4
Fibonacci Numbers

4.1 Fibonacci's Exercise

In the thirteenth century, the Italian mathematician Leonardo Fibonacci studied the following (not too realistic) question:

Leonardo Fibonacci

A farmer raises rabbits. Each rabbit gives birth to one rabbit when it turns 2 months old, and then to one rabbit each month thereafter. Rabbits never die, and we ignore male rabbits. How many rabbits will the farmer have in the nth month if he starts with one newborn rabbit?

It is easy to figure out the answer for small values of n. The farmer has 1 rabbit in the first month and 1 rabbit in the second month, since the rabbit has to be 2 months old before starting to reproduce. He has 2 rabbits during the third month, and 3 rabbits during the fourth, since his first rabbit delivered a new one after the second and one after the third. After 4 months, the second rabbit also delivers a new rabbit, so two new rabbits are added. This means that the farmer will have 5 rabbits during the fifth month.

It is easy to follow the multiplication of rabbits for any number of months if we notice that the number of new rabbits added each months is just the same as the number of rabbits who are at least 2 months old, i.e., who were already there in the previous month. In other words, to get the number of rabbits in the *next* month, we have to add the number of rabbits in the *previous* month to the number of rabbits in the *current* month. This makes it easy to compute the numbers one by one:

$$1,\ 1,\ 1+1=2,\ 2+1=3,\ 3+2=5,\ 5+3=8,\ 8+5=13,\ \dots$$

(It is quite likely that Fibonacci did not get his question as a real applied math problem; he played with numbers, noticed that this procedure gives numbers that were new to him but nevertheless had very interesting properties—as we'll see ourselves—and then tried to think of an "application.")

To write this as a formula, let us denote by F_n the number of rabbits during the nth month. Then we have, for $n = 2, 3, 4, \dots$,

$$F_{n+1} = F_n + F_{n-1}. \tag{4.1}$$

We also know that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$. It is convenient to define $F_0 = 0$; then equation (4.1) will remain valid for $n = 1$ as well. Using equation (4.1), we can easily determine any number of terms in this sequence of numbers:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ 1597, \dots .$$

The numbers in this sequence are called *Fibonacci numbers.*

We see that equation (4.1), together with the special values $F_0 = 0$ and $F_1 = 1$, uniquely determines the Fibonacci numbers. Thus we can consider (4.1), together with $F_0 = 0$ and $F_1 = 1$, as the definition of these numbers. This may seem a somewhat unusual definition: Instead of telling what F_n is (say, by a formula), we just give a rule that computes each Fibonacci number from the two previous numbers, and specify the first two values. Such a definition is called a *recurrence.* It is quite similar to induction in spirit (except that it is not a proof technique, but a definition method), and is sometimes also called *definition by induction.*

4.1.1 Why do we have to specify exactly two of the elements to begin with? Why not one or three?

Before trying to say more about these numbers, let us consider another counting problem:

> *A staircase has n steps. You walk up taking one or two at a time. How many ways can you go up?*

For $n = 1$, there is only 1 way. For $n = 2$, you have 2 choices: take one step twice or two once. For $n = 3$, you have 3 choices: three single steps, or one single followed by one double, or one double followed by one single.

Now stop and try to guess what the answer is in general! If you guessed that the number of ways to go up on a stair with n steps is n, you are wrong. The next case, $n = 4$, gives 5 possibilities $(1+1+1+1,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 2+2)$.

So instead of guessing, let's try the following strategy. Let's denote by J_n the answer. We try to figure out what J_{n+1} is, assuming we know the value of J_k for $1 \le k \le n$. If we start with a single step, we have J_n ways to go up the remaining n steps. If we start with a double step, we have J_{n-1} ways to go up the remaining $n-1$ steps. These are all the possibilities, and so

$$J_{n+1} = J_n + J_{n-1}.$$

This equation is the same as the equation we have used to compute the Fibonacci numbers F_n. Does this means that $F_n = J_n$? Of course not, as we see by looking at the beginning values: for example, $F_3 = 2$ but $J_3 = 3$. However, it is easy to observe that all that happens is that the J_n are shifted by one:

$$J_n = F_{n+1}.$$

This is valid for $n = 1, 2$, and then of course it is valid for every n, since the sequences $F_2, F_3, F_4, \ldots$ and $J_1, J_2, J_3, \ldots$ are computed by the same rule from their first two elements.

