quad g_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \quad n = 1, 2, \ldots .$$

Next we solve the y-equation. Of course, we want bounded solutions. For $\lambda = 0$ the only bounded solution is $y_0(r) = 1$ (the other independent solution in this case is $\ln r$, which is unbounded). For $\lambda = n^2$ the equation is

$$r^2 y''(r) + r y'(r) - n^2 y(r) = 0,$$

which is a *Cauchy–Euler equation* (see the Appendix) with general solution

$$y_n(r) = c_n r^{-n} + d_n r^n. \tag{4.56}$$

We can set $c_n = 0$ because we want bounded solutions on $0 \le r \le R$. Thus, setting $d_n = 1$ for all n, we have

$$y_n(r) = r^n, \quad n = 1, 2, \ldots .$$

In summary, we constructed solutions of the given boundary value problem of the form $u_0(r, \theta) = \text{constant} = a_0/2$, $u = u_n(r, \theta) = r^n(a_n \cos n\theta + b_n \sin n\theta)$, $n = 1, 2, \ldots$. To satisfy the boundary condition at $r = R$ we form the linear combination

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta). \tag{4.57}$$

The boundary condition $u(R, \theta) = f(\theta)$ then yields

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} R^n \Big(a_n \cos n\theta + b_n \sin n\theta \Big),$$

which is the full-range Fourier series for $f(\theta)$ (see Section 3.3). Therefore, the coefficients are given by

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \quad n = 0, 1, 2, \ldots, \tag{4.58}$$

$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \quad n = 1, 2, \ldots . \tag{4.59}$$

Hence, the solution to the BVP (4.52–4.54) is given by (4.57) with the coefficients given by (4.58–4.59).

As it turns out, this infinite series solution can be cleverly manipulated to obtain a simple formula, called Poisson's integral formula, for the solution. Let us substitute the coefficients given in formulas (4.58) and (4.59) (after changing the dummy integration variable from θ to ϕ) into the solution formula (4.57) to obtain

$$\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi \\ &\quad + \sum_{n=1}^{\infty}\frac{r^n}{\pi R^n}\int_0^{2\pi} f(\phi)(\cos n\phi\cos n\theta + \sin n\phi \sin n\theta)d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left(1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\phi)\right)d\phi. \end{aligned}$$

The infinite sum in the integrand can be determined exactly as follows. Recalling the identity $\cos\alpha = \frac{1}{2}(e^{i\alpha}+e^{-i\alpha})$, we can write

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\phi) = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{in(\theta-\phi)} + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n e^{-in(\theta-\phi)}.$$

But each series on the right side is a geometric series, and we know from calculus that

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \quad \text{provided that } \ |z|<1.$$

Therefore we get

$$\begin{aligned} 1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos n(\theta-\phi) &= 1 + \frac{re^{i(\theta-\phi)}}{R - re^{i(\theta-\phi)}} + \frac{re^{-i(\theta-\phi)}}{R - re^{-i(\theta-\phi)}} \\ &= \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta-\phi)}. \end{aligned}$$

Assembling the results, we have the following.

Theorem 4.19

(**Poisson's integral formula**) If f is a continuous function, then the BVP for Laplace's equation (4.52–4.54) on a circular domain of radius R is given by

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right)f(\phi)}{R^2 + r^2 - 2rR\cos(\theta-\phi)}d\phi. \quad \square \tag{4.60}$$

Remark 4.20

Poisson's integral formula is unwieldy for calculations. But it is an important theoretical tool. For example, if we use the formula to find the temperature at the origin, we immediately get

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)d\phi.$$

In other words, the temperature at the center of the disk is the average of the prescribed boundary temperatures. □

Remark 4.21

(**Exterior Dirichlet Problem**) By the same method, we can show that the bounded solution to the *exterior* Dirichlet problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r > R,\ 0 \le \theta \le 2\pi,$$
$$u(R,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi,$$

where f is continuous, is

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(r^2 - R^2\right) f(\phi)}{R^2 + r^2 - 2rR\cos(\theta - \phi)}\,d\phi. \quad \square$$

Solving Laplace's equation on portions of circles (wedges) and for annular domains is similar. Readers familiar with complex analysis may recall that some unusual-shaped domains can be mapped conformally onto a simple domain where Laplace's equation can be solved; conformal transformations preserve solutions to Laplace's equation, and so this method can result in a closed form solution, i.e., a formula, for the answer. Many elementary texts on complex analysis discuss this method (see, for example, Churchill (1960) for an elementary discussion).

As an aside, the most common method in practice for solving Laplace's equation on any bounded domain, regardless of its shape, is to use a numerical method. For example, in the *method of finite differences* (see Exercise 6 in Section 1.8) the derivatives are approximated by difference quotients, and the solution $u(x, y)$ is determined at discrete lattice points inside the region. This method is further discussed in Chapter 6. Another widely used method not discussed here is the *finite element method*, which also yields the approximate solution at discrete points of the domain.

General Results for Laplace's Equation

In the next few paragraphs we introduce some basic properties of solutions to Laplace's equation on bounded spatial domains in three dimensions. The results also apply to two dimensions. Many of the assertions follow from the **divergence theorem**, which states that for a smooth vector field $\boldsymbol{\phi} = \boldsymbol{\phi}(x, y, z)$,

$$\int_\Omega \operatorname{div}\boldsymbol{\phi}\,dV = \int_{\partial\Omega} \boldsymbol{\phi}\cdot\mathbf{n}\,dA. \tag{4.61}$$

The divergence theorem is a version of the fundamental theorem of calculus in three (or two) dimensions—it converts the integral of derivatives, a divergence, to an integral over the boundary. We know from elementary calculus that the divergence measures the local outflow per unit volume of a vector field $\boldsymbol{\phi}$, so the divergence theorem states that the net outflow, per unit volume, in a region Ω must equal the flux through the boundary (surface) $\partial\Omega$ of that region. Recall that the integral on the right is a flux integral through the surface oriented by the outward unit normal vector $\mathbf{n}$. Here we are taking a fluid flow interpretation in speaking of outflow, but the interpretation is valid for any smooth vector field. We always take Ω to be a nice region with a smooth boundary or with a boundary made up of finitely many smooth sections. There are some pathological domains on which the divergence theorem does not apply, but we do not consider those. Moreover, we always assume that the functions involved in our discussion are sufficiently smooth for the theorems to apply. For example, the functions should have two continuous partial derivatives in Ω, be continuous on $\Omega \cup \partial\Omega$, and have first partial derivatives that extend continuously to the boundary.

Two important integral identities, called Green's identities, follow from the divergence theorem. These identities are higher dimensional versions of Green's identity for Sturm–Liouville problems obtained in Section 4.2. To begin, note that if u is a scalar field and $\boldsymbol{\phi}$ is a vector field, then

$$\operatorname{div}(u\boldsymbol{\phi}) = u \operatorname{div}\boldsymbol{\phi} + \boldsymbol{\phi} \cdot \operatorname{grad} u$$

(see Exercise 4, Section 1.7). Integrating over the volume Ω and using the divergence theorem gives

$$\int_{\partial\Omega} u\,\boldsymbol{\phi} \cdot \mathbf{n}\, dA = \int_{\Omega} u \operatorname{div}\,\boldsymbol{\phi}\, dV + \int_{\Omega} \boldsymbol{\phi} \cdot \operatorname{grad}\, u\, dV.$$

Setting $\boldsymbol{\phi} = \operatorname{grad}\, v$ for a scalar function v then gives

$$\int_{\partial\Omega} u \operatorname{grad}\, v \cdot \mathbf{n}\, dA = \int_{\Omega} u\,\Delta v\, dV + \int_{\Omega} \operatorname{grad}\, v \cdot \operatorname{grad}\, u\, dV. \tag{4.62}$$

Here we have used the fact that $\operatorname{div}\,(\operatorname{grad}\, u) = \Delta u$, the Laplacian. In particular, if we set $v = u$, we obtain

$$\int_{\partial\Omega} u \operatorname{grad}\, u \cdot \mathbf{n}\, dA = \int_{\Omega} u\,\Delta u\, dV + \int_{\Omega} \operatorname{grad}\, u \cdot \operatorname{grad}\, u\, dV, \tag{4.63}$$

and this is **Green's first identity**. Note that in one dimension this identity is the same as

$$\int_a^b yy'' dx = -yy' \,|_a^b - \int_a^b \left(y'\right)^2 dx,$$

which is just the integration by part formula.

To obtain Green's second identity, interchange v and u in equation (4.62) and subtract that result from (4.62) to get

$$\int_{\Omega} u\Delta v \, dV = \int_{\Omega} v\Delta u \, dV + \int_{\partial\Omega} (u \,\mathrm{grad}\; v - v \,\mathrm{div}\; u) \cdot \mathbf{n} \, dA. \tag{4.64}$$

This is **Green's second identity**; it can be regarded as an integration by parts formula for the Laplacian Δ; note how the Laplacian is taken off v and put on u and a boundary term is produced.

Notation. We often use the notation

$$\frac{du}{dn} \equiv \mathbf{n} \cdot \mathrm{grad}\; u$$

to denote the *normal derivative*, i.e., the derivative of u on a boundary in the direction of the outward unit normal. □

Green's identities can be used to prove many interesting facts about problems involving Laplace's equation. To begin, we show that solutions to the Dirichlet problem are unique.

Theorem 4.22

A solution to the Dirichlet problem

$$\Delta u = 0 \;\text{ in } \Omega,$$
$$u = f(x, y, z) \;\text{ on } \partial\Omega,$$

is unique. □

By way of contradiction, assume that there are two solutions, u and v. Then the difference $w \equiv u - v$ must satisfy the problem

$$\Delta w = 0 \;\text{ in }\; \Omega,$$
$$w = 0 \;\text{ on } \partial\Omega.$$

Multiply this PDE by w and integrate over the region Ω to get

$$\int_{\Omega} w\Delta w \, dV = 0.$$

Green's first identity (4.63) implies

$$\int_{\partial\Omega} w \,\mathrm{grad}\; w \cdot \mathbf{n} \, dA - \int_{\Omega} \mathrm{grad}\; w \cdot \mathrm{grad}\; w \, dV = 0.$$

The first integral on the left is zero because $w = 0$ on the boundary. Therefore,

$$\int_{\Omega} \text{grad } w \cdot \text{grad } w \, dV = 0.$$

Because the integrand is never negative, it follows that grad $w = 0$, or $w =$ constant in Ω. Because $w = 0$ on the boundary, the constant must be zero, and so $w = 0$ in $\Omega \cup \partial\Omega$. This means that $u = v$ in $\Omega \cup \partial\Omega$. So there cannot be two solutions. The reader should recognize the proof above as an "energy argument" of the type introduced in Section 4.3 for SLPs. □

Another interesting property of harmonic functions satisfying Dirichlet-type boundary conditions is that they minimize the **energy integral**

$$E(w) \equiv \int_{\Omega} \text{grad } w \cdot \text{grad } w \, dV.$$

In electrostatics, for example, where w is the electric field potential, the energy E is the energy stored in the electrostatic field.

