12
Euler's Formula

12.1 A Planet Under Attack

In Chapter 11 we considered problems that can be cast in the language of graph theory: If we draw some special graphs in the plane, into how many parts do these graphs divide the plane? Indeed, we start with a set of lines; we consider the intersections of the given lines as nodes of the graph, and the segments arising on these lines as the edges of our graph. (For the time being, let us forget about the infinite half-lines. We'll come back to the connection between graphs and sets of lines later.)

More generally, we study a *planar map*: a graph that is drawn in the plane so that its edges are nonintersecting continuous curves. We also assume that the graph is connected. Such a graph divides the plane into certain parts, called *countries*. Exactly one country is infinite, the other countries are finite.

A very important result, discovered by Euler, tells us that we may determine the number of countries in a connected planar map if we know the number of nodes and edges of the graph. Euler's Formula is the following:

Theorem 12.1.1 *number of countries + number of nodes = number of edges + 2.*

Proof. To make the proof of this theorem more plausible, we'll tell a little story. This does not jeopardize the mathematical correctness of our proof.

Let us consider the given planar map as the map of a water system of a planet with a single very low continent. We consider the edges not as

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boundaries between countries but as dams, with watchtowers at the nodes. So the enclosed areas are not countries, but basins. The outermost "basin" is the sea, and all the other "basins" are dry (Figure 12.1). One advantage of this formulation is that we can allow a cut-edge in the graph, which we can think of as a kind of dam, or pier; this could not be considered as a boundary of two countries, since on both sides of it we would have the same "country" (in this case, the sea).

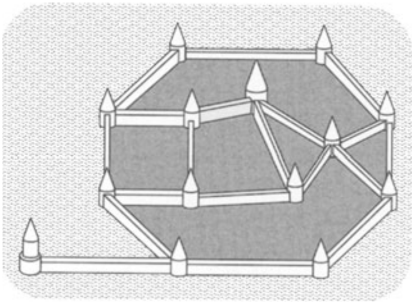

FIGURE 12.1. A graph of dams and watchtowers. There are 14 watchtowers, 7 basins (including the see), and 19 dams. Euler's Formula checks our: $14 + 7 = 19 + 2$.

One day, an enemy attacks the island, and the defenders decide to flood it with seawater by blowing up certain dams. The defenders are hoping to defeat the attack and return to their island, so they try to blow up the smallest possible number of dams. They figured out the following procedure: They blow up one dam at a time, and then only in the case if one side of the dam is already flooded, and the other side is dry. After the destruction of this dam the ocean fills up the previously dry basin with seawater. Notice that all the other dams (edges) around this basin are intact at this stage (because whenever a dam is blown up, the basins on both sides of it are flooded), so the seawater fills up only this particular basin. In Figure 12.2 we have indicated by numbers one possible order in which the dams can be blown up to flood the whole island.

Let us count the number of destroyed and intact dams. We denote the number of watchtowers (nodes) by v, the number of dams (edges) by e, and the number of basins, including the ocean, by f (we'll give an explanation later why are we using these letters). To flood all the $f - 1$ basins of the island, we had to destroy exactly $f - 1$ dams.

To count the surviving dams, let us look at the graph remaining after the explosions (Figure 12.3). First, one can notice that it contains no cycles, because the interior of any cycle would have remained dry. A second observation is that the remaining system of dams forms a connected graph, since every dam that was blown up was an edge of a cycle (the boundary

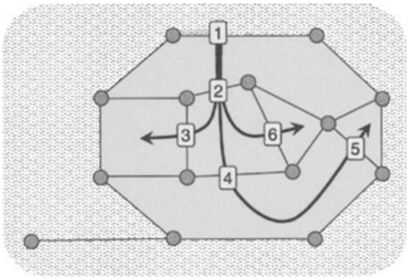

FIGURE 12.2. Flooding the island. To flood 6 basins, 6 dams must be blown up.

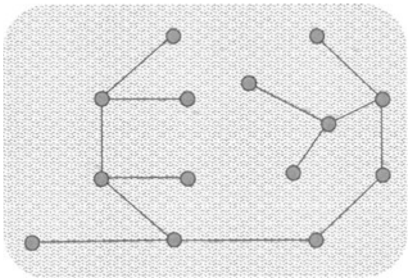

FIGURE 12.3. The island flooded. 13 dams remain intact, forming a tree.

of the basin that was flooded by this last explosion), and we know from exercise 7.2.5(b) that omitting such an edge would not destroy the connectivity of our graph. So the resulting graph after the explosions is connected and does not contain any cycle; therefore, it is a tree.

