3

Binomial Coefficients and Pascal's Triangle

3.1 The Binomial Theorem

In Chapter 1 we introduced the numbers $\binom{n}{k}$ and called them *binomial coefficients*. It is time to explain this strange name: it comes from a very important formula in algebra involving them, which we discuss next.

The issue is to compute powers of the simple algebraic expression $(x+y)$. We start with small examples:

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2,\\ (x+y)^3 &= (x+y)\cdot(x+y)^2 = (x+y)\cdot(x^2+2xy+y^2)\\ &= x^3 + 3x^2y + 3xy^2 + y^3,\end{aligned}$$

and continuing like this,

$$(x+y)^4 = (x+y)\cdot(x+y)^3 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

These coefficients are familiar! We have seen them, e.g., in exercise 1.8.2, as the numbers $\binom{n}{k}$. Let us make this observation precise. We illustrate the argument for the next value of n, namely $n = 5$, but it works in general.

Think of expanding

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

so that we get rid of all parentheses. We get each term in the expansion by selecting one of the two terms in each factor, and multiplying them. If we choose x, say, 2 times, then we must choose y 3 times, and so we get x^2y^3. How many times do we get this same term? Clearly, as many times as the number of ways to select the three factors that supply y (the remaining factors supply x). Thus we have to choose three factors out of 5, which can be done in $\binom{5}{3}$ ways.

Hence the expansion of $(x+y)^5$ looks like this:

$$(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.$$

We can apply this argument in general to obtain the *Binomial Theorem*:

Theorem 3.1.1 (The Binomial Theorem) *The coefficient of $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is $\binom{n}{k}$. In other words, we have the identity*

$$(x+y)^n = \binom{n}{0}x^n+\binom{n}{1}x^{n-1}y+\binom{n}{2}x^{n-2}y^2+\cdots+\binom{n}{n-1}xy^{n-1}+\binom{n}{n}y^n.$$

This important identity was discovered by the famous Persian poet and mathematician Omar Khayyam (1044?–1123?). Its name comes from the Greek word *binome* for an expression consisting of two terms, in this case, $x+y$. The appearance of the numbers $\binom{n}{k}$ in this theorem is the source of their name: *binomial coefficients.*

The Binomial Theorem can be applied in many ways to get identities concerning binomial coefficients. For example, let us substitute $x = y = 1$. Then we get identity (1.9):

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}. \tag{3.1}$$

Later on, we are going to see trickier applications of this idea. For the time being, another twist on it is contained in exercise (3.1.2).

3.1.1 Give a proof of the Binomial Theorem by induction, based on (1.8).

3.1.2 (a) Prove the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots = 0.$$

(The sum ends with $\binom{n}{n} = 1$, with the sign of the last term depending on the parity of n.)

(b) This identity is obvious if n is odd. Why?

3.1.3 Prove the identity in Exercise 3.1.2, using a combinatorial interpretation of the positive and negative terms.

3.2 Distributing Presents

Suppose we have n different presents, which we want to distribute to k children, where for some reason, we are told how many presents each child should get. So Adam should get n_{Adam} presents, Barbara, n_{Barbara} presents, etc. In a mathematically convenient (though not very friendly) way, we call the children $1, 2, \ldots, k$; thus we are given the numbers (nonnegative integers) $n_1, n_2, \ldots, n_k$. We assume that $n_1 + n_2 + \cdots + n_k = n$, else there is no way to distribute all the presents and give each child the right number of them.

The question is, of course, how many ways can these presents be distributed?

We can organize the distribution of presents as follows. We lay out the presents in a single row of length n. The first child comes and takes the first n_1 presents, starting from the left. Then the second comes and takes the next n_2; then the third takes the next n_3 presents etc. Child k gets the last n_k presents.

It is clear that we can determine who gets what by choosing the order in which the presents are laid out. There are $n!$ ways to order the presents. But of course, the number $n!$ overcounts the number of ways to distribute the presents, since many of these orderings lead to the same results (that is, every child gets the same set of presents). The question is, how many?

So let us start with a given distribution of presents, and let's ask the children to lay out the presents for us, nicely in a row, starting with the first child, then continuing with the second, third, etc. This way we get back *one* possible ordering that leads to the current distribution. The first child can lay out his presents in $n_1!$ possible orders; no matter which order he chooses, the second child can lay out her presents in $n_2!$ possible ways, etc. So the number of ways the presents can be laid out (given the distribution of the presents to the children) is a product of factorials:

$$n_1! \cdot n_2! \cdots n_k!\,.$$

Thus the number of ways of distributing the presents is

$$\frac{n!}{n_1! n_2! \cdots n_k!}\,.$$

3.2.1 We can describe the procedure of distributing the presents as follows. First, we select n_1 presents and give them to the first child. This can be done in $\binom{n}{n_1}$ ways. Then we select n_2 presents from the remaining $n - n_1$ and give them to the second child, etc.

Complete this argument and show that it leads to the same result as the previous one.

3.2.2 The following special cases should be familiar from previous problems and theorems. Explain why.

(a) $n = k$, $n_1 = n_2 = \cdots = n_k = 1$;

(b) $n_1 = n_2 = \cdots = n_{k-1} = 1$, $n_k = n - k + 1$;

(c) $k = 2$;

(d) $k = 3$, $n = 6$, $n_1 = n_2 = n_3 = 2$.