Theorem 4.23

(**Dirichlet's principle**) Suppose u satisfies

$$\Delta u = 0 \quad \text{in } \Omega; \qquad u = f \quad \text{on } \partial\Omega.$$

Then $E(u) \leq E(w)$ for all w satisfying $w = f$ on $\partial\Omega$. In other words, of all functions that satisfy the boundary condition, the solution to Laplace's equation is the one that minimizes the energy. □

To prove Dirichlet's principle, let $w = u + v$ where $v = 0$ on the boundary $\partial\Omega$. Then

$$\begin{aligned} E(w) &= E(u+v) \\ &= \int_{\Omega} \text{grad } (u+v) \cdot \text{grad } (u+v) \, dV \\ &= \int_{\Omega} \text{grad } u \cdot \text{grad } u \, dV + 2 \int_{\Omega} \text{grad } u \cdot \text{grad } v \, dV \\ &\quad + \int_{\Omega} \text{grad } v \cdot \text{grad } v \, dV \\ &= E(u) + E(v) + 2 \int_{\Omega} \text{grad } u \cdot \text{grad } v \, dV. \end{aligned}$$

But the integral term on the right is zero by equation (4.62). And because $E(v) \geq 0$, we have $E(u) \leq E(w)$, which completes the proof. □

EXERCISES

1. Consider the pure boundary value problem for Laplace's equation given by

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, \quad 0 < x < l, \ 0 < y < 1, \\
u(0, y) &= 0, \quad u(l, y) = 0, \quad 0 < y < 1, \\
u(x, 0) &= 0, \quad u(x, 1) = G(x), \quad 0 < x < l.
\end{aligned}
$$

   Use the separation of variables method to find an infinite series representation of the solution. Here, take $u(x, y) = \phi(x)\psi(y)$ and identify a boundary value problem for $\phi(x)$; proceed as in other separation of variables problems.

2. Find an infinite series representation for the solution to the equilibrium problem

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, \quad 0 < x < a, \ 0 < y < b, \\
u_x(0, y) = u_x(a, y) &= 0, \quad 0 < y < b, \\
u(x, 0) = f(x), \ u(x, b) &= 0, \quad 0 < x < a.
\end{aligned}
$$

   Interpret this problem in the context of steady heat flow.

3. Solve the boundary value problem

$$
\begin{aligned}
\Delta u &= 0, \quad r < R, \quad 0 \le \theta < 2\pi, \\
u(R, \theta) &= 4 + 3 \sin \theta, \quad 0 \le \theta < 2\pi.
\end{aligned}
$$

4. Show that the solution $u = u(r, \theta)$ to the *exterior* boundary value problem

$$
\begin{aligned}
\Delta u &= 0, \quad r > R, \ 0 \le \theta < 2\pi, \\
u(R, \theta) &= f(\theta), \ 0 \le \theta < 2\pi,
\end{aligned}
$$

   is given by

$$
u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(r^2 - R^2\right) f(\phi)}{R^2 + r^2 - 2rR\cos(\theta - \phi)} d\phi.
$$

   Hint: Follow the example in the book but choose the other bounded solution in equation (4.56).

5. Solve

$$
\begin{aligned}
\Delta u &= 0, \quad r > 1, \ 0 \le \theta < 2\pi, \\
u(1, \theta) &= \cos \theta, \quad 0 < \theta < 2\pi.
\end{aligned}
$$

6. Find an infinite series solution to the following boundary value problem on a wedge:

$$\Delta u = 0, \quad r < R,\ 0 \le \theta < \pi/2,$$
$$u(R,\theta) = f(\theta), \quad 0 < \theta < \pi/2,$$
$$u(r,0) = 0, \quad u(r,\pi/2) = 0, \quad 0 < r < R.$$

7. Solve the boundary value problem in Exercise 5 with the boundary conditions along the edges $\theta = 0$, $\theta = \pi/2$ replaced by

$$u(r,0) = 0, \quad u_\theta(r,\pi/2) = 0, \quad 0 < r < R.$$

8. Solve the Dirichlet problem on an annulus:

$$\Delta u = 0, \quad 1 < r < 2,\ 0 \le \theta < 2\pi,$$
$$u(1,\theta) = \sin\theta, \quad 0 < \theta < 2\pi,$$
$$u(2,\theta) = \cos\theta, \quad 0 < \theta < 2\pi.$$

9. Find a bounded solution to the exterior boundary value problem

$$\Delta u = 0, \quad r > R,$$
$$u = 1 + 2\sin\theta \text{ on } r = R.$$

10. Prove Dirichlet's principle for the Neumann problem: Let

$$E(w) \equiv \frac{1}{2}\int_\Omega \operatorname{grad} w \cdot \operatorname{grad} w \, dV - \int_{\partial\Omega} hw \, dA.$$

If u is the solution to the BVP

$$\Delta u = 0 \ \text{ in } \ \Omega,$$
$$\mathbf{n} \cdot \operatorname{grad} u = h \ \text{ on } \partial\Omega,$$

then $E(u) \le E(w)$ for all w sufficiently smooth. Observe that the average value of h on the boundary is zero.

11. Show that a necessary condition for the Neuman problem

$$\Delta u = f \ \text{ in } \ \Omega,$$
$$\mathbf{n} \cdot \operatorname{grad} u = h \ \text{ on } \partial\Omega,$$

to have a solution is that

$$\int_\Omega f dV = \int_{\partial\Omega} g \, dA.$$

Interpret this result in the context of heat conduction.

12. Find radial solutions $u = u(\rho)$ of the equation

$$\Delta u = k^2 u$$

in spherical coordinates. Hint: Let $u = \rho w$.

13. In three dimensions consider the radiation boundary value problem

$$\Delta u - cu = 0 \text{ in } \Omega,$$
$$\mathbf{n} \cdot \operatorname{grad} u + au = 0 \text{ on } \partial\Omega,$$

where $a, c > 0$ and Ω is a bounded region. Show that the only solution is $u = 0$. Hint: Use an energy argument. Multiply the PDE by u, integrate, and then use Green's first identity.

14. Use the last exercise to show that solutions to the boundary value problem

$$\Delta u - cu = g(x, y, z) \quad (x, y, z) \in \Omega,$$
$$n \cdot \operatorname{grad} u + au = f(x, y, z) \text{ on } \partial\Omega,$$

are unique.

15. Suppose $u = u(x, y, z)$ satisfies the Neumann problem

$$\Delta u = 0, \text{ in } \Omega,$$
$$\mathbf{n} \cdot \operatorname{grad} u = 0 \text{ on } \partial\Omega.$$

Show that u must be constant on Ω.

16. Suppose $u = u(x, y, z)$ satisfies the Robin problem

$$\Delta u = 0 \text{ in } \Omega,$$
$$\frac{du}{dn} + au = 0 \text{ on } \partial\Omega.$$

Show that solutions are unique if $a > 0$.

17. Show that the Poisson integral formula can be written in vector form as

$$u(\mathbf{x}) = \frac{R^2 - |\mathbf{x}|^2}{2\pi R} \int_{|\mathbf{y}|=R} \frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} ds,$$

where the integral is a line integral with respect to arclength over the circle $|\mathbf{y}| = R$.

18. Show that the Neumann problem on the unit disk, $\Delta u = 0$ in $r < 1$, $u = \sin^2\theta$ on $r = 1$, does not have a solution.

19. Consider the Dirichlet problem between two spheres: $\Delta u = 0$ in $R_1 < \rho < R_2$, and $u = a$ on $\rho = R_1$, $u = b$ on $\rho = R_2$. Find the solution and sketch graphs of u vs. ρ in the cases $a < b$ and $b < a$.

4.5 Cooling of a Sphere

In this section we present the solution to a classical problem in three dimensions, the cooling of a sphere. The symmetries in the problem permits us to reduce the dimension of the problem to one spatial dimension (the radius) and time, and we can follow the same procedures as in previous sections.

The problem is this: Given a sphere whose initial temperature depends only on the distance from the center (e.g., a constant initial temperature) and whose boundary is kept at a constant temperature, predict the temperature at any point inside the sphere at a later time. This is the problem, for example, of determining the temperature in a spherically-shaped object, e.g., a potato, that has been placed in a hot oven. The reader could conjecture that this problem is important for medical examiners who want to determine the time of an individual's death by measuring temperatures in brain tissue. Early researchers, notably Kelvin, used this problem to determine the age of the earth based on conjectures about its initial temperature and its temperature today.

This problem is reminiscent of Newton's law of cooling, which is encountered in ordinary differential equations texts. It states that the rate at which a body cools is proportional to the difference of its temperature and the temperature of the environment. Quantitatively, if $T = T(t)$ is the temperature of a body and T_{e} is the temperature of its environment, then $T'(t) = -h(T - T_{\mathrm{e}})$, where h is the constant heat loss coefficient. But the reader should note that this law applies only in the case that the body has a *uniform*, *homogeneous temperature*. In the PDE problem we are considering, the temperature may vary radially throughout the body.

