# Chapter 3
# The Simplex Method

The idea of the simplex method is to proceed from one basic feasible solution (that is, one extreme point) of the constraint set of a problem in standard form to another, in such a way as to continually decrease the value of the objective function until a minimum is reached. The results of Chap. 2 assure us that it is sufficient to consider only basic feasible solutions in our search for an optimal feasible solution. This chapter demonstrates that an efficient method for moving among basic solutions to the minimum can be constructed.

In the first five sections of this chapter the simplex machinery is developed from a careful examination of the system of linear equations that defines the constraints and the basic feasible solutions of the system. This approach, which focuses on individual variables and their relation to the system, is probably the simplest, but unfortunately is not easily expressed in compact form. In the last few sections of the chapter, the simplex method is viewed from a matrix theoretic approach, which focuses on all variables together. This more sophisticated viewpoint leads to a compact notational representation, increased insight into the simplex process, and to alternative methods for implementation.

## 3.1 Pivots

To obtain a firm grasp of the simplex procedure, it is essential that one first understand the process of pivoting in a set of simultaneous linear equations. There are two dual interpretations of the pivot procedure.



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### *First Interpretation*

Consider the set of simultaneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned} \tag{3.1}$$

where $m \leqslant n$. In matrix form we write this as

$$\mathbf{Ax} = \mathbf{b}. \tag{3.2}$$

In the space $E^n$ we interpret this as a collection of $m$ linear relations that must be satisfied by a vector $\mathbf{x}$. Thus denoting by $\mathbf{a}^i$ the $i$th row of $\mathbf{A}$ we may express (3.1) as:

$$\begin{aligned} \mathbf{a}^1\mathbf{x} &= b_1 \\ \mathbf{a}^2\mathbf{x} &= b_2 \\ &\vdots \\ \mathbf{a^m}\mathbf{x} &= b_m. \end{aligned} \tag{3.3}$$

This corresponds to the most natural interpretation of (3.1) as a set of $m$ equations.

If $m < n$ and the equations are linearly independent, then there is not a unique solution but a whole linear variety of solutions (see Appendix B). A unique solution results, however, if $n - m$ additional independent linear equations are adjoined. For example, we might specify $n - m$ equations of the form $\mathbf{e}^k\mathbf{x} = 0$, where $\mathbf{e}^k$ is the $k$th unit vector (which is equivalent to $x_k = 0$), in which case we obtain a basic solution to (3.1). Different basic solutions are obtained by imposing different additional equations of this special form.

If Eq. (3.3) are linearly independent, we may replace a given equation by any nonzero multiple of itself plus any linear combination of the other equations in the system. This leads to the well-known Gaussian reduction schemes, whereby multiples of equations are systematically subtracted from one another to yield either a triangular or canonical form. It is well known, and easily proved, that if the first $m$ columns of A are linearly independent, the system (3.1) can, by a sequence of such multiplications and subtractions, be converted to the following *canonical form*:

$$\begin{aligned} x_1 \qquad & +\bar{a}_{1(m+1)}x_{m+1} + \bar{a}_{1(m+2)}x_{m+2} + \cdots + \bar{a}_{1n}x_n = \bar{a}_{10} \\ x_2 \qquad & +\bar{a}_{2(m+1)}x_{m+1} + \bar{a}_{2(m+2)}x_{m+2} + \cdots + \bar{a}_{2n}x_n = \bar{a}_{20} \\ & \qquad \vdots \qquad\qquad\qquad\qquad \vdots \\ x_m \qquad & +\bar{a}_{m(m+1)}x_{m+1} + \bar{a}_{m(m+2)}x_{m+2} + \cdots + \bar{a}_{mn}x_n = \bar{a}_{m0}. \end{aligned} \tag{3.4}$$

Corresponding to this canonical representation of the system, the variables $x_1$, $x_2$, $\ldots$, $x_m$ are called *basic* and the other variables are *nonbasic*. The corresponding basic solution is then:

$$x_1 = \bar{a}_{10},\ x_2 = \bar{a}_{20},\ \ldots,\ x_m = \bar{a}_{m0},\ x_{m+1} = 0,\ \ldots,\ x_n = 0,$$

or in vector form: $\mathbf{x} = (\bar{\mathbf{a}}_0, \mathbf{0})$ where $\bar{\mathbf{a}}_0$ is $m$-dimensional and $\mathbf{0}$ is the $(n-m)$-dimensional zero vector.

Actually, we relax our definition somewhat and consider a system to be in *canonical form* if, among the $n$ variables, there are $m$ basic ones with the property that each appears in only one equation, its coefficient in that equation is unity, and no two of these $m$ variables appear in any one equation. This is equivalent to saying that a system is in canonical form if by some reordering of the equations and the variables it takes the form (3.4).

Also it is customary, from the dictates of economy, to represent the system (3.4) by its corresponding array of coefficients or *tableau*:

$$\begin{array}{cccccccccc}
x_1 & x_2 & x_3 & \cdots & x_m & x_{m+1} & x_{m+2} & \cdots & x_n & \\
1 & 0 & 0 & \cdots & 0 & \bar{a}_{1(m+1)} & \bar{a}_{1(m+2)} & \cdots & \bar{a}_{1n} & \bar{a}_{10} \\
0 & 1 & 0 & \cdots & 0 & \bar{a}_{2(m+1)} & \bar{a}_{2(m+2)} & \cdots & \cdot & \bar{a}_{20} \\
0 & 0 & 1 & \cdots & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 1 & \bar{a}_{m(m+1)} & \bar{a}_{m(m+2)} & \cdots & \bar{a}_{mn} & \bar{a}_{m0}
\end{array} \tag{3.5}$$

The question solved by pivoting is this: given a system in canonical form, suppose a basic variable is to be made nonbasic and a nonbasic variable is to be made basic; what is the new canonical form corresponding to the new set of basic variables? The procedure is quite simple. Suppose in the canonical system (3.4) we wish to replace the basic variable $x_p$, $1 \leqslant p \leqslant m$, by the nonbasic variable $x_q$. This can be done if and only if $\bar{a}_{pq}$ is nonzero; it is accomplished by dividing row $p$ by $\bar{a}_{pq}$ to get a unit coefficient for $x_q$ in the $p$th equation, and then subtracting suitable multiples of row $p$ from each of the other rows in order to get a zero coefficient for $x_q$ in all other equations. This transforms the $q$th column of the tableau so that it is zero except in its $p$th entry (which is unity) and does not affect the columns of the other basic variables. Denoting the coefficients of the new system in canonical form by $\bar{a}'_{ij}$, we have explicitly

$$\begin{cases} \bar{a}'_{ij} = \bar{a}_{ij} - \frac{\bar{a}_{iq}}{\bar{a}_{pq}}\bar{a}_{pj}, & i \neq p \\ \bar{a}'_{pj} = \frac{\bar{a}_{pj}}{\bar{a}_{pq}}. \end{cases} \tag{3.6}$$

Equation (3.6) are the pivot equations that arise frequently in linear programming. The element $\bar{a}_{pq}$ in the original system is said to be the *pivot element*.

*Example 1.* Consider the system in canonical form:

$$\begin{aligned} x_1 \qquad\qquad + x_4 + x_5 - x_6 &= 5 \\ x_2 \qquad + 2x_4 - 3x_5 + x_6 &= 3 \\ x_3 - x_4 + 2x_5 - x_6 &= -1. \end{aligned}$$

Let us find the basic solution having basic variables $x_4,\ x_5,\ x_6$. We set up the coefficient array below:

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | (1) | 1 | −1 | 5 |
| 0 | 1 | 0 | 2 | −3 | 1 | 3 |
| 0 | 0 | 1 | −1 | 2 | −1 | −1 |

The circle indicated is our first pivot element and corresponds to the replacement of $x_1$ by $x_4$ as a basic variable. After pivoting we obtain the array

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | −1 | 5 |
| −2 | 1 | 0 | 0 | (−5) | 3 | −7 |
| 1 | 0 | 1 | 0 | 3 | −2 | 4 |

and again we have circled the next pivot element indicating our intention to replace $x_2$ by $x_5$. We then obtain

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|
| 3/5 | 1/5 | 0 | 1 | 0 | −2/5 | 18/5 |
| 2/5 | −1/5 | 0 | 0 | 1 | −3/5 | 7/5 |
| −1/5 | 3/5 | 1 | 0 | 0 | (−1/5) | −1/5 |

Continuing, there results

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | |
|---|---|---|---|---|---|---|
| 1 | −1 | −2 | 1 | 0 | 0 | 4 |
| 1 | −2 | −3 | 0 | 1 | 0 | 2 |
| 1 | −3 | −5 | 0 | 0 | 1 | 1 |

From this last canonical form we obtain the new basic solution

$$x_4 = 4, \quad x_5 = 2, \quad x_6 = 1.$$

## *Second Interpretation*

The set of simultaneous equations represented by (3.1) and (3.2) can be interpreted in $E^m$ as a vector equation. Denoting the columns of A by $\mathbf{a}_1,\ \mathbf{a}_2,\ \ldots,\ \mathbf{a_n}$ we write (3.1) as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_\mathbf{n}\mathbf{a_n} = \mathbf{b}. \tag{3.7}$$

In this interpretation we seek to express $\mathbf{b}$ as a linear combination of the $\mathbf{a}_i$'s.

If $m < n$ and the vectors $\mathbf{a}_j$ span $E^m$ then there is not a unique solution but a whole family of solutions. The vector $\mathbf{b}$ has a unique representation, however, as a linear combination of a given linearly independent subset of these vectors. The corresponding solution with $(n-m)$ $x_j$ variables set equal to zero is a basic solution to (3.1).

Suppose now that we start again with a system in the canonical form corresponding to the tableau:

$$\begin{array}{cccccccccc}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \cdots & \mathbf{a}_m & \mathbf{a}_{m+1} & \mathbf{a}_{m+2} & \cdots & \mathbf{a}_n & \mathbf{b} \\
1 & 0 & 0 & \cdots & 0 & \bar{a}_{1(m+1)} & \bar{a}_{1(m+2)} & \cdots & \bar{a}_{1n} & \bar{a}_{10} \\
0 & 1 & 0 & \cdots & 0 & \bar{a}_{2(m+1)} & \bar{a}_{2(m+2)} & \cdots & \cdot & \bar{a}_{20} \\
0 & 0 & 1 & \cdots & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 1 & \bar{a}_{m(m+1)} & \bar{a}_{m(m+2)} & \cdots & \bar{a}_{mn} & \bar{a}_{m0}
\end{array} \tag{3.8}$$

In this case the first $m$ vectors form a basis. Furthermore, every other vector represented in the tableau can be expressed as a linear combination of these basis vectors by simply reading the coefficients down the corresponding column. Thus

$$\mathbf{a}_j = \bar{a}_{1j}\mathbf{a}_1 + \bar{a}_{2j}\mathbf{a}_2 + \cdots + \bar{a}_{mj}\mathbf{a}_m. \tag{3.9}$$

The tableau can be interpreted as giving the representations of the vectors $\mathbf{a}_j$ in terms of the basis; the $j$th column of the tableau is the representation for the vector $\mathbf{a}_j$. In particular, the expression for $\mathbf{b}$ in terms of the basis is given in the last column.

We now consider the operation of replacing one member of the basis by another vector not already in the basis. Suppose for example we wish to replace the basis vector $\mathbf{a}_p$, $1 \leqslant p \leqslant m$, by the vector $\mathbf{a}_q$. Provided that the first $m$ vectors with $\mathbf{a}_p$ replaced by $\mathbf{a}_q$ are linearly independent these vectors constitute a basis and every vector can be expressed as a linear combination of this new basis. To find the new representations of the vectors we must update the tableau. The linear independence condition holds if and only if $\bar{a}_{pq} \neq 0$.

Any vector $\mathbf{a}_j$ can be expressed in terms of the old array through (3.9). For $\mathbf{a}_q$ we have

$$\mathbf{a}_q = \sum_{\substack{i=1 \\ i \neq p}}^{m} \bar{a}_{iq}\mathbf{a}_i + \bar{a}_{pq}\mathbf{a}_p$$

from which we may solve for $\mathbf{a}_p$,

$$\mathbf{a}_p = \frac{1}{\bar{a}_{pq}}\mathbf{a}_q - \sum_{\substack{i=1 \\ i \neq p}}^{m} \frac{\bar{a}_{iq}}{\bar{a}_{pq}}\mathbf{a}_i. \tag{3.10}$$

Substituting (3.10) into (3.9) we obtain:

$$\mathbf{a}_j = \sum_{\substack{i=1 \\ i \neq p}}^{m} \left( \bar{a}_{ij} - \frac{\bar{a}_{iq}}{\bar{a}_{pq}} \bar{a}_{pj} \right) \mathbf{a}_i + \frac{\bar{a}_{pj}}{\bar{a}_{pq}} \mathbf{a}_q. \tag{3.11}$$

Denoting the coefficients of the new tableau, which give the linear combinations, by $\bar{a}'_{ij}$ we obtain immediately from (3.11)

$$\begin{cases} \bar{a}'_{ij} = \bar{a}_{ij} - \frac{\bar{a}_{iq}}{\bar{a}_{pq}} \bar{a}_{pj}, i \neq p \\ \bar{a}'_{pj} = \frac{\bar{a}_{pj}}{\bar{a}_{pq}}. \end{cases} \tag{3.12}$$

These formulas are identical to (3.6).

If a system of equations is not originally given in canonical form, we may put it into canonical form by adjoining the $m$ unit vectors to the tableau and, starting with these vectors as the basis, successively replace each of them with columns of A using the pivot operation.

*Example 2.* Suppose we wish to solve the simultaneous equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ 2x_1 - 3x_2 + x_3 &= 3 \\ -x_1 + 2x_2 - x_3 &= -1. \end{aligned}$$

To obtain an original basis, we form the augmented tableau

| $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | −1 | 5 |
| 0 | 1 | 0 | 2 | −3 | 1 | 3 |
| 0 | 0 | 1 | −1 | 2 | −1 | −1 |

and replace $\mathbf{e}_1$ by $\mathbf{a}_1$, $\mathbf{e}_2$ by $\mathbf{a}_2$, and $\mathbf{e}_3$ by $\mathbf{a}_3$. The required operations are identical to those of Example 1.

## 3.2 Adjacent Extreme Points

In Chap. 2 it was discovered that it is only necessary to consider basic feasible solutions to the system

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0} \tag{3.13}$$

when solving a linear program, and in the previous section it was demonstrated that the pivot operation can generate a new basic solution from an old one by replacing one basic variable by a nonbasic variable. It is clear, however, that although the pivot operation takes one basic solution into another, the nonnegativity of the solution will not in general be preserved. Special conditions must be satisfied in order that a pivot operation maintain feasibility. In this section we show how it is possible to select pivots so that we may transfer from one basic feasible solution to another.

We show that although it is not possible to arbitrarily specify the pair of variables whose roles are to be interchanged and expect to maintain the nonnegativity condition, it is possible to arbitrarily specify which nonbasic variable is to become basic and then determine which basic variable should become nonbasic. As is conventional, we base our derivation on the vector interpretation of the linear equations although the dual interpretation could alternatively be used.

### *Nondegeneracy Assumption*

Many arguments in linear programming are substantially simplified upon the introduction of the following.

> **Nondegeneracy Assumption:** Every basic feasible solution of (3.13) is a nondegenerate basic feasible solution.

This assumption is invoked throughout our development of the simplex method, since when it does not hold the simplex method can break down if it is not suitably amended. The assumption, however, should be regarded as one made primarily for convenience, since all arguments can be extended to include degeneracy, and the simplex method itself can be easily modified to account for it.