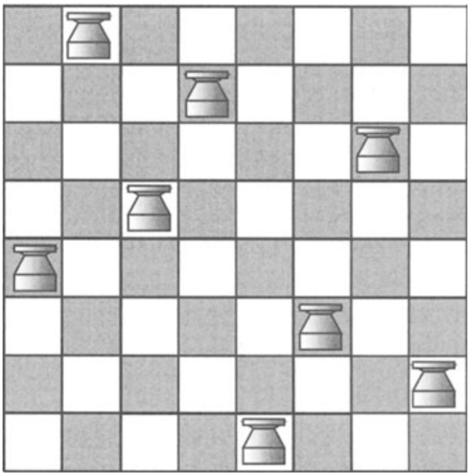

FIGURE 3.1. Placing 8 nonattacking rooks on a chessboard.

3.2.3 (a) How many ways can you place n rooks on a chessboard so that no two attack each other (Figure 3.1)? We assume that the rooks are identical, so interchanging two rooks does not count as a separate placement.

(b) How many ways can you do this if you have 4 wooden and 4 marble rooks?

(c) How many ways can you do this if all the 8 rooks are different?

3.3 Anagrams

Have you played with anagrams? One selects a word (say, COMBINATORICS) and tries to compose from its letters meaningful or, even better, funny words or expressions.

How many anagrams can you build from a given word? If you try to answer this question by playing around with the letters, you will realize that the question is badly posed; it is difficult to draw the line between meaningful and nonmeaningful anagrams. For example, it could easily happen that A CROC BIT SIMON. And it may be true that Napoleon always wanted a TOMB IN CORSICA. It is questionable, but certainly grammatically correct, to assert that COB IS ROMANTIC. Some universities may have a course on MAC IN ROBOTICS.

But one would have to write a book to introduce an exciting character, ROBIN COSMICAT, who enforces a COSMIC RIOT BAN, while appealing TO COSMIC BRAIN.

And it would be terribly difficult to explain an anagram like MTBIRASCIONOC.

To avoid this controversy, let's accept everything; i.e., we don't require the anagram to be meaningful (or even pronounceable). Of course, the production of anagrams then becomes uninteresting; but at least we can tell how many of them there are!

3.3.1 How many anagrams can you make from the word COMBINATORICS?

3.3.2 Which word gives rise to more anagrams: COMBINATORICS or COMBINATORICA? (The latter is the Latin name of the subject.)

3.3.3 Which word with 13 letters gives rise to the most anagrams? Which word gives rise to the least?

So let's see the general answer to the question of counting anagrams. If you have solved the problems above, it should be clear that the number of anagrams of an n-letter word depends on how many times letters of the word are repeated. So suppose that there are k letters $A, B, C, \ldots Z$ in the alphabet, and the word contains letter A n_1 times (this could be 0), letter B, n_2 times, etc., letter Z, n_k times. Clearly, $n_1 + n_2 + \cdots + n_k = n$.

Now, to form an anagram, we have to select n_1 positions for letter A, n_2 positions for letter B, etc., n_k positions for letter Z. Having formulated it this way, we can see that this is nothing but the question of distributing n presents to k children when it is prescribed how many presents each child gets. Thus we know from the previous section that the answer is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

3.3.4 It is clear that STATUS and LETTER have the same number of anagrams (in fact, $6!/(2!2!) = 180$). We say that these words are "essentially the same" (at least as far as counting anagrams goes): They have two letters repeated twice and two letters occurring only once. We call two words "essentially different", if they are not "essentially the same."

(a) How many 6-letter words are there, if, to begin with, we consider any two words different if they are not completely identical? (As before, the words don't have to be meaningful. The alphabet has 26 letters.)

(b) How many words with 6 letters are "essentially the same" as the word LETTER?

(c) How many "essentially different" 6-letter words are there?

(d) Try to find a general answer to question (c) (that is, how many "essentially different" words are there with n letters?). If you can't find the answer, read the following section and return to this exercise afterwards.

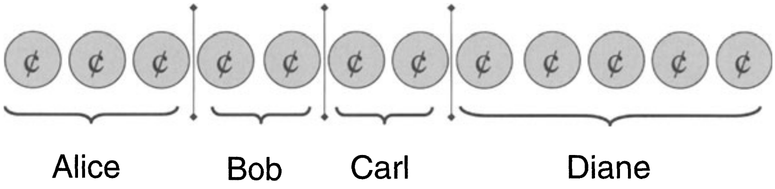

FIGURE 3.2. How to distribute n pennies to k children?

3.4 Distributing Money

Instead of distributing presents, let's distribute money. Let us formulate the question in general: We have n pennies that we want to distribute among k kids. Each child must get at least one penny (and, of course, an integer number of pennies). How many ways can we distribute the money?

Before answering this question, we must clarify the difference between distributing money and distributing presents. If you are distributing presents, you have to decide not only how many presents each child gets, but also *which* of the different presents the child gets. If you are distributing money, only the quantity matters. In other words, presents are *distinguishable* while pennies are not. (A question like that in section 3.2 where we specify in advance how many presents a given child gets would be trivial for money: There is only one way to distribute n pennies so that the first child gets n_1, the second child gets n_2, etc.)

