

Chapter 2

Basic Properties of Linear Programs

2.1 Introduction

A linear program (LP) is an optimization problem in which the objective function is linear in the unknowns and the constraints consist of linear equalities and linear inequalities. The exact form of these constraints may differ from one problem to another, but as shown below, any linear program can be transformed into the following *standard form*:

$$\begin{aligned}
 &\text{minimize} && c_1x_1 + c_2x_2 + \dots + c_nx_n \\
 &\text{subject to} && a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\
 &&& a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\
 &&& \vdots \\
 &&& a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\
 &\text{and} && x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0,
 \end{aligned} \tag{2.1}$$

where the b_i 's, c_i 's and a_{ij} 's are fixed real constants, and the x_i 's are real numbers to be determined. We always assume that each equation has been multiplied by minus unity, if necessary, so that each $b_i \geq 0$.

In more compact vector notation,¹ this standard problem becomes

$$\begin{aligned}
 &\text{minimize} && \mathbf{c}^T \mathbf{x} \\
 &\text{subject to} && \mathbf{A} \mathbf{x} = \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0}.
 \end{aligned} \tag{2.2}$$

Here \mathbf{x} is an n -dimensional column vector, \mathbf{c}^T is an n -dimensional row vector, \mathbf{A} is an $m \times n$ matrix, and \mathbf{b} is an m -dimensional column vector. The vector inequality $\mathbf{x} \geq \mathbf{0}$ means that each component of \mathbf{x} is nonnegative.

¹ See Appendix A for a description of the vector notation used throughout this book.

Before giving some examples of areas in which linear programming problems arise naturally, we indicate how various other forms of linear programs can be converted to the standard form.

Example 1 (Slack Variables). Consider the problem

$$\begin{aligned} & \text{minimize} && c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ & \text{subject to} && a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\ & && a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\ & && \vdots \\ & && a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\ & \text{and} && x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, \end{aligned}$$

In this case the constraint set is determined entirely by linear inequalities. The problem may be alternatively expressed as

$$\begin{aligned} & \text{minimize} && c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ & \text{subject to} && a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 = b_1 \\ & && a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + y_2 = b_2 \\ & && \vdots \\ & && a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + y_m = b_m \\ & \text{and} && x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0, \\ & \text{and} && y_1 \geq 0, y_2 \geq 0, \dots, y_m \geq 0. \end{aligned}$$

The new positive variables y_i introduced to convert the inequalities to equalities are called *slack variables* (or more loosely, *slacks*). By considering the problem as one having $n + m$ unknowns $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m$, the problem takes the standard form. The $m \times (n + m)$ matrix that now describes the linear equality constraints is of the special form $[\mathbf{A}, \mathbf{I}]$ (that is, its columns can be partitioned into two sets; the first n columns make up the original \mathbf{A} matrix and the last m columns make up an $m \times m$ identity matrix).

Example 2 (Surplus Variables). If the linear inequalities of Example 1 are reversed so that a typical inequality is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i,$$

it is clear that this is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - y_i = b_i$$

with $y_i \geq 0$. Variables, such as y_i , adjoined in this fashion to convert a “greater than or equal to” inequality to equality are called *surplus variables*.

It should be clear that by suitably multiplying by minus unity, and adjoining slack and surplus variables, any set of linear inequalities can be converted to standard form if the unknown variables are restricted to be nonnegative.

Example 3 (Free Variables—First Method). If a linear program is given in standard form except that one or more of the unknown variables is not required to be nonnegative, the problem can be transformed to standard form by either of two simple techniques.

To describe the first technique, suppose in (2.1), for example, that the restriction $x_1 \geq 0$ is not present and hence x_1 is free to take on either positive or negative values. We then write

$$x_1 = u_1 - v_1, \quad (2.3)$$

where we require $u_1 \geq 0$ and $v_1 \geq 0$. If we substitute $u_1 - v_1$ for x_1 everywhere in (2.1), the linearity of the constraints is preserved and all variables are now required to be nonnegative. The problem is then expressed in terms of the $n + 1$ variables $u_1, v_1, x_2, x_3, \dots, x_n$.

There is obviously a certain degree of redundancy introduced by this technique, however, since a constant added to u_1 and v_1 does not change x_1 (that is, the representation of a given value x_1 is not unique). Nevertheless, this does not hinder the simplex method of solution.

Example 4 (Free Variables—Second Method). A second approach for converting to standard form when x_1 is unconstrained in sign is to eliminate x_1 together with one of the constraint equations. Take any one of the m equations in (2.1) which has a nonzero coefficient for x_1 . Say, for example,

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i, \quad (2.4)$$

where $a_{i1} \neq 0$. Then x_1 can be expressed as a linear combination of the other variables plus a constant. If this expression is substituted for x_1 everywhere in (2.1), we are led to a new problem of exactly the same form but expressed in terms of the variables x_2, x_3, \dots, x_n only. Furthermore, the i th equation, used to determine x_1 , is now identically zero and it too can be eliminated. This substitution scheme is valid since any combination of nonnegative variables x_2, x_3, \dots, x_n leads to a feasible x_1 from (2.4), since the sign of x_1 is unrestricted. As a result of this simplification, we obtain a standard linear program having $n - 1$ variables and $m - 1$ constraint equations. The value of the variable x_1 can be determined after solution through (2.4).