### *Determination of Vector to Leave Basis*

Suppose we have the basic feasible solution $\mathbf{x} = (x_1,\ x_2,\ \ldots,\ x_m,\ 0, 0,\ \ldots,\ 0)$ or, equivalently, the representation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_m\mathbf{a}_m = \mathbf{b}. \tag{3.14}$$

Under the nondegeneracy assumption, $x_j > 0,\ i = 1, 2,\ \ldots,\ m$. Suppose also that we have decided to bring into the representation the vector $\mathbf{a}_q,\ q > m$. We have available a representation of $\mathbf{a}_q$ in terms of the current basis

$$\mathbf{a}_q = \bar{a}_{1q}\mathbf{a}_1 + \bar{a}_{2q}\mathbf{a}_2 + \cdots + \bar{a}_{mq}\mathbf{a}_m. \tag{3.15}$$

Multiplying (3.15) by a variable $\varepsilon \geqslant 0$ and subtracting from (3.14), we have

$$(x_1 - \varepsilon\bar{a}_{1q})\mathbf{a}_1 + (x_2 - \varepsilon\bar{a}_{2q})\mathbf{a}_2 + \cdots + (x_m - \varepsilon\bar{a}_{mq})\mathbf{a}_m + \varepsilon\mathbf{a}_q = \mathbf{b}. \tag{3.16}$$

Thus, for any $\varepsilon \geqslant 0$ (3.16) gives $\mathbf{b}$ as a linear combination of at most $m + 1$ vectors. For $\varepsilon = 0$ we have the old basic feasible solution. As $\varepsilon$ is increased from zero, the coefficient of $\mathbf{a}_q$ increases, and it is clear that for small enough $\varepsilon$, (3.16) gives a feasible but nonbasic solution. The coefficients of the other vectors will either increase or decrease linearly as $\varepsilon$ is increased. If any decrease, we may set $\varepsilon$ equal to the value corresponding to the first place where one (or more) of the coefficients vanishes. That is

$$\varepsilon = \min_i \{x_j/\bar{a}_{iq} : \bar{a}_{iq} > 0\}. \tag{3.17}$$

In this case we have a new basic feasible solution, with the vector $\mathbf{a}_q$ replacing the vector $\mathbf{a}_p$, where $p$ corresponds to the minimizing index in (3.17). If the minimum in (3.17) is achieved by more than a single index $i$, then the new solution is degenerate and any of the vectors with zero component can be regarded as the one that left the basis.

If none of the $\bar{a}_{iq}$'s are positive, then all coefficients in the representation (3.16) increase (or remain constant) as $\varepsilon$ is increased, and no new basic feasible solution is obtained. We observe, however, that in this case, where none of the $\bar{a}_{iq}$'s are positive, there are feasible solutions to (3.13) having arbitrarily large coefficients. This means that the set $K$ of feasible solutions to (3.13) is unbounded, and this special case, as we shall see, is of special significance in the simplex procedure.

In summary, we have deduced that, given a basic feasible solution and an arbitrary vector $\mathbf{a}_q$, there is either a new basic feasible solution having $\mathbf{a}_q$ in its basis and one of the original vectors removed, or the set of feasible solutions is unbounded.

Let us consider how the calculation of this section can be displayed in our tableau. We assume that corresponding to the constraints

$$\mathbf{Ax} = \mathbf{b}, \ \ \mathbf{x} \geqslant \mathbf{0},$$

we have a tableau of the form (3.8). Note that the tableau may be the result of several pivot operations applied to the original tableau, but in any event, it represents a solution with basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$. We assume that $\bar{a}_{10}, \bar{a}_{20}, \ldots, \bar{a}_{m0}$ are nonnegative, so that the corresponding basic solution $x_1 = \bar{a}_{10}, x_2 = \bar{a}_{20}, \ldots, x_m = \bar{a}_{m0}$ is feasible. We wish to bring into the basis the vector $\mathbf{a}_q$, $q > m$, and maintain feasibility. In order to determine which element in the $q$th column to use as the pivot (and hence which vector in the basis will leave), we use (3.17) and compute the ratios $x_i/\bar{a}_{iq} = \bar{a}_{i0}/\bar{a}_{iq}$, $i = 1, 2, \ldots, m$, select the smallest nonnegative ratio, and pivot on the corresponding $\bar{a}_{iq}$.

*Example 3.* Consider the system

| $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{a}_4$ | $\mathbf{a}_5$ | $\mathbf{a}_6$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 2 | 4 | 6 | 4 |
| 0 | 1 | 0 | 1 | 2 | 3 | 3 |
| 0 | 0 | 1 | −1 | 2 | 1 | 1 |

which has basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ yielding a basic feasible solution $\mathbf{x} = (4, 3, 1, 0, 0, 0)$. Suppose we elect to bring $\mathbf{a}_4$ into the basis. To determine which element in the fourth column is the appropriate pivot, we compute the three ratios:

$$4/2 = 2,\ 3/1 = 3,\ 1/-1 = -1$$

and select the smallest nonnegative one. This gives 2 as the pivot element. The new tableau is

| $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{a}_4$ | $\mathbf{a}_5$ | $\mathbf{a}_6$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|
| 1/2 | 0 | 0 | 1 | 2 | 3 | 2 |
| −1/2 | 1 | 0 | 0 | 0 | 0 | 1 |
| 1/2 | 0 | 1 | 0 | 4 | 4 | 3 |

with corresponding basic feasible solution $\mathbf{x} = (0, 1, 3, 2, 0,\ 0)$.

Our derivation of the method for selecting the pivot in a given column that will yield a new feasible solution has been based on the vector interpretation of the equation $\mathbf{Ax} = \mathbf{b}$. An alternative derivation can be constructed by considering the dual approach that is based on the rows of the tableau rather than the columns. Briefly, the argument runs like this: if we decide to pivot on $\bar{a}_{pq}$, then we first divide the $p$th row by the pivot element $\bar{a}_{pq}$ to change it to unity. In order that the new $\bar{a}_{p0}$ remain positive, it is clear that we must have $\bar{a}_{pq} > 0$. Next we subtract multiples of the $p$th row from each other row in order to obtain zeros in the $q$th column. In this process the new elements in the last column must remain nonnegative—if the pivot was properly selected. The full operation is to subtract, from the $i$th row, $\bar{a}_{iq}/\bar{a}_{pq}$ times the $p$th row. This yields a new solution obtained directly from the last column:

$$x'_i = x_i - \frac{\bar{a}_{iq}}{\bar{a}_{pq}} x_p.$$

For this to remain nonnegative, it follows that $x_p/\bar{a}_{pq} \leqslant x_i/\bar{a}_{iq}$, and hence again we are led to the conclusion that we select $p$ as the index $i$ minimizing $x_i/\bar{a}_{iq}$.

## *Geometrical Interpretations*

Corresponding to the two interpretations of pivoting and extreme points developed algebraically, are two geometrical interpretations. The first is in *activity space*, the space where $\mathbf{x}$ is represented. This is perhaps the most natural space to consider, and it was used in Sect. 2.5. Here the feasible region is shown directly as a convex set, and basic feasible solutions are extreme points. Adjacent extreme points are points that lie on a common edge.

The second geometrical interpretation is in *requirements space*, the space where the columns of $\mathbf{A}$ and $\mathbf{b}$ are represented. The fundamental relation is

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots + \mathbf{a}_n x_n = \mathbf{b}.$$

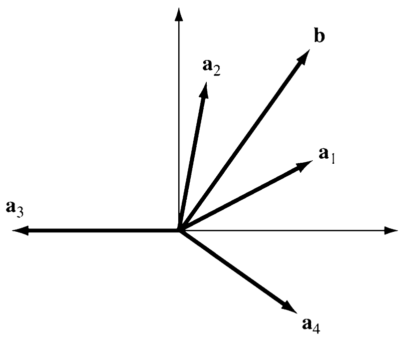


**Fig. 3.1** Constraint representation in requirements space

An example for $m = 2$, $n = 4$ is shown in Fig. 3.1. A feasible solution defines a representation of $\mathbf{b}$ as a positive combination of the $\mathbf{a}_i$'s. A basic feasible solution will use only $m$ positive weights. In the figure a basic feasible solution can be constructed with positive weights on $\mathbf{a}_1$ and $\mathbf{a}_2$ because $\mathbf{b}$ lies between them. A basic feasible solution cannot be constructed with positive weights on $\mathbf{a}_1$ and $\mathbf{a}_4$. Suppose we start with $\mathbf{a}_1$ and $\mathbf{a}_2$ as the initial basis. Then an adjacent basis is found by bringing in some other vector. If $\mathbf{a}_3$ is brought in, then clearly $\mathbf{a}_2$ must go out. On the other hand, if $\mathbf{a}_4$ is brought in, $\mathbf{a}_1$ must go out.

## 3.3 Determining a Minimum Feasible Solution

In the last section we showed how it is possible to pivot from one basic feasible solution to another (or determine that the solution set is unbounded) by arbitrarily selecting a column to pivot on and then appropriately selecting the pivot in that column. The idea of the simplex method is to select the column so that the resulting new basic feasible solution will yield a lower value to the objective function than the previous one. This then provides the final link in the simplex procedure. By an elementary calculation, which is derived below, it is possible to determine which vector should enter the basis so that the objective value is reduced, and by another simple calculation, derived in the previous section, it is possible to then determine which vector should leave in order to maintain feasibility.

Suppose we have a basic feasible solution

$$(\mathbf{x}_\mathbf{B},\ \mathbf{0}) = (\bar{a}_{10},\ \bar{a}_{20},\ \ldots,\ \bar{a}_{m0}, 0, 0,\ \ldots,\ 0)$$

together with a tableau having an identity matrix appearing in the first $m$ columns as shown in tableau (3.8). The value of the objective function corresponding to any solution $\mathbf{x}$ is

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n, \tag{3.18}$$

and hence for the basic solution, the corresponding value is

$$z_0 = \mathbf{c}_{\mathbf{B}}^T \mathbf{x}_{\mathbf{B}}, \tag{3.19}$$

where $\mathbf{c}_{\mathbf{B}}^T = [c_1, c_2, \ldots, c_m]$.

Although it is natural to use the basic solution $(\mathbf{x}_{\mathbf{B}}, \mathbf{0})$ when we have the tableau (3.8), it is clear that if arbitrary values are assigned to $x_{m+1}, x_{m+2}, \ldots, x_n$, we can easily solve for the remaining variables as

$$\begin{aligned} x_1 &= \bar{a}_{10} - \sum_{j=m+1}^{n} \bar{a}_{1j} x_j \\ x_2 &= \bar{a}_{20} - \sum_{j=m+1}^{n} \bar{a}_{2j} x_j \\ &\vdots \\ x_m &= \bar{a}_{m0} - \sum_{j=m+1}^{n} \bar{a}_{mj} x_j. \end{aligned} \tag{3.20}$$

Using (3.20) we may eliminate $x_1, x_2, \ldots, x_m$ from the general formula (3.18). Doing this we obtain

$$\begin{aligned} z = \mathbf{c}^T \mathbf{x} = z_0 &+ (c_{m+1} - z_{m+1}) x_{m+1} \\ &+ (c_{m+2} - z_{m+2}) x_{m+2} + \cdots + (c_n - z_n) x_n \end{aligned} \tag{3.21}$$

where

$$z_j = \bar{a}_{1j} c_1 + \bar{a}_{2j} c_2 + \cdots + \bar{a}_{mj} c_m, \; m+1 \leqslant j \leqslant n, \tag{3.22}$$

which is the fundamental relation required to determine the pivot column. The important point is that this equation gives the values of the objective function $z$ for any solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ in terms of the variables $x_{m+1}, \ldots, x_n$. From it we can determine if there is any advantage in changing the basic solution by introducing one of the nonbasic variables. For example, if $c_j - z_j$ is negative for some $j$, $m+1 \leqslant j \leqslant n$, then increasing $x_j$ from zero to some positive value would decrease the total cost, and therefore would yield a better solution. The formula (3.21) and (3.22) automatically take into account the changes that would be required in the values of the basic variables $x_1, x_2, \ldots, x_m$ to accommodate the change in $x_j$.

Let us derive these relations from a different viewpoint. Let $\bar{\mathbf{a}}_j$ be the $j$th column of the tableau. Then any solution satisfies

$$x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_m \mathbf{e}_m = \bar{\mathbf{a}}_0 - x_{m+1} \bar{\mathbf{a}}_{m+1} - x_{m+2} \bar{\mathbf{a}}_{m+2} - \cdots - x_n \bar{\mathbf{a}}_n.$$

Taking the inner product of this vector equation with $\mathbf{c}_{\mathbf{B}}^T$, we have

$$\sum_{i=1}^{m} c_j x_j = \mathbf{c}_{\mathbf{B}}^T \bar{\mathbf{a}}_0 - \sum_{j=m+1}^{n} z_j x_j,$$

where $z_j = \mathbf{c}_\mathbf{B}^T \bar{\mathbf{a}}_j$. Thus, adding $\sum_{j=m+1}^{n} c_j x_j$ to both sides,

$$\mathbf{c}^T \mathbf{x} = z_0 + \sum_{j=m+1}^{n} (c_j - z_j) x_j \tag{3.23}$$

as before.

We now state the condition for improvement, which follows easily from the above observation, as a theorem.

***Theorem (Improvement of Basic Feasible Solution).*** *Given a nondegenerate basic feasible solution with corresponding objective value $z_0$, suppose that for some $j$ there holds $c_j - z_j < 0$. Then there is a feasible solution with objectivevalue $z < z_0$. If the column $\mathbf{a}_j$ can be substituted for some vector in the originalbasis to yield a new basic feasible solution, this new solution will have $z < z_0$. If $\mathbf{a}_j$ cannot be substituted to yield a basic feasible solution, then the solutionset $K$ is unbounded and the objective function can be made arbitrarily small (toward minus infinity).*

*Proof.* The result is an immediate consequence of the previous discussion. Let $(x_1, x_2, \ldots, x_m, 0, 0, \ldots, 0)$ be the basic feasible solution with objective value $z_0$ and suppose $c_{m+1} - z_{m+1} < 0$. Then, in any case, new feasible solutions can be constructed of the form $(x'_1, x'_2, \ldots, x'_m, x'_{m+1}, 0, 0, \ldots, 0)$ with $x'_{m+1} > 0$. Substituting this solution in (3.21) we have

$$z - z_0 = (c_{m+1} - z_{m+1}) x'_{m+1} < 0,$$

and hence $z < z_0$ for any such solution. It is clear that we desire to make $x'_{m+1}$ as large as possible. As $x'_{m+1}$ is increased, the other components increase, remain constant, or decrease. Thus $x'_{m+1}$ can be increased until one $x'_i = 0$, $i \leqslant m$, in which case we obtain a new basic feasible solution, or if none of the $x'_i$'s decrease, $x'_{m+1}$ can be increased without bound indicating an unbounded solution set and an objective value without lower bound. ∎

We see that if at any stage $c_j - z_j < 0$ for some $j$, it is possible to make $x_j$ positive and decrease the objective function. The final question remaining is whether $c_j - z_j \geqslant 0$ for all $j$ implies optimality.

***Optimality Condition Theorem.*** *If for some basic feasible solution $c_j - z_j \geqslant 0$ for all $j$, then that solution is optimal.*

*Proof.* This follows immediately from (3.21), since any other feasible solution must have $x_i \geqslant 0$ for all $i$, and hence the value $z$ of the objective will satisfy $z - z_0 \geqslant 0$. ∎

Since the constants $c_j - z_j$ play such a central role in the development of the simplex method, it is convenient to introduce the somewhat abbreviated notation $r_j = c_j - z_j$ and refer to the $r_j$'s as the *relative cost coefficients* or, alternatively, the *reduced cost coefficients* (both terms occur in common usage). These coefficients measure the cost of a variable relative to a given basis. (For notational convenience we extend the definition of relative cost coefficients to basic variables as well; the relative cost coefficient of a basic variable is zero.)