Even though the problem is quite different from the distribution of presents, we can solve it by imagining a similar distribution method. We line up the pennies (it does not matter in which order; they are all alike), and then let the first child begin to pick them up from left to right. After a while we stop him and let the second child pick up pennies, etc. (Figure 3.2). *The distribution of the money is determined by specifying where to start with a new child.*

Now, there are $n-1$ points (between consecutive pennies) where we can let a new child in, and we have to select $k-1$ of them (since the first child always starts at the beginning, we have no choice there). Thus we have to select a $(k-1)$-element subset from an $(n-1)$-element set. The number of ways of doing so is $\binom{n-1}{k-1}$.

To sum up, we have the following theorem:

Theorem 3.4.1 *The number of ways to distribute n identical pennies to k children so that each child gets at least one is $\binom{n-1}{k-1}$.*

It is quite surprising that the binomial coefficients give the answer here, in a quite nontrivial and unexpected way!

Let's also discuss the natural (though unfair) modification of this question, where we also allow distributions in which some children get no money at all; we consider even giving all the money to one child. With the following trick, we can reduce the problem of counting such distributions to the problem we just solved: We borrow 1 penny from each child, and then distribute the whole amount (i.e., $n+k$ pennies) to the children so that each child gets at least one penny. This way every child gets back the money we borrowed from him or her, and the lucky ones get some more. The "more" is exactly n pennies distributed to k children. We already know that the number of ways to distribute $n+k$ pennies to k children so that each child gets at least one penny is $\binom{n+k-1}{k-1}$. So we have the next result:

Theorem 3.4.2 *The number of ways to distribute n identical pennies to k children is $\binom{n+k-1}{k-1}$.*

3.4.1 In how many ways can you distribute n pennies to k children if each child is supposed to get at least 2?

3.4.2 We distribute n pennies to k boys and ℓ girls in such a way that (to be really unfair) we require that each of the girls gets at least one penny (but we do not insist on the same thing for the boys). In how many ways can we do this?

3.4.3 A group of k earls are playing cards. Originally, they each have p pennies. At the end of the game, they count how much money they have. They do not borrow from each other, so that each cannot loose more than his p pennies. How many possible results are there?

3.5 Pascal's Triangle

To study various properties of binomial coefficients, the following picture is very useful. We arrange all binomial coefficients into a triangular scheme: in the "zeroth" row we put $\binom{0}{0}$; in the first row, we put $\binom{1}{0}$ and $\binom{1}{1}$; in the second row, $\binom{2}{0}$, $\binom{2}{1}$, and $\binom{2}{2}$; etc. In general, the nth row contains the numbers $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$. We shift these rows so that their midpoints match; this way we get a pyramidlike scheme, called *Pascal's Triangle* (named after the French mathematician and philosopher Blaise Pascal, 1623–1662). The figure below shows only a finite piece of Pascal's Triangle.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}
$$

We can replace each binomial coefficient by its numerical value to get another version of Pascal's Triangle (going a little further down, to the eighth row):

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
 & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

3.5.1 Prove that Pascal's Triangle is symmetric with respect to the vertical line through its apex.

3.5.2 Prove that each row of Pascal's Triangle starts and ends with 1.

3.6 Identities in Pascal's Triangle

Looking at Pascal's Triangle, it is not hard to notice its most important property: Every number in it (other than the 1's on the boundary) is the sum of the two numbers immediately above it. This, in fact, is a property of the binomial coefficients we already met, namely, equation (1.8) in Section 1.8:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{3.2}$$

This property of Pascal's Triangle enables us to generate the triangle very fast, building it up row by row, using (3.2). It also gives us a tool to prove many properties of the binomial coefficients, as we shall see.

As a first application, let us give a new solution to exercise 3.1.2. There the task was to prove the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0, \tag{3.3}$$

using the Binomial Theorem. Now we give a proof based on (3.2): We can replace $\binom{n}{0}$ by $\binom{n-1}{0}$ (both are just 1), $\binom{n}{1}$ by $\binom{n-1}{0} + \binom{n-1}{1}$, $\binom{n}{2}$ by $\binom{n-1}{1} + \binom{n-1}{2}$, etc. Thus we get the sum

$$\binom{n-1}{0} - \left[\binom{n-1}{0} + \binom{n-1}{1}\right] + \left[\binom{n-1}{1} + \binom{n-1}{2}\right]$$

$$+\cdots+(-1)^{n-1}\left[\binom{n-1}{n-2}+\binom{n-1}{n-1}\right]+(-1)^n\binom{n-1}{n-1},$$

which is clearly 0, since the second term in each pair of brackets cancels with the first term in the next pair of brackets.

This method gives more than just a new proof of an identity we already know. What do we get if we start the same way, adding and subtracting binomial coefficients alternatingly, but stop earlier? Writing this as a formula, we take

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\cdots+(-1)^k\binom{n}{k}.$$

If we do the same trick as above, we get

$$\binom{n-1}{0}-\left[\binom{n-1}{0}+\binom{n-1}{1}\right]+\left[\binom{n-1}{1}+\binom{n-1}{2}\right]-\cdots$$
$$+(-1)^k\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right].$$

Here again every term cancels except the last one; so the result is $(-1)^k\binom{n-1}{k}$.

There are many other surprising relations satisfied by the numbers in Pascal's Triangle. For example, let's ask, what is the sum of the *squares* of elements in each row?