Example 5 (Specific Case). As a specific instance of the above technique consider the problem

$$\begin{aligned} &\text{minimize} && x_1 + 3x_2 + 4x_3 \\ &\text{subject to} && x_1 + 2x_2 + x_3 = 5 \\ &&& 2x_1 + 3x_2 + x_3 = 6 \\ &&& x_2 \geq 0, \quad x_3 \geq 0. \end{aligned}$$

Since x_1 is free, we solve for it from the first constraint, obtaining

$$x_1 = 5 - 2x_2 - x_3. \quad (2.5)$$

Substituting this into the objective and the second constraint, we obtain the equivalent problem (subtracting five from the objective)

$$\begin{aligned} &\text{minimize} && x_2 + 3x_3 \\ &\text{subject to} && x_2 + x_3 = 4 \\ &&& x_2 \geq 0, x_3 \geq 0, \end{aligned}$$

which is a problem in standard form. After the smaller problem is solved (the answer is $x_2 = 4$, $x_3 = 0$) the value for x_1 ($x_1 = -3$) can be found from (2.5).

2.2 Examples of Linear Programming Problems

Linear programming has long proved its merit as a significant model of numerous allocation problems and economic phenomena. The continuously expanding literature of applications repeatedly demonstrates the importance of linear programming as a general framework for problem formulation. In this section we present some classic examples of situations that have natural formulations.

Example 1 (The Diet Problem). How can we determine the most economical diet that satisfies the basic minimum nutritional requirements for good health? Such a problem might, for example, be faced by the dietitian of a large army. We assume that there are available at the market n different foods and that the j th food sells at a price c_j per unit. In addition there are m basic nutritional ingredients and, to achieve a balanced diet, each individual must receive at least b_i units of the i th nutrient per day. Finally, we assume that each unit of food j contains a_{ij} units of the i th nutrient.

If we denote by x_j the number of units of food j in the diet, the problem then is to select the x_j 's to minimize the total cost

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the nutritional constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i, \quad i = 1, \dots, m,$$

and the nonnegativity constraints

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0$$

on the food quantities.

This problem can be converted to standard form by subtracting a nonnegative surplus variable from the left side of each of the m linear inequalities. The diet problem is discussed further in Chap. 4.

Example 2 (Manufacturing Problem). Suppose we own a facility that is capable of manufacturing n different products, each of which may require various amounts

of m different resources. Each product can be produced at any level $x_j \geq 0$, $j = 1, 2, \dots, n$, and each unit of the j th product can sell for p_j dollars and needs a_{ij} units of the i th resource, $i = 1, 2, \dots, m$. Assuming linearity of the production facility, if we are given a set of m numbers b_1, b_2, \dots, b_m describing the available quantities of the m resources, and we wish to manufacture products at maximum revenue, our decision problem is a linear program to maximize

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

subject to the resource constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i, \quad i = 1, \dots, m$$

and the nonnegativity constraints on all production variables.

Example 3 (The Transportation Problem). Quantities a_1, a_2, \dots, a_m , respectively, of a certain product are to be shipped from each of m locations and received in amounts b_1, b_2, \dots, b_n , respectively, at each of n destinations. Associated with the shipping of a unit of product from origin i to destination j is a shipping cost c_{ij} . It is desired to determine the amounts x_{ij} to be shipped between each origin–destination pair $i = 1, 2, \dots, m; j = 1, 2, \dots, n$; so as to satisfy the shipping requirements and minimize the total cost of transportation.

To formulate this problem as a linear programming problem, we set up the array shown below:

$$\begin{array}{cccc|c} x_{11} & x_{12} & \cdots & x_{1n} & a_1 \\ x_{21} & x_{22} & \cdots & x_{2n} & a_2 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \hline x_{m1} & x_{m2} & \cdots & x_{mn} & a_m \\ b_1 & b_2 & \cdots & b_n & \end{array}$$

The i th row in this array defines the variables associated with the i th origin, while the j th column in this array defines the variables associated with the j th destination. The problem is to place nonnegative variables x_{ij} in this array so that the sum across the i th row is a_j , the sum down the j th column is b_j , and the weighted sum $\sum_{j=1}^n \sum_{i=1}^m c_{ij}x_{ij}$, representing the transportation cost, is minimized.

Thus, we have the linear programming problem:

$$\begin{array}{ll} \text{minimize} & \sum_{ij} c_{ij}x_{ij} \\ \text{subject to} & \sum_{j=1}^n x_{ij} = a_j \quad \text{for } i = 1, 2, \dots, m \end{array} \quad (2.6)$$

$$\sum_{i=1}^m x_{ij} = b_j \quad \text{for } j = 1, 2, \dots, n \quad (2.7)$$

$$x_{ij} \geq 0 \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n.$$

In order that the constraints (2.6) and (2.7) be consistent, we must, of course, assume that $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ which corresponds to assuming that the total amount shipped is equal to the total amount received.

The transportation problem is now clearly seen to be a linear programming problem in mn variables. The equations (2.6) and (2.7) can be combined and expressed in matrix form in the usual manner and this results in an $(m+n) \times (mn)$ coefficient matrix consisting of zeros and ones only.

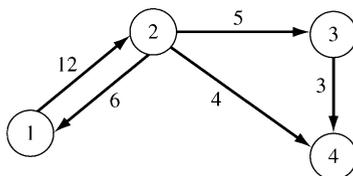


Fig. 2.1 A network with capacities

Example 4 (The Maximal Flow Problem). Consider a capacitated network (see Fig. 2.1, and Appendix D) in which two special nodes, called the *source* and the *sink*, are distinguished. Say they are nodes 1 and m , respectively. All other nodes must satisfy the strict conservation requirement; that is, the net flow into these nodes must be zero. However, the source may have a net outflow and the sink a net inflow. The outflow f of the source will equal the inflow of the sink as a consequence of the conservation at all other nodes. A set of arc flows satisfying these conditions is said to be a *flow* in the network of value f . The maximal flow problem is that of determining the maximal flow that can be established in such a network. When written out, it takes the form

$$\begin{aligned}
 & \text{minimize } f \\
 & \text{subject to } \sum_{j=1}^n x_{1j} - \sum_{j=1}^n x_{j1} - f = 0 \\
 & \quad \sum_{j=1}^n x_{ij} - \sum_{j=1}^n x_{ji} = 0, \quad i \neq 1, m \quad (2.8) \\
 & \quad \sum_{j=1}^n x_{mj} - \sum_{j=1}^n x_{jm} + f = 0 \\
 & \quad 0 \leq x_{ij} \leq k_{ij}, \quad \text{for all } i, j,
 \end{aligned}$$

where $k_{ij} = 0$ for those no-arc pairs (i, j) .