We conclude this section by giving an economic interpretation of the relative cost coefficients. Let us agree to interpret the linear program

$$\begin{aligned} &\text{minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

as a diet problem (see Sect. 2.2) where the nutritional requirements must be met exactly. A column of **A** gives the nutritional equivalent of a unit of a particular food. With a given basis consisting of, say, the first $m$ columns of **A**, the corresponding simplex tableau shows how any food (or more precisely, the nutritional content of any food) can be constructed as a combination of foods in the basis. For instance, if carrots are not in the basis we can, using the description given by the tableau, construct a *synthetic* carrot which is nutritionally equivalent to a carrot, by an appropriate combination of the foods in the basis.

In considering whether or not the solution represented by the current basis is optimal, we consider a certain food not in the basis—say carrots—and determine if it would be advantageous to bring it into the basis. This is very easily determined by examining the cost of carrots as compared with the cost of synthetic carrots. If carrots are food $j$, then the unit cost of carrots is $c_j$. The cost of a unit of synthetic carrots is, on the other hand,

$$z_j = \sum_{i=1}^{m} c_j \bar{a}_{ij}.$$

If $r_j = c_j - z_j < 0$, it is advantageous to use real carrots in place of synthetic carrots, and carrots should be brought into the basis.

In general each $z_j$ can be thought of as the price of a unit of the column $\mathbf{a}_j$ when constructed from the current basis. The difference between this synthetic price and the direct price of that column determines whether that column should enter the basis.

## 3.4 Computational Procedure: Simplex Method

In previous sections the theory, and indeed much of the technique, necessary for the detailed development of the simplex method has been established. It is only necessary to put it all together and illustrate it with examples.

In this section we assume that we begin with a basic feasible solution and that the tableau corresponding to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is in the canonical form for this solution. Methods for obtaining this first basic feasible solution, when one is not obvious, are described in the next section.

In addition to beginning with the array $\mathbf{A}\mathbf{x} = \mathbf{b}$ expressed in canonical form corresponding to a basic feasible solution, we append a row at the bottom consisting of the relative cost coefficients and the negative of the current cost. The result is a *simplex tableau*.

Thus, if we assume the basic variables are (in order) $x_1,\ x_2, \ldots,\ x_m$, the simplex tableau takes the initial form shown in Fig. 3.2.

The basic solution corresponding to this tableau is

$$x_j = \begin{cases} \bar{a}_{i0} & 0 \leqslant i \leqslant m \\ 0 & m+1 \leqslant i \leqslant n \end{cases}$$

which we have assumed is feasible, that is, $\bar{a}_{i0} \geqslant 0,\ i = 1, 2,\ \ldots,\ m$. The corresponding value of the objective function is $z_0$.

| $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\cdots$ | $\mathbf{a}_m$ | $\mathbf{a}_{m+1}$ | $\mathbf{a}_{m+2}$ | $\cdots$ | $\mathbf{a}_j$ | $\cdots$ | $\mathbf{a}_n$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | $\cdots$ | 0 | $\bar{a}_{1(m+1)}$ | $\bar{a}_{1(m+2)}$ | $\cdots$ | $\bar{a}_{1j}$ | $\cdots$ | $\bar{a}_{1n}$ | $\bar{a}_{10}$ |
| 0 | 1 | | 0 | $\bar{a}_{2(m+1)}$ | $\bar{a}_{2(m+2)}$ | $\cdots$ | $\bar{a}_{2j}$ | $\cdots$ | $\bar{a}_{2n}$ | $\bar{a}_{10}$ |
| . | . | | . | . | . | | . | | . | . |
| . | . | | . | . | . | | . | | . | . |
| 0 | 0 | | 0 | $\bar{a}_{i(m+1)}$ | $\bar{a}_{i(m+2)}$ | $\cdots$ | $\bar{a}_{ij}$ | $\cdots$ | $\bar{a}_{in}$ | $\bar{a}_{i0}$ |
| . | . | | . | . | . | | . | | . | . |
| . | . | | . | . | . | | . | | . | . |
| 0 | 0 | | 1 | $\bar{a}_{m(m+1)}$ | $\bar{a}_{m(m+2)}$ | $\cdots$ | $\bar{a}_{mj}$ | $\cdots$ | $\bar{a}_{mn}$ | $\bar{a}_{m0}$ |
| 0 | 0 | | 0 | $r_{m+1}$ | $r_{m+2}$ | $\cdots$ | $r_j$ | $\cdots$ | $r_n$ | $-z_0$ |

**Fig. 3.2** Canonical simplex tableau

The relative cost coefficients $r_j$ indicate whether the value of the objective will increase or decrease if $x_j$ is pivoted into the solution. If these coefficients are all nonnegative, then the indicated solution is optimal. If some of them are negative, an improvement can be made (assuming nondegeneracy) by bringing the corresponding component into the solution. When more than one of the relative cost coefficients is negative, any one of them may be selected to determine in which column to pivot. Common practice is to select the most negative value. (See Exercise 13 for further discussion of this point.)

Some more discussion of the relative cost coefficients and the last row of the tableau is warranted. We may regard $z$ as an additional variable and

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n - z = 0$$

as another equation. A basic solution to the augmented system will have $m+1$ basic variables, but we can require that $z$ be one of them. For this reason it is not necessary to add a column corresponding to $z$, since it would always be $(0, 0,\ \ldots,\ 0, 1)$. Thus, initially, a last row consisting of the $c_j$'s and a right-hand side of zero can be appended to the standard array to represent this additional equation. Using standard pivot operations, the elements in this row corresponding to basic variables can be reduced to zero. This is equivalent to transforming the additional equation to the form

$$r_{m+1}x_{m+1} + r_{m+2}x_{m+2} + \cdots + r_nx_n - z = -z_0. \tag{3.24}$$

This must be equivalent to (3.23), and hence the $r_j$'s obtained are the relative cost coefficients. Thus, the last row can be treated operationally like any other row: just start with $c_j$'s and reduce the terms corresponding to basic variables to zero by row operations.

After a column $q$ is selected in which to pivot, the final selection of the pivot element is made by computing the ratio $\bar{a}_{i0}/\bar{a}_{iq}$ for the positive elements $\bar{a}_{iq}$, $i = 1, 2, \ldots, m$, of the $q$th column and selecting the element $p$ yielding the minimum ratio. Pivoting on this element will maintain feasibility as well as (assuming nondegeneracy) decrease the value of the objective function. If there are ties, any element yielding the minimum can be used. If there are no nonnegative elements in the column, the problem is unbounded. After updating the entire tableau with $\bar{a}_{pq}$ as pivot and transforming the last row in the same manner as all other rows (except row $q$), we obtain a new tableau in canonical form. The new value of the objective function again appears in the lower right-hand corner of the tableau.

The simplex algorithm can be summarized by the following steps:

*Step* 0. Form a tableau as in Fig. 3.2 corresponding to a basic feasible solution. The relative cost coefficients can be found by row reduction.
*Step* 1. If each $r_j \geqslant 0$, stop; the current basic feasible solution is optimal.
*Step* 2. Select $q$ such that $r_q < 0$ to determine which nonbasic variable is to become basic.
*Step* 3. Calculate the ratios $\bar{a}_{i0}/\bar{a}_{iq}$ for $\bar{a}_{iq} > 0$, $i = 1, 2, \ldots, m$. If no $\bar{a}_{iq} > 0$, stop; the problem is unbounded. Otherwise, select $p$ as the index $i$ corresponding to the minimum ratio.
*Step* 4. Pivot on the $pq$th element, updating all rows including the last. Return to Step 1.

Proof that the algorithm solves the problem (again assuming nondegeneracy) is essentially established by our previous development. The process terminates only if optimality is achieved or unboundedness is discovered. If neither condition is discovered at a given basic solution, then the objective is strictly decreased. Since there are only a finite number of possible basic feasible solutions, and no basis repeats because of the strictly decreasing objective, the algorithm must reach a basis satisfying one of the two terminating conditions.

*Example 1.* Maximize $3x_1 + x_2 + 3x_3$ subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leqslant 2 \\ x_1 + 2x_2 + 3x_3 &\leqslant 5 \\ 2x_1 + 2x_2 + x_3 &\leqslant 6 \\ x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 &\geqslant 0. \end{aligned}$$

To transform the problem into standard form so that the simplex procedure can be applied, we change the maximization to minimization by multiplying the objective function by minus one, and introduce three nonnegative slack variables $x_4$, $x_5$, $x_6$. We then have the initial tableau

| | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{a}_4$ | $\mathbf{a}_5$ | $\mathbf{a}_6$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|---|
| | ② | ① | 1 | 1 | 0 | 0 | 2 |
| | 1 | 2 | ③ | 0 | 1 | 0 | 5 |
| | 2 | 2 | 1 | 0 | 0 | 1 | 6 |
| $\mathbf{r}^T$ | −3 | −1 | −3 | 0 | 0 | 0 | 0 |

First tableau

The problem is already in canonical form with the three slack variables serving as the basic variables. We have at this point $r_j = c_j - z_j = c_j$, since the costs of the slacks are zero. Application of the criterion for selecting a column in which to pivot shows that any of the first three columns would yield an improved solution. In each of these columns the appropriate pivot element is determined by computing the ratios $\bar{a}_{i0}/\bar{a}_{ij}$ and selecting the smallest positive one. The three allowable pivots are all circled on the tableau. It is only necessary to determine one allowable pivot, and normally we would not bother to calculate them all. For hand calculation on problems of this size, however, we may wish to examine the allowable pivots and select one that will minimize (at least in the short run) the amount of division required. Thus for this example we select the second column and result in:

| | | | | | | |
|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 0 | 0 | 2 |
| −3 | 0 | ① | −2 | 1 | 0 | 1 |
| −2 | 0 | −1 | −2 | 0 | 1 | 2 |
| −1 | 0 | −2 | 1 | 0 | 0 | 2 |

Second tableau

We note that the objective function—we are using the negative of the original one—has decreased from zero to minus two. We now pivot on ①.

| | | | | | | |
|---|---|---|---|---|---|---|
| ⑤ | 1 | 0 | 3 | −1 | 0 | 1 |
| −3 | 0 | 1 | −2 | 1 | 0 | 1 |
| −5 | 0 | 0 | −4 | 1 | 1 | 3 |
| −7 | 0 | 0 | −3 | 2 | 0 | 4 |

Third tableau

The value of the objective function has now decreased to minus four and we may pivot in either the first or fourth column. We select ⑤.

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1/5 | 0 | 3/5 | −1/5 | 0 | 1/5 |
| 0 | 3/5 | 1 | −1/5 | 2/5 | 0 | 8/5 |
| 0 | 1 | 0 | −1 | 0 | 1 | 4 |
| 0 | 7/5 | 0 | 6/5 | 3/5 | 0 | 27/5 |

Fourth tableau

Since the last row has no negative elements, we conclude that the solution corresponding to the fourth tableau is optimal. Thus $x_1 = 1/5,\ x_2 = 0,\ x_3 = 8/5,\ x_4 = 0,\ x_5 = 0,\ x_6 = 4$ is the optimal solution with a corresponding value of the (negative) objective of $-(27/5)$.

### *Degeneracy*

It is possible that in the course of the simplex procedure, degenerate basic feasible solutions may occur. Often they can be handled as a nondegenerate basic feasible solution. However, it is possible that after a new column $q$ is selected to enter the basis, the minimum of the ratios $\bar{a}_{i0}/\bar{a}_{iq}$ may be zero, implying that the zero-valued basic variable is the one to go out. This means that the new variable $x_q$ will come in at zero value, the objective will not decrease, and the new basic feasible solution will also be degenerate. Conceivably, this process could continue for a series of steps until, finally, the original degenerate solution is again obtained. The result is a *cycle* that could be repeated indefinitely.

Methods have been developed to avoid such cycles (see Exercises 15–17 for a full discussion of one of them, which is based on perturbing the problem slightly so that zero-valued variables are actually small positive values, and Exercise 32 for Bland's rule, which is simpler). In practice, however, such procedures are found to be unnecessary. When degenerate solutions are encountered, the simplex procedure generally does not enter a cycle. However, anticycling procedures are simple, and many codes incorporate such a procedure for the sake of safety.

## 3.5 Finding a Basic Feasible Solution

A basic feasible solution is sometimes immediately available for linear programs. For example, in problems with constraints of the form

$$\mathbf{Ax} \leqslant \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0} \tag{3.25}$$

with $\mathbf{b} \geqslant \mathbf{0}$, a basic feasible solution to the corresponding standard form of the problem is provided by the slack variables. This provides a means for initiating the simplex procedure. The example in the last section was of this type. An initial basic feasible solution is not always apparent for other types of linear programs, however, and it is necessary to develop a means for determining one so that the simplex method can be initiated. Interestingly (and fortunately), an auxiliary linear program and corresponding application of the simplex method can be used to determine the required initial solution.

By elementary straightforward operations the constraints of a linear programming problem can always be expressed in the form

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{0} \tag{3.26}$$

with $\mathbf{b} \geqslant \mathbf{0}$. In order to find a solution to (3.26) consider the artificial minimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} u_j \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{u} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \ \mathbf{u} \geqslant \mathbf{0} \end{aligned} \tag{3.27}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ is a vector of artificial variables. If there is a feasible solution to (3.26), then it is clear that (3.27) has a minimum value of zero with $\mathbf{u} = \mathbf{0}$. If (3.26) has no feasible solution, then the minimum value of (3.27) is greater than zero.

Now (3.27) is itself a linear program in the variables $\mathbf{x}$, $\mathbf{u}$, and the system is already in canonical form with basic feasible solution $\mathbf{u} = \mathbf{b}$. If (3.27) is solved using the simplex technique, a basic feasible solution is obtained at each step. If the minimum value of (3.27) is zero, then the final basic solution will have all $u_j = 0$, and hence barring degeneracy, the final solution will have no $u_j$ variables basic. If in the final solution some $u_j$ are both zero and basic, indicating a degenerate solution, these basic variables can be exchanged for nonbasic $x_j$ variables (again at zero level) to yield a basic feasible solution involving $x$ variables only. (However, the situation is more complex if A is not of full rank. See Exercise 21.)

*Example 1.* Find a basic feasible solution to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 4 \\ 3x_1 + 3x_2 + x_3 &= 3 \\ x_1 \geqslant 0, \ x_2 \geqslant 0, \ x_3 &\geqslant 0. \end{aligned}$$

We introduce artificial variables $x_4 \geqslant 0$, $x_5 \geqslant 0$ and an objective function $x_4 + x_5$. The initial tableau is

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $\mathbf{b}$ |
|---|---|---|---|---|---|---|
| | 2 | 1 | 2 | 1 | 0 | 4 |
| | 3 | 3 | 1 | 0 | 1 | 3 |
| $\mathbf{c}^T$ | 0 | 0 | 0 | 1 | 1 | 0 |

Initial tableau

A basic feasible solution to the expanded system is given by the artificial variables. To initiate the simplex procedure we must update the last row so that it has zero components under the basic variables. This yields:

| | | | | | | |
|---|---|---|---|---|---|---|
| | 2 | 1 | 2 | 1 | 0 | 4 |
| | ③ | 3 | 1 | 0 | 1 | 3 |
| $\mathbf{r}^T$ | −5 | −4 | −3 | 0 | 0 | −7 |

First tableau

Pivoting in the column having the most negative bottom row component as indicated, we obtain:

| | | | | | |
|---|---|---|---|---|---|
| 0 | −1 | (4/3) | 1 | −2/3 | 2 |
| 1 | 1 | 1/3 | 0 | 1/3 | 1 |
| 0 | 1 | −4/3 | 0 | 5/3 | −2 |

Second tableau

In the second tableau there is only one choice for pivot, and it leads to the final tableau shown.

| | | | | | |
|---|---|---|---|---|---|
| 0 | −3/4 | 1 | 3/4 | −1/2 | 3/2 |
| 1 | 5/4 | 0 | −1/4 | 1/2 | 1/2 |
| 0 | 0 | 0 | 1 | 1 | 0 |

Final tableau

Both of the artificial variables have been driven out of the basis, thus reducing the value of the objective function to zero and leading to the basic feasible solution to the original problem

$$x_1 = 1/2, \quad x_2 = 0, \quad x_3 = 3/2.$$

Using artificial variables, we attack a general linear programming problem by use of the *two-phase method*. This method consists simply of a *phase I* in which artificial variables are introduced as above and a basic feasible solution is found (or it is determined that no feasible solutions exist); and a *phase II* in which, using the basic feasible solution resulting from phase I, the original objective function is minimized. During phase II the artificial variables and the objective function of phase I are omitted. Of course, in phase I artificial variables need be introduced only in those equations that do not contain slack variables.