Let's experiment by computing the sum of the squares of elements in the first few rows:

$$\begin{aligned} 1^2 &= 1,\\ 1^2+1^2 &= 2,\\ 1^2+2^2+1^2 &= 6,\\ 1^2+3^2+3^2+1^2 &= 20,\\ 1^2+4^2+6^2+4^2+1^2 &= 70. \end{aligned}$$

We may recognize these numbers as the numbers in the middle column of Pascal's Triangle. Of course, only every second row contains an entry in the middle column, so the last value above, the sum of squares in the fourth row is the middle element in the eighth row. So the examples above suggest the following identity:

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{n-1}^2+\binom{n}{n}^2=\binom{2n}{n}. \qquad (3.4)$$

Of course, the few experiments above do not prove that this identity always holds, so we need a proof.

We will give an interpretation of both sides of the identity as the result of a counting problem; it will turn out that they count the same things, so they are equal. It is obvious what the right-hand side counts: the number of subsets of size n of a set of size $2n$. It will be convenient to choose the set $S = \{1, 2, \ldots, 2n\}$ as our $2n$-element set.

The combinatorial interpretation of the left-hand side is not so easy. Consider a typical term, say $\binom{n}{k}^2$. We claim that this is the number of n-element subsets of $\{1, 2, \ldots, 2n\}$ that contain exactly k elements from $\{1, 2, \ldots, n\}$ (the first half of our set S). In fact, how do we choose such an n-element subset of S? We choose k elements from $\{1, 2, \ldots, n\}$ and then $n-k$ elements from $\{n+1, n+2, \ldots, 2n\}$. The first can be done in $\binom{n}{k}$ ways; no matter which k-element subset of $\{1, 2, \ldots, n\}$ we selected, we have $\binom{n}{n-k}$ ways to choose the other part. Thus the number of ways to choose an n-element subset of S having k elements from $\{1, 2, \ldots, n\}$ is

$$\binom{n}{k} \cdot \binom{n}{n-k} = \binom{n}{k}^2$$

(by the symmetry of Pascal's Triangle).

Now, to get the total number of n-element subsets of S, we have to sum these numbers for all values of $k = 0, 1, \ldots, n$. This proves identity (3.4).

3.6.1 Give a proof of the formula (1.9),

$$1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n,$$

along the lines of the proof of (3.3). (One could expect that, as with the "alternating" sum, we could get a nice formula for the sum obtained by stopping earlier, like $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$. But this is not the case: No simpler expression is known for this sum in general.)

3.6.2 By the Binomial Theorem, the right-hand side in identity (3.4) is the coefficient of $x^n y^n$ in the expansion of $(x+y)^{2n}$. Write $(x+y)^{2n}$ in the form $(x+y)^n (x+y)^n$, expand both factors $(x+y)^n$ using the Binomial Theorem, and then try to figure out the coefficient of $x^n y^n$ in the product. Show that this gives another proof of identity (3.4).

3.6.3 Prove the following identity:

$$\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k}\binom{m}{0} = \binom{n+m}{k}.$$

You can use a combinatorial interpretation of both sides, as in the proof of (3.4) above, or the Binomial Theorem as in the previous exercise.

Here is another relation between the numbers in Pascal's Triangle. Let us start with the first element in the nth row, and sum the elements moving down diagonally to the right (Figure 3.3). For example, starting with the first element in the second row, we get

$$\begin{aligned} 1 &= 1, \\ 1+3 &= 4, \\ 1+3+6 &= 10, \\ 1+3+6+10 &= 20, \\ 1+3+6+10+15 &= 35. \end{aligned}$$

These numbers are just the numbers in the next skew line of the table!

$$\begin{array}{c}
1 \\
1 \quad 1 \\
\mathbf{1} \quad 2 \quad 1 \\
1 \quad \mathbf{3} \quad 3 \quad 1 \\
1 \quad 4 \quad \mathbf{6} \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad \mathbf{10} \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad \mathbf{15} \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad \mathbf{21} \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad \boxed{\mathbf{56}} \quad 28 \quad 8 \quad 1
\end{array}$$

FIGURE 3.3. Adding up entries in Pascal's Triangle diagonally.

If we want to put this in a formula, we get

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}. \tag{3.5}$$

To *prove* this identity, we use induction on k. If $k = 0$, the identity just says that $1 = 1$, so it is trivially true. (We can check it also for $k = 1$, even though this is not necessary. Anyway, it says that $1 + (n+1) = n+2$.)

So suppose that the identity (3.5) is true for a given value of k, and we want to prove that it also holds for $k+1$ in place of k. In other words, we want to prove that

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{k} + \binom{n+k+1}{k+1} = \binom{n+k+2}{k+1}.$$

Here the sum of the first k terms on the left-hand side is $\binom{n+k+1}{k}$ by the induction hypothesis, and so the left-hand side is equal to

$$\binom{n+k+1}{k} + \binom{n+k+1}{k+1}.$$

But this is indeed equal to $\binom{n+k+2}{k+1}$ by the fundamental property (3.2) of Pascal's Triangle. This completes the proof by induction.

3.6.4 Suppose that you want to choose a $(k+1)$-element subset of the $(n+k+1)$-element set $\{1, 2, \ldots, n+k+1\}$. You decide to do this by choosing first the largest element, then the rest. Show that counting the number of ways to choose the subset this way, you get a combinatorial proof of identity (3.5).