For simplicity, we consider a sphere of radius $\rho = \pi$ whose initial temperature is $T_0 =$ constant. We assume the boundary is maintained at zero degrees for all time $t > 0$. In general, the temperature u inside the sphere depends on three spherical coordinates, ρ, θ, ϕ, and time. But reflection reveals that the temperature will depend only on the distance ρ from the center of the sphere and on time; that is, $u = u(\rho, t)$. There are no variations in the initial or boundary conditions that would cause gradients to change in the θ and ϕ coordinate directions. Evidently, the temperature must satisfy the heat equation

$$u_t = k\Delta u,$$

where k is the diffusivity and Δ is the Laplacian (see Section 1.7). Because u does not depend on ϕ and θ, the Laplacian takes on a particularly simple form:

$$\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho}$$

(see Section 1.8). Therefore, we can formulate the model as

$$u_t = k\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right), \quad 0 < \rho < \pi,\ t > 0, \tag{4.65}$$

$$u(\pi, t) = 0, \quad t > 0, \tag{4.66}$$

$$u(\rho, 0) = T_0, \quad 0 \le \rho < \pi. \tag{4.67}$$

There is an implied boundary condition at $\rho = 0$, namely that the temperature should remain bounded.

To solve problem (4.65–4.67) we assume $u(\rho, t) = y(\rho)g(t)$. Substituting into the PDE and separating variables gives, in the usual way,

$$\frac{g'(t)}{kg(t)} = \frac{y''(\rho) + \frac{2}{\rho}y'(\rho)}{y(\rho)} = -\lambda,$$

where $-\lambda$ is the separation constant. Substituting $u(\rho, t) = y(\rho)g(t)$ into the boundary condition (4.66) gives $y(\pi) = 0$, and the boundedness of u at $\rho = 0$ forces $y(0)$ to be bounded. Therefore, we get the boundary value problem

$$-y''(\rho) - \frac{2}{\rho}y'(\rho) = \lambda y(\rho), \quad 0 < \rho < \pi, \tag{4.68}$$

$$y(0)\ \text{bounded}, \quad y(\pi) = 0. \tag{4.69}$$

This is a *singular* SLP.

If $\lambda = 0$, then the equation is Cauchy–Euler, having the form $-y'' - \frac{2}{\rho}y' = 0$. The general solution is $y = a/\rho + b$. The boundedness condition at zero forces $a = 0$, and $y(\pi) = 0$ implies $b = 0$. Therefore $\lambda = 0$ is not an eigenvalue. If $\lambda \ne 0$, it may not be clear to the reader how to solve the variable-coefficient equation (4.68). However, if we introduce the new dependent variable $Y = Y(\rho)$ defined by

$$Y(\rho) = \rho y(\rho),$$

then (4.68) transforms into the familiar equation

$$-Y''(\rho) = \lambda Y(\rho).$$

If $\lambda > 0$, then $Y = a\cos\sqrt{\lambda}\rho + b\sin\sqrt{\lambda}\rho$, which gives

$$y(\rho) = \frac{1}{\rho}\left(a\cos\sqrt{\lambda}\rho + b\sin\sqrt{\lambda}\rho\right).$$

We must have $a = 0$ because $\cos\sqrt{\lambda}\rho/\rho$ is unbounded at $\rho = 0$. Therefore, applying the boundary condition $y(\pi) = 0$ then gives

$$b\sin\sqrt{\lambda}\pi = 0,$$

which in turn yields the eigenvalues

$$\lambda = \lambda_n = n^2, \quad n = 1, 2, \ldots .$$

The eigenfunctions are

$$y_n(\rho) = \frac{\sin n\rho}{\rho}, \quad n = 1, 2, \ldots$$

We leave it as an exercise to show that there are no negative eigenvalues (see Exercise 1).

The corresponding solutions to the time equation for $g(t)$ are easily found to be $g_n(t) = c_n e^{-n^2 kt}$. Thus we have determined infinitely many solutions of the form

$$u_n(\rho, t) = c_n e^{-n^2 kt} \frac{\sin n\rho}{\rho}, \quad n = 1, 2, \ldots,$$

that satisfy the PDE and the boundary conditions. To satisfy the initial condition (4.67) we form the linear combination

$$u(\rho, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 kt} \frac{\sin n\rho}{\rho}.$$

Applying the initial condition gives

$$u(\rho, 0) = T_0 = \sum_{n=1}^{\infty} c_n \frac{\sin n\rho}{\rho},$$

or

$$T_0 \rho = \sum_{n=1}^{\infty} c_n \sin n\rho.$$

The right side of this equation is the Fourier sine series of the function $T_0\rho$ on the interval $[0, \pi]$. Therefore, the Fourier coefficients c_n are given by

$$c_n = \frac{2}{\pi} \int_0^{\pi} T_0 \rho \sin n\rho \, d\rho.$$

The integral can be calculated by hand using integration by parts. We get

$$c_n = (-1)^{n+1} \frac{2T_0}{n}.$$

In summary, we have derived the solution formula for the model (4.65–4.67):

$$u(\rho, t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2T_0}{n} e^{-n^2 kt} \frac{\sin n\rho}{\rho}. \tag{4.70}$$

This formula gives the temperature at time t at a distance ρ from the center of the sphere.

EXERCISES

1. Show that the eigenvalue problem (4.68–4.69) has no negative eigenvalues.

2. In the cooling of the sphere problem take $R = \pi$ inches, $T_0 = 37$ degrees Celsius, and $k = 5.58$ inches-squared per hour. Using (4.70), plot the temperature at the center of the sphere as a function of time; take $t = 0$ to $t = 1$ hour.

3. This problem deals with the cooling of a sphere with a radiation boundary condition. A spherical ball of radius R has a diffusivity k and initial temperature $u = f(\rho)$, $0 \leq \rho \leq R$, which depends only on the distance ρ from the center. Heat radiates from the surface of the sphere via the law

$$u_\rho(R,t) = -hu(R,t),$$

where h is a positive constant and $Rh < 1$. Find a formula for the temperature $u(\rho, t)$ in the sphere. Hint: Find the eigenvalues λ_n as the positive roots of the equation

$$\tan R\lambda = \frac{R\lambda}{1 - Rh}.$$

Next, take $R = \pi$ inches, $k = 5.58$ inches-squared per hour, $f(\rho) = 37$ degrees Celsius, and $h = 0.1$ per inch, and calculate a four term approximate solution to the temperature function.

4. Consider a sphere of unit radius on which there is a prescribed potential u depending only on the spherical coordinate angle ϕ (see Section 1.8). This exercise deals with using separation of variables to find an expression for the bounded potential that satisfies Laplace's equation inside the sphere. The boundary value problem is

$$\Delta u = 0, \;\; 0 < \rho < 1, \; 0 < \phi < \pi; \;\; u(1, \phi) = f(\phi), \;\; 0 \leq \phi \leq \pi,$$

where $u = u(\rho, \phi)$. Notice that u will not depend on the polar angle θ because of the symmetry in the boundary condition.

(a) Assume $u = R(\rho)Y(\phi)$ and derive the two equations

$$-\Big((1-x^2)y'\Big)' = \lambda y, \;\; -1 < x < 1; \quad \Big(\rho^2 R'\Big)' - \lambda R = 0,$$

where $x = \cos\phi$ and $y(x) = Y(\arccos x)$.

(b) The equation for y in part (a) is *Legendre's differential equation*, and it has bounded, continuous solutions on $[-1, 1]$ when

$$\lambda = \lambda_n = n(n+1), \quad y = y_n(x) = P_n(x), \quad n = 0, 1, 2, \ldots,$$

where the P_n are polynomial functions called the **Legendre polynomials**. The Legendre polynomials are orthogonal on $[-1, 1]$, and they are given by Rodrigues' formula

$$P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} \left[\left(x^2 - 1 \right)^n \right].$$

Go through a formal calculation and derive a formula for the solution to the given BVP in the form

$$u(\rho, \phi) = \sum_{n=0}^{\infty} c_n \rho^n P_n(\cos \phi),$$

where

$$c_n = \frac{1}{||P_n||^2} \int_0^{\pi} f(\phi) P_n(\cos \phi) \sin \phi \, d\phi.$$

(c) Find the first four Legendre polynomials using Rodrigues' formula.

(d) If $f(\phi) = \sin \phi$, find an approximate solution.

5. Estimate the age of the earth. Use the data in Exercise 5 of Section 2.4.

4.6 Diffusion in a Disk

In the last section we solved a diffusion problem in spherical coordinates and found, assuming angular symmetry, that it was a routine application of the separation of variables method. In this section we solve a diffusion problem in polar coordinates; we will find the task is not so routine because we obtain a differential equation (**Bessel's equation**) that may be unfamiliar to the reader. A computer algebra system may help illuminate some of the steps along the way. (The instructor may find that this problem is well suited for presentation in a computer laboratory setting.).

Consider a circular, planar disk of radius R whose initial temperature is a function of the radius alone and whose boundary is held at zero degrees. We are interested in how heat diffuses throughout the disk. Intuition dictates that the temperature u in the disk depends only on time and the distance r from the center, or $u = u(r, t)$. We can make this assumption because there is nothing in the initial or boundary condition to cause heat to diffuse in an angular direction—heat flows only along rays emanating from the origin.

We know that u must satisfy the two-dimensional heat equation $u_t = k\Delta u$ in $0 < r < R$, where k is the diffusivity and Δ is the Laplacian in polar coordinates, given in Section 1.8. Therefore the PDE is

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r\right), \quad 0 < r < R,\ t > 0. \tag{4.71}$$

The boundary condition is

$$u(R,t) = 0, \quad t > 0, \tag{4.72}$$

and the initial condition is

$$u(r,0) = f(r), \quad 0 \le r < R, \tag{4.73}$$

where f is a given radial temperature distribution. It is understood that the temperature should be bounded at $r = 0$. Equations (4.71–4.73) define the initial boundary value problem we want to solve.

Now separate variables. Taking $u(r,t) = y(r)g(t)$, the PDE splits into

$$\frac{g'(t)}{kg(t)} = \frac{y''(r) + \frac{1}{r}y'(r)}{y(r)} = -\lambda.$$

The equation for $g(t)$ is solved in the usual way to obtain

$$g(t) = e^{-\lambda kt}.$$

The radial equation is

$$y''(r) + \frac{1}{r}y'(r) = -\lambda y(r). \tag{4.74}$$

Upon multiplying by r, we can rewrite this equation as

$$-\Big(ry'(r)\Big)' = \lambda r y(r), \tag{4.75}$$

which is **Bessel's equation**. Condition (4.72) leads to the boundary condition

$$y(R) = 0, \tag{4.76}$$

and we impose the boundedness requirement

$$y(0) \quad \text{bounded}. \tag{4.77}$$

Note that the ODE (4.75), along with the boundary conditions (4.76–4.77), is a *singular* Sturm–Liouville problem.