*Example 2.* Consider the problem

$$\begin{aligned} \text{minimize} \quad & 4x_1 + x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + x_2 + 2x_3 = 4 \\ & 3x_1 + 3x_2 + x_3 = 3 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{aligned}$$

There is no basic feasible solution apparent, so we use the two-phase method. The first phase was done in Example 1 for these constraints, so we shall not repeat it here. We give only the final tableau with the columns corresponding to the artificial variables deleted, since they are not used in phase II. We use the new cost function in place of the old one. Temporarily writing $\mathbf{c}^T$ in the bottom row we have

| | $x_1$ | $x_2$ | $x_3$ | $\mathbf{b}$ |
|---|---|---|---|---|
| | 0 | −3/4 | 1 | 3/2 |
| | 1 | 5/4 | 0 | 1/2 |
| $\mathbf{c}^T$ | 4 | 1 | 1 | 0 |

Initial tableau

Transforming the last row so that zeros appear in the basic columns, we have

| | | | |
|---|---|---|---|
| 0 | −3/4 | 1 | 3/2 |
| 1 | (5/4) | 0 | 1/2 |
| 0 | −13/4 | 0 | −7/2 |

First tableau

| | | | |
|---|---|---|---|
| 3/5 | 0 | 1 | 9/5 |
| 4/5 | 1 | 0 | 2/5 |
| 13/5 | 0 | 0 | −11/5 |

Second tableau

and hence the optimal solution is $x_1 = 0,\ x_2 = 2/5,\ x_3 = 9/5$.

*Example 3 (A Free Variable Problem).*

$$\begin{array}{ll} \text{minimize} & -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\ \text{subject to} & -x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 7 \\ & -x_1 + 2x_2 + 3x_3 + x_4 + x_5 = 6 \\ & -x_1 + x_2 + x_3 + 2x_4 + x_5 = 4 \\ & x_1 \text{ free},\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 \geqslant 0,\ x_5 \geqslant 0. \end{array}$$

Since $x_1$ is free, it can be eliminated, as described in Chap. 2, by solving for $x_1$ in terms of the other variables from the first equation and substituting everywhere else. This can all be done with the simplex tableau as follows:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | **b** |
|---|---|---|---|---|---|---|
| | −(1) | 1 | 2 | 1 | 2 | 7 |
| | −1 | 2 | 3 | 1 | 1 | 6 |
| | −1 | 1 | 1 | 2 | 1 | 4 |
| $\mathbf{c}^T$ | −2 | 4 | 7 | 1 | 5 | 0 |

Initial tableau

We select any nonzero element in the first column to pivot on—this will eliminate $x_1$.

| | | | | | |
|---|---|---|---|---|---|
| 1 | −1 | −2 | −1 | −2 | −7 |
| 0 | 1 | 1 | 0 | −1 | −1 |
| 0 | 0 | −1 | 1 | −1 | −3 |
| 0 | 2 | 3 | −1 | 1 | −14 |

Equivalent problem

We now save the first row for future reference, but our linear program only involves the sub-tableau indicated. There is no obvious basic feasible solution for this problem, so we introduce artificial variables $x_6$ and $x_7$.

| | $x_2$ | $x_3$ | $x_4$, | $x_5$ | $x_6$ | $x_7$ | **b** |
|---|---|---|---|---|---|---|---|
| | −1 | −1 | 0 | 1 | 1 | 0 | 1 |
| | 0 | 1 | −1 | 1 | 0 | 1 | 3 |
| $\mathbf{c}^T$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Initial tableau for phase I

Transforming the last row appropriately we obtain

| | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | **b** |
|---|---|---|---|---|---|---|---|
| | −1 | −1 | 0 | ① | 1 | 0 | 1 |
| | 0 | 1 | −1 | 1 | 0 | 1 | 3 |
| $\mathbf{r}^T$ | 1 | 0 | 1 | −2 | 0 | 0 | −4 |

First tableau—phase I

| $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | **b** |
|---|---|---|---|---|---|---|
| −1 | −1 | 0 | 1 | 1 | 0 | 1 |
| ① | 2 | −1 | 0 | −1 | 1 | 2 |
| −1 | −2 | 1 | 0 | 2 | 0 | −2 |

Second tableau—phase I

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 1 | −1 | 1 | 0 | 1 | 3 |
| 1 | 2 | −1 | 0 | −1 | 1 | 2 |
| 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Final tableau—phase I

Now we go back to the equivalent reduced problem

| | $x_2$ | $x_3$ | $x_4$ | $x_5$ | **b** |
|---|---|---|---|---|---|
| | 0 | 1 | −1 | 1 | 3 |
| | 1 | 2 | −1 | 0 | 2 |
| $\mathbf{c}^T$ | 2 | 3 | −1 | 1 | −14 |

Initial tableau—phase II

Transforming the last row appropriately we proceed with:

| | | | | |
|---|---|---|---|---|
| 0 | 1 | −1 | 1 | 3 |
| 0 | ② | −1 | 0 | 2 |
| 0 | −2 | 2 | 0 | −21 |

First tableau—phase II

| | | | | |
|---|---|---|---|---|
| −1/2 | 0 | −1/2 | 1 | 2 |
| 1/2 | 1 | −1/2 | 0 | 1 |
| 1 | 0 | 1 | 0 | −19 |

Final tableau—phase II

The solution $x_3 = 1,\ x_5 = 2$ can be inserted in the expression for $x_1$ giving

$$x_1 = -7 + 2 \cdot 1 + 2 \cdot 2 = -1;$$

thus the final solution is

$$x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 0,\ x_5 = 2.$$

## 3.6 Matrix Form of the Simplex Method

Although the elementary pivot transformations associated with the simplex method are in many respects most easily discernible in the tableau format, with attention focused on the individual elements, there is much insight to be gained by studying a matrix interpretation of the procedure. The vector-matrix relationships that exist between the various rows and columns of the tableau lead, however, not only to increased understanding but also, in a rather direct way, to the *revised simplex procedure* which in many cases can result in considerable computational advantage. The matrix formulation is also a natural setting for the discussion of dual linear programs and other topics related to linear programming.

A preliminary observation in the development is that the tableau at any point in the simplex procedure can be determined solely by a knowledge of which variables are basic. As before we denote by **B** the submatrix of the original A matrix consisting of the $m$ columns of A corresponding to the basic variables. These columns are linearly independent and hence the columns of **B** form a basis for $E^m$. We refer to **B** as the basis matrix.

As usual, let us assume that **B** consists of the first $m$ columns of A. Then by partitioning $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{c}^T$ as

$$\mathbf{A} = [\mathbf{B},\ \mathbf{D}]$$
$$\mathbf{x} = (\mathbf{x_B},\ \mathbf{x_D}), \quad \mathbf{c}^T = \left[\mathbf{c}_\mathbf{B}^T,\ \mathbf{c}_\mathbf{D}^T\right],$$

the standard linear program becomes

$$\begin{aligned} &\text{minimize} && \mathbf{c}_\mathbf{B}^T\mathbf{x_B} + \mathbf{c}_\mathbf{D}^T\mathbf{x_D} \\ &\text{subject to} && \mathbf{Bx_B} + \mathbf{Dx_D} = \mathbf{b} \\ &&& \mathbf{x_B} \geqslant \mathbf{0},\ \mathbf{x_D} \geqslant \mathbf{0}. \end{aligned} \tag{3.28}$$

The basic solution, which we assume is also feasible, corresponding to the basis **B** is $\mathbf{x} = (\mathbf{x_B},\ 0)$ where $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$. The basic solution results from setting $\mathbf{x_D} = 0$. However, for any value of $\mathbf{x_D}$ the necessary value of $\mathbf{x_B}$ can be computed from (3.28) as

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Dx_D}, \tag{3.29}$$

and this general expression when substituted in the cost function yields

$$\begin{aligned} z &= \mathbf{c}_\mathbf{B}^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{Dx_D}) + \mathbf{c}_\mathbf{D}^T\mathbf{x_D} \\ &= \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D})\mathbf{x_D}, \end{aligned} \tag{3.30}$$

which expresses the cost of any solution to (3.28) in terms of $\mathbf{x_D}$. Thus

$$\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D} \tag{3.31}$$

is the relative cost vector (for nonbasic variables). It is the components of this vector that are used to determine which vector to bring into the basis.

Having derived the vector expression for the relative cost it is now possible to write the simplex tableau in matrix form. The initial tableau takes the form

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \\ \hdashline \mathbf{c}^T & 0 \end{array}\right] = \left[\begin{array}{c:c:c} \mathbf{B} & \mathbf{D} & \mathbf{b} \\ \hdashline \mathbf{c}_\mathbf{B}^T & \mathbf{c}_\mathbf{D}^T & 0 \end{array}\right], \tag{3.32}$$

which is not in general in canonical form and does not correspond to a point in the simplex procedure. If the matrix **B** is used as a basis, then the corresponding tableau becomes

$$\mathbf{T} = \left[\begin{array}{c:c:c} \mathbf{I} & \mathbf{B}^{-1}\mathbf{D} & \mathbf{B}^{-1}\mathbf{b} \\ \hdashline \mathbf{0} & \mathbf{c}_\mathbf{D}^T - \mathbf{C}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D} & -\mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{b} \end{array}\right], \tag{3.33}$$

which is the matrix form we desire.

## *The Revised Simplex Method and LU Decomposition

Extensive experience with the simplex procedure applied to problems from various fields, and having various values of $n$ and $m$, has indicated that the method can be expected to converge to an optimum solution in about $m$, or perhaps $3m/2$, pivot operations. (Except in the worst case. See Chap. 5.) Thus, particularly if $m$ is much smaller than $n$, that is, if the matrix **A** has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization.

Since the other columns are not explicitly used, it appears that the work expended in calculating the elements in these columns after each pivot is, in some sense, wasted effort. The revised simplex method is a scheme for ordering the computations required of the simplex method so that unnecessary calculations are avoided. In fact, even if pivoting is eventually required in all columns, but $m$ is small compared to $n$, the revised simplex method can frequently save computational effort.

The revised form of the simplex method is this: Given the inverse $\mathbf{B}^{-1}$ of a current basis, and the current solution $\mathbf{x}_\mathbf{B} = \bar{\mathbf{a}}_0 = \mathbf{B}^{-1}\mathbf{b}$,

*Step* 1. Calculate the current relative cost coefficients $\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}\mathbf{D}$. This can best be done by first calculating $\mathbf{y}^T = \mathbf{c}_\mathbf{B}^T\mathbf{B}^{-1}$ and then the relative cost vector $\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{y}^T\mathbf{D}$. If $\mathbf{r}_\mathbf{D} \geqslant \mathbf{0}$ stop; the current solution is optimal.

*Step* 2. Determine which vector $\mathbf{a}_q$ is to enter the basis by selecting the most negative cost coefficient; and calculate $\bar{\mathbf{a}}_q = \mathbf{B}^{-1}\mathbf{a}_q$ which gives the vector $\mathbf{a}_q$ expressed in terms of the current basis.

*Step* 3. If no $\bar{a}_{iq} > 0$, stop; the problem is unbounded. Otherwise, calculate the ratios $\bar{a}_{i0}/\bar{a}_{iq}$ for $\bar{a}_{iq} > 0$ to determine which vector is to leave the basis.

*Step* 4. Update $\mathbf{B}^{-1}$ and the current solution $\mathbf{B}^{-1}\mathbf{b}$. Return to Step 1.

Updating of $\mathbf{B}^{-1}$ is accomplished by the usual pivot operations applied to an array consisting of $\mathbf{B}^{-1}$ and $\bar{\mathbf{a}}_q$, where the pivot is the appropriate element in $\bar{\mathbf{a}}_q$. Of course $\mathbf{B}^{-1}\mathbf{b}$ may be updated at the same time by adjoining it as another column.

One may go one step further in the matrix interpretation of the simplex method and note that execution of a single simplex cycle is not explicitly dependent on having $\mathbf{B}^{-1}$ but rather on the ability to solve linear systems with $\mathbf{B}$ as the coefficient matrix. A decomposition of $\mathbf{B} = \mathbf{LU}$ can be updated where $\mathbf{L}$ is a lower triangular matrix and $\mathbf{U}$ is an upper triangular matrix. Then each of the linear systems can be solved by solving two triangular systems.

## 3.7 Simplex Method for Transportation Problems

The transportation problem was stated briefly in Chap. 2. We restate it here. There are $m$ origins that contain various amounts of a commodity that must be shipped to $n$ destinations to meet demand requirements. Specifically, origin $i$ contains an amount $a_i$, and destination $j$ has a requirement of amount $b_j$. It is assumed that the system is *balanced* in the sense that total supply equals total demand. That is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j. \tag{3.34}$$

The numbers $a_i$ and $b_j$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, are assumed to be nonnegative, and in many applications they are in fact nonnegative integers. There is a unit cost $c_{ij}$ associated with the shipping of the commodity from origin $i$ to destination $j$. The problem is to find the shipping pattern between origins and destinations that satisfies all the requirements and minimizes the total shipping cost.

In mathematical terms the above problem can be expressed as finding a set of $x_{ij}$' s, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, to

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i \quad \text{for} \quad i = 1, 2, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = b_j \quad \text{for} \quad j = 1, 2, \ldots, n \\ & x_{ij} \geqslant 0 \quad \text{for all} \quad i \text{ and } j. \end{aligned} \tag{3.35}$$

This mathematical problem, together with the assumption (3.34), is the general transportation problem. In the shipping context, the variables $x_{ij}$ represent the amounts of the commodity shipped from origin $i$ to destination $j$.

The structure of the problem can be seen more clearly by writing the constraint equations in standard form:

$$
\begin{array}{ccccccccccccc}
x_{11} + x_{12} + \cdots + x_{1n} & & & & & & & & = a_1 \\
& & x_{21} + x_{22} + \cdots + x_{2n} & & & & & & = a_2 \\
& & & & & & & & \vdots \\
& & & & & & x_{m1} + x_{m2} + \cdots + x_{mn} & & = a_m \\
\hline
x_{11} & & + x_{21} & & & & x_{m1} & & = b_1 \\
& x_{12} & & + x_{22} & & & & + x_{m2} & = b_2 \\
& & & & & & & & \vdots \\
& & x_{1n} & & & + x_{2n} & & + x_{mn} & = b_n
\end{array}
\tag{3.36}
$$

The structure is perhaps even more evident when the coefficient matrix **A** of the system of equations above is expressed in vector-matrix notation as

$$
\mathbf{A} = \begin{bmatrix} \mathbf{1}^T & & & \\ & \mathbf{1}^T & & \\ & & \ddots & \\ & & & \mathbf{1}^T \\ \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \end{bmatrix}, \tag{3.37}
$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ is $n$-dimensional, and where each $\mathbf{I}$ is an $n \times n$ identity matrix.