3.7 A Bird's-Eye View of Pascal's Triangle

Let's imagine that we are looking at Pascal's Triangle from a distance. Or to put it differently, we are not interested in the exact numerical values of the entries, but rather in their order of magnitude, rise and fall, and other global properties. The first such property of Pascal's Triangle is its symmetry (with respect to the vertical line through its apex), which we already know.

Another property one observes is that *along any row, the entries increase until the middle, and then decrease.* If n is even, there is a unique middle element in the nth row, and this is the largest; if n is odd, then there are two equal middle elements, which are largest.

So let us *prove* that the entries increase until the middle (then they begin to decrease by the symmetry of the table). We want to compare two consecutive entries:

$$\binom{n}{k} \;?\; \binom{n}{k+1}.$$

If we use the formula in Theorem 1.8.1, we can write this as

$$\frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \;?\; \frac{n(n-1)\cdots(n-k)}{(k+1)k\cdots 1}.$$

There are many common factors on both sides that are positive, and so we can simplify. We get the really simple comparison

$$1 \;?\; \frac{n-k}{k+1}.$$

Rearranging, we get

$$k \;?\; \frac{n-1}{2}.$$

So if $k < (n-1)/2$, then $\binom{n}{k} < \binom{n}{k+1}$; if $k = (n-1)/2$, then $\binom{n}{k} = \binom{n}{k+1}$ (this is the case of the two entries in the middle if n is odd); and if $k > (n-1)/2$, then $\binom{n}{k} > \binom{n}{k+1}$.

It will be useful later that this computation also describes by *how much* consecutive elements increase or decrease. If we start from the left, the second entry (namely, n) is larger by a factor of n than the first; the third (namely, $n(n-1)/2$) is larger by a factor of $(n-1)/2$ than the second. In general,

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1}. \tag{3.6}$$

3.7.1 For which values of n and k is $\binom{n}{k+1}$ twice the previous entry in Pascal's Triangle?

3.7.2 Instead of the ratio, look at the difference of two consecutive entries in Pascal's Triangle:

$$\binom{n}{k+1} - \binom{n}{k}.$$

For which value of k is this difference largest?

We know that each row of Pascal's Triangle is symmetric. We also know that the entries start with 1, rise to the middle, and then fall back to 1. Can we say more about their shape?

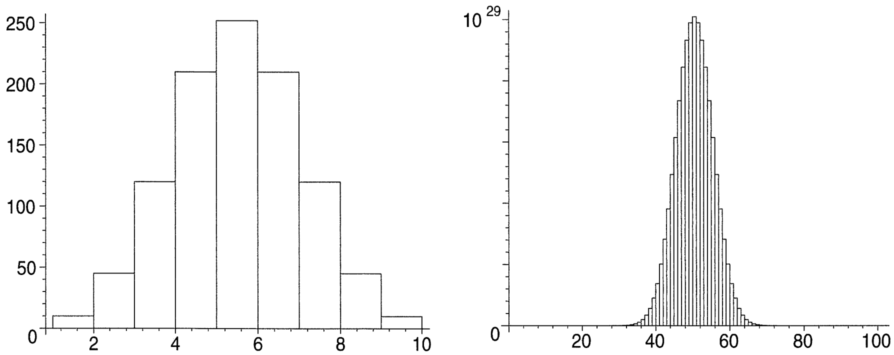

FIGURE 3.4. Bar chart of the nth row of Pascal's Triangle, for $n = 10$ and $n = 100$.

Figure 3.4 shows the graph of the numbers $\binom{n}{k}$ $(k = 0, 1, \dots, n)$ for the values $n = 10$ and $n = 100$. We can make several further observations.

— First, the largest number gets very large.

— Second, not only do these numbers increase to the middle and then decrease, but the middle ones are substantially larger than those at the beginning and end. For $n = 100$, we see bars only in the range $\binom{100}{25}, \binom{100}{26}, \dots, \binom{100}{75}$; the numbers outside this range are so small compared to the largest that they do not show in the figure.

— Third, we can observe that the shape of the graph is quite similar for different values of n.

Let's look more carefully at these observations. For the discussions that follow, we shall assume that n is even (for odd values of n, the results would be quite similar, except that one would have to word them differently). If

n is even, then we already know that the largest entry in the nth row is the middle number $\binom{n}{n/2}$, and all other entries are smaller.

How large is the largest number in the nth row of Pascal's Triangle? We know immediately an upper bound on this number:

$$\binom{n}{n/2} < 2^n,$$

since 2^n is the sum of all entries in the row. It only takes a little more sophistication to get a lower bound:

$$\binom{n}{n/2} > \frac{2^n}{n+1},$$

since $2^n/(n+1)$ is the average of the numbers in the row, and the largest number is certainly at least as large as the average.

These bounds already give a pretty good idea about the size of $\binom{n}{n/2}$; in particular, they show that this number gets very large. Take, say, $n = 500$. Then we get

$$\frac{2^{500}}{501} < \binom{500}{250} < 2^{500}.$$

If we want to know the number of digits of $\binom{500}{250}$, we just have to take the logarithm (in base 10) of it. From the bounds above, we get

$$500 \lg 2 - \lg 501 = 147.8151601 \cdots < \lg \binom{500}{250} < 500 \lg 2 = 150.5149978 \ldots .$$

This inequality gives the number of digits with a small error: If we guess that it is 150, then we are off by at most 2 (actually, 150 is the true value).