Example 5 (A Warehousing Problem). Consider the problem of operating a warehouse, by buying and selling the stock of a certain commodity, in order to maximize profit over a certain length of time. The warehouse has a fixed capacity C , and there is a cost r per unit for holding stock for one period. The price, p_i , of the commodity is known to fluctuate over a number of time periods—say months, indexed by i . In any period the same price holds for both purchase or sale. The warehouse is originally empty and is required to be empty at the end of the last period.

To formulate this problem, variables are introduced for each time period. In particular, let x_i denote the level of stock in the warehouse at the beginning of period i . Let u_i denote the amount bought during period i , and let s_i denote the amount sold during period i . If there are n periods, the problem is

$$\begin{aligned}
 &\text{maximize } \sum_{i=1}^n (p_i(s_i - u_i) - rx_i) \\
 &\text{subject to } \begin{aligned}
 x_{i+1} &= x_i + u_i - s_i & i = 1, 2, \dots, n - 1 \\
 0 &= x_n + u_n - s_n \\
 x_i + z_i &= C & i = 2, \dots, n \\
 x_1 = 0, x_i \geq 0, u_i \geq 0, & & s_i \geq 0, z_i \geq 0,
 \end{aligned}
 \end{aligned}$$

where z_i is a slack variable. If the constraints are written out explicitly for the case $n = 3$, they take the form

$-u_1 + s_1$	$+x_2$		$=0$
	$-x_2 - u_2 + s_2$	$+x_3$	$=0$
	x_2	$+z_2$	$=C$
		$-x_3 - u_3 + s_3$	$=0$
		x_3	$+z_3 = C$

Note that the coefficient matrix can be partitioned into blocks corresponding to the variables of the different time periods. The only blocks that have nonzero entries are the diagonal ones and the ones immediately above the diagonal. This structure is typical of problems involving time.

Example 6 (Linear Classifier and Support Vector Machine). Suppose several d -dimensional data points are classified into two distinct classes. For example, two-dimensional data points may be grade averages in science and humanities for different students. We also know the academic major of each student, as being in science or humanities, which serves as the classification. In general we have vectors $\mathbf{a}_i \in E^d$ for $i = 1, 2, \dots, n_1$ and vectors $\mathbf{b}_j \in E^d$ for $j = 1, 2, \dots, n_2$. We wish to find a hyperplane that separates the \mathbf{a}_i 's from the \mathbf{b}_j 's. Mathematically we wish to find $\mathbf{y} \in E^d$ and a number β such that

$$\begin{aligned}
 \mathbf{a}_i^T \mathbf{y} + \beta &\geq 1 \quad \text{for all } i \\
 \mathbf{b}_j^T \mathbf{y} + \beta &\leq -1 \quad \text{for all } j,
 \end{aligned}$$

where $\{\mathbf{x} : \mathbf{x}^T \mathbf{y} + \beta = 0\}$ is the desired hyperplane, and the separation is defined by the $+1$ and -1 . This is a linear program. See Fig. 2.2.

Example 7 (Combinatorial Auction). Suppose there are m mutually exclusive potential states and only one of them will be true at maturity. For example, the states may correspond to the winning horse in a race of m horses, or the value of a stock index, falling within m intervals. An auction organizer who establishes a *parimutuel* auction is prepared to issue contracts specifying subsets of the m possibilities that pay \$1 if the final state is one of those designated by the contract, and zero otherwise. There are n participants who may place orders with the organizer for the purchase of such contracts. An order by the j th participant consists of an m -vector $\mathbf{a}_j = (a_{1j}, a_{2j}, \dots, a_{mj})^T$ where each component is either 0 or 1, a one indicating a desire to be paid if the corresponding state occurs.

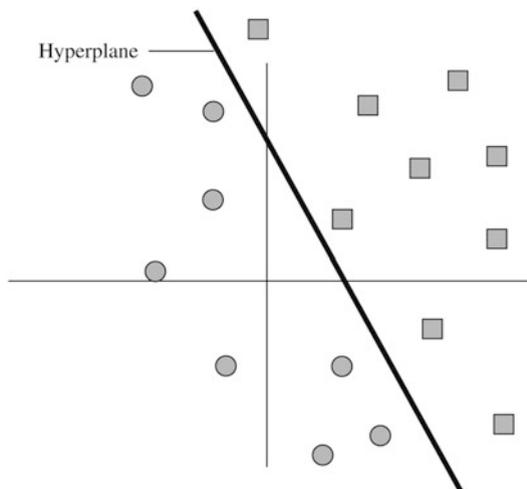


Fig. 2.2 Support vector for data classification

Accompanying the order is a number π_j which is the price limit the participant is willing to pay for one unit of the order. Finally, the participant also declares the maximum number q_j of units he or she is willing to accept under these terms.

The auction organizer, after receiving these various orders, must decide how many contracts to fill. Let x_j be the (real) number of units awarded to the j th order. Then the j th participant will pay $\pi_j x_j$. The total amount paid by all participants is $\boldsymbol{\pi}^T \mathbf{x}$, where \mathbf{x} is the vector of x_j 's and $\boldsymbol{\pi}$ is the vector of prices.

