Chapter 10
Radiation Emission and Scattering

Topics The radiation field. Multipole expansion. Electric dipole radiation. Magnetic dipole radiation.

Basic equations of this chapter:
Fields in the radiation zone of a point-like source at $r = 0$ having an electric dipole moment $\mathbf{p}(t)$:

$$\mathbf{E}(\mathbf{r},t) = \frac{[\ddot{\mathbf{p}}(t_{\rm ret})\times\hat{\mathbf{r}}]\times\hat{\mathbf{r}}}{rc^2}\,, \qquad \mathbf{B}(\mathbf{r},t) = \hat{\mathbf{r}}\times\mathbf{E} \tag{10.1}$$

where $t_{\rm ret} = t - r/c$.
Instantaneous radiation power from the electric dipole source and its angular distribution

$$P_{\rm rad} = \frac{2}{3c^3}|\ddot{\mathbf{p}}|^2\,, \qquad \frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}\Omega} = \frac{3P_{\rm rad}}{4\pi}\sin^2\theta\,, \tag{10.2}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{r}$, and the infinitesimal solid angle $\mathrm{d}\Omega = 2\pi\sin\theta\mathrm{d}\theta$.
Analogous formulas for the fields and the power of a magnetic dipole $\mathbf{m}(t)$:

$$\mathbf{E}(\mathbf{r},t) = -\frac{[\ddot{\mathbf{m}}(t_{\rm ret})\times\hat{\mathbf{r}}]}{rc^2}\,, \qquad \mathbf{B}(\mathbf{r},t) = \hat{\mathbf{r}}\times\mathbf{E}\,, \tag{10.3}$$

$$P_{\rm rad} = \frac{2}{3c^3}|\ddot{\mathbf{m}}|^2\,, \qquad \frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}\Omega} = \frac{3P_{\rm rad}}{4\pi}\sin^2\theta\,. \tag{10.4}$$

10.1 Cyclotron Radiation

An electron moves in the xy plane in the presence of a constant and uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$. The initial velocity is $v_0 \ll c$, so that the motion is non-

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relativistic and the electron moves on a circular orbit of radius $r_L = v_0/\omega_L$ and frequency $\omega_L = eB_0/m_e c$ (Larmor frequency).
a) Describe the radiation emitted by the electron in the dipole approximation specifying its frequency, its polarization for radiation observed along the z axis, and along a direction lying in the xy plane, and the total irradiated power P_{rad}. Discuss the validity of the dipole approximation.
b) The electron gradually loses energy because of the emitted radiation. Use the equation $P_{rad} = -dU/dt$, where U is the total energy of the electron, to show that the electron actually spirals toward the "center" of its orbit. Evaluate the time constant τ of the energy loss, assuming $\tau \gg \omega_L^{-1}$, and provide a numerical estimate.
c) The spiral motion cannot occur if we consider the Lorentz force $\mathbf{f}_L = -(e/c)\boldsymbol{v}\times\mathbf{B}$ as the only force acting on the electron. Show that a spiral motion can be obtained by adding a friction force $\mathbf{f}_{fr}$ proportional to the electron velocity.

10.2 Atomic Collapse

In the classical model for the hydrogen atom, an electron travels in a circular orbit of radius a_0 around the proton.
a) Evaluate the frequency ω of the radiation emitted by the orbiting electron, and the emitted radiation power, both as functions of a_0.
b) Use the results of point **a)** to show that, classically, the electron would collapse on the nucleus, and find the decay time assuming $a_0 = 0.53\times10^{-8}$ cm (Bohr radius, actually obtained from quantum considerations) .

10.3 Radiative Damping of the Elastically Bound Electron

The motion of a classical, elastically bound electron in the absence of external fields is described by the equation

$$\frac{d^2\mathbf{r}}{dt^2} + \eta\frac{d\mathbf{r}}{dt} + \omega_0^2\mathbf{r} = 0\,, \tag{10.5}$$

where the vector $\mathbf{r}$ is the distance of the electron from its equilibrium position, η is a friction coefficient, and ω_0 is the undamped angular frequency. We assume that at time $t = 0$ the electron is located at $\mathbf{r}(0) = \mathbf{s}_0$, with zero initial velocity.
a) As a first step, find the solution of (10.5) assuming $\eta = 0$, and evaluate the cycle-averaged emitted radiation power P_{rad} due to the electron acceleration.
b) Assuming the oscillation amplitude to decay due to the radiative energy loss, estimated the decay time τ using the result of point **a)** for the emitted power P_{rad}. Determine under which conditions τ is much longer than one oscillation period.

Now assume $\eta \neq 0$, with $\eta \ll \omega_0$, in Eq.(10.5). In the following, neglect quantities of the order $(\eta/\omega_0)^2$ or higher.
c) Describe the motion of the electron and determine, *a posteriori*, the value of η that reproduces the radiative damping.