Using Stirling's formula (Theorem 2.2.1), one can get an even better approximation of this largest entry. We know that

$$\binom{n}{n/2} = \frac{n!}{(n/2)!(n/2)!}.$$

Here, by the Stirling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad (n/2)! \sim \sqrt{\pi n}\left(\frac{n}{2e}\right)^{n/2},$$

and so

$$\binom{n}{n/2} \sim \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\pi n \left(\frac{n}{2e}\right)^n} = \sqrt{\frac{2}{\pi n}}\, 2^n. \tag{3.7}$$

So we know that the largest entry in the nth row of Pascal's Triangle is in the middle, and we know approximately how large this element is. We also know that going either left or right, the elements begin to drop. How fast do they drop? Figure 3.4 suggests that starting from the middle, the binomial coefficients drop just by a little at the beginning, but pretty soon this accelerates.

Looking from the other end, we see this even more clearly. Let us consider, say, row 57 (just to take a non-round number for a change). The first few elements are

$$1,\ 57,\ 1596,\ 29260,\ 395010,\ 4187106,\ 36288252,\ 264385836,\ 1652411475,$$

$$8996462475,\ 43183019880,\ 184509266760,\ 707285522580,\ \ldots$$

and the ratios between consecutive entries are:

$$57,\ 28,\ 18.33,\ 13.5,\ 10.6,\ 8.67,\ 7.29,\ 6.25,\ 5.44,\ 4.8,\ 4.27,\ 3.83,\ \ldots$$

While the entries are growing fast, these ratios get smaller and smaller, and we know that when we reach the middle, they have to turn less than 1 (since the entries themselves begin to decrease). But what are these ratios? We computed them above, and found that

$$\frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n-k}{k+1}.$$

If we write this as

$$\frac{n-k}{k+1} = \frac{n+1}{k+1} - 1,$$

then we see immediately that the ratio of two consecutive binomial coefficients decreases as k increases.

3.8 An Eagle's-Eye View: Fine Details

Let us ask a more quantitative question about the shape of a row in Pascal's Triangle: Which binomial coefficient in this row is (for example) half of the largest?

We consider the case where n is even; then we can write $n = 2m$, where m is a positive integer. The largest, middle entry in the nth row is $\binom{2m}{m}$. Consider the binomial coefficient that is t steps from the middle. It does not matter whether we go left or right, so take, say, $\binom{2m}{m-t}$. We want to compare it with the largest coefficient.

The following formula describes the rate at which the binomial coefficients drop:

$$\binom{2m}{m-t} \Big/ \binom{2m}{m} \approx e^{-t^2/m}. \tag{3.8}$$

The graph of right-hand side of (3.8) (as a function of t) is shown in Figure 3.5 for $m = 50$. This is the famous *Gauss curve* (sometimes also called the

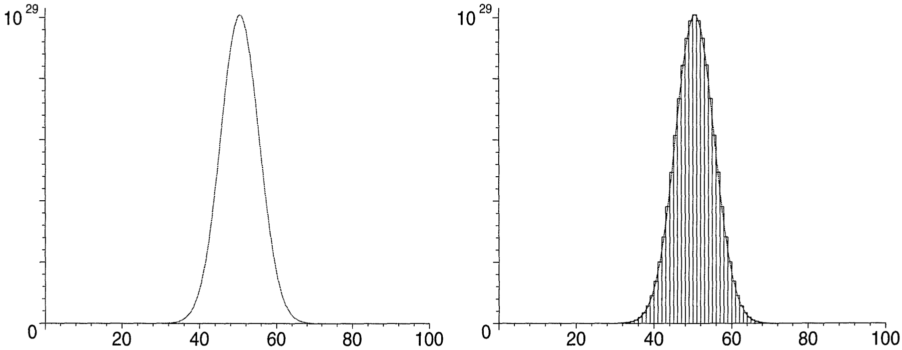

FIGURE 3.5. The Gauss curve $e^{-t^2/m}$ for $m = 50$, and the chart of binomial coefficients in the 100th row of Pascal's Triangle.

"bell curve"). Plotting many types of statistics gives a similar picture. In Figure 3.5 we show the curve alone and also overlaid with the binomial coefficients, to show the excellent fit.

Equation (3.8) is not an exact equation, and to make it a precise mathematical statement, we need to tell how large the error can be. Below, we shall derive the following inequalities:

$$e^{-t^2/(m-t+1)} \le \binom{2m}{m-t} \Big/ \binom{2m}{m} \le e^{-t^2/(m+t)}. \tag{3.9}$$

The upper and lower bounds in this formula are quite similar to the (imprecise) approximation $e^{-t^2/m}$ given in (3.8), and it is easy to see that the latter value is between them. The right hand side of (3.8) in fact gives a better approximation than either the upper or the lower bound. For example, suppose that we want to estimate the ratio $\binom{100}{40} / \binom{100}{50}$, which is $0.1362\dots$. From (3.9) we get

$$0.08724 \le \binom{100}{40} \Big/ \binom{100}{50} \le 0.1889,$$

while the approximation given in (3.8) is $0.1353\dots$, much closer to the truth. Using heavier calculus (analysis) would give tighter bounds; we give here only as much as we can without appealing to calculus.