If the outcome is the i th state, the auction organizer must pay out a total of $\sum_{j=1}^n a_{ij} x_j = (\mathbf{A}\mathbf{x})_i$. The organizer would like to maximize profit in the worst possible case, and does this by solving the problem

$$\begin{aligned} &\text{maximize } \boldsymbol{\pi}^T \mathbf{x} - \max_i (\mathbf{A}\mathbf{x})_i \\ &\text{subject to } \mathbf{0} \leq \mathbf{x} \leq \mathbf{q}. \end{aligned}$$

This problem can be expressed alternatively as selecting \mathbf{x} and scalar s to

$$\begin{aligned} &\text{maximize } \boldsymbol{\pi}^T \mathbf{x} - s \\ &\text{subject to } \mathbf{A}\mathbf{x} - \mathbf{1}s \leq \mathbf{0} \\ &\quad \mathbf{0} \leq \mathbf{x} \leq \mathbf{q} \end{aligned}$$

where $\mathbf{1}$ is the vector of all 1's. Notice that the profit will always be nonnegative, since $\mathbf{x} = \mathbf{0}$ is feasible.

2.3 Basic Solutions

Consider the system of equalities

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{2.9}$$

where \mathbf{x} is an n -vector, \mathbf{b} an m -vector, and \mathbf{A} is an $m \times n$ matrix. Suppose that from the n columns of \mathbf{A} we select a set of m linearly independent columns (such a set exists if the rank of \mathbf{A} is m). For notational simplicity assume that we select the first m columns of \mathbf{A} and denote the $m \times m$ matrix determined by these columns by \mathbf{B} . The matrix \mathbf{B} is then nonsingular and we may uniquely solve the equation.

$$\mathbf{B}\mathbf{x}_B = \mathbf{b} \tag{2.10}$$

for the m -vector \mathbf{x}_B . By putting $\mathbf{x} = (\mathbf{x}_B, \mathbf{0})$ (that is, setting the first m components of \mathbf{x} equal to those of \mathbf{x}_B and the remaining components equal to zero), we obtain a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$. This leads to the following definition.

Definition. Given the set of m simultaneous linear equations in n unknowns (2.9), let \mathbf{B} be any nonsingular $m \times m$ submatrix made up of columns of \mathbf{A} . Then, if all $n - m$ components of \mathbf{x} not associated with columns of \mathbf{B} are set equal to zero, the solution to the resulting set of equations is said to be a *basic solution* to (2.9) with respect to the basis \mathbf{B} . The components of \mathbf{x} associated with columns of \mathbf{B} are called *basic variables*.

In the above definition we refer to \mathbf{B} as a basis, since \mathbf{B} consists of m linearly independent columns that can be regarded as a basis for the space E^m . The basic solution corresponds to an expression for the vector \mathbf{b} as a linear combination of these basis vectors. This interpretation is discussed further in the next section.

In general, of course, Eq. (2.9) may have no basic solutions. However, we may avoid trivialities and difficulties of a nonessential nature by making certain elementary assumptions regarding the structure of the matrix \mathbf{A} . First, we usually assume that $n > m$, that is, the number of variables x_j exceeds the number of equality constraints. Second, we usually assume that the rows of \mathbf{A} are linearly independent, corresponding to linear independence of the m equations. A linear dependency among the rows of \mathbf{A} would lead either to contradictory constraints and hence no solutions to (2.9), or to a redundancy that could be eliminated. Formally, we explicitly make the following assumption in our development, unless noted otherwise.

Full Rank Assumption. The $m \times n$ matrix \mathbf{A} has $m < n$, and the m rows of \mathbf{A} are linearly independent.

Under the above assumption, the system (2.9) will always have a solution and, in fact, it will always have at least one basic solution.

The basic variables in a basic solution are not necessarily all nonzero. This is noted by the following definition.

Definition. If one or more of the basic variables in a basic solution has value zero, that solution is said to be a *degenerate basic solution*.

We note that in a nondegenerate basic solution the basic variables, and hence the basis \mathbf{B} , can be immediately identified from the positive components of the solution. There is ambiguity associated with a degenerate basic solution, however, since the zero-valued basic and some of nonbasic variables can be interchanged.

So far in the discussion of basic solutions we have treated only the equality constraint (2.9) and have made no reference to positivity constraints on the variables. Similar definitions apply when these constraints are also considered. Thus, consider now the system of constraints

$$\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \quad (2.11)$$

which represent the constraints of a linear program in standard form.

Definition. A vector \mathbf{x} satisfying (2.11) is said to be *feasible* for these constraints. A feasible solution to the constraints (2.11) that is also basic is said to be a *basic feasible solution*; if this solution is also a degenerate basic solution, it is called a *degenerate basic feasible solution*.

2.4 The Fundamental Theorem of Linear Programming

In this section, through the fundamental theorem of linear programming, we establish the primary importance of basic feasible solutions in solving linear programs. The method of proof of the theorem is in many respects as important as the result itself, since it represents the beginning of the development of the simplex method. The theorem (due to Carathéodory) itself shows that it is necessary only to consider basic feasible solutions when seeking an optimal solution to a linear program because the optimal value is always achieved at such a solution.

Corresponding to a linear program in standard form

$$\begin{aligned} & \text{minimize } \mathbf{c}^T \mathbf{x} \\ & \text{subject to } \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0} \end{aligned} \quad (2.12)$$

a feasible solution to the constraints that achieves the minimum value of the objective function subject to those constraints is said to be an *optimal feasible solution*. If this solution is basic, it is an *optimal basic feasible solution*.

Fundamental Theorem of Linear Programming. Given a linear program in standard form (2.12) where \mathbf{A} is an $m \times n$ matrix of rank m ,

- i) if there is a feasible solution, there is a basic feasible solution;
- ii) if there is an optimal feasible solution, there is an optimal basic feasible solution.