In practice it is usually unnecessary to write out the constraint equations of the transportation problem in the explicit form (3.36). A specific transportation problem is generally defined by simply presenting the data in compact form, such as:

$$
\mathbf{a} = (a_1,\ a_2, \ldots,\ a_m)
$$

$$
\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}.
$$

$$
\mathbf{b} = (b_1,\ b_2, \ldots,\ b_n)
$$

The solution can also be represented by an $m \times n$ array, and as we shall see, all computations can be made on arrays of a similar dimension.

*Example 1.* As an example, which will be solved completely in a later section, a specific transportation problem with four origins and five destinations is defined by

$$
\mathbf{a} = (30,\ 80,\ 10,\ 60)
$$

$$
\mathbf{C} = \begin{bmatrix} 3 & 4 & 6 & 8 & 9 \\ 2 & 2 & 4 & 5 & 5 \\ 2 & 2 & 2 & 3 & 2 \\ 3 & 3 & 2 & 4 & 2 \end{bmatrix}.
$$

$$
\mathbf{b} = (10,\ 50,\ 20,\ 80,\ 20)
$$

Note that the balance requirement is satisfied, since the sum of the supply and the demand are both 180.

### *Finding a Basic Feasible Solution*

A first step in the study of the structure of the transportation problem is to show that there is always a feasible solution, thus establishing that the problem is well defined. A feasible solution can be found by allocating shipments from origins to destinations in proportion to supply and demand requirements. Specifically, let $S$ be equal to the total supply (which is also equal to the total demand). Then let $x_{ij} = a_i b_j / S$ for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. The reader can easily verify that this is a feasible solution. We also note that the solutions are bounded, since each $x_{ij}$ is bounded by $a_i$ (and by $b_j$). A bounded program with a feasible solution has an optimal solution. Thus, a transportation problem always has an optimal solution.

A second step in the study of the structure of the transportation problem is based on a simple examination of the constraint equations. Clearly there are $m$ equations corresponding to origin constraints and $n$ equations corresponding to destination constraints—a total of $n + m$. However, it is easily noted that the sum of the origin equations is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i, \tag{3.38}$$

and the sum of the destination equations is

$$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j. \tag{3.39}$$

The left-hand sides of these equations are equal. Since they were formed by two distinct linear combinations of the original equations, it follows that the equations in the original system are not independent. The right-hand sides of (3.38) and (3.39) are equal by the assumption that the system is balanced, and therefore the two equations are, in fact, consistent. However, it is clear that the original system of equations is redundant. This means that one of the constraints can be eliminated without changing the set of feasible solutions. Indeed, *any* one of the constraints can be chosen as the one to be eliminated, for it can be reconstructed from those remaining. It follows that a basis for the transportation problem consists of $m + n - 1$ vectors, and a nondegenerate basic feasible solution consists of $m + n - 1$ variables. The simple solution found earlier in this section is clearly not a basic solution.

There is a straightforward way to compute an initial basic feasible solution to a transportation problem. The method is worth studying at this stage because it introduces the computational process that is the foundation for the general solution technique based on the simplex method. It also begins to illustrate the fundamental property of the structure of transportation problems.

### The Northwest Corner Rule

This procedure is conducted on the *solution array* shown below:

| | | | | | |
|---|---|---|---|---|---|
| $x_{11}$ | $x_{12}$ | $x_{13}$ | $\cdots$ | $x_{1n}$ | $a_1$ |
| $x_{21}$ | $x_{22}$ | $x_{23}$ | $\cdots$ | $x_{2n}$ | $a_2$ |
| $\vdots$ | | | | | $\vdots$ |
| $x_{m1}$ | $x_{m2}$ | $x_{m3}$ | $\cdots$ | $x_{mn}$ | $a_m$ |
| $b_1$ | $b_2$ | $b_3$ | $\cdots$ | $b_n$ | |

(3.40)

The individual elements of the array appear in *cells* and represent a solution. An empty cell denotes a value of zero.

Beginning with all empty cells, the procedure is given by the following steps:

*Step* 1. Start with the cell in the upper left-hand corner.
*Step* 2. Allocate the maximum feasible amount consistent with row and column sum requirements involving that cell. (At least one of these requirements will then be met.)
*Step* 3. Move one cell to the right if there is any remaining row requirement (supply). Otherwise move one cell down. If all requirements are met, stop; otherwise go to Step 2.

The procedure is called the *Northwest Corner Rule* because at each step it selects the cell in the upper left-hand corner of the subarray consisting of current nonzero row and column requirements.

*Example 1.* A basic feasible solution constructed by the Northwest corner Rule is shown below for Example 1 of the last section.

| | | | | | |
|---|---|---|---|---|---|
| 10 | 20 | | | | 30 |
| | 30 | 20 | 30 | | 80 |
| | | | 10 | | 10 |
| | | | 40 | 20 | 60 |
| 10 | 50 | 20 | 80 | 20 | |

(3.41)

In the first step, at the upper left-hand corner, a maximum of 10 units could be allocated, since that is all that was required by column 1. This left $30 - 10 = 20$ units required in the first row. Next, moving to the second cell in the top row, the remaining 20 units were allocated. At this point the row 1 requirement is met, and it is necessary to move down to the second row. The reader should be able to follow the remaining steps easily.

There is the possibility that at some point both the row and column requirements corresponding to a cell may be met. The next entry will then be a zero, indicating a degenerate basic solution. In such a case there is a choice as to where to place the zero. One can either move right or move down to enter the zero. Two examples of degenerate solutions to a problem are shown below:

| | | | | |
|---|---|---|---|---|
| 30 | | | | 30 |
| 20 | 20 | | | 40 |
| | 0 | 20 | | 20 |
| | | 20 | 40 | 60 |
| 50 | 20 | 40 | 40 | |

| | | | | |
|---|---|---|---|---|
| 30 | | | | 30 |
| 20 | 20 | 0 | | 40 |
| | | 20 | | 20 |
| | | 20 | 40 | 60 |
| 50 | 20 | 40 | 40 | |

It should be clear that the Northwest Corner Rule can be used to obtain different basic feasible solutions by first permuting the rows and columns of the array before the procedure is applied. Or equivalently, one can do this indirectly by starting the procedure at an arbitrary cell and then considering successive rows and columns in an arbitrary order.

## *Basis Triangularity*

We now establish the most important structural property of the transportation problem: the triangularity of all bases. This property simplifies the process of solution of a system of equations whose coefficient matrix corresponds to a basis, and thus leads to efficient implementation of the simplex method.

The concept of upper and lower triangular matrices was introduced in connection with Gaussian elimination methods, see Appendix C. It is useful at this point to generalize slightly the notion of upper and lower triangularity.

**Definition.** A nonsingular square matrix $\mathbf{M}$ is said to be *triangular* if by a permutation of its rows and columns it can be put in the form of a lower triangular matrix.

There is a simple and useful procedure for determining whether a given matrix $\mathbf{M}$ is triangular:

*Step* 1. Find a row with exactly one nonzero entry.
*Step* 2. Form a submatrix of the matrix used in Step 1 by crossing out the row found in Step 1 and the column corresponding to the nonzero entry in that row. Return to Step 1 with this submatrix.
If this procedure can be continued until all rows have been eliminated, then the matrix is triangular. It can be put in lower triangular form explicitly by arranging the rows and columns in the order that was determined by the procedure.

*Example 1.* Shown below on the left is a matrix before the above procedure is applied to it. Indicated along the edges of this matrix is the order in which the rows and columns are indexed according to the procedure. Shown at the right is the same matrix when its rows and columns are permuted according to the order found.

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 4 & 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 2 & 1 & 7 & 2 & 1 & 3 \\ 2 & 3 & 2 & 0 & 0 & 3 \\ 0 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} 3 \\ 6 \\ 2 \\ 1 \\ 5 \\ 4 \\ \end{matrix} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 5 & 1 & 4 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 3 & 2 & 3 & 2 & 0 \\ 2 & 1 & 2 & 3 & 7 & 1 \end{bmatrix}$$
$$\begin{matrix} 4 & 2 & 1 & 6 & 3 & 5 \end{matrix}$$

Triangularization

We are now prepared to derive the most important structural property of the transportation problem.

***Basis Triangularity Theorem.*** *Every basis of the transportation problem is triangular.*

*Proof.* Refer to the system of constraints (3.36). Let us change the sign of the top half of the system; then the coefficient matrix of the system consists of entries that are either +1, −1, or 0. Following the result of the theorem in Sect. 3.7, delete any one of the equations to eliminate the redundancy. From the resulting coefficient matrix, form a basis **B** by selecting a nonsingular subset of $m + n - 1$ columns.

Each column of **B** contains at most two nonzero entries, a + 1 and a − 1. Thus there are at most $2(m+n-1)$ nonzero entries in the basis. However, if every column contained two nonzero entries, then the sum of all rows would be zero, contradicting the nonsingularity of **B**. Thus at least one column of **B** must contain only one nonzero entry. This means that the total number of nonzero entries in **B** is less than $2(m + n - 1)$. It then follows that there must be a row with only one nonzero entry; for if every row had two or more nonzero entries, the total number would be at least $2(m+n-1)$. This means that the first step of the procedure for verifying triangularity is satisfied. A similar argument can be applied to the submatrix of **B** obtained by crossing out the row with the single nonzero entry and the column corresponding to that entry; that submatrix must also contain a row with a single nonzero entry. This argument can be continued, establishing that the basis **B** is triangular. ∎

*Example 2.* As an illustration of the Basis Triangularity Theorem, consider the basis selected by the Northwest Corner Rule in Example 1. This basis is represented below, except that only the basic variables are indicated, not their values.

| | | | | | |
|---|---|---|---|---|---|
| $x_{11}$ | $x_{12}$ | | | | 30 |
| | $x_{22}$ | $x_{23}$ | $x_{24}$ | | 80 |
| | | | $x_{34}$ | | 10 |
| | | | $x_{44}$ | $x_{45}$ | 60 |
| 10 | 50 | 20 | 80 | 20 | |

A row in a basis matrix corresponds to an equation in the original system and is associated with a constraint either on a row or column sum in the solution array. In this example the equation corresponding to the first column sum contains only one basis variable, $x_{11}$. The value of this variable can be found immediately to be 10. The next equation corresponds to the first row sum. The corresponding variable is $x_{12}$, which can be found to be 20, since $x_{11}$ is known. Progression in this manner through the basis variables is equivalent to back substitution.

The importance of triangularity is, of course, the associated method of *back substitution* for the solution of a triangular system of equations, as discussed in Appendix C. Moreover, since any basis matrix is triangular and all nonzero elements are equal to one (or minus one if the signs of some equations are changed), it follows that the process of back substitution will simply involve repeated additions and subtractions of the given row and column sums. No multiplication is required. It therefore follows that if the original row and column totals are integers, the values of all basic variables will be integers. This is an important result, which we summarize by a corollary to the Basis Triangularity Theorem.

***Corollary.*** *If the row and column sums of a transportation problem are integers, then the basic variables in any basic solution are integers.*

## *The Transportation Simplex Method*

Now that the structural properties of the transportation problem have been developed, it is a relatively straightforward task to work out the details of the simplex method for the transportation problem. A major objective is to exploit fully the triangularity property of bases in order to achieve both computational efficiency and a compact representation of the method. The method used is actually a direct adaptation of the version of the revised simplex method presented in the first part of Sect. 3.6. The basis is never inverted; instead, its triangular form is used directly to solve for all required variables.

### Simplex Multipliers

Simplex multipliers are associated with the constraint equations. In this case we partition the vector of multipliers as $\mathbf{y} = (\mathbf{u},\ \mathbf{v})$. Here, $u_i$ represents the multiplier associated with the $i$th row sum constraint, and $v_j$ represents the multiplier associated with the $j$th column sum constraint. Since one of the constraints is redundant, an arbitrary value may be assigned to any one of the multipliers (see Exercise 4, Chap. 4). For notational simplicity we shall at this point set $v_n = 0$.

Given a basis $\mathbf{B}$, the simplex multipliers are found to be the solution to the equation $\mathbf{y}^T\mathbf{B} = \mathbf{c}_\mathbf{B}^T$. To determine the explicit form of these equations, we again refer to the original system of constraints (3.36). If $x_{ij}$ is basic, then the corresponding column from $\mathbf{A}$ will be included in $\mathbf{B}$. This column has exactly two +1 entries: one in the $i$th position of the top portion and one in the $j$th position of the bottom portion. This column thus generates the simplex multiplier equation $u_i + v_j = c_{ij}$, since $u_i$ and $v_j$ are the corresponding components of the multiplier vector. Overall, the simplex multiplier equations are

$$u_i + v_j = c_{ij}, \tag{3.42}$$

for all $i,\ j$ for which $x_{ij}$ is basic. The coefficient matrix of this system is the transpose of the basis matrix and hence it is triangular. Thus, this system can be solved by back substitution. This is similar to the procedure for finding the values of basic variables and, accordingly, as another corollary of the Triangular Basis Theorem, an integer property holds for simplex multipliers.

***Corollary.*** *If the unit costs $c_{ij}$ of a transportation problem are all integers, then (assuming one simplex multiplier is set arbitrarily equal to an integer) the simplex multipliers associated with any basis are integers.*

Once the simplex multipliers are known, the relative cost coefficients for nonbasic variables can be found in the usual manner as $\mathbf{r}_\mathbf{D}^T = \mathbf{c}_\mathbf{D}^T - \mathbf{y}^T\mathbf{D}$. In this case the relative cost coefficients are

$$r_{ij} = c_{ij} - u_i - v_j \quad \text{for} \quad \begin{array}{l} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n. \end{array} \tag{3.43}$$

This relation is valid for basic variables as well if we define relative cost coefficients for them—having value zero.

Given a basis, computation of the simplex multipliers is quite similar to the calculation of the values of the basic variables. The calculation is easily carried out on an array of the form shown below, where the circled elements correspond to the positions of the basic variables in the current basis.

| | | | | | |
|---|---|---|---|---|---|
| $c_{11}$ | $\textcircled{c_{12}}$ | $c_{13}$ | $\cdots$ | $c_{1n}$ | $u_1$ |
| $c_{21}$ | $\textcircled{c_{22}}$ | $c_{23}$ | $\cdots$ | $c_{2n}$ | $u_2$ |
| $\vdots$ | | | | $\vdots$ | $\vdots$ |
| $c_{m1}$ | | $\cdots$ | | $\textcircled{c_{mn}}$ | $u_m$ |
| $v_1$ | $v_2$ | | $\cdots$ | $v_n$ | |

In this case the main part of the array, with the coefficients $c_{ij}$, remains fixed, and we calculate the extra column and row corresponding to $\mathbf{u}$ and $\mathbf{v}$.

The procedure for calculating the simplex multipliers is this:

*Step* 1. Assign an arbitrary value to any one of the multipliers.

*Step* 2. Scan the rows and columns of the array until a circled element $c_{ij}$ is found such that either $u_i$ or $v_j$ (but not both) has already been determined.

*Step* 3. Compute the undetermined $u_i$ or $v_j$ from the equation $c_{ij} = u_i + v_j$. If all multipliers are determined, stop. Otherwise, return to Step 2.

The triangularity of the basis guarantees that this procedure can be carried through to determine all the simplex multipliers.