It is easily seen that $\lambda = 0$ is not an eigenvalue; in this case the general solution of (4.75) is $y = a + b\ln r$; and $b = 0$ by boundedness and $a = 0$ by the condition $y(R) = 0$. We leave it as an exercise using an energy argument to show that there are no negative eigenvalues (Exercise 1). Therefore we consider the case where λ is positive. In the last section we found a simple substitution for a similar equation $y'' + (2/\rho)y' = \lambda y$ that transformed it to a familiar equation; but Bessel's equation (4.74) is slightly different (by just a factor of 2 in one term), and a simple change of variables does not work. A successful approach involves the use of power series methods for differential equations; that is, one assume a solution of the form

$$y(r) = \sum_{n=0}^{\infty} a_n r^n$$

and substitute into (4.75) to determine the coefficients a_0, a_1, a_2,.... This calculation, which we do not perform, ultimately leads to two linearly independent solutions of (4.74), denoted in the literature by $J_0(\sqrt{\lambda}\, r)$ and $Y_0(\sqrt{\lambda}\, r)$; so the general of (4.75) is

$$y(r) = c_1 J_0(\sqrt{\lambda} r) + c_2 Y_0(\sqrt{\lambda} r)$$

where c_1 and c_2 are arbitrary constants.

Remark 4.24

Maple and MATLAB, for example, denote these special functions $J_0(z)$ and $Y_0(z)$ by BesselJ, BesselY and besselj, bessely, respectively. These are the **Bessel functions** of order zero. Their properties, graphs, and values are catalogued in these computer algebra systems as well as handbooks. □

Plots of the Bessel functions are shown in Figure 4.4. It is clear that $Y_0(z)$ blows up near $z = 0$; in fact, it can be shown that $Y_0(z)$ behaves like $\ln z$ for small, positive z. Therefore, we discard the second term in the solution to maintain boundedness; so we set $c_2 = 0$. Therefore, we have the bounded solution to the differential equation (4.75) as

$$y(r) = c_1 J_0(\sqrt{\lambda} r). \tag{4.78}$$

The Bessel function $J_0(z)$ is a nicely behaved oscillatory function. It is similar to the cosine function, but its oscillations decay as z increases. It has infinitely many zeros z_n, $n = 1, 2, 3, \ldots$; the first few are $z_1 = 2.405$, $z_2 = 5.520$, $z_3 = 8.654$, $z_4 = 11.790$. Its series representation can be found as

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n}} z^{2n}.$$

One can also show that for large z,

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4}\right),$$

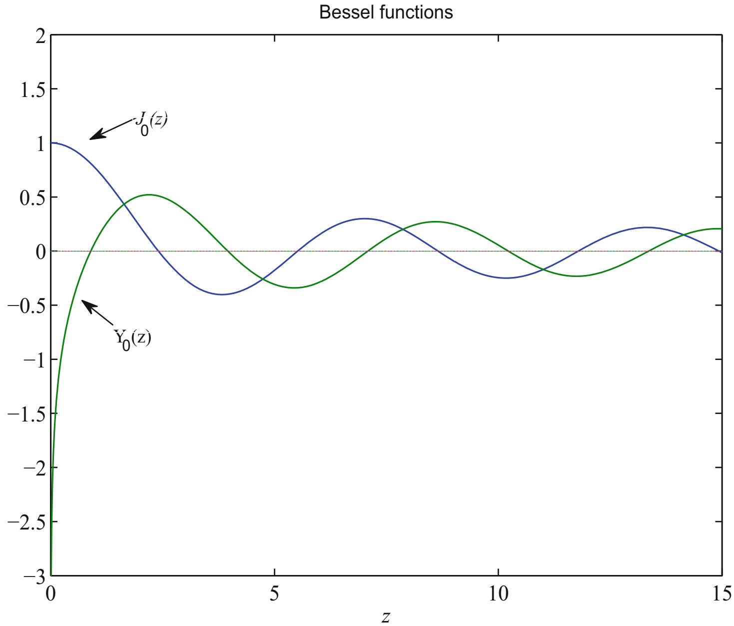

Figure 4.4 MATLAB plots of the Bessel functions $J_0(z)$ and $Y_0(z)$ for $0 \leq z \leq 15$

which shows its "decaying cosine" behavior.

Now we apply the boundary condition (4.76) to (4.78), and we obtain

$$y(R) = c_1 J_0(\sqrt{\lambda}R) = 0.$$

Thus

$$\sqrt{\lambda}R = z_n \ , \quad z = 1, 2, 3, \ldots,$$

where the z_n are the zeros of J_0. Consequently, the eigenvalues are

$$\lambda_n = \frac{z_n^2}{R^2}, \quad n = 1, 2, 3, \ldots,$$

and the corresponding eigenfunctions are

$$y_n(r) = J_0\left(\frac{z_n r}{R}\right), \quad n = 1, 2, 3, \ldots.$$

In summary, we have constructed solutions $g_n(t)y_n(r)$ of the form

$$e^{-\lambda_n kt} J_0\left(\frac{z_n r}{R}\right), \quad n = 1, 2, 3, \ldots,$$

that satisfy the PDE and the boundary conditions.

To satisfy the initial condition (4.73) we form the linear combination

$$u(r,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} J_0\left(\frac{z_n r}{R}\right). \tag{4.79}$$

The initial condition forces

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{z_n r}{R}\right).$$

To find the c_n we use *orthogonality*. This last equation is a Fourier–Bessel expansion, and the Bessel functions $J_0(\frac{z_n r}{R})$ satisfy the orthogonality condition

$$\int_0^R J_0\left(\frac{z_n r}{R}\right) J_0\left(\frac{z_m r}{R}\right) r dr = 0, \quad n \neq m. \tag{4.80}$$

(See Exercise 2.) The factor r in the integrand is a weight function, and the Bessel functions $J_0(\frac{z_n r}{R})$ are orthogonal with respect to r. The orthogonality allows us to determine the coefficients in the standard way: Multiply equation (4.80) by $rJ_0\left(\frac{z_m r}{R}\right)$ and integrate from $r = 0$ to $r = R$, interchange the summation and the integral, and then use the weighted orthogonality relation to collapse the infinite sum to one term. We obtain

$$\int_0^R f(r) J_0\left(\frac{z_n r}{R}\right) r dr = c_m \| J_0\left(\frac{z_m r}{R}\right) \|^2,$$

where

$$\| J_0\left(\frac{z_n r}{R}\right) \|^2 = \int_0^R J_0\left(\frac{z_m r}{R}\right)^2 r dr.$$

Therefore, the coefficients are given by

$$c_n = \frac{\int_0^R f(r) J_0\left(\frac{z_n r}{R}\right) r dr}{\| J_0\left(\frac{z_n r}{R}\right) \|^2}. \tag{4.81}$$

In conclusion, the formal solution to the initial boundary value problem (4.71–4.73) is given by (4.79) with coefficients given by (4.81).

In the exercises the reader is asked to calculate this solution numerically in special cases.

We conclude by making some brief remarks about special functions. The Bessel functions J_0 and Y_0 encountered above are just two examples of large classes of Bessel functions. Problems in cylindrical geometry, for example, the vibrations of a drum head, lead to such functions. Moreover, other special functions (Legendre polynomials, Laguerre polynomials, and others) occur in a similar way for problems in other coordinate systems and domains. That is, they occur as solutions to eigenvalue problems for ordinary differential equations that arise from the separation of variables process. Generally, special functions arising in this manner possess orthogonality properties that give the expansion of the solution in terms of those functions.

EXERCISES

1. Show that the eigenvalue problem (4.75–4.77) has no negative eigenvalues. Hint: Use an energy argument—multiply the ODE by y and integrate from $r = 0$ to $r = R$; use integration by parts and use the boundedness at $r = 0$ to get the boundary term to vanish.

2. Derive the weighted orthogonality relation (4.80). Hint: Proceed as in Section 4.4 for regular Sturm–Liouville problems.

3. For the initial boundary value problem (4.71–4.73) take $R = 1$, $k = 0.25$, and $f(r) = 5r^3(1 - r)$. Use a computer algebra program to determine a 3-term approximation to the solution using (4.79). Sketch radial temperature profiles $u(r, t)$ for several fixed values of t.

4. When $R = 1$ sketch the first four eigenfunctions $J_0(\frac{z_n r}{R})$, $n = 1, 2, 3, 4$.

5. (Computer algebra project) Consider the boundary value problem

$$y'' + axy = 0, \quad 0 < x < L,$$
$$y'(0) = 0, \quad y(L) = 0,$$

where a is a positive constant. Find the smallest positive value of L for which the boundary value problem has a nontrivial solution. The solution is that L can be calculated from the equation

$$\frac{2}{3}\sqrt{a}\, L^{2/3} = 1.86635.$$

Hint: In your software program use the differential equation solver to find the general solution as a linear combination of the two Bessel functions $J_{1/3}$ and $Y_{1/3}$; use the left boundary condition in limiting form to determine one constant in terms of the other, and then use the right boundary condition to determine the appropriate value of L.

4.7 Sources on Bounded Domains

Readers should be familiar with methods for solving ordinary differential equations with source, or forcing, terms. This topic is facilitated by having the notion of a general solution to the equation. The *general solution* to the second-order linear equation

$$Ly \equiv y'' + a(x)y' + b(x)y = f(x)$$

is $y(x) = y_h(x) + y_p(x)$ where y_h is the general solution to the homogeneous equation, $Ly_h = 0$, and y_p is any particular solution to the nonhomogeneous equation, $Ly_p = f(x)$. For simple source terms, particular solutions are usually found by the method of undetermined coefficients (judicious guessing) or by a general formula called the variation of parameters formula. (See Appendix A).

For partial differential equations the situation is more complicated. We introduce several techniques to deal with both nonhomogeneous equations and nonhomogeneous boundary conditions on bounded spatial domains for the heat equation, wave equation, and Laplace's equation.