To derive (3.9), we start with transforming the ratio in the middle; or rather, we take its reciprocal, which is larger than 1 and therefore a bit easier to work with:

$$\binom{2m}{m} \Big/ \binom{2m}{m-t} = \frac{(2m)!}{m!m!} \Big/ \frac{(2m)!}{(m-t)!(m+t)!} = \frac{(m-t)!(m+t)!}{m!m!}$$
$$= \frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)\cdots(m-t+1)}.$$

So we have some sort of a formula for this ratio, but how useful is it? How do we tell, for example, for which value of t this ratio becomes larger than 2? We can certainly write this as a formula:

$$\frac{(m+t)(m+t-1)\cdots(m+1)}{m(m-1)\cdots(m-t+1)} > 2. \tag{3.10}$$

We could try to solve this inequality for t (similar to the proof that the entries are increasing to the middle), but it would be too complicated to solve. So even to answer such a simple question about binomial coefficients like, how far from the middle do they drop to half of the maximum? needs more work, and we have to do some arithmetic trickery. We divide the first factor of the numerator by the first factor of the denominator, the second factor by the second factor etc., to get

$$\frac{m+t}{m} \cdot \frac{m+t-1}{m-1} \cdots \frac{m+1}{m-t+1}.$$

This product is still not easy to handle, but we have met similar ones in Section 2.5! There the trick was to take the logarithm, and this works here just as well. We get

$$\ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}\right).$$

Just as in Section 2.5, we can estimate the logarithms on the left-hand side using the inequalities in Lemma 2.5.1. Let's start with deriving an upper bound. For a typical term in the sum we have

$$\ln\left(\frac{m+t-k}{m-k}\right) \le \frac{m+t-k}{m-k} - 1 = \frac{t}{m-k},$$

and so

$$\ln\left(\frac{m+t}{m}\right) + \ln\left(\frac{m+t-1}{m-1}\right) + \cdots + \ln\left(\frac{m+1}{m-t+1}\right)$$
$$\le \frac{t}{m} + \frac{t}{m-1} + \cdots + \frac{t}{m-t+1}.$$

Can we bring this sum into a closed form? No, but we can use another trick from section 2.5. We replace each denominator by $m-t+1$, since this is the smallest; then we increase all the fractions (except the last one, which we don't change) and get an upper bound:

$$\frac{t}{m} + \frac{t}{m-1} + \cdots + \frac{t}{m-t+1} \le \frac{t}{m-t+1} + \frac{t}{m-t+1} + \cdots + \frac{t}{m-t+1}$$
$$= \frac{t^2}{m-t+1}.$$

Remember, this is an upper bound on the *logarithm* of the ratio $\binom{2n}{m} \Big/ \binom{2m}{m-t}$; to get an upper bound on the ratio itself, we have to apply the exponential function. Then we have another step to undo: we have to take the reciprocal, to get the lower bound in (3.9).

The upper bound in (3.9) can be derived using similar steps; the details are left to the reader as an exercise 3.8.2.

Let us return to our earlier question: We want to know when (for which value of t) the quotient in (3.9) will be larger than 2. We might need similar information for other numbers instead of 2, so let's try to answer the question for a general number $C > 1$. Thus we want to know for which value of t we get

$$\binom{2m}{m} \Big/ \binom{2m}{m-t} > C. \tag{3.11}$$

From (3.8) we know that the left-hand side is about $e^{t^2/m}$, so we start with solving the equation

$$e^{t^2/m} = C.$$

The exponential function on the left looks nasty, but the good old logarithm helps again: We get

$$\frac{t^2}{m} = \ln C,$$

which is now easy to solve:

$$t = \sqrt{m \ln C}.$$

So we expect that if t is larger than this, than (3.11) holds. But of course we have to be aware of the fact that this is only an approximation, not a precise result! Instead of (3.8), we can use the exact inequalities (3.9) to get the following lemma:

Lemma 3.8.1 *If $t \geq \sqrt{m \ln C} + \ln C$, then (3.11) holds; if $t \leq \sqrt{m \ln C} - \ln C$, then (3.11) does not hold.*

The derivation of these conditions from (3.9) is similar to the derivation of the approximate result from (3.8) and is left to the reader as exercise 3.8.3 (difficult!).

In important applications of binomial coefficients (one of which, the law of large numbers, will be discussed in Chapter 5) we also need to know a good bound on the sum of the smallest binomial coefficients, compared with the sum of all of them. Luckily, our previous observations and lemmas enable us to get an answer with some computation but without substantial new ideas.

Lemma 3.8.2 *Let* $0 \le k \le m$ *and* $c = \binom{2m}{k} \Big/ \binom{2m}{m}$. *Then*

$$\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{k-1} < \frac{c}{2} \cdot 2^{2m}. \tag{3.12}$$

To digest the meaning of this, choose $m = 500$, and let's try to see how many binomial coefficients in the 1000th row we have to add up (starting with $\binom{1000}{0}$) to reach 0.5% of the total. Lemma 3.8.2 tells us that if we choose $0 \le k \le 500$ so that $\binom{1000}{k} \Big/ \binom{1000}{500} < 1/100$, then adding up the first k binomial coefficients gives a sum less than 0.5% of the total. In turn, Lemma 3.8.1 tells us a k that will be certainly good: any $k \le 500 - \sqrt{500 \ln 100} - \ln 100 = 447.4$. So the first 447 entries in the 1000th row of Pascal's Triangle make up less than 0.5% of the total sum. By the symmetry of Pascal's Triangle, the last 447 add up to another less than 0.5%. The middle 107 terms account for 99% of the total.