*Example 1.* Consider the cost array of Example 1 of Sect. 5.1, which is shown below with the circled elements corresponding to a basic feasible solution (found by the Northwest Corner Rule). Only these numbers are used in the calculation of the multipliers.

$$\begin{bmatrix} ③ & ④ & 6 & 8 & 9 \\ 2 & ② & ④ & ⑤ & 5 \\ 2 & 2 & 2 & ③ & 2 \\ 3 & 3 & 2 & ④ & ② \end{bmatrix}.$$

We first arbitrarily set $v_5 = 0$. We then scan the cells, searching for a circled element for which only one multiplier must be determined. This is the bottom right corner element, and it gives $u_4 = 2$. Then, from the equation $4 = 2 + v_4$, $v_4$ is found to be 2. Next, $u_3$ and $u_2$ are determined, then $v_3$ and $v_2$, and finally $u_1$ and $v_1$. The result is shown below:

| | | | | | | $u$ |
|---|---|---|---|---|---|---|
| | ③ | ④ | 6 | 8 | 9 | 5 |
| | 2 | ② | ④ | ⑤ | 5 | 3 |
| | 2 | 2 | 2 | ③ | 2 | 1 |
| | 3 | 3 | 2 | ④ | ② | 2 |
| $v$ | −2 | −1 | 1 | 2 | 0 | |

### Cycle of Change

In accordance with the general simplex procedure, if a nonbasic variable has an associated relative cost coefficient that is negative, then that variable is a candidate for entry into the basis. As the value of this variable is gradually increased, the values of the current basic variables will change continuously in order to maintain feasibility. Then, as usual, the value of the new variable is increased precisely to the point where one of the old basic variables is driven to zero.

We must work out the details of how the values of the current basic variables change as a new variable is entered. If the new basic vector is $\mathbf{d}$, then the change in the other variables is given by $-\mathbf{B}^{-1}\mathbf{d}$, where $\mathbf{B}$ is the current basis. Hence, once again we are faced with a problem of solving a system associated with the triangular basis, and once again the solution has special properties. In the next theorem recall that $\mathbf{A}$ is defined by (3.37).

***Theorem.*** *Let* $\mathbf{B}$ *be a basis from* $\mathbf{A}$ *(ignoring one row), and let* $\mathbf{d}$ *be another column. Then the components of the vector* $\mathbf{w} = \mathbf{B}^{-1}\mathbf{d}$ *are either* $0, +1$, *or* $-1$.

*Proof.* Let $\mathbf{w}$ be the solution to the equation $\mathbf{Bw} = \mathbf{d}$. Then $\mathbf{w}$ is the representation of $\mathbf{d}$ in terms of the basis. This equation can be solved by Cramer's rule as

$$w_k = \frac{\det \mathbf{B}_k}{\det \mathbf{B}},$$

where $\mathbf{B}_k$ is the matrix obtained by replacing the $k$th column of $\mathbf{B}$ by $\mathbf{d}$. Both $\mathbf{B}$ and $\mathbf{B}_k$ are submatrices of the original constraint matrix $\mathbf{A}$. The matrix $\mathbf{B}$ may be put in triangular form with all diagonal elements equal to +1. Hence, accounting for the sign change that may result from the combined row and column interchanges, $\det \mathbf{B} = +1$ or $-1$. Likewise, it can be shown (see Exercise 3) that $\det \mathbf{B}_k = 0, +1$, or $-1$. We conclude that each component of $\mathbf{w}$ is either $0,\ +1$, or $-1$. ∎

The implication of the above result is that when a new variable is added to the solution at a unit level, the current basic variables will each change by $+1,\ -1$, or 0. If the new variable has a value $\theta$, then, correspondingly, the basic variables change by $+\theta,\ -\theta$, or 0. It is therefore only necessary to determine the signs of change for each basic variable.

The determination of these signs is again accomplished by row and column scanning. Operationally, one assigns a + to the cell of the entering variable to represent a change of $+\theta$, where $\theta$ is yet to be determined. Then +'s, −'s, and 0's are assigned, one by one, to the cells of some basic variables, indicating changes of $+\theta,\ -\theta$, or 0 to maintain a solution. As usual, after each step there will always be an equation that uniquely determines the sign to be assigned to another basic variable. The result will be a sequence of pluses and minuses assigned to cells that form a cycle leading from the cell of the entering variable back to that cell. In essence, the new change is part of a cycle of redistribution of the commodity flow in the transportation system.

Once the sequence of +'s, −'s, and 0's is determined, the new basic feasible solution is found by setting the level of the change $\theta$. This is set so as to drive one of the old basic variables to zero. One must simply examine those basic variables for which a minus sign has been assigned, for these are the ones that will decrease as the new variable is introduced. Then $\theta$ is set equal to the smallest magnitude of these variables. This value is added to all cells that have a + assigned to them and subtracted from all cells that have a − assigned. The result will be the new basic feasible solution.

The procedure is illustrated by the following example.

*Example 2.* A completed solution array is shown below:

| | | | | | |
|---|---|---|---|---|---|
| | | $10^0$ | | | 10 |
| | | $20^-$ | | $10^+$ | 30 |
| $20^+$ | $10^0$ | | | $30^-$ | 60 |
| $10^0$ | | | | | 10 |
| $10^-$ | | + | $40^0$ | | 50 |
| 40 | 10 | 30 | 40 | 40 | |

In this example $x_{53}$ is the entering variable, so a plus sign is assigned there. The signs of the other cells were determined in the order $x_{13},\ x_{23},\ x_{25},\ x_{35},\ x_{32},\ x_{31},\ x_{41},\ x_{51},\ x_{54}$. The smallest variable with a minus assigned to it is $x_{51} = 10$. Thus we set $\theta = 10$.

### The Transportation Simplex Algorithm

It is now possible to put together the components developed to this point in the form of a complete revised simplex procedure for the transportation problem. The steps are:

*Step* 1. Compute an initial basic feasible solution using the Northwest Corner Rule or some other method.
*Step* 2. Compute the simplex multipliers and the relative cost coefficients. If all relative cost coefficients are nonnegative, stop; the solution is optimal. Otherwise, go to Step 3.
*Step* 3. Select a nonbasic variable corresponding to a negative cost coefficient to enter the basis (usually the one corresponding to the most negative cost coefficient). Compute the cycle of change and set $\theta$ equal to the smallest basic variable with a minus assigned to it. Update the solution. Go to Step 2.

*Example 3.* We can now completely solve the problem that was introduced in Example 1 of the first section. The requirements and a first basic feasible solution obtained by the Northwest Corner Rule are shown below. The plus and minus signs indicated on the array should be ignored at this point, since they cannot be computed until the next step is completed.

| | | | | | |
|---|---|---|---|---|---|
| 10 | 20 | | | | 30 |
| | 30 | $20^-$ | $30^+$ | | 80 |
| | | | $10^0$ | | 10 |
| | | + | $40^-$ | $20^0$ | 60 |
| 10 | 50 | 20 | 80 | 20 | |

The cost coefficients of the problem are shown in the array below, with the circled cells corresponding to the current basic variables. The simplex multipliers, computed by row and column scanning, are shown as well.

| | | | | | |
|---|---|---|---|---|---|
| ③ | ④ | 6 | 8 | 9 | 5 |
| 2 | ② | ④ | ⑤ | 5 | 3 |
| 2 | 2 | 2 | ③ | 2 | 1 |
| 3 | 3 | 2 | ④ | ② | 2 |
| −2 | −1 | 1 | 2 | 0 | |

The relative cost coefficients are found by subtracting $u_j + v_j$ from $c_{ij}$. In this case the only negative result is in cell 4,3; so variable $x_{43}$ will be brought into the basis. Thus a + is entered into this cell in the original array, and the cycle of zeros and plus and minus signs is determined as shown in that array. (It is not necessary to continue scanning once a complete cycle is determined.)

The smallest basic variable with a minus sign is 20 and, accordingly, 20 is added or subtracted from elements of the cycle as indicated by the signs. This leads to the new basic feasible solution shown in the array below:

| | | | | | |
|---|---|---|---|---|---|
| 10 | 20 | | | | 30 |
| | 30 | | 50 | | 80 |
| | | | 10 | | 10 |
| | | 20 | 20 | 20 | 60 |
| 10 | 50 | 20 | 80 | 20 | |

The new simplex multipliers corresponding to the new basis are computed, and the cost array is revised as shown below. In this case all relative cost coefficients are positive, indicating that the current solution is optimal.

| | | | | | |
|---|---|---|---|---|---|
| ③ | ④ | 6 | 8 | 9 | 5 |
| 2 | ② | 4 | ⑤ | 5 | 3 |
| 2 | 2 | 2 | ③ | 2 | 1 |
| 3 | 3 | ② | ④ | ② | 2 |
| −2 | −1 | 0 | 2 | 0 | |

## Degeneracy

As in all linear programming problems, degeneracy, corresponding to a basic variable having the value zero, can occur in the transportation problem. If degeneracy is encountered in the simplex procedure, it can be handled quite easily by introduction of the standard perturbation method (see Exercise 15, Chap. 3). In this method a zero-valued basic variable is assigned the value $\varepsilon$ and is then treated in the usual way. If it later leaves the basis, then the $\varepsilon$ can be dropped.

*Example 4.* To illustrate the method of dealing with degeneracy, consider a modification of Example 3, with the fourth row sum changed from 60 to 20 and the fourth column sum changed from 80 to 40. Then the initial basic feasible solution found by the Northwest Corner Rule is degenerate. An $\varepsilon$ is placed in the array for the zero-valued basic variable as shown below:

| | | | | | |
|---|---|---|---|---|---|
| 10 | 20 | | | | 30 |
| | 30 | $20^-$ | $30^+$ | | 80 |
| | | | $10^0$ | | 10 |
| | | + | $\varepsilon^-$ | $20^0$ | 20 |
| 10 | 50 | 20 | 40 | 20 | |

The relative cost coefficients will be the same as in Example 3, and hence again $x_{43}$ should be chosen to enter, and the cycle of change is the same as before. In this case, however, the change is only $\varepsilon$, and variable $x_{44}$ leaves the basis. The new relative cost coefficients are all positive, indicating that the new solution is optimal. Now the $\varepsilon$ can be dropped to yield the final solution (which is, itself, degenerate in this case).

| | | | | | |
|---|---|---|---|---|---|
| 10 | 20 | | | | 30 |
| | 30 | 20 | 30 | | 80 |
| | | | 10 | | 10 |
| | | $\varepsilon$ | | 20 | 20 |
| 10 | 50 | 20 | 40 | 20 | |

## *3.8 Decomposition

Large linear programming problems usually have some special structural form that can (and should) be exploited to develop efficient computational procedures. One common structure is where there are a number of separate activity areas that are linked through common resource constraints. An example is provided by a multidivisional firm attempting to minimize the total cost of its operations. The divisions of the firm must each meet internal requirements that do not interact with the constraints of other divisions; but in addition there are common resources that must be shared among divisions and thereby represent linking constraints.

A problem of this form can be solved by the Dantzig-Wolfe decomposition method described in this section. The method is an iterative process where at each step a number of separate subproblems are solved. The subproblems are themselves linear programs within the separate areas (or within divisions in the example of the firm). The objective functions of these subproblems are varied from iteration to iteration and are determined by a separate calculation based on the results of the previous iteration. This action coordinates the individual subproblems so that, ultimately, the solution to the overall problem is solved. The method can be derived as a special version of the revised simplex method, where the subproblems correspond to evaluation of reduced cost coefficients for the main problem.

To describe the method we consider the linear program in standard form

$$\begin{aligned} &\text{minimize} \quad \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geqslant \mathbf{0}. \end{aligned} \tag{3.44}$$

Suppose, for purposes of this entire section, that the $\mathbf{A}$ matrix has the special "block-angular" structure:

$$\mathbf{A} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \cdots & \mathbf{L}_N \\ \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_N \end{bmatrix} \tag{3.45}$$

By partitioning the vectors $\mathbf{x}$, $\mathbf{c}^T$, and $\mathbf{b}$ consistent with this partition of $\mathbf{A}$, the problem can be rewritten as

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{N} \mathbf{c}_i^T \mathbf{x}_i \\ \text{subject to} \quad & \sum_{i=1}^{N} \mathbf{L}_i \mathbf{x}_i = \mathbf{b}_0 \\ & \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i \\ & \mathbf{x}_i \geqslant \mathbf{0}, \ i = 1, \ldots, N. \end{aligned} \tag{3.46}$$

This may be viewed as a problem of minimizing the total cost of $N$ different linear programs that are independent except for the first constraint, which is a linking constraint of, say, dimension $m$.

Each of the subproblems is of the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}_i^T \mathbf{x}_i \\ \text{subject to} \quad & \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i, \ \mathbf{x}_i \geqslant \mathbf{0}. \end{aligned}$$

The constraint set for the $i$th subproblem is $S_i = \{\mathbf{x}_i : \mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i, \ \mathbf{x}_i \geqslant \mathbf{0}\}$. As for any linear program, this constraint set $S_i$ is a polytope and can be expressed as the intersection of a finite number of closed half-spaces. There is no guarantee that each $S_i$ is bounded, even if the original linear program (3.44) has a bounded constraint set. We shall assume for simplicity, however, that each of the polytopes $S_i$, $i = 1, \ldots, N$ is indeed bounded and hence is a polyhedron. One may guarantee that this assumption is satisfied by placing artificial (large) upper bounds on each $\mathbf{x}_i$.

Under the boundedness assumption, each polyhedron $S_i$ consists entirely of points that are convex combinations of its extreme points. Thus, if the extreme points of $S_i$ are $\{\mathbf{x}_{i1}, \mathbf{x}_{i2}, \ldots, \mathbf{x}_{iK_i}\}$, then any point $\mathbf{x}_i \in S_i$ can be expressed in the form

$$\begin{aligned} & \mathbf{x}_i = \sum_{j=1}^{K_i} \alpha_{ij} \mathbf{x}_{ij}, \\ \text{where} \ & \sum_{j=1}^{K_i} \alpha_{ij} = 1 \\ \text{and} \ & \alpha_{ij} \geqslant 0, \quad j = 1, \ldots, K_i. \end{aligned} \tag{3.47}$$

The $\alpha_{ij}$'s are the weighting coefficients of the extreme points.

We now convert the original linear program to an equivalent *master problem*, of which the objective is to find the optimal weighting coefficients for each polyhedron, $S_i$. Corresponding to each extreme point $\mathbf{x}_{ij}$ in $S_i$, define $p_{ij} = \mathbf{c}_i^T \mathbf{x}_{ij}$ and $\mathbf{q}_{ij} = \mathbf{L}_i \mathbf{x}_{ij}$. Clearly $p_{ij}$ is the equivalent cost of the extreme point $\mathbf{x}_{ij}$, and $\mathbf{q}_{ij}$ is its equivalent activity vector in the linking constraints.

Then the original linear program (3.44) is equivalent, using (3.47), to the *master problem*:

$$\begin{aligned}
&\text{minimize} \quad \sum_{i=1}^{N} \sum_{j=1}^{K_i} p_{ij}\alpha_{ij} \\
&\text{subject to} \quad \sum_{i=1}^{N} \sum_{j=1}^{K_i} \mathbf{q}_{ij}\alpha_{ij} = \mathbf{b}_0 \\
&\left.\begin{aligned} &\sum_{j=1}^{K_i} \alpha_{ij} = 1 \\ &\alpha_{ij} \geqslant 0, \quad j = 1, \ldots, K_i \end{aligned}\right\} i = 1, \ldots, N.
\end{aligned} \tag{3.48}$$

This master problem has variables

$$\boldsymbol{\alpha}^T = (\alpha_{11}, \ldots, \alpha_{1K_1}, \alpha_{21}, \ldots, \alpha_{2K_2}, \ldots, \alpha_{N1}, \ldots, \alpha_{NK_N})$$

and can be expressed more compactly as

$$\begin{aligned}
&\text{minimize} \quad \mathbf{p}^T\boldsymbol{\alpha} \\
&\text{subject to} \quad \mathbf{Q}\boldsymbol{\alpha} = \mathbf{g}, \boldsymbol{\alpha} \geqslant \mathbf{0}
\end{aligned} \tag{3.49}$$

where $\mathbf{g}^T = [\mathbf{b}_0^T, 1, 1, \ldots, 1]$; the element of $\mathbf{p}$ associated with $\alpha_{ij}$ is $p_{ij}$; and the column of $\mathbf{Q}$ associated with $\alpha_{ij}$ is

$$\begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix},$$

with $\mathbf{e}_i$ denoting the $i$th unit vector in $E^N$.