One obvious approach is to use the linearity of the equation and boundary conditions to split a problem into simpler forms, essentially breaking apart the inhomogenieties in the problem. Consider the initial BVP (P) for the heat equation,

$$(P): \begin{cases} u_t - ku_{xx} = f(x,t), & a < x < b, \quad t > 0, \\ u(a,t) = b_1(t), \ u(b,t) = b_2(t), & t > 0, \\ u(x,0) = g(x), & t > 0. \end{cases}$$

Now consider the two problems

$$(P_1): \begin{cases} v_t - kv_{xx} = 0, & a < x < b, \quad t > 0, \\ v(a,t) = b_1(t), \ v(b,t) = b_2(t), & t > 0, \\ v(x,0) = g(x), & t > 0, \end{cases}$$

and

$$(P_2): \begin{cases} w_t - kw_{xx} = f(x,t), & a < x < b, \quad t > 0, \\ w(a,t) = 0, \ w(b,t) = 0, & t > 0, \\ w(x,0) = 0, & t > 0. \end{cases}$$

(P_1) is the same problem as (P) with no source term, and (P_2) is the same problem as (P) with homogeneous boundary conditions. Using linearity, it is easy to see that that the solution u to (P) is

$$u(x,t) = v(x,t) + w(x,t),$$

where v and w are the solutions to (P_1) and (P_2), respectively. Therefore, we have broken down (P) into two simpler problems. The initial condition is handled by Fourier series.

We now treat several cases. First we use Duhamel's principle (see Chapter 2 for the unbounded domain case) to show that source terms can put into initial conditions. An example is reviewed for the heat equation, but the procedure works equally well for the wave equation by placing the source in the intial condition for velocity.

Example 4.25

(**Duhamel's Principle**) Consider a simple diffusion model given by

$$u_t - ku_{xx} = f(x,t), \quad 0 < x < \pi, \ t > 0, \tag{4.82}$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0, \tag{4.83}$$

$$u(x,0) = 0, \quad 0 < x < \pi. \tag{4.84}$$

Interpreted in the context of heat flow, $u = u(x,t)$ is the temperature of a rod whose initial temperature is zero and whose ends are maintained at zero degrees. It is the heat source $f(x,t)$ that is driving the system.

The simplest way to solve (4.82–4.84) is to use Duhamel's principle as formulated in Section 2.5 for initial value problems on infinite domains. On bounded domains the principle is the same. It states that the solution of (4.82–4.84) is given by

$$u(x,t) = \int_0^t w(x, t-\tau, \tau)\,d\tau,$$

where $w = w(x,t,\tau)$ is the solution to the homogeneous problem

$$w_t - kw_{xx} = 0, \quad 0 < x < \pi, \ t > 0,$$
$$w(0,t,\tau) = w(\pi,t,\tau) = 0, \quad t > 0,$$
$$w(x,0,\tau) = f(x,\tau), \quad 0 < x < \pi.$$

Recall that τ is a parameter. We have already solved this problem in Section 4.1. We have

$$w(x,t,\tau) = \sum_{n=1}^{\infty} c_n e^{-n^2kt} \sin nx,$$

where

$$c_n = c_n(\tau) = \frac{2}{\pi}\int_0^{\pi} f(x,\tau)\sin nx\,dx.$$

Notice that the Fourier coefficients c_n depend on the parameter τ. So, the solution to (4.82–4.84) is

$$u(x,t) = \int_0^t \left(\sum_{n=1}^{\infty} c_n(\tau)e^{-n^2k(t-\tau)}\sin nx\right)d\tau. \tag{4.85}$$

For example, solve (4.82–4.84) when $f(x,t) = \sin x$. Easily the Fourier coefficients are given by $c_n = 0, \ \ n > 1$, $c_1 = 1$. Then the solution is

$$\begin{aligned} u(x,t) &= \int_0^t c_1 e^{-k(t-\tau)} \sin x d\tau \\ &= \frac{1}{k}\left(1 - e^{-kt}\right)\sin x. \quad \square \end{aligned}$$

Remark 4.26

In the preceding example, the limit of the solution as $t \to \infty$ is

$$\lim_{t\to\infty} u(x,t) = (1/k)\sin x.$$

Notice that this is the same as the steady-state solution, i.e., the time-independent solution of the system

$$\begin{aligned} u_t - ku_{xx} &= \sin x, \quad 0 < x < \pi, \ t > 0, \\ u(0,t) &= u(\pi,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad 0 < x < \pi, \end{aligned}$$

found by solving $-U''(x) = \sin x,\ U(0) = U(\pi) = 0$.

This is true in general. If the source term in (4.82–4.84) in the heat equation depends only on x, or $f(x,t) = F(x)$, then as $t \to \infty$ the solution to (4.82–4.84) approaches the steady-state solution found by solving $-U''(x) = F(x),\ U(0) = U(\pi) = 0$. □

Equations with sources can also be solved by an **eigenfunction method**. The first step is to find the eigenfunctions of the Sturm–Liouville problem associated with the *homogeneous* problem, with no sources.

Example 4.27

In the case of (4.82–4.84) the eigenvalues and eigenfunctions of the homogeneous problems are (see Section 4.1)

$$\lambda_n = n^2, \quad y_n(x) = \sin nx, \quad n = 1, 2, \ldots.$$

Then we assume a solution of the *nonhomogeneous problem* (4.82–4.84) of the form

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t)\sin nx, \tag{4.86}$$

where the $g_n(t)$ are to be found. We determine the $g_n(t)$ by substituting this expression for u into the PDE (4.82), along with the expression for the eigenfunction expansion for f, namely,

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin nx.$$

In this last expansion, the $f_n(t)$ are known because f is known; the $f_n(t)$ are the Fourier coefficients

$$f_n(t) = \frac{2}{\pi}\int_0^{\pi} f(x,t)\sin nx dx. \tag{4.87}$$

Carrying out this substitutions, we get

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty} g_n(t)\sin nx - k\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty} g_n(t)\sin nx = \sum_{n=1}^{\infty} f_n(t)\sin nx,$$

or

$$\sum_{n=1}^{\infty} g_n'(t)\sin nx + k\sum_{n=1}^{\infty} n^2 g_n(t)\sin nx = \sum_{n=1}^{\infty} f_n(t)\sin nx.$$

Collecting the coefficients of the independent eigenfunctions $\sin nx$, we obtain

$$g_n'(t) + n^2 k g_n(t) = f_n(t),$$

which is a linear ordinary differential equation for the unknown $g_n(t)$. It can be solved in the standard way by multiplying by the integrating factor e^{n^2kt} and then integrating both sides. After some straightforward manipulation we obtain

$$g_n(t) = g_n(0)e^{-n^2kt} + \int_0^t f_n(\tau)e^{-n^2k(t-\tau)}d\tau.$$

To obtain the values of $g_n(0)$ we substitute the initial condition into (4.86) to get

$$\sum_{n=1}^{\infty} g_n(0)\sin nx = 0,$$

which implies $g_n(0) = 0$ for all n. Therefore, we have determined the $g_n(t)$, and the solution to the problem (4.82–4.84) is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\int_0^t f_n(\tau)e^{-n^2k(t-\tau)}d\tau\right)\sin nx, \tag{4.88}$$

where the $f_n(t)$ are given by (4.87). This solution formula (4.88) is the same as the solution formula (4.85) obtained by Duhamel's principle. □

In summary, in the eigenfunction method for solving problems with sources we expand both the source f and the assumed solution u in terms of the eigenfunctions of the homogeneous problem. Substituting these expansions into the PDE leads to ODEs for the coefficients in the eigenfunction expansion for u. Solving these ODEs for the coefficients then gives the solution u as an eigenfunction expansion. Whereas Duhamel's principle is applied to initial value problems (evolution problems), the eigenfunction expansion method can be applied to all types problems, including equilibrium problems associated with Laplace's equation.

To apply the separation of variables method we need homogeneous boundary conditions. If boundary conditions are nonhomogeneous, we can often make a change of the dependent variable that leads to homogeneous ones.

Example 4.28

(**Homogenizing the boundary conditions**) Consider the problem

$$\begin{gathered} u_t - 3u_{xx} = 0, \;\; 0 < x < 1, \; t > 0, \\ u(0,t) = 2e^{-t}, \;\; u(1,t) = 1, \\ u(x,0) = x^2, \;\; 0 < x < 1. \end{gathered}$$

For the diffusion equation we can always make a transformation of the dependent function to force zero boundary conditions at the expense of introducing a source term in the PDE. Define $w(x,t)$ to be the function $u(x,t)$ minus a *linear function of* x that satisfies the boundary conditions. That is, take

$$w(x,t) = u(x,t) - \left(2e^{-t} + \left(1 - 2e^{-t}\right)x\right).$$

Then w solves the problem

$$\begin{gathered} w_t - 3w_{xx} = 2e^{-t}(1-x), \;\; 0 < x < 1, \; t > 0, \\ w(0,t) = w(1,t) = 0, \;\; t > 0, \\ w(x,0) = x^2 + x, \;\; 0 < x < 1. \end{gathered}$$

This problem can be solved using the eigenfunction as in the last example. □

Remark 4.29

(**Steady heat source**) For the heat equation with a steady source,

$$\begin{gathered} u_t - ku_{xx} = f(x), \;\; a < x < b, \; t > 0, \\ u(a,t) = A, \;\; u(b,t) = B, \;\; t > 0, \\ u(x,0) = g(x), \;\; a < x < b, \end{gathered}$$

where A and B are constant, the equilibrium temperature $U = U(x)$ satisfies

$$-kU'' = f(x), \quad U(a) = A, \;\; U(b) = B.$$

The substitution $w(x,t) = u(x,t) - U(x)$ leads to the homogeneous problem

$$\begin{gathered} w_t - kw_{xx} = 0, \;\; a < x < b, \; t > 0, \\ w(a,t) = 0, \;\; w(b,t) = 0, \;\; t > 0, \\ u(x,0) = G(x) \equiv g(x) - U(x), \;\; a < x < b. \end{gathered}$$

This problem may be solved using separation of variables. □

Remark 4.30

It can be generally stated that if the heat equation has boundary conditions of the form

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = g_1(t), \quad \beta_1 u(l,t) + \beta_2 u_x(l,t) = g_2(t),$$

then the transformation

$$u = w + U(x,t) \equiv w + a(t)\left(1 - \frac{x}{l}\right) + b(t)\frac{x}{l}$$

where $a(t)$ and $b(t)$ are chosen so that $U(x,t)$ satisfies the boundary conditions, leads to a problem for w with homogeneous boundary conditions. The source term and initial condition may be altered. □