Proof of (i). Denote the columns of \mathbf{A} by $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. Suppose $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is a feasible solution. Then, in terms of the columns of \mathbf{A} , this solution satisfies:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}.$$

Assume that exactly p of the variables x_i are greater than zero, and for convenience, that they are the first p variables. Thus

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_p\mathbf{a}_p = \mathbf{b}. \quad (2.13)$$

There are now two cases, corresponding as to whether the set $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ is linearly independent or linearly dependent.

CASE 1: Assume $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ are linearly independent. Then clearly, $p \leq m$. If $p = m$, the solution is basic and the proof is complete. If $p < m$, then, since \mathbf{A} has rank m , $m - p$ vectors can be found from the remaining $n - p$ vectors so that the resulting set of m vectors is linearly independent. (See Exercise 12.) Assigning the value zero to the corresponding $m - p$ variables yields a (degenerate) basic feasible solution.

CASE 2: Assume $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ are linearly dependent. Then there is a non-trivial linear combination of these vectors that is zero. Thus there are constants y_1, y_2, \dots, y_p , at least one of which can be assumed to be positive, such that

$$y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \dots + y_p\mathbf{a}_p = \mathbf{0}. \quad (2.14)$$

Multiplying this equation by a scalar ε and subtracting it from (2.13), we obtain

$$(x_1 - \varepsilon y_1)\mathbf{a}_1 + (x_2 - \varepsilon y_2)\mathbf{a}_2 + \dots + (x_p - \varepsilon y_p)\mathbf{a}_p = \mathbf{b}. \quad (2.15)$$

This equation holds for every ε , and for each ε the components $x_j - \varepsilon y_j$ correspond to a solution of the linear equalities—although they may violate $x_i - \varepsilon y_i \geq 0$. Denoting $\mathbf{y} = (y_1, y_2, \dots, y_p, 0, 0, \dots, 0)$, we see that for any ε

$$\mathbf{x} - \varepsilon \mathbf{y} \quad (2.16)$$

is a solution to the equalities. For $\varepsilon = 0$, this reduces to the original feasible solution. As ε is increased from zero, the various components increase, decrease, or remain constant, depending upon whether the corresponding y_i is negative, positive, or zero. Since we assume at least one y_i is positive, at least one component will decrease as ε is increased. We increase ε to the first point where one or more components become zero. Specifically, we set

$$\varepsilon = \min\{x_i/y_i : y_i > 0\}.$$

For this value of ε the solution given by (2.16) is feasible and has at most $p - 1$ positive variables. Repeating this process if necessary, we can eliminate positive variables until we have a feasible solution with corresponding columns that are linearly independent. At that point Case 1 applies. ■

Proof of (ii). Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ be an optimal feasible solution and, as in the proof of (i) above, suppose there are exactly p positive variables x_1, x_2, \dots, x_p . Again there are two cases; and Case 1, corresponding to linear independence, is exactly the same as before.

Case 2 also goes exactly the same as before, but it must be shown that for any ε the solution (2.16) is optimal. To show this, note that the value of the solution $\mathbf{x} - \varepsilon \mathbf{y}$ is

$$\mathbf{c}^T \mathbf{x} - \varepsilon \mathbf{c}^T \mathbf{y}. \quad (2.17)$$

For ε sufficiently small in magnitude, $\mathbf{x} - \varepsilon \mathbf{y}$ is a feasible solution for positive or negative values of ε . Thus we conclude that $\mathbf{c}^T \mathbf{y} = 0$. For, if $\mathbf{c}^T \mathbf{y} \neq 0$, an ε of small magnitude and proper sign could be determined so as to render (2.17) smaller than $\mathbf{c}^T \mathbf{x}$ while maintaining feasibility. This would violate the assumption of optimality of \mathbf{x} and hence we must have $\mathbf{c}^T \mathbf{y} = 0$.

Having established that the new feasible solution with fewer positive components is also optimal, the remainder of the proof may be completed exactly as in part (i). ■

This theorem reduces the task of solving a linear program to that of searching over basic feasible solutions. Since for a problem having n variables and m constraints there are at most

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

basic solutions (corresponding to the number of ways of selecting m of n columns), there are only a finite number of possibilities. Thus the fundamental theorem yields an obvious, but terribly inefficient, finite search technique. By expanding upon the technique of proof as well as the statement of the fundamental theorem, the efficient simplex procedure is derived.

It should be noted that the proof of the fundamental theorem given above is of a simple algebraic character. In the next section the geometric interpretation of this theorem is explored in terms of the general theory of convex sets. Although the geometric interpretation is aesthetically pleasing and theoretically important, the reader should bear in mind, lest one be diverted by the somewhat more advanced arguments employed, the underlying elementary level of the fundamental theorem.

2.5 Relations to Convexity

Our development to this point, including the above proof of the fundamental theorem, has been based only on elementary properties of systems of linear equations. These results, however, have interesting interpretations in terms of the theory of convex sets that can lead not only to an alternative derivation of the fundamental theorem, but also to a clearer geometric understanding of the result. The main link between the algebraic and geometric theories is the formal relation between basic feasible solutions of linear inequalities in standard form and extreme points of polytopes. We establish this correspondence as follows. The reader is referred to Appendix B for a more complete summary of concepts related to convexity, but the definition of an extreme point is stated here.

Definition. A point \mathbf{x} in a convex set C is said to be an *extreme point* of C if there are no two distinct points \mathbf{x}_1 and \mathbf{x}_2 in C such that $\mathbf{x} = \alpha\mathbf{x}_1 + (1 - \alpha)\mathbf{x}_2$ for some α , $0 < \alpha < 1$.

An extreme point is thus a point that does not lie strictly within a line segment connecting two other points of the set. The extreme points of a triangle, for example, are its three vertices.

Theorem (Equivalence of Extreme Points and Basic Solutions). Let \mathbf{A} be an $m \times n$ matrix of rank m and \mathbf{b} an m -vector. Let K be the convex polytope consisting of all n -vectors \mathbf{x} satisfying

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}. \quad (2.18)$$

A vector \mathbf{x} is an extreme point of K if and only if \mathbf{x} is a basic feasible solution to (2.18).