Now we apply the important fact that if a tree has v nodes, than it has $v-1$ edges.

Summarizing what we have learned, we know that $f-1$ dams were blown up and $v-1$ dams survived. So the number of edges is the sum of these two numbers. Expressing this fact as an equation yields $(v-1)+(f-1)=e$. Rearranging, we get

$$f+v=e+2,$$

and this is just Euler's Formula. □

12.1.1 Into how many parts do the diagonals divide a convex n-gon (we assume that no 3 diagonals go through the same point)?

12.1.2 On a circular island we build n straight dams going from sea to sea, so that every two intersect but no three go through the same point. Use Euler's Formula to determine how many parts we get. As a hint look at Figure 12.4 (the solution of this exercise will be the third method for solving the same problem).

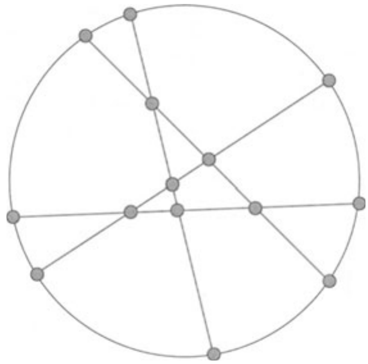

FIGURE 12.4. The planar map given by n straight lines.

12.2 Planar Graphs

Which graphs can be drawn as planar maps? This question is important not only because we want to know to which graphs we can apply Euler's Formula, but also in many applications of graph theory, for example, placing a network on a printed circuit board.

A graph is called *planar* if it can be drawn as a map in the plane, that is, if we can represent its nodes by different points in the plane, and its edges by curves connecting the appropriate points in such a way that these curves don't intersect each other (except, of course, when two edges have a common endpoint, in which case the two corresponding curves will have this one common point).

Are there graphs that are not planar? As a nice application of Euler's Formula, we can prove the following:

Theorem 12.2.1 *The complete graph K_5 on five nodes is not a planar graph.*

One could prove this by distinguishing a large number of cases and using various intuitive but potentially misleading properties of curves in the plane. But we are able to give an elegant proof now, using Euler's Formula.

Proof. Our proof is indirect: Assuming that K_5 can be drawn in the plane without any edges crossing, we get a contradiction. (It is not surprising that we are not able to provide a figure, since the impossibility of such a figure is what we want to prove.) Let us compute the number of countries that we would have in such a drawing. We have 5 nodes and $\binom{5}{2} = 10$ edges; hence the number of countries is, by Euler's Formula, $10+2-5=7$. Every country has at least 3 edges on its boundary, so we must have at least $\frac{3 \cdot 7}{2} = 10.5$ edges. (We had to divide by 2, because we counted every edge for two different countries.) The assumption that K_5 is planar led us to a contradiction, namely, $10 > 10.5$, so our assumption must have been false, and the complete graph on 5 nodes (K_5) is not planar. □

One of the most interesting phenomena in mathematics occurs when in the proof of some result one can make use of theorems that at a first glance do not have any connection with the actual problem. Would any of you guess that one can use the nonplanarity of the complete graph on five points to give another proof of the starting exercise of Section 11.3? Given our five points in the plane, connect any two of them by a segment. The resulting graph is a complete graph on five vertices, which is not a planar graph, as we already know; therefore, we can find two segments that intersect each other. The four endpoints of these two segments form a quadrilateral whose diagonals intersect; therefore, this quadrilateral is convex.

As another application of Euler's Formula, let's answer the following question: How many edges can a planar map with n nodes have?

Theorem 12.2.2 *A planar graph on n nodes has at most $3n - 6$ edges.*

Proof. The derivation of this bound is quite similar to our argument above showing that K_5 is not a planar graph. Let the graph have n nodes, e edges, and f faces. We know by Euler's Formula that

$$n + f = e + 2.$$

We obtain another relation among these numbers if we count edges face by face. Each face has at least 3 edges on its boundary, so we count at least $3f$ edges. Every edge is counted twice (it is on the border of two faces), so the number of edges is at least $3f/2$. In other words, $f \leq \frac{2}{3}e$. Using this with Euler's Formula, we get

$$e + 2 = n + f \leq n + \frac{2}{3}e,$$

which after rearrangement gives $e \leq 3n - 6$. □

12.2.1 Is the graph obtained by omitting an edge of K_5 planar?

12.2.2 There are three houses and three wells. Can we build a path from every house to every well so that these paths do not cross? (The paths are not necessarily straight lines.)

