Chapter 11
Electromagnetic Waves in Matter

Topics Wave equation in continuous media. Classical model of the electron, bound and free electrons. Frequency-dependent conductivity $\sigma(\omega)$ and dielectric permittivity $\varepsilon(\omega)$ for harmonic fields. Relation between $\sigma(\omega)$ and $\varepsilon(\omega)$. Transverse and longitudinal waves. The refraction index. Propagation of monochromatic waves in matter. Dispersion relations. Reflection and transmission at a plane interface: Snell's law, Fresnel's formulas, total reflection, Brewster's angle. Anisotropic media.

Basic equations of this chapter:
Wave equation for the electric field:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\partial_t^2\mathbf{E} - \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) = \frac{4\pi}{c^2}\mathbf{J} = \frac{4\pi}{c^2}\partial_t^2\mathbf{P}\,. \tag{11.1}$$

(Notice that $\mathbf{J} = \partial_t\mathbf{P}$.)
Definition of $\sigma(\omega)$, $\chi(\omega)$ and $\varepsilon(\omega)$ for harmonic fields $\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\tilde{\mathbf{E}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right]$, $\mathbf{J}(\mathbf{r},t) = \mathrm{Re}\left[\tilde{\mathbf{J}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right]$, $\mathbf{P}(\mathbf{r},t) = \mathrm{Re}\left[\tilde{\mathbf{P}}(\mathbf{r})\mathrm{e}^{-\mathrm{i}\omega t}\right]$:

$$\tilde{\mathbf{J}} = \sigma(\omega)\tilde{\mathbf{E}}\,, \qquad \tilde{\mathbf{P}} = \chi(\omega)\tilde{\mathbf{E}}\,, \tag{11.2}$$

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega)\,, \qquad \chi(\omega) = \frac{\mathrm{i}\sigma(\omega)}{\omega} \quad (\omega \neq 0)\,. \tag{11.3}$$

Dispersion relation in a medium and refraction index $\mathsf{n}(\omega)$:

$$\frac{k^2c^2}{\omega^2} = \varepsilon(\omega) = \mathsf{n}^2(\omega)\,. \tag{11.4}$$

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11.1 Wave Propagation in a Conductor at High and Low Frequencies

In a classical treatment, a metal has n_e conduction electrons per unit volume, whose equations of motion in the presence of an external electric field $\mathbf{E}(\mathbf{r},t)$ are

$$m_e \frac{d\boldsymbol{v}}{dt} = -e\mathbf{E}(\mathbf{r},t) - m_e \eta \boldsymbol{v} , \qquad \boldsymbol{v} = \frac{d\mathbf{r}}{dt} , \tag{11.5}$$

where $-e$ and m_e are electron charge and mass, respectively, and η is a constant describing friction.

a) Determine the complex conductivity of the metal, $\sigma = \sigma(\omega)$, as a function of the angular frequency ω of the electric field, and the values of ω for which σ is either purely real or purely imaginary. Discuss these limits for a good conductor, whose DC conductivity (i.e., its conductivity for static fields) has values of the order of $\sigma_{DC} \sim 5 \times 10^{17}\ \mathrm{s}^{-1}$.

Now consider a monochromatic, plane EM wave, linearly polarized along the y axis and traveling in the positive direction along the x axis of a Cartesian coordinate system. The wave is incident on a conductor filling the $x > 0$ half-space, while we have vacuum in the $x < 0$ half-space.

b) Consider both cases of σ purely real and purely imaginary, and determine the frequency ranges in which the wave is evanescent inside the metal.

c) Find the time-averaged EM energy flux through the metal surface and show that it is equal to the amount of energy dissipated inside the metal.