Suppose that at some stage of the revised simplex method for the master problem we know the basis $\mathbf{B}$ and corresponding simplex multipliers $\mathbf{y}^T = \mathbf{p}_{\mathbf{B}}^T\mathbf{B}^{-1}$. The corresponding relative cost vector is $\mathbf{r}_{\mathbf{D}}^T = \mathbf{c}_{\mathbf{D}}^T - \mathbf{y}^T\mathbf{D}$, having components

$$r_{ij} = p_{ij} - \mathbf{y}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix}. \tag{3.50}$$

It is not necessary to calculate all the $r_{ij}$'s; it is only necessary to determine the minimal $r_{ij}$. If the minimal value is nonnegative, the current solution is optimal and the process terminates. If, on the other hand, the minimal element is negative, the corresponding column should enter the basis.

The search for the minimal element in (3.50) is normally made with respect to nonbasic columns only. The search can be formally extended to include basic columns as well, however, since for basic elements

$$p_{ij} - \mathbf{y}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix} = 0.$$

The extra zero values do not influence the subsequent procedure, since a new column will enter only if the minimal value is less than zero.

We therefore define $r^*$ as the minimum relative cost coefficient for *all* possible basis vectors. That is,

$$r^* = \underset{i\in\{1,\ldots,N\}}{\text{minimum}} \left\{ r_i^* = \underset{j\in\{1,\ldots,K_i\}}{\text{minimum}} \{ p_{ij} - \mathbf{y}^T \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix} \} \right\}.$$

Using the definitions of $p_{ij}$ and $\mathbf{q}_{ij}$, this becomes

$$r_i^* = \underset{j\in\{1,\ldots,K_i\}}{\text{minimum}} \left\{ \mathbf{c}_i^T \mathbf{x}_{ij} - \mathbf{y}_0^T \mathbf{L}_j \mathbf{x}_{ij} - y_{m+i} \right\}, \tag{3.51}$$

where $\mathbf{y}_0$ is the vector made up of the first $m$ elements of $\mathbf{y}$, $m$ being the number of rows of $\mathbf{L}_j$ [the number of linking constraints in (3.47)].

The minimization problem in (3.51) is actually solved by the $i$th *subproblem*:

$$\begin{aligned} \text{minimize} \quad & (\mathbf{c}_i^T - \mathbf{y}_0^T \mathbf{L}_j)\mathbf{x}_j \\ \text{subject to} \quad & \mathbf{A}_j \mathbf{x}_j = \mathbf{b}_j, \ \mathbf{x}_j \geqslant \mathbf{0} \end{aligned} \tag{3.52}$$

This follows from the fact that $y_{m+i}$ is independent of the extreme point index $j$ (since $\mathbf{y}$ is fixed during the determination of the $r_j$'s), and that the solution of (3.52) must be that extreme point of $S_i$, say $\mathbf{x}_{ik}$, of minimum cost, using the adjusted cost coefficients $\mathbf{c}_i^T - \mathbf{y}_0^T \mathbf{L}_j$.

Thus, an algorithm for this special version of the revised simplex method applied to the master problem is the following: Given a basis $\mathbf{B}$

*Step* 1. Calculate the current basic solution $\mathbf{x_B}$, and solve $\mathbf{y}^T\mathbf{B} = \mathbf{c}_\mathbf{B}^T$ for $\mathbf{y}$.
*Step* 2. For each $i = 1, 2, \ldots, N$, determine the optimal solution $\mathbf{x}_i^*$ of the $i$th subproblem (3.52) and calculate

$$r_i^* = (\mathbf{c}_i^T - \mathbf{y}_0^T \mathbf{L}_j)\mathbf{x}_i^* - \mathbf{y}_{m+i}. \tag{3.53}$$

If all $r_i^* > 0$, stop; the current solution is optimal.
*Step* 3. Determine which column is to enter the basis by selecting the minimal $r_i^*$.
*Step* 4. Update the basis of the master problem as usual.

This algorithm has an interesting economic interpretation in the context of a multidivisional firm minimizing its total cost of operations as described earlier. Division $i$'s activities are internally constrained by $\mathbf{A}\mathbf{x}_i = \mathbf{b}_j$, and the common resources $\mathbf{b}_0$ impose linking constraints. At Step 1 of the algorithm, the firm's central management formulates its current master plan, which is perhaps suboptimal, and announces a new set of prices that each division must use to revise its recommended strategy at Step 2. In particular, $-\mathbf{y}_0$ reflects the new prices that higher management has placed on the common resources. The division that reports the greatest rate of potential cost improvement has its recommendations incorporated in the new master plan at Step 3, and the process is repeated. If no cost improvement is possible, central management settles on the current master plan.

## 3.9 Summary

The simplex method is founded on the fact that the optimal value of a linear program, if finite, is always attained at a basic feasible solution. Using this foundation there are two ways in which to visualize the simplex process. The first is to view the process as one of continuous change. One starts with a basic feasible solution and imagines that some nonbasic variable is increased slowly from zero. As the value of this variable is increased, the values of the current basic variables are continuously adjusted so that the overall vector continues to satisfy the system of linear equality constraints. The change in the objective function due to a unit change in this nonbasic variable, taking into account the corresponding required changes in the values of the basic variables, is the relative cost coefficient associated with the nonbasic variable. If this coefficient is negative, then the objective value will be continuously improved as the value of this nonbasic variable is increased, and therefore one increases the variable as far as possible, to the point where further increase would violate feasibility. At this point the value of one of the basic variables is zero, and that variable is declared nonbasic, while the nonbasic variable that was increased is declared basic.

The other viewpoint is more discrete in nature. Realizing that only basic feasible solutions need be considered, various bases are selected and the corresponding basic solutions are calculated by solving the associated set of linear equations. The logic for the systematic selection of new bases again involves the relative cost coefficients and, of course, is derived largely from the first, continuous, viewpoint.

Problems of special structure are important both for applications and for theory. The transportation problem represents an important class of linear programs with structural properties that lead to an efficient implementation of the simplex method. The most important property of the transportation problem is that any basis is triangular. This means that the basic variables can be found, one by one, directly by back substitution, and the basis need never be inverted. Likewise, the simplex multipliers can be found by back substitution, since they solve a set of equations involving the transpose of the basis. Moreover, when any basis matrix is triangular and all nonzero elements are equal to one (or minus one if the signs of some equations are changed), it follows that the process of back substitution will simply involve repeated additions and subtractions of the given row and column sums. No multiplication or division is required. It therefore follows that if the original right-hand side are integers, the values of all basic variables will be integers. Hence, an optimal basic solution, where each entry is integral, always exists; that is, there is no gap between continuous linear program and integer linear program (or the integrality gap is zero). The transportation problem can be generalized to a minimum cost flow problem in a network. This leads to the interpretation of a simplex basis as corresponding to a spanning tree in the network; see Appendix D.

Many linear programming methods have implemented a *Presolver* procedure to eliminate redundant or duplicate constraints and/or value fixed variables, and to check possible constraint inconsistency and unboundedness. This typically results in problem size reduction and possible infeasibility detection.

## 3.10 Exercises

1. Using pivoting, solve the simultaneous equations

$$3x_1 + 2x_2 = 5$$
$$5x_1 + x_2 = 9.$$

2. Using pivoting, solve the simultaneous equations

$$x_1 + 2x_2 + x_3 = 7$$
$$2x_1 - x_2 + 2x_3 = 6$$
$$x_1 + x_2 + 3x_3 = 12.$$

3. Solve the equations in Exercise 2 by Gaussian elimination as described in Appendix C.
4. Suppose $\mathbf{B}$ is an $m \times m$ square nonsingular matrix, and let the tableau $\mathbf{T}$ be constructed, $\mathbf{T} = [\mathbf{I}, \mathbf{B}]$ where I is the $m \times m$ identity matrix. Suppose that pivot operations are performed on this tableau so that it takes the form $[\mathbf{C}, \mathbf{I}]$. Show that $\mathbf{C} = \mathbf{B}^{-1}$.
5. Show that if the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are a basis in $E^m$, the vectors $\mathbf{a}_1$, $\mathbf{a}_2, \ldots, \mathbf{a}_{p-1}$,
$\mathbf{a_q}, \mathbf{a}_{p+1}, \ldots, \mathbf{a}_m$ also are a basis if and only if $\bar{a}_{pq} \neq 0$, where $\bar{a}_{pq}$ is defined by the tableau (3.5).
6. If $r_j > 0$ for every $j$ corresponding to a variable $x_j$ that is not basic, show that the corresponding basic feasible solution is the unique optimal solution.
7. Show that a degenerate basic feasible solution may be optimal without satisfying $r_j \geqslant 0$ for all $j$.
8.

(a) Using the simplex procedure, solve

$$\begin{array}{ll} \text{maximize} & -x_1 + x_2 \\ \text{subject to} & x_1 - x_2 \leqslant 2 \\ & x_1 + x_2 \leqslant 6 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{array}$$

(b) Draw a graphical representation of the problem in $x_1, x_2$ space and indicate the path of the simplex steps.

(c) Repeat for the problem

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & -2x_1 + x_2 \leqslant 1 \\ & x_1 - x_2 \leqslant 1 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0. \end{array}$$

9. Using the simplex procedure, solve the spare-parts manufacturer's problem (Exercise 4, Chap. 2).
10. Using the simplex procedure, solve

$$\begin{array}{lrl} \text{minimize} & 2x_1 + 4x_2 + x_3 + x_4 & \\ \text{subject to} & x_1 + 3x_2 \qquad\quad + x_4 & \leqslant 4 \\ & 2x_1 + \quad x_2 \qquad\qquad & \leqslant 3 \\ & x_2 + 4x_3 + x_4 & \leqslant 3 \\ & x_1 \geqslant 0 \qquad i = 1, 2, 3, 4. & \end{array}$$

11. For the linear program of Exercise 10
(a) How much can the elements of $\mathbf{b} = (4, 3, 3)$ be changed without changing the optimal basis?
(b) How much can the elements of $\mathbf{c} = (2, 4, 1, 1)$ be changed without changing the optimal basis?
(c) What happens to the optimal cost for small changes in $\mathbf{b}$?
(d) What happens to the optimal cost for small changes in $\mathbf{c}$?
12. Consider the problem

$$\begin{array}{lrl} \text{minimize} & x_1 - 3x_2 - 0.4x_3 & \\ \text{subject to} & 3x_1 - x_2 + 2x_3 & \leqslant 7 \\ & -2x_1 + 4x_2 \qquad & \leqslant 12 \\ & -4x_1 + 3x_2 + \ 3x_3 & \leqslant 14 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \ x_3 \geqslant 0. & \end{array}$$

(a) Find an optimal solution.
(b) How many optimal basic feasible solutions are there?
(c) Show that if $c_4 + \frac{1}{3}a_{14} + \frac{4}{5}a_{24} \geqslant 0$, then another activity $x_4$ can be introduced with cost coefficient $c_1$ and activity vector $(a_{14},\ a_{24},\ a_{34})$ without changing the optimal solution.
13. Rather than select the variable corresponding to the most negative relative cost coefficient as the variable to enter the basis, it has been suggested that a better criterion would be to select that variable which, when pivoted in, will produce the greatest improvement in the objective function. Show that this criterion leads to selecting the variable $x_k$ corresponding to the index $k$ minimizing $\max_{i, \bar{a}_{ik}>0} r_k \bar{a}_{i0} / \bar{a}_{ik}$.
14. In the ordinary simplex method one new vector is brought into the basis and one removed at every step. Consider the possibility of bringing two new vectors into the basis and removing two at each stage. Develop a complete procedure that operates in this fashion.
15. *Degeneracy*. If a basic feasible solution is degenerate, it is then theoretically possible that a sequence of degenerate basic feasible solutions will be generated that endlessly cycles without making progress. It is the purpose of this exercise and the next two to develop a technique that can be applied to the simplex method to avoid this *cycling*.

Corresponding to the linear system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = [\mathbf{a}_1, \ \mathbf{a}_2, \ \ldots, \ \mathbf{a}_n]$ define the perturbed system $\mathbf{Ax} = \mathbf{b}(\varepsilon)$ where $\mathbf{b}(\varepsilon) = \mathbf{b} + \varepsilon\mathbf{a}_1 + \varepsilon^2\mathbf{a}_2 + \cdots + \varepsilon^n\mathbf{a}_n, \ \varepsilon > 0$. Show that if there is a basic feasible solution (possibly degenerate) to the unperturbed system with basis $\mathbf{B} = [\mathbf{a}_1, \ \mathbf{a}_2, \ \ldots, \ \mathbf{a}_m]$, then corresponding to the same basis, there is a nondegenerate basic feasible solution to the perturbed system for some range of $\varepsilon > 0$.

16. Show that corresponding to any basic feasible solution to the perturbed system of Exercise 15, which is nondegenerate for some range of $\varepsilon > 0$, and to a vector $\mathbf{a}_k$ not in the basis, there is a unique vector $\mathbf{a}_j$ in the basis which when replaced by $\mathbf{a}_k$ leads to a basic feasible solution; and that solution is nondegenerate for a range of $\varepsilon > 0$.
17. Show that the tableau associated with a basic feasible solution of the perturbed system of Exercise 15, and which is nondegenerate for a range of $\varepsilon > 0$, is identical with that of the unperturbed system except in the column under $\mathbf{b}(\varepsilon)$. Show how the proper pivot in a given column to preserve feasibility of the perturbed system can be determined from the tableau of the unperturbed system. Conclude that the simplex method will avoid cycling if whenever there is a choice in the pivot element of a column $k$, arising from a tie in the minimum of $\bar{a}_{i0}/\bar{a}_{ik}$ among the elements $i \in I_0$, the tie is resolved by finding the minimum of $\bar{a}_{i1}/\bar{a}_{ik}, \ i \in I_0$. If there still remainties among elements $i \in I$, the process is repeated with $\bar{a}_{i2}/\bar{a}_{ik}$, etc., until there is a unique element.
18. Using the two-phase simplex procedure solve

(a)

$$\begin{array}{rl} \text{minimize} & -3x_1 + x_2 + 3x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9 \\ & x_1 - \ x_2 + 2x_3 - \ x_4 = 6 \\ & x_1 \geqslant 0, \ i = 1, 2, 3, 4. \end{array}$$

(b)

$$\begin{array}{rl} \text{minimize} & x_1 + 6x_2 - 7x_3 + x_4 + 5x_5 \\ \text{subject to} & 5x_1 - 4x_2 + 13x_3 - 2x_4 + x_5 = 20 \\ & x_1 - \ x_2 + \ 5x_3 - \ x_4 + x_5 = 8 \\ & x_1 \geqslant 0, \ i = 1, 2, 3.4, 5. \end{array}$$

19. Solve the oil refinery problem (Exercise 3, Chap. 2).
20. Show that in the phase I procedure of a problem that has feasible solutions, if an artificial variable becomes nonbasic, it need never again be made basic. Thus, when an artificial variable becomes nonbasic its column can be eliminated from future tableaus.
21. Suppose the phase I procedure is applied to the system $\mathbf{Ax} = \mathbf{b}, \ \mathbf{x} \geqslant \mathbf{0}$, and that the resulting tableau (ignoring the cost row) has the form

| $x_1\ x_2 \cdots x_k$ | $x_{k+1} \cdots x_n$ | $y_1\ y_2 \cdots y_k$ | $y_{k+1} \cdots y_m$ | |
|---|---|---|---|---|
| 1 | | | $0 \ \cdots \ 0$ | $\bar{b}_1$ |
| 1 | $\mathbf{R}_1$ | $\mathbf{S}_1$ | $0 \ \cdots \ 0$ | $\vdots$ |
| | | | $\vdots$ | |
| 1 | | | $0 \ \cdots \ 0$ | $\bar{b}_k$ |
| $0 \ 0 \cdots \ 0$ | | | 1 | 0 |
| $\vdots$ | $\mathbf{R}_2$ | $\mathbf{S}_2$ | 1 | $\vdots$ |
| $0 \ \cdots \ 0$ | | | 1 | 0 |

This corresponds to having $m - k$ basic artificial variables at zero level.