Proof. Suppose first that $\mathbf{x} = (x_1, x_2, \dots, x_m, 0, 0, \dots, 0)$ is a basic feasible solution to (2.18). Then

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_m\mathbf{a}_m = \mathbf{b},$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$, the first m columns of \mathbf{A} , are linearly independent. Suppose that \mathbf{x} could be expressed as a convex combination of two other points in K ; say, $\mathbf{x} = \alpha\mathbf{y} + (1 - \alpha)\mathbf{z}$, $0 < \alpha < 1$, $\mathbf{y} \neq \mathbf{z}$. Since all components of \mathbf{x} , \mathbf{y} , \mathbf{z} are nonnegative and since $0 < \alpha < 1$, it follows immediately that the last $n - m$ components of \mathbf{y} and \mathbf{z} are zero. Thus, in particular, we have

$$y_1\mathbf{a}_1 + y_2\mathbf{a}_2 + \dots + y_m\mathbf{a}_m = \mathbf{b}$$

and

$$z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \dots + z_m\mathbf{a}_m = \mathbf{b}.$$

Since the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ are linearly independent, however, it follows that $\mathbf{x} = \mathbf{y} = \mathbf{z}$ and hence \mathbf{x} is an extreme point of K .

Conversely, assume that \mathbf{x} is an extreme point of K . Let us assume that the nonzero components of \mathbf{x} are the first k components. Then

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_k\mathbf{a}_k = \mathbf{b},$$

with $x_i > 0$, $i = 1, 2, \dots, k$. To show that \mathbf{x} is a basic feasible solution it must be shown that the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly independent. We do this by contradiction. Suppose $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly dependent. Then there is a nontrivial linear combination that is zero:

$$y_1 \mathbf{a}_1 + y_2 \mathbf{a}_2 + \dots + y_k \mathbf{a}_k = \mathbf{0}.$$

Define the n -vector $\mathbf{y} = (y_1, y_2, \dots, y_k, 0, 0, \dots, 0)$. Since $x_i > 0$, $1 \leq i \leq k$, it is possible to select ε such that

$$\mathbf{x} + \varepsilon \mathbf{y} \geq \mathbf{0}, \quad \mathbf{x} - \varepsilon \mathbf{y} \geq \mathbf{0}.$$

We then have $\mathbf{x} = \frac{1}{2}(\mathbf{x} + \varepsilon \mathbf{y}) + \frac{1}{2}(\mathbf{x} - \varepsilon \mathbf{y})$ which expresses \mathbf{x} as a convex combination of two distinct vectors in K . This cannot occur, since \mathbf{x} is an extreme point of K . Thus $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k$ are linearly independent and \mathbf{x} is a basic feasible solution. (Although if $k < m$, it is a degenerate basic feasible solution.) ■

This correspondence between extreme points and basic feasible solutions enables us to prove certain geometric properties of the convex polytope K defining the constraint set of a linear programming problem.

Corollary 1. *If the convex set K corresponding to (2.18) is nonempty, it has at least one extreme point.*

Proof. This follows from the first part of the Fundamental Theorem and the Equivalence Theorem above. ■

Corollary 2. *If there is a finite optimal solution to a linear programming problem, there is a finite optimal solution which is an extreme point of the constraint set.*

Corollary 3. *The constraint set K corresponding to (2.18) possesses at most a finite number of extreme points.*

Proof. There are obviously only a finite number of basic solutions obtained by selecting m basis vectors from the n columns of \mathbf{A} . The extreme points of K are a subset of these basic solutions. ■

Finally, we come to the special case which occurs most frequently in practice and which in some sense is characteristic of well-formulated linear programs—the case where the constraint set K is nonempty and bounded. In this case we combine the results of the Equivalence Theorem and Corollary 3 above to obtain the following corollary.

Corollary 4. *If the convex polytope K corresponding to (2.18) is bounded, then K is a convex polyhedron, that is, K consists of points that are convex combinations of a finite number of points.*

Some of these results are illustrated by the following examples:

Example 1. Consider the constraint set in E^3 defined by

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

This set is illustrated in Fig. 2.3. It has three extreme points, corresponding to the three basic solutions to $x_1 + x_2 + x_3 = 1$.

Example 2. Consider the constraint set in E^3 defined by

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 &= 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

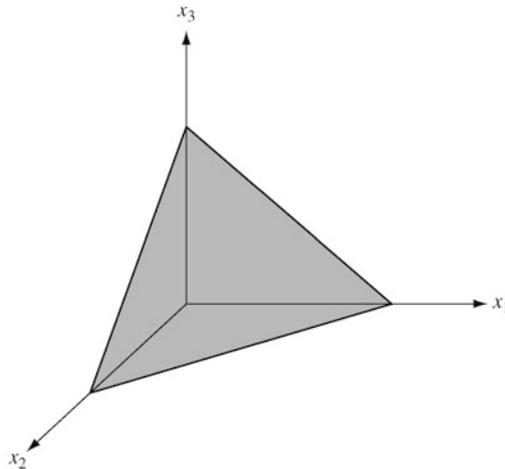


Fig. 2.3 Feasible set for Example 1

This set is illustrated in Fig. 2.4. It has two extreme points, corresponding to the two basic feasible solutions. Note that the system of equations itself has three basic solutions, $(2, -1, 0)$, $(1/2, 0, 1/2)$, $(0, 1/3, 2/3)$, the first of which is not feasible.