4.1.2 We have n dollars to spend. Every day we buy either a candy for 1 dollar or an ice cream for 2 dollars. In how many ways can we spend the money?

4.1.3 How many subsets does the set $\{1, 2, \ldots, n\}$ have that contain no two consecutive integers?

4.2 Lots of Identities

There are many interesting relations valid for the Fibonacci numbers. For example, what is the sum of the first n Fibonacci numbers? We have

$$\begin{aligned}0&=0,\\ 0+1&=1,\\ 0+1+1&=2,\\ 0+1+1+2&=4,\\ 0+1+1+2+3&=7,\\ 0+1+1+2+3+5&=12,\\ 0+1+1+2+3+5+8&=20,\\ 0+1+1+2+3+5+8+13&=33.\end{aligned}$$

Staring at these numbers for a while, it is not hard to recognize that by adding 1 to the right-hand sides we get Fibonacci numbers; in fact, we get Fibonacci numbers two steps after the last summand. As a formula, we have

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Of course, at this point this is only a *conjecture*, an unproven mathematical statement we believe to be true. To prove it, we use induction on n (since the Fibonacci numbers are defined by recurrence, induction is the natural and often only proof method at hand).

We have already checked the validity of the statement for $n = 0$ and 1. Suppose that we know that the identity holds for the sum of the first $n-1$ Fibonacci numbers. Consider the sum of the first n Fibonacci numbers:

$$F_0 + F_1 + \cdots + F_n = (F_0 + F_1 + \cdots + F_{n-1}) + F_n = (F_{n+1} - 1) + F_n,$$

by the induction hypothesis. But now we can use the recurrence equation for the Fibonacci numbers to get

$$(F_{n+1} - 1) + F_n = F_{n+2} - 1.$$

This completes the induction proof.

4.2.1 Prove that F_{3n} is even.

4.2.2 Prove that F_{5n} is divisible by 5.

4.2.3 Prove the following identities.

(a) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

(b) $F_0 - F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$.

(c) $F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}$.

(d) $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

4.2.4 We want to extend the Fibonacci numbers in the other direction; i.e., we want to define F_n for negative values of n. We want to do this so that the basic recurrence (4.1) remain valid. So from $F_{-1} + F_0 = F_1$ we get $F_{-1} = 1$; then from $F_{-2} + F_{-1} = F_0$ we get $F_{-2} = -1$, etc. How are these "Fibonacci numbers with negative indices" related to those with positive indices? Find several values, conjecture, and then prove the answer.

Now we state a little more difficult identity:

$$F_n^2 + F_{n-1}^2 = F_{2n-1}. \tag{4.2}$$

It is easy to check this for many values of n, and we can be convinced that it is true, but to prove it is a bit more difficult. Why is this more difficult than previous identities? Because if we want to prove it by induction (we don't really have other means at this point), then on the right-hand side we have only every other Fibonacci number, and so we don't know how to apply the recursion there.

One way to fix this is to find a similar formula for F_{2n}, and prove both of them by induction. With some luck (or deep intuition?) you can conjecture the following:

$$F_{n+1}F_n + F_nF_{n-1} = F_{2n}. \tag{4.3}$$

Again, it is easy to check that this holds for many small values of n. To prove (4.3), let us use the basic recurrence (4.1) twice:

$$\begin{aligned} F_{n+1}F_n + F_nF_{n-1} &= (F_n + F_{n-1})F_n + (F_{n-1} + F_{n-2})F_{n-1} \\ &= \left(F_n^2 + F_{n-1}^2\right) + (F_nF_{n-1} + F_{n-1}F_{n-2}) \end{aligned}$$

(apply (4.2) to the first term and induction to the second term)

$$= F_{2n-1} + F_{2n-2} = F_{2n}.$$

The proof of (4.2) is similar:

$$\begin{aligned} F_n^2 + F_{n-1}^2 &= (F_{n-1} + F_{n-2})^2 + F_{n-1}^2 \\ &= (F_{n-1}^2 + F_{n-2}^2) + 2F_{n-1}F_{n-2} + F_{n-1}^2 \\ &= \left(F_{n-1}^2 + F_{n-2}^2\right) + F_{n-1}(F_{n-2} + F_{n-1}) + F_{n-1}F_{n-2} \\ &= \left(F_{n-1}^2 + F_{n-2}^2\right) + F_nF_{n-1} + F_{n-1}F_{n-2} \end{aligned}$$

(apply induction to the first term and (4.3) to the second term)

$$= F_{2n-3} + F_{2n-2} = F_{2n-1}.$$

Wait a minute! What kind of trickery is this? We use (4.3) in the proof of (4.2), and then (4.2) in the proof of (4.3)? Relax, the argument is OK: It is just that the two induction proofs have to go simultaneously. If we know that both (4.3) and (4.2) are true for a certain value of n, then we prove (4.2) for the next value (if you look at the proof, you can see that it uses smaller values of n only), and then use this and the induction hypothesis again to prove (4.3).