(a) Show that any nonzero element in $\mathbf{R}_2$ can be used as a pivot to eliminate a basic artificial variable, thus yielding a similar tableau but with $k$ increased by one.

(b) Suppose that the process in (a) has been repeated to the point where $\mathbf{R}_2 = \mathbf{0}$. Show that the original system is redundant, and show how phase II may proceed by eliminating the bottom rows.

(c) Use the above method to solve the linear program

$$\begin{array}{ll} \text{minimize} & 2x_1 + 6x_2 + x_3 + x_4 \\ \text{subject to} & x_1 + 2x_2 \qquad\quad + x_4 = 6 \\ & x_1 + 2x_2 + x_3 + x_4 = 7 \\ & x_1 + 3x_2 - x_3 + 2x_4 = 7 \\ & x_1 + x_2 + x_3 \qquad\quad = 5 \\ & x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0, \quad x_4 \geqslant 0. \end{array}$$

22. Find a basic feasible solution to

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 3 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 12 \\ x_1 + 4x_2 + 2x_3 + x_4 &= 9 \\ x_1 \geqslant 0, \quad i &= 1, 2, 3, 4. \end{aligned}$$

23. Consider the system of linear inequalities $\mathbf{Ax} \geqslant \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$ with $\mathbf{b} \geqslant \mathbf{0}$. This system can be transformed to standard form by the introduction of $m$ surplus variables so that it becomes $\mathbf{Ax}–\mathbf{y} = \mathbf{b}$, $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{y} \geqslant \mathbf{0}$. Let $b_k = \max_i b_i$ and consider the new system in standard form obtained by adding the $k$th row to the negative of every other row. Show that the new system requires the addition of only a single artificial variable to obtain an initial basic feasible solution.
Use this technique to find a basic feasible solution to the system.

$$\begin{aligned} x_1 + 2x_2 + x_3 &\geqslant 4 \\ 2x_1 + x_2 + x_3 &\geqslant 5 \\ 2x_1 + 3x_2 + 2x_3 &\geqslant 6 \\ x_j \geqslant 0, \ i = 1, 2, 3. \end{aligned}$$

24. It is possible to combine the two phases of the two-phase method into a single procedure by the *big-M method*. Given the linear program in standard form

$$\begin{aligned} &\text{minimize} && \mathbf{c}^T\mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geqslant \mathbf{0}, \end{aligned}$$

one forms the approximating problem

$$\begin{aligned} &\text{minimize} && \mathbf{c}^T\mathbf{x} + M\sum_{i=1}^{m} u_i \\ &\text{subject to} && \mathbf{A}\mathbf{x} + \mathbf{u} = \mathbf{b} \\ & && \mathbf{x} \geqslant \mathbf{0},\ \mathbf{u} \geqslant \mathbf{0}. \end{aligned}$$

In this problem $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ is a vector of artificial variables and $M$ is a large constant. The term $M\sum_{i=1}^{m} u_j$ serves as a penalty term for nonzero $u_i$'s.

If this problem is solved by the simplex method, show the following:

(a) If an optimal solution is found with $\mathbf{y} = \mathbf{0}$, then the corresponding $\mathbf{x}$ is an optimal basic feasible solution to the original problem.

(b) If for every $M > 0$ an optimal solution is found with $\mathbf{y} \neq \mathbf{0}$, then the original problem is infeasible.

(c) If for every $M > 0$ the approximating problem is unbounded, then the original problem is either unbounded or infeasible.

(d) Suppose now that the original problem has a finite optimal value $V(\infty)$. Let $V(M)$ be the optimal value of the approximating problem. Show that $V(M) \leqslant V(\infty)$.

(e) Show that for $M_1 \leqslant M_2$ we have $V(M_1) \leqslant V(M_2)$.

(f) Show that there is a value $M_0$ such that for $M \geqslant M_0$, $V(M) = V(\infty)$, and hence conclude that the big-$M$ method will produce the right solution for large enough values of $M$.

25. Using the revised simplex method find a basic feasible solution to

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 3 \\ 2x_1 + 4x_2 + x_3 + 2x_4 &= 12 \\ x_1 + 4x_2 + 2x_3 + x_4 &= 9 \\ x_1 \geqslant 0,\ i &= 1, 2, 3, 4. \end{aligned}$$

26. The following tableau is an intermediate stage in the solution of a minimization problem:

| | $\mathbf{y}_1$ | $\mathbf{y}_2$ | $\mathbf{y}_3$ | $\mathbf{y}_4$ | $\mathbf{y}_5$ | $\mathbf{y}_6$ | $\mathbf{y}_0$ |
|---|---|---|---|---|---|---|---|
| | 1 | 2/3 | 0 | 0 | 4/3 | 0 | 4 |
| | 0 | −7/3 | 3 | 1 | −2/3 | 0 | 2 |
| | 0 | −2/3 | −2 | 0 | 2/3 | 1 | 2 |
| $\mathbf{r}^T$ | 0 | 8/3 | −11 | 0 | 4/3 | 0 | −8 |

(a) Determine the next pivot element.
(b) Given that the inverse of the current basis is

$$\mathbf{B}^{-1} = [\mathbf{a}_1,\ \mathbf{a}_4,\ \mathbf{a}_6]^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

and the corresponding cost coefficients are

$$\mathbf{c}_{\mathbf{B}}^T = (c_1,\ c_4,\ c_6) = (-1,\ -3, 1),$$

find the original problem.

27. In many applications of linear programming it may be sufficient, for practical purposes, to obtain a solution for which the value of the objective function is within a predetermined tolerance $\varepsilon$ from the minimum value $z^*$. Stopping the simplex algorithm at such a solution rather than searching for the true minimum may considerably reduce the computations.
(a) Consider a linear programming problem for which the sum of the variables is known to be bounded above by $s$. Let $z_0$ denote the current value of the objective function at some stage of the simplex algorithm, $(c_j - z_j)$ the corresponding relative cost coefficients, and

$$M = \max(z_j - c_j)j.$$

Show that if $M \leqslant \varepsilon/s$, then $z_0 - z^* \leq \varepsilon$.
(b) Consider the transportation problem described in Sect. 2.2 (Example 3). Assuming this problem is solved by the simplex method and it is sufficient to obtain a solution within $\varepsilon$ tolerance from the optimal value of the objective function, specify a stopping criterion for the algorithm in terms of $\varepsilon$ and the parameters of the problem.

28. A matrix **A** is said to be *totally unimodular* if the determinant of every square submatrix formed from it has value 0, +1, or −1

 (a) Show that the matrix **A** defining the equality constraints of a transportation problem is totally unimodular.
 (b) In the system of equations **Ax** = **b**, assume that **A** is totally unimodular and that all elements of **A** and **b** are integers. Show that all basic solutions have integer components.

29. For the arrays below:

 (a) Compute the basic solutions indicated. (*Note*: They may be infeasible.)
 (b) Write the equations for the basic variables, corresponding to the indicated basic solutions, in lower triangular form.

| | | | |
|---|---|---|---|
| | $x$ | $x$ | 10 |
| | $x$ | | 20 |
| $x$ | | $x$ | 30 |
| 20 | 20 | 20 | |

| | | | |
|---|---|---|---|
| $x$ | | $x$ | 10 |
| | $x$ | | 20 |
| | $x$ | $x$ | 30 |
| 20 | 20 | 20 | |

30. For the arrays of cost coefficients below, the circled positions indicate basic variables.

    (a) Compute the simplex multipliers.
    (b) Write the equations for the simplex multipliers in upper triangular form, and compare with Part(b) of Exercise 4.

| | | |
|---|---|---|
| 3 | ⑥ | ⑦ |
| 2 | ④ | 3 |
| ① | 5 | ② |

| | | |
|---|---|---|
| ③ | 6 | ⑦ |
| 2 | ④ | 3 |
| 1 | ⑤ | ② |

31. Consider the modified transportation problem where there is more available at origins than is required at destinations (i.e., $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$).

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{m} \sum_{i=1}^{n} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} \leqslant a_i, \quad i = 1, 2, \ldots, m \\ & \sum_{i=1}^{n} x_{ij} = b_j, \quad j = 1, 2, \ldots, n \\ & x_{ij} \geqslant 0, \ \text{for all } i, \ j. \end{aligned}$$

    (a) Show how to convert it to an ordinary transportation problem.
    (b) Suppose there is a storage cost of $s_i$ per unit at origin $i$ for goods not transported to a destination. Repeat Part(a) with this assumption.

32. Solve the following transportation problem, which is an original example of Hitchcock.

$$\mathbf{a} = \begin{pmatrix} 25 & 25 & 50 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 15 & 20 & 30 & 35 \end{pmatrix} \qquad \mathbf{C} = \begin{bmatrix} 10 & 5 & 6 & 7 \\ 8 & 2 & 7 & 6 \\ 9 & 3 & 4 & 8 \end{bmatrix}$$

33. In a transportation problem, suppose that two rows or two columns of the cost coefficient array differ by a constant. Show that the problem can be reduced by combining those rows or columns.
34. The transportation problem is often solved more quickly by carefully selecting the starting basic feasible solution. The *matrix minimum* technique for finding a starting solution is: (3.34) Find the lowest cost unallocated cell in the array, and allocate the maximum possible to it, (3.35) Reduce the corresponding row and column requirements, and drop the row or column having zero remaining requirement. Go back to Step 1 unless all remaining requirements are zero.

    (a) Show that this procedure yields a basic feasible solution.
    (b) Apply the method to Exercise 7.

35. *The caterer problem.* A caterer is booked to cater a banquet each evening for the next $T$ days. He requires $r_t$ clean napkins on the $t$th day for $t = 1, 2, \ldots, T$. He may send dirty napkins to the laundry, which has two speeds of service—fast and slow. The napkins sent to the fast service will be ready for the next day's banquet; those sent to the slow service will be ready for the banquet 2 days later. Fast and slow service cost $c_1$ and $c_2$ per napkin, respectively, with $c_1 > c_2$. The caterer may also purchase new napkins at any time at cost $c_0$. He has an initial stock of $s$ napkins and wishes to minimize the total cost of supplying fresh napkins.

    (a) Formulate the problem as a transportation problem. (*Hint*: Use $T+1$ sources and $T$ destinations.)
    (b) Using the values $T = 4$, $s = 200$, $r_1 = 100$, $r_2 = 130$, $r_3 = 150$, $r_4 = 140$, $c_1 = 6$, $c_2 = 4$, $c_0 = 12$, solve the problem.

36. *The marriage assignment problem.* A group of $n$ men and $n$ women live on an island. The amount of happiness that the $i$th man and the $j$th woman derive by spending a fraction $x_{ij}$ of their lives together is $c_{ij}x_{ij}$. What is the nature of the living arrangements that maximizes the total happiness of the islanders?
37. *Anticycling Rule.* A remarkably simple procedure for avoiding cycling was developed by Bland, and we discuss it here.
    ***Bland's Rule.*** *In the simplex method*:

(a) *Select the column to enter the basis by* $j = \min\{j : r_j < 0\}$; *that is, select the lowest indexed favorable column.*
(b) *In case ties occur in the criterion for determining which column is to leave the basis, select the one with lowest index.*

We can prove by contradiction that the use of Bland's rule prohibits cycling. Suppose that cycling occurs. During the cycle a finite number of columns enter and leave the basis. Each of these columns enters at level zero, and the cost function does not change.

Delete all rows and columns that do not contain pivots during a cycle, obtaining a new linear program that also cycles. Assume that this reduced linear program has $m$ rows and $n$ columns. Consider the solution stage where column $n$ is about to leave the basis, being replaced by column $p$. The corresponding tableau is as follows (where the entries shown are explained below):

$$\begin{array}{l|ccccc} & \mathbf{a}_1 \cdots & \mathbf{a}_p & \cdots & \mathbf{a}_n & \mathbf{b} \\ & & \leqslant 0 & & 0 & 0 \\ & & \leqslant 0 & & 0 & 0 \\ & & \vdots & & \vdots & \vdots \\ & & > 0 & & 1 & 0 \\ \hline \mathbf{c}^T & & < 0 & & 0 & 0 \end{array}$$

Without loss of generality, we assume that the current basis consists of the last $m$ columns. In fact, we may define the reduced linear program in terms of this tableau, calling the current coefficient array $\mathbf{A}$ and the current relative cost vector $\mathbf{c}$. In this tableau we pivot on $a_{mp}$, so $a_{mp} > 0$. By Part(b) of Bland's rule,

$\mathbf{a}_n$ can leave the basis only if there are no ties in the ratio test, and since $\mathbf{b} = \mathbf{0}$ because all rows are in the cycle, it follows that $a_{ip} \leqslant 0$ for all $i \neq m$.
Now consider the situation when column $n$ is about to reenter the basis. Part(a) of Bland's rule ensures that $r_n < 0$ and $r_j \geqslant 0$ for all $i \neq n$. Apply the formula $r_i = c_i - \mathbf{y}^T\mathbf{a}_i$ to the last $m$ columns to show that each component of $\mathbf{y}$ except $y_m$ is nonpositive; and $y_m > 0$. Then use this to show that $r_p = c_p - \mathbf{y}^T\mathbf{a}_p < c_p < 0$, contradicting $r_p \geqslant 0$.

38. Use the Dantzig-Wolfe decomposition method to solve

$$\begin{array}{ll} \text{minimize} & -4x_1 - x_2 - 3x_3 - 2x_4 \\ \text{subject to} & 2x_1 + 2x_2 + x_3 + 2x_4 \leqslant 6 \\ & x_2 + 2x_3 + 3x_4 \leqslant 4 \\ & 2x_1 + x_2 \leqslant 5 \\ & x_2 \leqslant 1 \\ & -x_3 + 2x_4 \leqslant 2 \\ & x_3 + 2x_4 \leqslant 6 \\ & x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0,\ x_4 \geqslant 0. \end{array}$$

## References

3.1–3.5 All of this is now standard material contained in most courses in linear programming. See the references cited at the end of Chap. 2. For the original work in this area, see Dantzig [D2] for development of the simplex method; Orden [O2] for the artificial basis technique; Dantzig, Orden and Wolfe [D8], Orchard-Hays [O1], and Dantzig [D4] for the revised simplex method; and Charnes and Lemke [C3] and Dantzig [D5] for upper bounds. The synthetic carrot interpretation is due to Gale [G2].

3.6 The idea of using **LU** decomposition for the simplex method is due to Bartels and Golub [B2]. See also Bartels [B1]. For a nice simple introduction to Gaussian elimination, see Forsythe and Moler [F15]. For an expository treatment of modern computer implementation issues of linear programming, see Murtagh [M9].

3.7 The transportation problem in its present form was first formulated by Hitchcock [H11]. Koopmans [K8] also contributed significantly to the early development of the problem. The simplex method for the transportation problem was developed by Dantzig [D3]. Most textbooks on linear programming include a discussion of the transportation problem. See especially Simonnard [S6], Murty [M11], and Bazaraa and Jarvis [B5]. The method of changing basis is often called the *stepping stone method*. The assignment problem has a long and interesting history. The important fact that the integer problem is solved by a standard linear programming problem follows from a theorem of Birkhoff [B16], which states that the extreme points of the set of feasible assignments are permutation matrices.

3.8 For a more comprehensive description of the Dantzig and Wolfe [D11] decomposition method, see Dantzig [D6].
The degeneracy technique discussed in Exercises 15–17 is due to Charnes [C2]. The anticycling method of Exercise 35 is due to Bland [B19]. For the state of the art in Simplex solvers see Bixby [B18].