Proof. To prove this lemma, let us write $k = m - t$, and compare the sum on the left-hand side of (3.12) with the sum

$$\binom{2m}{m-t} + \binom{2m}{m-t+1} + \cdots + \binom{2m}{m-1}. \tag{3.13}$$

Let us denote the sum $\binom{2m}{0} + \binom{2m}{1} + \cdots + \binom{2m}{m-t-1}$ by A, and the sum $\binom{2m}{m-t} + \binom{2m}{m-t+1} + \cdots + \binom{2m}{m-1}$ by B.

We have

$$\binom{2m}{m-t} = c\binom{2m}{m}$$

by the definition of c. This implies that

$$\binom{2m}{m-t-1} < c\binom{2m}{m-1},$$

since we know that binomial coefficients drop by a larger factor from $\binom{2m}{m-t}$ to $\binom{2m}{m-t-1}$ than they do from $\binom{2m}{m}$ to $\binom{2m}{m-1}$. Repeating the same argument,[1] we get that

$$\binom{2m}{m-t-i} < c\binom{2m}{m-i}$$

for every $i \ge 0$.

Hence it follows that the sum of any t consecutive binomial coefficients is less than c times the sum of the next t (as long as these are all on the left hand side of Pascal's Triangle). Going back from $\binom{2m}{m-1}$, the first block of t binomial coefficients adds up to A (by the definition of A); the next block

[1] In other words, using induction.

of t adds up to less than cA, the next block to less than c^2A, etc. Adding up, we get that

$$B < cA + c^2A + c^3A \dots$$

On the right-hand side we only have to sum $\lceil (m-t)/t \rceil$ terms, but we are generous and let the summation run to infinity! The geometric series on the right-hand side adds up to $\frac{c}{1-c}A$, so we get that

$$B < \frac{c}{1-c}A.$$

We need another inequality involving A and B, but this is easy:

$$B + A < \frac{1}{2}2^{2m}$$

(since the sum on the left-hand side includes only the left-hand side of Pascal's Triangle, and the middle element is not even counted). From these two inequalities we get

$$B < \frac{c}{1-c}A < \frac{c}{1-c}\left(\frac{1}{2}2^{2m} - B\right),$$

and hence

$$\left(1 + \frac{c}{1-c}\right)B < \frac{c}{1-c}\frac{1}{2}2^{2m}.$$

Multiplying by $1-c$ gives that $B < c\frac{1}{2}2^{2m}$, which proves the lemma. □

3.8.1 (a) Check that the approximation in (3.8) is always between the lower and upper bounds given in (3.9).

(b) Let $2m = 100$ and $t = 10$. By what percentage is the upper bound in (3.9) larger than the lower bound?

3.8.2 Prove the upper bound in (3.9).

3.8.3 Complete the proof of Lemma 3.8.1.

Review Exercises

3.8.4 Find all values of n and k for which $\binom{n}{k+1} = 3\binom{n}{k}$.

3.8.5 Find the value of k for which $k\binom{99}{k}$ is largest.

3.8.6 In city with a regular "chessboard" street plan, the North-South streets are called 1st Street, 2nd Street, ..., 20th Street, and the East-West streets are called 1st Avenue, 2nd Avenue, ..., 10th Avenue. What is the minimum number of blocks you have to walk to get from the corner of 1st Street and 1st Avenue to the corner of 20th Street and 10th Avenue? In how many ways can you get there walking this minimum number of blocks?

3.8.7 In how many ways can you read off the word MATHEMATICS from the following tables:

M	*A*	*T*	*H*	*E*	*M*
A	*T*	*H*	*E*	*M*	*A*
T	*H*	*E*	*M*	*A*	*T*
H	*E*	*M*	*A*	*T*	*I*
E	*M*	*A*	*T*	*I*	*C*
M	*A*	*T*	*I*	*C*	*S*

M	*A*	*T*	*H*		
A	*T*	*H*	*E*		
T	*H*		*M*	*A*	
H	*E*		*A*	*T*	*I*
	M	*A*	*T*	*I*	*C*
			I	*C*	*S*

3.8.8 Prove the following identities:

$$\sum_{k=0}^{m}(-1)^k\binom{n}{k} = (-1)^m\binom{n-1}{m};$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m} = \binom{n}{m}2^{n-m}.$$

3.8.9 Prove the following inequalities:

$$\frac{n^k}{k^k} \le \binom{n}{k} \le \frac{n^k}{k!}.$$

3.8.10 In how many ways can you distribute n pennies to k children if each child is supposed to get at least 5?

3.8.11 Prove that if we move straight down in Pascal's Triangle (visiting every other row), then the numbers we see are increasing.

3.8.12 Prove that

$$1+\binom{n}{1}2+\binom{n}{2}4+\cdots+\binom{n}{n-1}2^{n-1}+\binom{n}{n}2^n = 3^n.$$

Try to find a combinatorial proof.

3.8.13 Suppose that you want to choose a $(2k+1)$-element subset of the n-element set $\{1, 2, \dots, n\}$. You decide to do this by choosing first the middle element, then the k elements to its left, then the k elements to its right. Formulate the combinatorial identity you get from this.

3.8.14 Let n be a positive integer divisible by 3. Use Stirling's formula to find the approximate value of $\binom{n}{n/3}$.

3.8.15 Prove that $\binom{n}{10} \sim \frac{n^{10}}{10!}$.