Example 3. Consider the constraint set in E^2 defined in terms of the inequalities

$$\begin{aligned} x_1 + \frac{8}{3}x_2 &\leq 4 \\ x_1 + x_2 &\leq 2 \\ 2x_1 &\leq 3 \\ x_1 \geq 0, x_2 &\geq 0. \end{aligned}$$

This set is illustrated in Fig. 2.5. We see by inspection that this set has five extreme points. In order to compare this example with our general results we must introduce slack variables to yield the equivalent set in E^5 :

$$\begin{aligned} x_1 + \frac{8}{3}x_2 + x_3 &= 4 \\ x_1 + x_2 + x_4 &= 2 \\ 2x_1 + x_5 &= 3 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0. \end{aligned}$$

A basic solution for this system is obtained by setting any two variables to zero and solving for the remaining three. As indicated in Fig. 2.5, each edge of the figure corresponds to one variable being zero, and the extreme points are the points where two variables are zero.

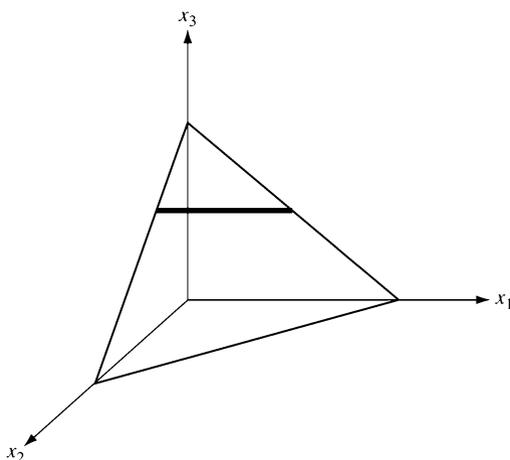


Fig. 2.4 Feasible set for Example 2

The last example illustrates that even when not expressed in standard form the extreme points of the set defined by the constraints of a linear program correspond to the possible solution points. This can be illustrated more directly by including the objective function in the figure as well. Suppose, for example, that in Example 3 the objective function to be minimized is $-2x_1 - x_2$. The set of points satisfying $-2x_1 - x_2 = z$ for fixed z is a line. As z varies, different parallel lines are obtained as shown in Fig. 2.6. The optimal value of the linear program is the smallest value of z for which the corresponding line has a point in common with the feasible set. It should be reasonably clear, at least in two dimensions, that the points of solution will always include an extreme point. In the figure this occurs at the point $(3/2, 1/2)$ with $z = -7/2$.

2.6 Exercises

1. Convert the following problems to standard form:

(a) minimize $x + 2y + 3z$
 subject to $2 \leq x + y \leq 3$
 $4 \leq x + z \leq 5$
 $x \geq 0, y \geq 0, z \geq 0$.

(b) minimize $x + y + z$
 subject to $x + 2y + 3z = 10$
 $x \geq 1, y \geq 2, z \geq 1$.

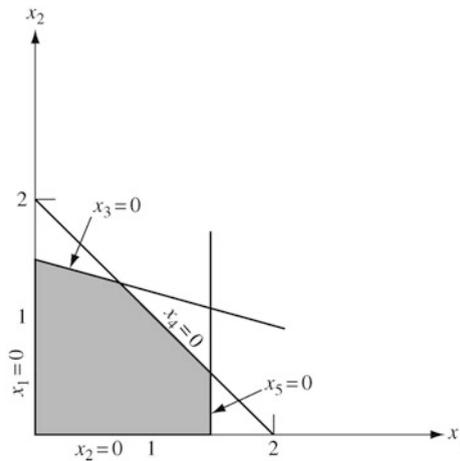


Fig. 2.5 Feasible set for Example 3

2. A manufacturer wishes to produce an alloy that is, by weight, 30 % metal A and 70 % metal B. Five alloys are available at various prices as indicated below:

Alloy	1	2	3	4	5
%A	10	25	50	75	95
% B	90	75	50	25	5
Price/lb	\$ 5	\$ 4	\$ 3	\$ 2	\$ 1.50

The desired alloy will be produced by combining some of the other alloys. The manufacturer wishes to find the amounts of the various alloys needed and to determine the least expensive combination. Formulate this problem as a linear program.

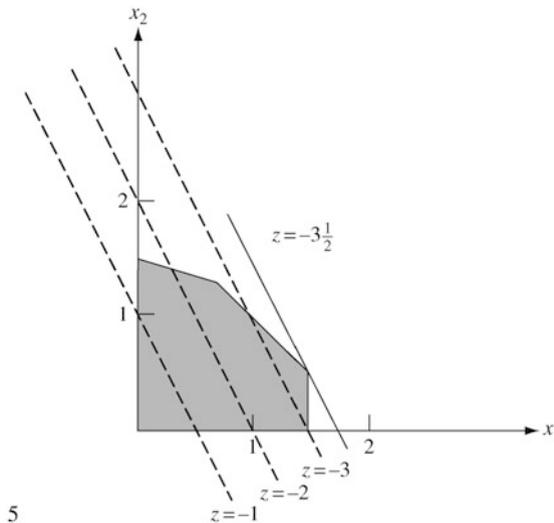


Fig. 2.6 Illustration of extreme point solution

- An oil refinery has two sources of crude oil: a light crude that costs \$35/barrel and a heavy crude that costs \$30/barrel. The refinery produces gasoline, heating oil, and jet fuel from crude in the amounts per barrel indicated in the following table:

	Gasoline	Heating oil	Jet fuel
Light crude	0.3	0.2	0.3
Heavy crude	0.3	0.4	0.2

The refinery has contracted to supply 900,000 barrels of gasoline, 800,000 barrels of heating oil, and 500,000 barrels of jet fuel. The refinery wishes to find the amounts of light and heavy crude to purchase so as to be able to meet its obligations at minimum cost. Formulate this problem as a linear program.