Example 4.31

(**Wave equation**) Consider the nonhomogeneous wave equation

$$\begin{aligned} u_{tt} &= u_{xx} + Ax, \quad 0 < x < 1, \ t > 0, \\ u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 < x < 1. \end{aligned}$$

Here there is an equilibrium solution $U = U(x)$ satisfying

$$-U'' = Ax, \quad U(0) = U(1) = 0.$$

It is easily found by integrating twice that $U(x) = \frac{A}{6}x\left(1-x^2\right)$. Then, $w(x,t) = u(x,t) - U(x)$ satisfies

$$\begin{aligned} w_{tt} &= w_{xx}, \quad 0 < x < 1, \ t > 0, \\ w(0,t) &= 0, \quad w(1,t) = 0, \quad t > 0, \\ w(x,0) &= -U(x), \quad w_t(x,0) = 0, \quad 0 < x < 1. \end{aligned}$$

Now the problem can be solved using separation of variables. □

Example 4.32

Consider the wave equation with nonhomogeneous boundary conditions:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < l, \ t > 0, \\ u(0,t) &= a(t), \quad u(l,t) = b(t), \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 0, \quad 0 < x < l. \end{aligned}$$

The reader can easily verify that the change of variables

$$w(x,t) = u(x,t) - \left(b(t) + \frac{b(t) - a(t)}{l} x\right)$$

leads to a problem with homogeneous boundary conditions for w. Nonhomogeneous terms are introduced in the resulting PDE and initial conditions. □

EXERCISES

1. Use Duhamel's principle as formulated for the wave equation in Section 2.5 to find the solution to

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t), \quad 0 < x < \pi, \ t > 0, \\ u(0,t) = u(\pi,t) &= 0, \quad t > 0, \\ u(x,0) = u_t(x,0) &= 0, \quad 0 < x < \pi. \end{aligned}$$

2. Use the eigenfunction method to solve the problem in Exercise 1. Hint: In the eigenfunction method you will have to solve a second-order nonhomogeneous ODE, which can be done using variation of parameters.

3. In the problem defined by (4.82–4.84) assume that the heat source does not depend on time, i.e., $f(x,t) = F(x)$. By calculating the τ-integral, show how the solution (4.85) simplifies in this case. Find a formula for the solution when the heat source is $F(x) = x(\pi - x)$ and then find the limiting temperature profile as $t \to \infty$. Show that this limiting temperature profile is the same as the steady-state solution, that is, the solution $v = v(x)$ to

$$-kv''(x) = x(\pi - x), \quad 0 < x < \pi; \quad v(0) = v(\pi) = 0.$$

4. Solve

$$\begin{aligned} u_t &= u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad u(1,t) = A, \quad t > 0, \\ u(x,0) &= \cos x, \quad 0 < x < 1. \end{aligned}$$

5. Solve

$$\begin{aligned} u_t &= u_{xx} + Q, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad u(1,t) = 2u_0, \quad t > 0, \\ u(x,0) &= u_0(1 - \cos \pi x), \quad 0 < x < 1. \end{aligned}$$

6. Transform the problem

$$u_t = ku_{xx}, \quad 0 < x < 1, \ t > 0,$$
$$u(0,t) = \cos t, \ \ hu(1,t) + u_x(1,t) = 1, \ \ t > 0,$$
$$u(x,0) = \sin \pi x + x, \ \ 0 < x < 1,$$

into a problem with homogeneous boundary conditions. Give a physical interpretation of the boundary conditions.

7. Solve the problem

$$u_t = ku_{xx} + \sin 3\pi x, \quad 0 < x < 1, \ t > 0,$$
$$u(0,t) = u(1,t) = 0, \ \ t > 0,$$
$$u(x,0) = \sin \pi x, \ \ 0 < x < 1.$$

8. Solve the wave equation with a constant gravitational force:

$$u_{tt} = c^2 u_{xx} - g, \ \ 0 < x < 1, \ \ t > 0,$$
$$u(0,t) = u(1,t) = 0, \ \ t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \ \ 0 < x < 1.$$

9. Solve

$$u_{tt} = u_{xx} + q, \ \ 0 < x < 1, \ \ t > 0,$$
$$u(0,t) = 0, \ \ u(1,t) = \sin t, \ \ t > 0,$$
$$u(x,0) = x(1-x), \ \ u_t(x,0) = 0, \ \ 0 < x < 1.$$

10. Use Duhamel's principle to find a bounded solution to

$$u_t = \Delta u + f(r,t), \ \ 0 < r < R, \ t > 0,$$
$$u(R,t) = 0, \ \ t > 0,$$
$$u(r,0) = 0, \ \ 0 \le r < R.$$

11. Find the solution of the diffusion problem on a disk:

$$u_t = \frac{1}{r}\left(ru_r\right)_r + f(r), \ \ r < R, \ t > 0,$$
$$u(R,t) = u_0, \ \ t > 0,$$
$$u(r,0) = g(r), \ \ r < R.$$

4.8 Poisson's Equation*

One of the fundamental equations in electrostatics and in steady heat flow is Poisson's equation with a Dirichlet boundary condition:

$$\Delta u = u_{xx} + u_{yy} = f(x,y), \quad (x,y) \in \Omega,$$
$$u(x,y) = 0, \quad (x,y) \in \partial\Omega.$$

A basic result is that the nonhomogeneous source term $f(x,y)$ can be put into the boundary condition by changing the dependent variable; conversely, a nonhomogeneous boundary condition can be put into a source term in the PDE.

1. If $\Delta u = f$ on Ω and $u = g$ on $\partial\Omega$, let $u(x,y) = w(x,y) + v(x,y)$ where v is *any particular solution* of $\Delta v = f$. Then $\Delta u = \Delta w + \Delta v = f$. So, $\Delta w = 0$. On the boundary $\partial\Omega$, $u = w + v = g$, where v is the particular solution above. So $w = g - v$ on $\partial\Omega$. In summary,

$$\Delta w = 0 \text{ in } \Omega; \;\; w = g - v \text{ on } \partial\Omega.$$

2. Let $\Delta u = f$ in Ω and $u = g$ on $\partial\Omega$. Let v be *any* function that satisfies the the boundary condition. Let $u(x,y) = w(x,y) + v(x,y)$ and $\Delta u = \Delta w + \Delta v = f$, or $\Delta w = f - \Delta v$. On the boundary $\partial\Omega$, we have $u = w + v$ or $g = w + g$. Thus

$$\Delta w = f - \Delta v \text{ in } \Omega; \;\; w = 0 \text{ on } \partial\Omega.$$

In summary, this result states that *the Dirichlet problem for Poisson's equation can be solved if the Dirichlet problem for Laplace's equation can be solved.*

Example 4.33

For simple regions Ω, finding a particular solution is easy if $f(x,y)$ is a polynomial of degree n. Just assume a particular solution in the form of an $n+2$ degree polynomial. For example, consider Poisson's equation $\Delta u = -2$ on a rectangle $0 < x < a$, $0 < y < b$, with zero Dirichlet boundary conditions. Take

$$v(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2,$$

a second degree polynomial. Substituting into the PDE gives $2D + 2F = -2$ or $D + F = -1$. So choose $F = 0$ and $D = -1$. The other constants are arbitrary. Choose $B = a$. Then, $v(x,y) = ax - x^2$ is a particular solution. Therefore $\Delta w = 0$ in Ω and $w = -v(x,y)$ on $\partial\Omega$; in particular, $w(0,y) = w(a,y) = 0$ and

$w(x,0) = w(x,b) = x^2 - ax$. This problem for w can be solved by separation of variables because it has homogeneous boundary conditions at $x = 0$ and $x = a$. $\square$

Example 4.34

On a disk of radius R, solve the boundary value problem

$$\Delta u = 2, \quad 0 < r < R,$$
$$u(R,\theta) = 0, \quad 0 \le \theta \le 2\pi.$$

In polar coordinates, the PDE is

$$\frac{1}{r}\Big(ru_r\Big)_r + \frac{1}{r^2}u_{\theta\theta} = 2.$$

Because the boundary condition and source are independent of θ, this problem has a solution that depends only on r. By direct integration of the PDE,

$$u(r,\theta) = \frac{1}{2}r^2 + c_1 \ln r + c_2,$$

where c_1 and c_2 are arbitrary. Clearly $c_1 = 0$ by boundedness. Then $u(R,\theta) = 0$ forces $c_2 = -R^2/2$, giving

$$u(r,\theta) = \frac{1}{2}\Big(r^2 - R^2\Big). \quad \square$$

Eigenvalue Problems

A problem of great importance is to determine the eigenvalues of spatial differential operators on bounded domain.

Example 4.35

(**Eigenvalues of the Laplacian**) Consider the two-dimensional problem

$$-\Delta u = \lambda u, \quad \mathbf{x} \in \Omega, \tag{4.89}$$
$$u = 0, \quad \mathbf{x} \in \partial\Omega. \tag{4.90}$$

Generally, Ω is a two dimensional domain with boundaries along the coordinate directions.To find the **eigenvalues** λ we use separation of variables. The negative sign on the Laplacian gives positive eigenvalues. Specifically, we take $\Omega: 0 < x < \pi,\ 0 < y < \pi$. Then the PDE is $-(u_{xx} + u_{yy}) = \lambda u$ for $u = u(x,y)$. The Dirichlet boundary condition becomes $u(0,y) = u(\pi,y) = 0$ for $0 \le y \le \pi$, and $u(x,0) = u(x,\pi) = 0$ for $0 \le x \le \pi$. We assume a separable solution of the form $u(x,y) = X(x)Y(y)$ for some functions X and Y to be determined. Substituting gives

$$-X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y),$$

which can be rewritten as

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y) + \lambda Y(y)}{Y(y)},$$

with a function of x on one side and a function of y on the other. Therefore,

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y) + \lambda Y(y)}{Y(y)} = \mu$$

for some separation constant μ. Hence, the PDE separates into two ODEs for the two spatial variables:

$$-X''(x) = \mu X(x), \quad -Y''(y) = (\lambda - \mu)Y(y).$$

Next we substitute the assumed form of u into the boundary conditions to obtain

$$X(0) = X(\pi) = 0 \text{ and } Y(0) = Y(\pi) = 0.$$

Consequently we have obtained two boundary value problems of Sturm–Liouville type,

$$-X''(x) = \mu X(x), \quad X(0) = X(\pi) = 0,$$
$$-Y''(y) = (\lambda - \mu)Y(y), \quad Y(0) = Y(\pi) = 0.$$