12.3 Euler's Formula for Polyhedra

There is still an apparently irrelevant question to deal with. Why did we denote the number of countries by f? Well, this is the starting letter of the word *face*. When Euler was trying to find "his" formula, he was studying polyhedra (solids bounded by plane polygons) like the cube, pyramids, and prisms. Let us count for some polyhedra the number of faces, edges, and vertices (Table 12.1).

Polyhedron	# of vertices	# of edges	# of faces
cube	8	12	6
tetrahedron	4	6	4
triangular prism	6	9	5
pentagonal prism	10	15	7
pentagonal pyramid	6	10	6
dodecahedron	20	30	12
icosahedron	12	30	20

TABLE 12.1.

(You don't know what the dodecahedron and icosahedron are? These are two very pretty regular polyhedra; their faces are regular pentagons and triangles, respectively. They can be seen in Figure 12.5.)

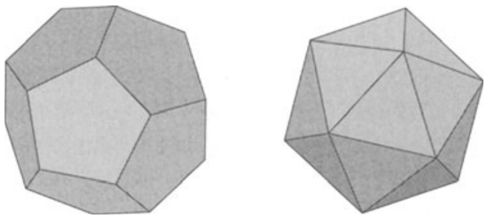

FIGURE 12.5. Two regular polyhedra: the dodecahedron and the icosahedron.

Staring at these numbers for a little while, one discovers that in every case the following relation holds:

$$\textit{number of faces} + \textit{number of vertices} = \textit{number of edges} + 2.$$

This formula strongly resembles Euler's Formula; the only difference is that instead of nodes, we speak of vertices, and instead of countries, here we speak of faces. This similarity is not a coincidence; we may get the formula for polyhedra from the formula for planar maps very easily as follows. Imagine that our polyhedron is made out of rubber. Punch a hole into one of the faces and blow it up like balloon. The most familiar solids will be blown up to spheres (for instance the cube and prism). But we have to be careful here: There are solids that cannot be blown up to a sphere. For instance, the "picture frame" shown on Figure 12.6 blows up to a "torus," similar to a life buoy (or doughnut). Be careful, the above relation holds only for solids that can be blown up to spheres! (Just to reassure you, all the convex solids can be blown up to spheres.) Now grab the rubber sphere at the side of the hole and stretch it until you get a huge rubber plane. If we paint the edges of the original solid with black ink, then we will see a map on the plane. The nodes of this map are the vertices of the solid, the edges are the edges of the solid, and the countries are the faces of the body. Therefore, if we use Euler's Formula for maps, we get Euler's Formula for polyhedra (Euler himself stated the theorem in the polyhedral form).

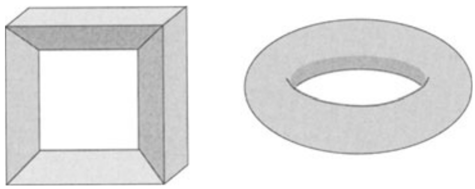

FIGURE 12.6. A nonconvex polyhedron that blows up to a torus.

Review Exercises

12.3.1 Is the complement of the cycle of length 6 (C_6) a planar graph?

12.3.2 Take a hexagon and add the three longest diagonals. Is the graph obtained this way planar?

12.3.3 Does the "picture frame" polyhedron in Figure 12.6 satisfy Euler's Formula?

12.3.4 Prove that a planar bipartite graph on n nodes has at most $2n-4$ edges.

12.3.5 Using Euler's Formula, show that the Petersen graph is not planar.

12.3.6 A convex polyhedron has only pentagonal and hexagonal faces. Prove that it has exactly 12 pentagonal faces.

12.3.7 Every face of a convex polyhedron has at least 5 vertices, and every vertex has degree 3. Prove that if the number of vertices is n, then the number of edges is at most $5(n-2)/3$.

12.3.8 Does the "picture frame" polyhedron in Figure 12.6 satisfy Euler's Formula?