10.4 Radiation Emitted by Orbiting Charges

Two identical point charges q rotate with constant angular velocity $\boldsymbol{\omega}$ on the circular orbit $x^2 + y^2 = R^2$ on the $z = 0$ plane of a Cartesian reference frame.
a) Write the most general trajectory for the charges both in polar coordinates $r_i = r_i(t)$, $\phi_i = \phi_i(t)$ and in Cartesian coordinates $x_i = x_i(t)$, $y_i = y_i(t)$ (where $i = 1,2$ labels the charge) and calculate the electric dipole moment of the system.
b) Characterize the dipole radiation emitted by the two-charge system, discussing how the power depends on the initial conditions, and finding the polarization of the radiation emitted along the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions.
c) Answer questions **a)** and **b)** in the case where the charges are orbiting with *opposite* angular velocity.
d) Now consider a system of *three* identical charges on the circular orbit with the same angular velocity. Find the initial conditions for which the radiation power is either zero or has its maximum.
e) Determine whether the magnetic dipole moment gives some contribution to the radiation, for each of the above specified cases.

10.5 Spin-Down Rate and Magnetic Field of a Pulsar

A pulsar is a neutron star with mass $M \approx 1.4 M_\odot \approx 2.8 \times 10^{33}$ g (where $M_\odot$ is the Sun mass), and radius $R \simeq 10\,\text{km} = 10^6\,\text{cm}$. The star rotates with angular velocity $\boldsymbol{\omega}$ and has a magnetic moment $\mathbf{m}$, which is, in general, *not* parallel to the rotation axis. [1]
a) Describe the radiation emitted by the pulsar, and find the total radiated power, assuming that the angle between the magnetic moment and the rotation axis is α, as in Fig. 10.1.
b) Find the "spindown rate" (decay constant of the rotation) of the pulsar, assuming that energy loss is due to radiation only.

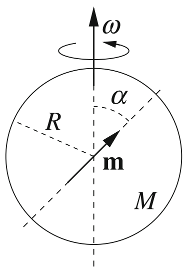

Fig. 10.1

c) Explain how, from the knowledge of mass, radius, rotation period T, and time derivative $\mathrm{d}T/\mathrm{d}t$ of the pulsar one can estimate the magnetic field at the pulsar surface. Give a numerical approximation based on the results of observations [3] which give $T = 7.476551 \pm 3$ s and $\dot{T} = (2.8 \pm 1.4) \times 10^{-11} \simeq 10^{-3}$ s/year (for simplicity assume that **m** is perpendicular.

10.6 A Bent Dipole Antenna

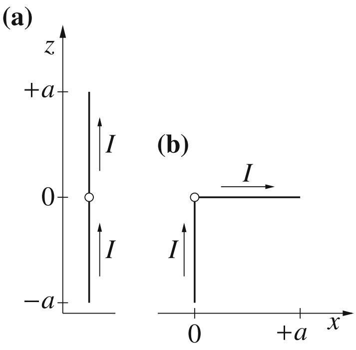

Fig. 10.2

A dipole antenna consists of two identical conductive elements, usually two metal rods, each of length a and resistance R. The driving current is applied between the two halves of the antenna, so that the current flows as shown in Fig. 10.2a). For a "short" antenna ($a \ll \lambda = 2\pi c/\omega$) the current can be approximately specified as [2]

$$I = I(z,t) = \mathrm{Re}\left[I_0 \left(1 - \frac{|z|}{a} \right) \mathrm{e}^{-\mathrm{i}\omega t} \right]. \qquad (10.6)$$

The dependence of the current oscillation amplitude on z is shown in Fig. 10.3. Calculate

a) the cycle-averaged the dissipated power P_{diss};

b) the linear charge density q_ℓ on the rods of the antenna, and the antenna electric dipole moment **p**;

c) the cycle-averaged radiated power P_{rad} and the ratio $P_{\mathrm{rad}}/P_{\mathrm{diss}}$.

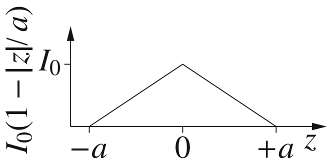

Fig. 10.3

d) Find the directions along which there no radiation is observed.

Now assume that the upper rod of the dipole antenna is bent by 90°, so that it is parallel to the x axis, as shown in Fig. 10.2b), without perturbing either the current or the charge density anywhere in the two rods.

e) Answer questions **a)**, **b)** and **c)** again for the bent antenna, pointing out the differences with the straight antenna.

10.7 A Receiving Circular Antenna

A receiving circular antenna is a circular coil of radius a and resistance R. The amplitude of the received signal is proportional to the current induced in the antenna by an incoming EM wave (Fig. 10.4).