The first problem has been solved earlier in this chapter and we have eigenpairs

$$X_n(x) = \sin nx, \quad \mu_n = n^2, \quad n = 1, 2, 3, ...$$

Then the Y problem becomes

$$-Y''(y) = \left(\lambda - n^2\right)Y(y), \quad Y(0) = Y(\pi) = 0.$$

This problem is also of Sturm–Liouville type and will have nontrivial solutions $Y_k(y) = \sin ky$ when $\lambda - n^2 = k^2$, for $k = 1, 2, 3, ...$ Therefore, double indexing λ, we get eigenvalues and eigenfunctions for the negative Laplacian with Dirichlet boundary conditions as

$$u_{n,k}(x,y) = \sin nx \sin ky, \quad \lambda_{n,k} = n^2 + k^2, \quad n, k = 1, 2, 3, ...$$

Observe that there are infinitely many positive, real eigenvalues, whose limit is infinity, and the corresponding eigenfunctions are orthogonal, that is,

$$\int_\Omega (\sin nx \sin ky)(\sin mx \sin ly)dxdy = 0 \quad \text{if} \quad n \neq m \text{ or } k \neq l. \quad \square$$

This example illustrates what to expect from the Dirichlet problem (4.89–4.90). In fact, the following properties hold, where we use the vector notation $\mathbf{x} = (x, y)$.

1. The eigenvalues are real.
2. There are infinitely many eigenvalues that can be ordered as $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ with $\lambda_n \to +\infty$ as $n \to \infty$.
3. Eigenfunctions corresponding to distinct eigenvalues are orthogonal with inner product $(u, v) = \int_\Omega u(\mathbf{x})v(\mathbf{x})d\mathbf{x}$.
4. The eigenfunctions $u_n(\mathbf{x})$ form a complete orthogonal set in the sense that any square-integrable function $f(\mathbf{x})$ on Ω can be uniquely represented in its generalized Fourier series

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} c_n u_n(\mathbf{x}), \quad c_n = \frac{\left(f, u_n\right)}{||u_n||^2},$$

where the c_n are the Fourier coefficients, and the norm is $||u_n|| = \sqrt{\left(u_n, u_n\right)}$. Convergence of the series is in the $L^2(\Omega)$ sense, meaning

$$\int_\Omega \left(f(\mathbf{x}) - \sum_{n=1}^{N} c_n u_n(\mathbf{x})\right)^2 d\mathbf{x} \to 0 \quad \text{as} \quad N \to \infty.$$

The results are exactly the same when the Dirichlet boundary condition (4.90) is replaced by a Neumann boundary condition $\frac{du}{dn} = 0$, with the exception that $\lambda = 0$ is also an eigenvalue, with eigenfunction $u_0(x) = \text{constant}$. When a Robin boundary condition $\frac{du}{dn} + a(\mathbf{x})u = 0$ is imposed, again there is a zero eigenvalue with constant eigenfunction provided $a(x) \geq 0$. If $a(x)$ fails to be nonnegative, then there may also be negative eigenvalues.

Remark 4.36

The previous results extend to the higher dimension Sturm–Liouville problem

$$-\text{div}\,(p\,\text{grad}\,u) + qu = \lambda w u, \quad \mathbf{x} \in \Omega.$$

Here, $w = w(\mathbf{x}) > 0$, $p = p(\mathbf{x}) > 0$, $q = q(\mathbf{x})$, p, q and w are continuous on $\overline{\Omega}$, and p has continuous first partial derivatives on Ω. The boundary conditions may be Dirichlet ($u = 0$ on $\partial\Omega$) Neumann $\left(\frac{du}{dn} = 0 \text{ on } \partial\Omega\right)$, or Robin $\left(\frac{du}{dn} + a(\mathbf{x})u = 0 \text{ on } \partial\Omega, \text{ with } a(x) \geq 0\right)$. In these cases the eigenfunctions u and v are orthogonal with respect to the inner product

$$(u, v)_w = \int_\Omega u(\mathbf{x})v(\mathbf{x})w(\mathbf{x})\,d\mathbf{x},$$

and $\|u\|_w = \sqrt{(u,u)_w}$. The Fourier series takes the form

$$f(\mathbf{x}) = \sum_n^{\infty} c_n u_n(\mathbf{x}), \quad c_n = \frac{\left(f, u_n\right)_w}{||u_n||_w^2},$$

where convergence is

$$\int_\Omega \left(f(\mathbf{x}) - \sum_{n=1}^{N} c_n u_n(\mathbf{x}) \right)^2 w(\mathbf{x})\, d\mathbf{x} \to 0 \quad \text{as} \quad N \to \infty. \quad \square$$

Just as for one-dimensional Sturm–Liouville problems, the completeness of the set of eigenfunctions allows us to solve the nonhomogeneous problem.

Example 4.37

(**Poisson's equation**) Consider the problem

$$-\Delta u = \rho(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{4.91}$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega. \tag{4.92}$$

This problem can be interpreted in either one, two, or three dimensions, and the weight function is 1. Let λ_n, $u_n(\mathbf{x})$, $n = 1, 2, 3 \ldots$ be eigenvalue-eigenfunction pairs for the negative Laplacian, i.e.,

$$-\Delta u = \lambda u, \quad \mathbf{x} \in \Omega,$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega.$$

Next, assume a solution to (4.91–4.92) of the form[9]

$$u(\mathbf{x}) = \sum c_n u_n(\mathbf{x}),$$

where the c_n are to be determined. Further, expand ρ in terms of the eigenfunctions as

$$\rho(\mathbf{x}) = \sum \rho_n u_n(\mathbf{x}), \quad \rho_n = \frac{\left(\rho, u_n\right)}{\|u_n\|^2}.$$

Substituting into the PDE (4.91) gives

$$-\Delta u = \sum -c_n \Delta u_n = \sum c_n \lambda_n u_n = \sum \rho_n u_n.$$

Therefore

$$c_n \lambda_n = \rho_n, \quad n = 1, 2, 3, \ldots.$$

[9] All sums are over $n = 1, 2, 3, \ldots$.

If $\lambda_n \neq 0$ for all n, then

$$c_n = \frac{\rho_n}{\lambda_n}$$

for all n. Therefore, we have the solution representation

$$u(\mathbf{x}) = \sum \frac{\rho_n}{\lambda_n} u_n(\mathbf{x}).$$

It is instructive to write this as

$$\begin{aligned} u(\mathbf{x}) &= \sum \frac{\rho_n}{\lambda_n} u_n(\mathbf{x}) \\ &= \sum \frac{(\rho, u_n)}{\lambda_n \|u_n\|^2} u_n(\mathbf{x}) \\ &= \sum \frac{1}{\lambda_n \|u_n\|^2} \int_\Omega \rho(\mathbf{y}) u_n(\mathbf{y}) d\mathbf{y} u_n(\mathbf{x}) \\ &= \int_\Omega \left(\sum \frac{u_n(\mathbf{y}) u_n(\mathbf{x})}{\lambda_n \|u_n\|^2} \right) \rho(y) dy. \end{aligned}$$

If we define the **Green's function** by

$$G(\mathbf{x}, \mathbf{y}) = \sum \frac{u_n(\mathbf{y}) u_n(\mathbf{x})}{\lambda_n \|u_n\|^2},$$

Then the solution may be written simply as

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y}.$$

Thus, the solution is a weighted average of contributions at every point of the domain. A different interpretation is that the right side, an integral operator with kernel G, is the *inverse operator* of $-\Delta$ with a Dirichlet boundary condition. In other words the solution to $-\Delta u = \rho$ is $u = (-\Delta)^{-1}\rho$. □

Example 4.38

(**Neumann problem**) If we consider the Neumann problem

$$\begin{aligned} -\Delta u &= \rho(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ \frac{du}{dn}(\mathbf{x}) &= 0, \quad \mathbf{x} \in \partial\Omega, \end{aligned}$$

then $\lambda_1 = 0$ is an eigenvalue of the negative Laplacian with nontrivial eigenfunction $u_1(x) = 1$. Then the same calculation as above repeats and we obtain

$$c_n \lambda_n = \rho_n, \quad n = 1, 2, 3, \ldots.$$

Now $\lambda_n \neq 0$ for all $n > 1$, and we get

$$c_n = \frac{\rho_n}{\lambda_n}, \quad n > 1.$$

However, in the case $\lambda_1 = 0$, we get $c_1 \cdot 0 = \rho_1$. If $\rho_1 \neq 0$, then this problem has no solution. However, if $\rho_1 = 0$, that is, $(\rho, u_1) = 0$, then c_1 is arbitrary. Therefore,

$$\begin{aligned} u(\mathbf{x}) &= c_1 + \sum_{n=2}^{\infty} \frac{\rho_n}{\lambda_n} u_n(\mathbf{x}) \\ &= c_1 + \int_{\Omega} \left(\sum_{n=2}^{\infty} \frac{u_n(\mathbf{y}) u_n(\mathbf{x})}{\lambda_n \|u_n\|^2} \right) \rho(y) dy, \quad c_1 \text{ arbitrary.} \end{aligned}$$

In summary, if ρ is orthogonal to the eigenfunction $u_1(\mathbf{x}) = 1$, there are infinitely many solutions. If ρ is not orthogonal to $u_1(\mathbf{x}) = 1$, then there is no solution. □

This latter result can be generalized as follows.

Theorem 4.39

(**Fredholm alternative**) Consider the boundary value problem

$$-\text{div}\,(p(\mathbf{x})\,\text{grad}\,u) + q(\mathbf{x})u = \mu u + f(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

with homogeneous Dirichlet, Neumann, or Robin boundary conditions, where the coefficient functions satisfy the conditions in Remark 4.32, and f is a given function. Then,

(a) If μ is not an eigenvalue of the corresponding homogeneous problem ($f = 0$), then there is a unique solution for all functions f with $\int_{\Omega} f(\mathbf{x}) d\mathbf{x} < \infty$.