11.2 Energy Densities in a Free Electron Gas

A plane, monochromatic, transverse electromagnetic wave propagates in a medium containing n_e free electrons per unit volume. The electrons move with negligible friction. Calculate

a) the dispersion relation of the wave, the phase (v_φ) and group (v_g) velocities, and the relation between the amplitudes of the electric (E_0) and magnetic (B_0) fields;

b) the EM energy density u_{EM} (averaged over an oscillation period) as a function of E_0;

c) the kinetic energy density u_K (averaged over an oscillation period), defined as $u_K = n_e m_e \langle \boldsymbol{v}^2 \rangle /2$, where $\boldsymbol{v}$ is the electron oscillation velocity, and the *total* energy density $u = u_{EM} + u_K$.

d) Assume that the medium fills the half-space $x > 0$, while we have vacuum in the half-space $x < 0$. An EM wave, propagating along the x axis, enters the medium. Assume that both v_g and v_φ are real quantities. Use the above results to verify the conservation of the energy flux, expressed by the relation

$$c(u_\mathrm{i} - u_\mathrm{r}) = v_\mathrm{g} u_\mathrm{t} , \tag{11.6}$$

where u_i, u_r and u_t are the total energy densities for the incident, reflected and transmitted waves, respectively.

11.3 Longitudinal Waves

Consider a *longitudinal* monochromatic plane wave, propagating in a medium along the x axis of a Cartesian reference frame. "Longitudinal" means that the electric field $\mathbf{E}$ of the wave is parallel to the wavevector $\mathbf{k}$. Assume that the electric and magnetic fields of the wave are

$$\mathbf{E} = \mathbf{E}(x,t) = \hat{\mathbf{x}} E_0 \,\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t} , \qquad \mathbf{B} \equiv 0 , \tag{11.7}$$

respectively, and that the optical properties of the medium are described by a given frequency-dependent dielectric permittivity $\varepsilon_\mathrm{r}(\omega)$.

a) Show that the possible frequencies for the wave (11.7) correspond to zeros of the dielectric permittivity, $\varepsilon_\mathrm{r}(\omega) = 0$.

b) Find the charge and current densities in the medium associated to the presence of the wave fields (11.7).

c) Assuming that the optical properties of the medium are determined by n_e classical electrons per unit volume, bound to atoms by an elastic force $-m_\mathrm{e}\omega_0^2\mathbf{r}$, determine $\varepsilon_\mathrm{r}(\omega)$ and the dispersion relation for the longitudinal wave.

11.4 Transmission and Reflection by a Thin Conducting Foil

A plane wave of frequency $\omega = 2\pi c/\lambda$ strikes at normal incidence a thin metal foil of thickness $d \ll \lambda$. At the limit of an infinitely thin foil, the volume electron density in space can be approximated as $n_\mathrm{v}(x) = n_\mathrm{e} d\,\delta(x)$, where n_e is the volume electron density in the conductor, so that $n_\mathrm{e} d$ is the surface electron density on the foil, and $\delta(x)$ is the Dirac delta function. Analogously, the volume current density in space can be approximated as $\mathbf{J}(x,t) = \mathbf{K}(t)\,\delta(x)$, where $\mathbf{K}(t)$ is the surface current density on the foil.

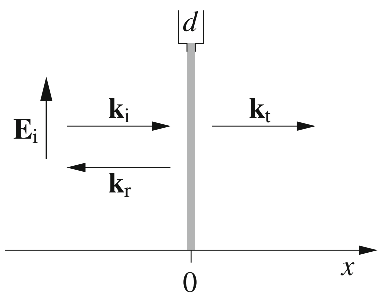

Fig. 11.1

a) Prove the following relations for the field components parallel to the foil surface

$$E_{\|}(0^+) - E_{\|}(0^-) = 0, \qquad B_{\|}(0^+) - B_{\|}(0^-) = \frac{4\pi}{c} K . \tag{11.8}$$