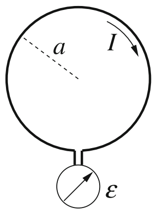

Fig. 10.4

a) Assume that the incoming signal is a monochromatic, linearly polarized wave of wavelength $\lambda \gg a$, and electric field amplitude E_0. Find how the antenna must be oriented with respect to the wave vector $\mathbf{k}$ and to the polarization in order to detect the maximum signal, and evaluate the signal amplitude.
b) In a receiving linear antenna the signal is approximately proportional to $E_{\|}\ell$, where $E_{\|}$ is the component of the electric field of the wave parallel to the antenna, and ℓ is the length of the antenna. Old portable TV sets were provided with both a linear and a circular antenna, typical dimensions were $\ell \simeq 50\,\text{cm}$ and $a \simeq \ell/2$. Which antenna is best suited to detect EM waves with λ in the 10^2–10^3 cm range?
c) Calculate the power P_{rad} scattered by the antenna, and the ratio $P_{\text{rad}}/P_{\text{diss}}$, where P_{diss} is the power dissipated in the antenna by Joule heating.

10.8 Polarization of Scattered Radiation

An EM wave impinges on a particle that acquires an electric dipole moment $\mathbf{p} = \alpha\mathbf{E}$, where $\mathbf{E}$ is the electric field of the wave at the position of the particle. Assume that the size of the particle is much smaller than the wavelength of the incoming wave.
a) Find the polarization of the scattered radiation as a function of the polarization of the incoming wave, and of the angle between the directions of observation and propagation.
b) If the incoming radiation is *unpolarized*, what can be said about the polarization of the scattered radiation?

10.9 Polarization Effects on Thomson Scattering

An electron is in the field of an elliptically polarized plane wave of frequency ω propagating along the z axis of a Cartesian reference frame. The electric field of the wave can be written as

$$\mathbf{E} = E_0\left[\hat{\mathbf{x}}\cos\theta\cos(kz-\omega t) + \hat{\mathbf{y}}\sin\theta\sin(kz-\omega t)\right], \tag{10.7}$$

where θ is a constant real number with $0 \leqslant \theta \leqslant \pi/2$. such that we have linear polarization along the x axis for $\theta = 0$, linear polarization along the y axis for $\theta = \pi/2$, and circular polarization for $\theta = \pi/4$.

First, neglecting the effects of the magnetic force $-e\boldsymbol{v}\times\mathbf{B}/c$,

a) characterize the radiation scattered by the electron by determining the frequency and the polarization observed along each axis (x, y, z), and find a direction along which the radiation is circularly polarized;

b) calculate the total (cycle-averaged) scattered power and discuss its dependence on θ;

Now consider the effect of the magnetic force on the scattering process.

c) Evaluate the $-e\boldsymbol{v}\times\mathbf{B}/c$ term by calculating the $\mathbf{B}$ field from (10.7) and using the result of point **a)** for $\boldsymbol{v}$. Discuss the direction and frequency of the magnetic force and its dependence on θ as well.

d) Discuss how the scattering of the incident wave is modified by the magnetic force by specifying which new frequencies are observed, in which direction and with which polarization, and the modification of the scattered power.

10.10 Scattering and Interference

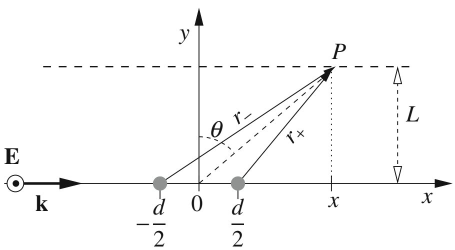

Fig. 10.5

A monochromatic plane wave propagates along the x axis of a Cartesian coordinate system. The wave is linearly polarized in the $\hat{\mathbf{z}}$ direction, and has wavelength λ. Two identical, point-like scatterers are placed on the x axis at $x = \pm d/2$, respectively, as in Fig. 10.5. The dipole moment of each scatterer is $\mathbf{p} = \alpha\mathbf{E}$, where $\mathbf{E}$ is the electric field of the incoming wave at the scatterer position. The intensity I_s of the scattered radiation is measured on the $y = L$ plane, with both $L \gg d$ and $L \gg \lambda$.

a) Evaluate the phase difference $\Delta\phi$ between the two scattered waves in a generic point $P \equiv (x, L, 0)$, with L a constant, as a function of the observation angle $\theta = \arctan(x/L)$, as shown in Fig. 10.5.

b) Study the scattered intensity distribution $I_s = I_s(\theta)$ as a function of kd, where $\mathbf{k}$ is the wave vector of the incoming wave. Determine for which values of kd interference fringes appear.