This trick is called *simultaneous induction*, and it is a useful method to make induction more powerful.

4.2.5 Prove that the following recurrence can be used to compute Fibonacci numbers of odd index, without computing those with even index:

$$F_{2n+1} = 3F_{2n-1} - F_{2n-3}.$$

Use this identity to prove (4.2) without the trick of simultaneous induction. Give a similar proof of (4.3).

4.2.6 Mark the first entry of any row of Pascal's triangle (this is a 1). Move one step east and one step northeast, and mark the entry there. Repeat this until you get out of the triangle. Compute the sum of the entries you marked.

(a) What numbers do you get if you start from different rows? First conjecture, than prove your answer.

(b) Formulate this fact as an identity involving binomial coefficients.

Suppose that Fibonacci's farmer starts with A newborn rabbits. At the end of the first month (when there is no natural population increase yet), he buys $B-A$ newborn rabbits so that he has B rabbits. From here on, rabbits begin to multiply, and so he has $A+B$ rabbits after the second month, $A+2B$ rabbits after the third month, etc. How many rabbits will he have after the nth month? Mathematically, we define a sequence $E_0, E_1, E_2, \dots$ by $E_0 = A$, $E_1 = B$, and from then on, $E_{n+1} = E_n + E_{n-1}$ (the rabbits multiply by the same rule of biology; just the starting numbers are different).

For every two numbers A and B, we have this "modified Fibonacci sequence." How different are they from the real Fibonacci sequence? Do we have to study them separately for every choice of A and B?

It turns out that the numbers E_n can be expressed quite easily in terms of the Fibonacci numbers F_n. To see this, let us compute a few beginning values of the sequence E_n (of course, the result will contain the starting values A and B as parameters).

$$E_0 = A, \quad E_1 = B, \quad E_2 = A + B, \quad E_3 = B + (A + B) = A + 2B,$$
$$E_4 = (A + B) + (A + 2B) = 2A + 3B,$$
$$E_5 = (A + 2B) + (2A + 3B) = 3A + 5B,$$
$$E_6 = (2A + 3B) + (3A + 5B) = 5A + 8B,$$
$$E_7 = (3A + 5B) + (5A + 8B) = 8A + 13B, \dots .$$

It is easy to recognize what is going on: Each E_n is the sum of a multiple of A and a multiple of B, and the coefficients are ordinary Fibonacci numbers! For a formula, we can conjecture

$$E_n = F_{n-1}A + F_nB. \tag{4.4}$$

Of course, we have not proved this formula; but once we write it up, its proof is so easy (by induction on n, of course) that it is left to the reader as Exercise 4.3.10.

There is an important special case of this identity: We can start with two consecutive Fibonacci numbers $A = F_a$ and $B = F_{a+1}$. Then the sequence E_n is just the Fibonacci sequence, but shifted to the left. Hence we get the following identity:

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b. \tag{4.5}$$

This is a powerful identity for the Fibonacci numbers, which can be used to derive many others; some applications follow as exercises.

4.2.7 Give a proof of (4.2) and (4.3), based on (4.5).

4.2.8 Use (4.5) to prove the following generalization of Exercises 4.2.1 and 4.2.2: If n is a multiple of k, then F_n is a multiple of F_k.

4.2.9 Cut a chessboard into 4 pieces as shown in Figure 4.1 and assemble a 5×13 rectangle from them. Does this prove that $5 \cdot 13 = 8^2$? Where are we cheating? What does this have to do with Fibonacci numbers?

4.3 A Formula for the Fibonacci Numbers

How large are the Fibonacci numbers? Is there a simple formula that expresses F_n as a function of n?