(b) If μ is an eigenvalue of the homogenous problem, then there is no solution or infinitely many solutions, depending upon the function f.

The proof of this theorem is straightforward and follows exactly the format of the proofs in Section 4.2, namely, to expand u and f in eigenfunction expansions and solve for the coefficients of u.

Green's Function for Infinite Domains

Solving Poisson's equation on bounded domains can, in theory, be done by the eigenfunction method discussed above. In practice, the domain must be simple for success. For infinite domains we illustrate a physical approach based on point sources. Recall that (Section 2.1), for the heat equation, we found the solution to a problem with a unit, point source and then superimposed those solutions over a distribution of sources to obtain the solution to the Cauchy problem. For Poisson's equation we appeal to electrostatics in two dimensions. (Equally, one could think in terms of unit heat sources.) A unit positive charge placed at the origin induces an electric field whose potential is given by

$$U(x,y) = -\frac{1}{2\pi}\ln r, \quad r = \sqrt{x^2+y^2}$$

which satisfies Laplace's equation. The potential is not defined at $(0,0)$. The equipotential curves are clearly circles centered at the origin, and consistent with the fact that for a unit charge, the flux through an arbitrary circle C_r of radius r is

$$\text{flux} = \int_{C_r} \text{grad } u \cdot \mathbf{n}\, ds = \int_0^{2\pi} u_r r \, dr d\theta = 1.$$

By simple translation, the potential response at (x,y) caused by a unit point charge at any point (ξ,η) is

$$G(x,y,\xi,\eta) = -\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2+(y-\eta)^2}, \tag{4.93}$$

which is Green's function in $\mathbb{R}^2$. Arguing exactly as for the heat equation, we can solve Poisson's equation

$$u_{xx}+u_{yy} = f(x,y), \quad (x,y)\in\mathbb{R}^2 \tag{4.94}$$

by treating the source term f as a distributed set of point sources at (ξ,η) of magnitude $f(\xi,\eta)$ for all ξ and η. By linearity we can superimpose these point source solutions $f(\xi,\eta)G(x,y,\xi,\eta)$ over all space to obtain

$$u(x,y) = \int_{\mathbb{R}^2} G(x,y,\xi,\eta)f(\xi,\eta)\, d\xi d\eta,$$

which is the solution to (4.94). Here, we assume that the source vanishes sufficiently rapid at infinity.

Example 4.40

(**Green's function in upper-half plane**) For infinite domains having boundaries with simple geometry, we can modify the Green's function above to derive a solution. Consider Poisson's equation in the upper-half plane:

$$u_{xx}+u_{yy} = f(x,y), \quad x\in\mathbb{R},\ y>0, \tag{4.95}$$

$$u(x,0) = 0, \quad x\in\mathbb{R}. \tag{4.96}$$

For a point charge, $G(x, y, \xi, \eta)$ the Green's function (4.93) does not satisfy the boundary condition at $y = 0$. So we introduce a negative point charge at the point $(\xi, -\eta)$ to counter the positive point charge at (ξ, η). The Green's function for this *image charge* is $G(x, y, \xi, -\eta)$. See Figure 4.5. Therefore the net potential at an arbitrary point (x, y) in $y > 0$ is the sum of the two potentials from both charges, or

$$G(x, y, \xi, \eta) = -\frac{1}{2\pi} \ln R + \frac{1}{2\pi} \ln \overline{R} = \frac{1}{2\pi} \ln \frac{\overline{R}}{R},$$

where R and $\overline{R}$ are the distances shown in the figure, which are easily calculated by the distance formula. Easily G satisfies the zero boundary condition at $y = 0$. Therefore, G is Green's function for the upper-half plane. Again, by superposition, the solution to (4.95–4.96) is

$$u(x, y) = \frac{1}{2\pi} \int_{y>0} \ln \left(\frac{\overline{R}}{R} \right) f(\xi, \eta) \, d\xi d\eta. \quad \square$$

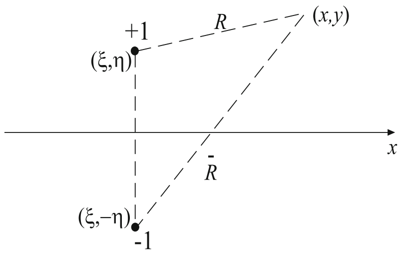

Figure 4.5 A unit (+1) charge placed at (ξ, η) and its negative image charge (-1) at $(\xi, -\eta)$.. The distances between those charges and an arbitrary point (x, y) are R and $\overline{R}$, respectively, which are functions of x, y, ξ, η

The procedure in this example is called the **method of images**, and it can be applied for simple domains in both 2 and 3 dimensions. See Strauss (1992) or Haberman (2013), for example, for a thorough discussion of Green's functions.

EXERCISES

1. Solve the Dirichlet problem on a disk of radius $r = 2$:

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 2, \quad u(x, y) = \sin\theta, \quad x^2 + y^2 = 2.$$

2. On a disk of radius R solve

$$\Delta u = -A \text{ in } 0 \leq r < R, \quad u = 1 \text{ on } r = R.$$

3. Solve the Poisson equation $\Delta u = 1$ on the unit disk with boundary condition $u(1, \theta) = \sin\theta$.

4. Solve $\Delta u = 1$ on the annulus $R_0 < r < R_1$ with $u = 0$ at $r = R_0$, $r = R_1$.

5. On the disk of radius R in the plane solve

$$\Delta u = f(r), \quad r < 1, \quad u = u_0, \quad r = R,$$

where u_0 is a constant.

6. Solve the boundary value problem

$$\begin{aligned} \Delta u &= f(r), \quad 0 < r < R, \ 0 \leq \theta < 2\pi, \\ u(R, \theta) &= g(\theta), \quad 0 \leq \theta < 2\pi. \end{aligned}$$

7. In $\mathbb{R}^2$ find the eigenvalues and eigenfunctions of the Neumann problem

$$-\Delta u = \lambda u, \quad \mathbf{x} \in \Omega, \quad \frac{du}{dn} = 0, \quad \mathbf{x} \in \partial\Omega,$$

where Ω is the rectangle $0 < x < \pi, \ 0 < y < 1$.

8. Use the eigenfunction method to solve the Dirichlet problem for the Poisson equation:

$$\begin{aligned} u_{xx} + u_{yy} &= f(x, y), \quad 0 < x < \pi, \ 0 < y < 1, \\ u(0, y) &= u(\pi, y) = 0, \quad 0 < y < 1, \\ u(x, 0) &= u(x, 1) = 0, \quad 0 < x < \pi. \end{aligned}$$

Hint: Assume $u(x, t) = \sum_{n=1}^{\infty} g_n(y) \sin nx$. Substitute into the PDE and boundary conditions at $y = 0$ and $y = 1$ to obtain

$$g_n''(y) - n^2 g_n(y) = f_n(y), \quad g_n(0) = g_n(1) = 0,$$

where $f_n(y) = \left(\frac{2}{\pi}\right) \int_0^{\pi} f(x, y) \sin nx \, dx$. Write the general solution in the form

$$g_n(y) = c_1 e^{ny} + c_2 e^{-ny} + \frac{2}{n} \int_0^y f_n(\xi) \sinh n(y - \xi) d\xi$$

and determine the constants c_1 and c_2.

9. Consider Laplace's equation on a unit sphere.

a) Assume a solution of the form $u(r,\theta,\phi) = S_n(\phi)r^n$, $n = 0, 1, 2, \ldots$, and show that $S_n(\phi)$, the *spherical harmonics*, satisfies the equation

$$S_n'' + \frac{\cos\phi}{\sin\phi} S_n' + n(n+1)S_n = 0.$$

b) Change the independent variable to $x = \cos\phi$ with $P_n(x) = S_n(\phi)$, and show

$$\left(1 - x^2\right) P_n'' - 2xP_n' + n(n+1)P_n = 0.$$

This is **Legendre's differential equation**.

c) Assume a power series solution of Legendre's equation of the form

$$P_n(x) = \sum_{k=0}^{\infty} a_k x^k,$$

and show that for each fixed n,

$$a_{k+2} = \frac{k(k+1) - n(n+1)}{(k+2)(k+1)} a_k, \quad k = 0, 1, 2, \ldots.$$

d) Show that for each fixed n, there is a polynomial solution. These are called the Legendre polynomials. Up to a constant multiple, find the first four Legendre polynomials $P_0(x), \ldots, P_3(x)$. [Note: It can be shown that the Legendre polynomials are orthogonal on $-1 < x < 1$.]

10. Consider the partial differential operator

$$Lu = -\mathrm{div}\,(p\,\mathrm{grad}\,u) + qu, \quad \mathbf{x} \in \Omega,$$

where $p = p(\mathbf{x}) > 0$, $q = q(\mathbf{x})$, p and q are continuous on $\overline{\Omega}$, and p has continuous first partial derivatives on $\overline{\Omega}$.

a) Prove the integration by parts formula

$$\int_\Omega vLu\,dx = \int_\Omega uLv\,dx + \int_{\partial\Omega} p\left(u\frac{dv}{dn} - v\frac{du}{dn}\right) dA.$$

b) Consider the eigenvalue problem $Lu = \lambda u$, $\mathbf{x} \in \Omega$, with a Dirichlet boundary condition $u = 0$, $\mathbf{x} \in \partial\Omega$. Prove that the eigenvalues are positive and that distinct eigenvalues have corresponding orthogonal eigenfunctions.

11. Use the method of images to find the Green's function in the first quadrant of the plane, $x > 0$, $y > 0$, with zero conditions on the boundaries $x = y = 0$. Hint: Place appropriate image charges in the second and fourth quadrants.

12. Find Green's function $G(r, \theta, \rho, \phi)$ for the unit circle with zero boundary condition. Hint: Find the potential at (r, θ) due to unit source charge (ρ, ϕ) inside the circle. Place an image charge outside the circle at $(1/\rho, \phi)$.

13. Show that Green's function for $\mathbb{R}^3$ is given by $G(\mathbf{x}, \mathbf{y}) = 1/4\pi R$, which is the potential at $\mathbf{x}$ caused by a point source at $\mathbf{y}$, and $R =\mid \mathbf{x} - \mathbf{y} \mid$. Hint: Find the radial solution to Laplace's equation that has unit flux across any sphere containing the origin.