b) Evaluate the EM field in the whole space as a function of the foil conductivity σ, with σ, in general, a complex scalar quantity. Assume a linear dependence of the current density $\mathbf{J}$ on the electric field $\mathbf{E}$, using the complex notation $\mathbf{J} = \mathrm{Re}\left(\tilde{\mathbf{J}}\,\mathrm{e}^{-\mathrm{i}\omega t}\right)$.
c) Now use the classical equation of motion for the electrons in the metal

$$m_\mathrm{e}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -e\mathbf{E} - m_\mathrm{e}\eta\boldsymbol{v}\,, \tag{11.9}$$

where η is a damping constant, to obtain an expression for σ, and evaluate the cycle-averaged absorbed power at the limits $\nu \gg \omega$ and $\nu \ll \omega$, respectively.
d) Verify the conservation of energy for the system by showing that the flux of EM energy into the foil equals the absorbed power.

11.5 Anti-reflection Coating

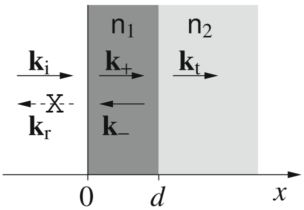

Fig. 11.2

A monochromatic plane EM wave of angular frequency ω travels in vacuum ($x < 0$) along the x direction of a Cartesian coordinate system. On the plane $x = 0$ the wave strikes normally a semi-infinite composite medium. The medium comprises a first layer, between the planes $x = 0$ and $x = d$, of real refractive index n_1, followed by a semi-infinite layer filling the half-space $x > d$, of real refractive index n_2, as shown in Fig. 11.2.

We want to determine the conditions on n_1 and d in order to have a *total* transmission of the incident wave, so that there is *no* reflected wave in the vacuum region. Proceed as follows:
a) write the general solution for the EM wave in each region of space;
b) write the relations between the amplitudes of the EM fields in each region due to matching conditions at the two interfaces;
c) having determined from point **b)** the relation between n_1, n_2 and d necessary to the absence of reflection, find the values of n_1 and d for which a solution exists in the $\mathsf{n}_2 = 1$ case.
d) How does the answer to point **c)** change if $\mathsf{n}_2 \neq 1$?

11.6 Birefringence and Waveplates

The refractive index of *anisotropic* crystals depends on both the propagation direction and the polarization of the incoming EM wave. We choose a Cartesian reference frame such that the $x = 0$ plane separates the investigated medium from vacuum. The wave vector of the incident wave, $\mathbf{k}_\mathrm{i}$, lies in the xy plane and forms an angle θ_i with the x axis, as shown in Fig. 11.3. In this context we consider a material whose refractive index has the values $\mathrm{n_s}$ for a wave polarized perpendicularly to the incidence xy plane (S polarization, from German *senkrecht*, perpendicular), and $\mathrm{n_p}$ for waves whose electric field lies in the xy plane (P polarization, from *parallel*). Here, both $\mathrm{n_s}$ and $\mathrm{n_p}$ are assumed to be real and positive, with $\mathrm{n_p} > \mathrm{n_s}$. The treatment of the opposite case, $\mathrm{n_s} > \mathrm{n_p}$, is straightforward.

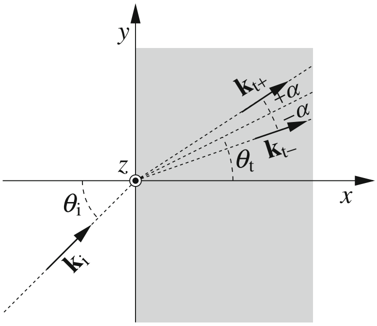

Fig. 11.3

a) Assume that the incoming wave is linearly polarized, and that its electric field forms an angle $\psi = \pi/4$ with the z axis, so that its polarization is a mixture of S and P polarizations. The incident ray splits into two refracted rays at different angles, $\theta_{\mathrm{t}\pm} = \theta_\mathrm{t} \pm \alpha$, as shown in Fig. 11.3, where $\mathbf{k}_{\mathrm{t}+}$ corresponds to S, and $\mathbf{k}_{\mathrm{t}-}$ to P polarization. Show how the values of n_s and n_p can be obtained from the measurements of θ_t and α. Assume that $\mathrm{n_p} = \bar{\mathrm{n}} + \delta\mathrm{n}$, and $\mathrm{n}_- = \bar{\mathrm{n}} - \delta\mathrm{n}$, with $\delta\mathrm{n}/\bar{\mathrm{n}} \ll 1$, and keep only first-order therms in $\delta\mathrm{n}/\bar{\mathrm{n}}$.