An easy way out, at least for the author of a book, is to state the answer right away:

Theorem 4.3.1 *The Fibonacci numbers are given by the formula*

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

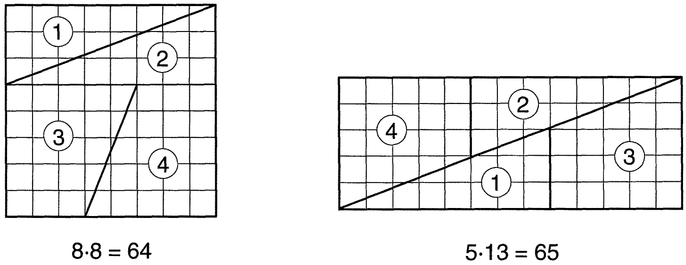

FIGURE 4.1. Proof of 64 = 65.

Proof. It is straightforward to check that this formula gives the right value for $n = 0, 1$, and then one can prove its validity for all n by induction. □

4.3.1 Prove Theorem 4.3.1 by induction on n.

Do you feel cheated by this proof? You should; while it is, of course, logically correct what we did, one would like to see more: How can one arrive at such a formula? What should we try to get a similar formula if we face a similar, but different, recurrence?

So let us forget Theorem 4.3.1 for a while and let us try to find a formula for F_n "from scratch."

One thing we can try is to experiment. The Fibonacci numbers grow quite fast; how fast? Let's grab our calculator and compute the ratio of consecutive Fibonacci numbers:

$$\frac{1}{1} = 1, \quad \frac{2}{1} = 2, \quad \frac{3}{2} = 1.5, \quad \frac{5}{3} = 1.666666667,$$
$$\frac{8}{5} = 1.600000000, \quad \frac{13}{8} = 1.625000000, \quad \frac{21}{13} = 1.615384615,$$
$$\frac{34}{21} = 1.619047619, \quad \frac{55}{34} = 1.617647059, \quad \frac{89}{55} = 1.618181818,$$
$$\frac{144}{89} = 1.617977528, \quad \frac{233}{144} = 1.618055556, \quad \frac{377}{233} = 1.618025751.$$

It seems that the ratio of consecutive Fibonacci numbers is very close to 1.618, at least if we ignore the first few values. This suggests that the Fibonacci numbers behave like a geometric progression (for a geometric progression, the ratio of any two consecutive elements would be exactly the same). So let's see whether there is any geometric progression that satisfies the same recurrence as the Fibonacci numbers. Let $G_n = c \cdot q^n$ be a geometric progression ($c, q \neq 0$). Then

$$G_{n+1} = G_n + G_{n-1}$$

translates into

$$c \cdot q^{n+1} = c \cdot q^n + c \cdot q^{n-1},$$

which after simplification becomes

$$q^2 = q + 1.$$

So both numbers c and n disappear.[1]

So we have a quadratic equation for q, which we can solve and get

$$q_1 = \frac{1+\sqrt{5}}{2} \approx 1.618034, \qquad q_2 = \frac{1-\sqrt{5}}{2} \approx -0.618034.$$

This gives us two kinds of geometric progressions that satisfy the same recurrence as the Fibonacci numbers:

$$G_n = c\left(\frac{1+\sqrt{5}}{2}\right)^n, \qquad G'_n = c\left(\frac{1-\sqrt{5}}{2}\right)^n$$

(where c is an arbitrary constant). Unfortunately, neither G_n nor G'_n gives the Fibonacci sequence: for one, $G_0 = G'_0 = c$, while $F_0 = 0$. But notice that the sequence $G_n - G'_n$ also satisfies the recurrence:

$$G_{n+1} - G'_{n+1} = (G_n + G_{n-1}) - (G'_n + G'_{n-1}) = (G_n - G'_n) + (G_{n-1} - G'_{n-1})$$

(using that G_n and G'_n satisfy the recurrence). So we have matched the first value F_0, since $G_0 - G'_0 = 0$. What about the next one? We have $G_1 - G'_1 = c\sqrt{5}$. We can match this with $F_1 = 1$ if we choose $c = 1/\sqrt{5}$.

Thus we have two sequences, F_n and $G_n - G'_n$, that both begin with the same two numbers and satisfy the same recurrence. So we can use the same rule to compute the numbers F_n as the numbers $G_n - G'_n$, and it follows that they must be the same: $F_n = G_n - G'_n$.

Now you can substitute for the values of G_n and G'_n and see that we got the formula in the theorem!