10.11 Optical Beats Generating a "Lighthouse Effect"

Two oscillating dipoles, $\mathbf{p}_-$ and $\mathbf{p}_+$, are located at $(0,-d/2,0)$ and $(0,+d/2,0)$, respectively, in a Cartesian reference frame. The two dipoles are parallel to the z axis and oscillate, with equal amplitude, at slightly different frequencies $\omega_\pm = \omega_0 \pm \delta\omega/2$, with $\delta\omega \ll \omega_0$. In complex representation we have $\mathbf{p}_\pm = \mathbf{p}_0\,\mathrm{e}^{-\mathrm{i}\omega_\pm t}$. The distance between the two dipoles is $d = \lambda_0/2 = \pi c/\omega_0$. The radiation emitted by the dipoles is observed at a point P at a distance $\mathbf{r}$ from the origin, with $r \gg \lambda_0$, on the $z = 0$ plane. Let ϕ be the angle between $\mathbf{r}$ and the x axis, as shown in Fig. 10.6.

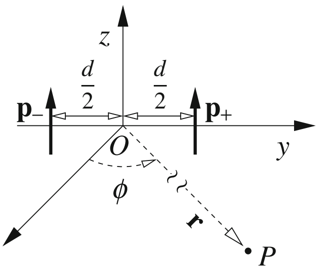

Fig. 10.6

a) Determine the direction of the electric field in P and its dependence on ϕ and $\omega_\pm$, up to the first order in $\delta\omega/\omega_0$.

The wave intensity in P is measured by two detectors with different temporal resolutions: the first detector measures the "instantaneous" flux averaged over an interval Δt such that $2\pi/\omega_0 \ll \Delta t \ll 2\pi/\delta\omega$, while the second detector averages over $\Delta t' \gg 2\pi/\delta\omega$.

b) Determine the dependence on the angle ϕ and the time t of the fluxes measured with the two detectors.

c) How do the above results change if the observation point is located in the $x = 0$ plane?

10.12 Radiation Friction Force

An accelerated point charge emits radiation. Considering for definiteness an electron performing a periodic (non-relativistic for simplicity) motion in an oscillating external field, there is a finite amount of energy leaving the electron as radiation, but on the average the external field produces no work. Thus, to account self-consistently for the energy lost as radiation, it is necessary to modify the Newton-Lorentz force by adding a new "friction" term $\mathbf{F}_{\mathrm{rad}}$ so that the mechanical work done by $\mathbf{F}_{\mathrm{rad}}$ equals the radiated energy.[1]

We thus write for the electron

$$m_\mathrm{e}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -e\left(\mathbf{E} + \frac{\boldsymbol{v}}{c}\times\mathbf{B}\right) + \mathbf{F}_{\mathrm{rad}}\,, \tag{10.8}$$

[1]From another viewpoint, $\mathbf{F}_{\mathrm{rad}}$ aims to describe the back-action or *reaction* of the self-generated EM fields on the accelerated charge.

and look for a suitable expression for $\mathbf{F}_{\mathrm{rad}}$ starting from the condition

$$\int_t^{t+T} \mathbf{F}_{\mathrm{rad}}(t) \cdot \boldsymbol{v}(t)\, \mathrm{d}t = -\int_t^{t+T} P_{\mathrm{rad}}(t)\, \mathrm{d}t\,, \tag{10.9}$$

where T is the period of the electron motion and $P_{\mathrm{rad}}(t)$ is the instantaneous radiated power, which is given by the Larmor formula

$$P_{\mathrm{rad}}(t) = \frac{2e^2}{3c^2}\left|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|^2 . \tag{10.10}$$

a) Show by direct substitution of the expression for F_{rad}

$$\mathbf{F}_{\mathrm{rad}} = m_{\mathrm{e}}\tau\,\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} \tag{10.11}$$

into (10.9), that the equation is verified, and find the expression of the constant τ, estimating its numerical value.

b) Determine the steady state solution of (10.8), where $\mathbf{F}_{\mathrm{rad}}$ is given by (10.11), for an electron in a uniform, oscillating electric field

$$\mathbf{E}(t) = \mathrm{Re}\left(-e\,\mathbf{E}_0\,\mathrm{e}^{-\mathrm{i}\omega t}\right) . \tag{10.12}$$

Compare the result with what obtained using the simple classical model an electron subject to a frictional force

$$m_{\mathrm{e}}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{ext}} - m_{\mathrm{e}}\eta\boldsymbol{v}\,. \tag{10.13}$$

References

1. C. Bernardini, C. Guaraldo, *Fisica del Nucleo* (Editori Riuniti, Roma, 1982)
2. J.D. Jackson, *Classical Electrodynamics*, §9.2 and 9.4, 3rd edn. (Wiley, New York, London, Sydney, 1998)
3. C. Kouveliotou et al., An X-ray pulsar with a superstrong magnetic field in the soft γ-ray repeater SGR1806-20. Nature **393**, 235–237 (1998)