- A small firm specializes in making five types of spare automobile parts. Each part is first cast from iron in the casting shop and then sent to the finishing shop where holes are drilled, surfaces are turned, and edges are ground. The required worker-hours (per 100 units) for each of the parts of the two shops are shown below:

Part	1	2	3	4	5
Casting	2	1	3	3	1
Finishing	3	2	2	1	1

The profits from the parts are \$30, \$20, \$40, \$25, and \$10 (per 100 units), respectively. The capacities of the casting and finishing shops over the next month are 700 and 1,000 worker-hours, respectively. Formulate the problem of

determining the quantities of each spare part to be made during the month so as to maximize profit.

5. Convert the following problem to standard form and solve:

$$\begin{aligned}
 &\text{maximize} && x_1 + 4x_2 + x_3 \\
 &\text{subject to} && 2x_1 - 2x_2 + x_3 = 4 \\
 &&& x_1 - x_3 = 1 \\
 &&& x_2 \geq 0, \quad x_3 \geq 0.
 \end{aligned}$$

6. A large textile firm has two manufacturing plants, two sources of raw material, and three market centers. The transportation costs between the sources and the plants and between the plants and the markets are as follows:

		Plant	
		A	B
Source	1	\$1/ton	\$1.50/ton
	2	\$2/ton	\$1.50/ton

		Market		
		1	2	3
Plant	A	\$4/ton	\$2/ton	\$1/ton
	B	\$3/ton	\$4/ton	\$2/ton

Ten tons are available from source 1 and 15 tons from source 2. The three market centers require 8 tons, 14 tons, and 3 tons. The plants have unlimited processing capacity.

- (a) Formulate the problem of finding the shipping patterns from sources to plants to markets that minimizes the total transportation cost.
 - (b) Reduce the problem to a single standard transportation problem with two sources and three destinations. (*Hint*: Find minimum cost paths from sources to markets.)
 - (c) Suppose that plant A has a processing capacity of 8 tons, and plant B has a processing capacity of 7 tons. Show how to reduce the problem to two separate standard transportation problems.
7. A businessman is considering an investment project. The project has a lifetime of 4 years, with cash flows of $-\$100,000$, $+\$50,000$, $+\$70,000$, and $+\$30,000$ in each of the 4 years, respectively. At any time he may borrow funds at the rates of 12 %, 22 %, and 34 % (total) for 1, 2, or 3 periods, respectively. He may loan funds at 10 % per period. He calculates the *present value* of a project as the maximum amount of money he would pay now, to another party, for the project, assuming that he has no cash on hand and must borrow and lend to pay the other party and operate the project while maintaining a nonnegative cash

balance after all debts are paid. Formulate the project valuation problem in a linear programming framework.

8. Convert the following problem to a linear program in standard form:

$$\begin{aligned} & \text{minimize } |x| + |y| + |z| \\ & \text{subject to } x + y \leq 1 \\ & \qquad \qquad 2x + z = 3. \end{aligned}$$

9. A class of piecewise linear functions can be represented as $f(\mathbf{x}) = \text{Maximum}(\mathbf{c}_1^T \mathbf{x} + d_1, \mathbf{c}_2^T \mathbf{x} + d_2, \dots, \mathbf{c}_p^T \mathbf{x} + d_p)$. For such a function f , consider the problem

$$\begin{aligned} & \text{minimize } f(\mathbf{x}) \\ & \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

Show how to convert this problem to a linear programming problem.

10. A small computer manufacturing company forecasts the demand over the next n months to be d_i , $i = 1, 2, \dots, n$. In any month it can produce r units, using *regular* production, at a cost of b dollars per unit. By using *overtime*, it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month. Formulate the problem of determining the production schedule that minimizes cost. (*Hint*: See Exercise 9.)
11. Discuss the situation of a linear program that has one or more columns of the \mathbf{A} matrix equal to zero. Consider both the case where the corresponding variables are required to be nonnegative and the case where some are free.
12. Suppose that the matrix $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ has rank m , and that for some $p < m$, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p$ are linearly independent. Show that $m - p$ vectors from the remaining $n - p$ vectors can be adjoined to form a set of m linearly independent vectors.
13. Suppose that \mathbf{x} is a feasible solution to the linear program (2.12), with \mathbf{A} an $m \times n$ matrix of rank m . Show that there is a feasible solution \mathbf{y} having the same value (that is, $\mathbf{c}^T \mathbf{y} = \mathbf{c}^T \mathbf{x}$) and having at most $m + 1$ positive components.
14. What are the basic solutions of Example 3, Sect. 2.5?
15. Let S be a convex set in E^n and S^* a convex set in E^m . Suppose \mathbf{T} is an $m \times n$ matrix that establishes a one-to-one correspondence between S and S^* , i.e., for every $\mathbf{s} \in S$ there is $\mathbf{s}^* \in S^*$ such that $\mathbf{T}\mathbf{s} = \mathbf{s}^*$, and for every $\mathbf{s}^* \in S^*$ there is a single $\mathbf{s} \in S$ such that $\mathbf{T}\mathbf{s} = \mathbf{s}^*$. Show that there is a one-to-one correspondence between extreme points of S and S^* .
16. Consider the two linear programming problems in Example 1, Sect. 2.1, one in E^n and the other in E^{n+m} . Show that there is a one-to-one correspondence between extreme points of these two problems.

References

- 2.1–2.4 The approach taken in this chapter, which is continued in the next, is the more or less standard approach to linear programming as presented in, for example, Dantzig [D6], Hadley [H1], Gass [G4], Simonnard [S6], Murty [M11], and Gale [G2]. Also see Bazaraa, Jarvis, and H. F. Sherali [B6], Bertsimas and Tsitsiklis [B13], Cottle, [C6], Dantzig and Thapa [D9, D10], Nash and Sofer [N1], Saigal [S1], and Vanderbei [V3].
- 2.5 An excellent discussion of this type can be found in Simonnard [S6].