The formula we just derived gives new kind of information about the Fibonacci numbers. The first base in the exponential expression is $q_1 = (1+\sqrt{5})/2 \approx 1.618034 > 1$, while the second base q_2 is between -1 and 0. Hence if n increases, then G_n will become very large, while $|G'_n| < \frac{1}{2}$ once $n \geq 2$, and in fact, G'_n becomes very small. This means that

$$F_n \approx G_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n,$$

[1] This disappearance of c and n from the equation could be expected. The reason behind it is that if we find a sequence that satisfies Fibonacci's recurrence, then we can multiply its elements by any other real number and get another sequence that satisfies the recurrence. This means that we should not get any condition on c. Further, if we have a sequence that satisfies Fibonacci's recurrence, then starting the sequence anywhere later, it will also satisfy the recurrence. This suggests that we should not get any condition on n.

where the term we ignore is less than $\frac{1}{2}$ if $n \geq 2$ (and tends to 0 if n tends to infinity); this implies that F_n is the integer nearest to G_n.

The base $\tau = (1+\sqrt{5})/2$ is a famous number: It is called the *golden ratio*, and it comes up all over mathematics; for example, it is the ratio between the diagonal and side of a regular pentagon. Another way to characterize it is the following: If $b/a = \tau$, then $(a+b)/b = \tau$. So if the ratio between the longer and shorter sides of a rectangle is τ, then cutting off a square, we are left with a rectangle that is similar to the original.

4.3.2 Define a sequence of integers L_n by $L_1 = 1$, $L_2 = 3$, and $L_{n+1} = L_n + L_{n-1}$. (These numbers are called *Lucas numbers.*) Show that L_n can be expressed in the form $a \cdot q_1^n + b \cdot q_2^n$ (where q_1 and q_2 are the same numbers as in the proof above), and find the values of a and b.

4.3.3 Define a sequence of integers I_n by $I_0 = 0$, $I_1 = 1$, and $I_{n+1} = 4I_n + I_{n-1}$. (a) Find a combinatorial counting problem to which the answer is I_n. (b) Find a formula for I_n.

4.3.4 Alice claims that she knows another formula for the Fibonacci numbers: $F_n = \lceil e^{n/2-1} \rceil$ for $n = 1, 2, \ldots$ (where $e = 2.718281828\ldots$ is, naturally, the base of the natural logarithm). Is she right?

Review Exercises

4.3.5 In how many ways can you cover a $2 \times n$ chessboard by dominoes?

4.3.6 How many subsets does the set $\{1, 2, \ldots, n\}$ have that contain no two consecutive integers if 0 and n also count as consecutive?

4.3.7 How many subsets does the set $\{1, 2, \ldots, n\}$ have that contain no three consecutive integers? Find a recurrence.

4.3.8 Which number is larger, 2^{100} or F_{100}?

4.3.9 Prove the following identities:

(a) $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$;

(b) $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$;

(c) $\binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \cdots + \binom{n}{n}F_n = F_{2n}$;

(d) $\binom{n}{0}F_1 + \binom{n}{1}F_2 + \binom{n}{2}F_3 + \cdots + \binom{n}{n}F_{n+1} = F_{2n+1}$.

4.3.10 Prove (4.4).

4.3.11 Is it true that if F_n is a prime, then n is a prime?

4.3.12 Consider a sequence of numbers $b_0, b_1, b_2, \ldots$ such that $b_0 = 0$, $b_1 = 1$, and $b_2, b_3, \ldots$ are defined by the recurrence

$$b_{k+1} = 3b_k - 2b_{k-1}.$$

Find the value of b_k.

4.3.13 Assume that the sequence $(a_0, a_1, a_2, \ldots)$ satisfies the recurrence

$$a_{n+1} = a_n + 2a_{n-1}.$$

We know that $a_0 = 4$ and $a_2 = 13$. What is a_5?

4.3.14 Recalling the Lucas numbers L_n introduced in Exercise 4.3.2, prove the following identities:

(a) $F_{2n} = F_n L_n$;

(b) $2F_{k+n} = F_k L_n + F_n L_k$;

(c) $2L_{k+n} = 5F_k F_n + L_k L_n$;

(d) $L_{4k} = L_{2k}^2 - 2$;

(e) $L_{4k+2} = L_{2k+1}^2 + 2$.

4.3.15 Prove that if n is a multiple of 4, then F_n is a multiple of 3.

4.3.16 (a) Prove that every positive integer can be written as the sum of different Fibonacci numbers.

(b) Prove even more: every positive integer can be written as the sum of different Fibonacci numbers, so that no two consecutive Fibonacci numbers are used.

(c) Show by an example that the representation in (a) is not unique, but also prove that the more restrictive representation in (b) is.