b) Now assume normal incidence ($\theta_\mathrm{i} = 0$), and that the electric field of the linearly polarized incoming wave, $\mathbf{E}_\mathrm{i}$, still forms an angle $\psi = \pi/4$ with the $\hat{\mathbf{z}}$ axis, as in Fig. 11.4. The crystal has a thickness $d \gg \lambda$. Find the values of d such that the light exiting the crystal is either *circularly* polarized, or linearly polarized, but rotated by $\pi/2$ with respect to the polarization of the incident light. Neglect the difference between the reflection coefficients for S and P polarizations.

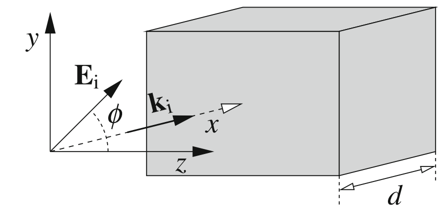

Fig. 11.4

11.7 Magnetic Birefringence and Faraday Effect

An EM plane wave of frequency ω travels in a medium in the presence of a static uniform magnetic field $\mathbf{B}_0 = B_0\,\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the z unit vector of a Cartesian reference frame. B_0 is much stronger than the magnetic field of the wave. The direction of the wave propagation is also parallel to $\hat{\mathbf{z}}$. The medium contains n_e bound electrons per unit volume, obeying the classical equations of motion

$$m_\mathrm{e}\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -e\left(\mathbf{E}+\frac{\boldsymbol{v}}{c}\times\mathbf{B}\right) - m_\mathrm{e}\omega_0^2\mathbf{r}\,, \qquad \boldsymbol{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,. \tag{11.10}$$

where m_e and $-e$ are the electron mass and charge, respectively.

a) Show that the propagation of the wave depends on its polarization by evaluating the refractive index for *circular* polarization, either left-handed or right-handed.

b) Now consider the propagation of a *linearly* polarized wave. Assume the electric field at $z = 0$ to be given by $\mathbf{E}_\mathrm{i}(z = 0, t) = \hat{\mathbf{x}}E_i\,\mathrm{e}^{-\mathrm{i}\omega t}$, and a relatively weak magnetic field so that $\omega \gg \omega_c$ and terms of order higher than ω_c/ω may be neglected. Find the electric field at the position $z = \ell$, showing that the polarization has *rotated* (Faraday effect).

11.8 Whistler Waves

Lightnings excite transverse EM signals which propagate in the ionosphere, mostly in the direction parallel to the Earth's magnetic field lines.

a) Show that, in a frequency range to be determined, and depending on the wave polarization, the dispersion relation for such signals has the form

$$\omega = \alpha k^2\,, \tag{11.11}$$

with α a constant depending on the free electron density n_e and the magnetic field B_0 (both assumed to be uniform for simplicity). Give a numerical estimate for the frequency range, knowing that typical values are $n_\mathrm{e} \approx 10^5\ \mathrm{cm}^{-3}$, and $B_0 \approx 0.5$ G.

b) Determine the group and phase velocities following from (11.11) as functions of ω, and compare them to c.

c) Suppose that a lightning locally excites a pulse having a frequency spectrum extending from a value ω_1 to $\omega_2 = 2\omega_1$, within the frequency range determined at point **a)**. Assuming the pulse to be "short" (in a sense to be clarified *a posteriori*), estimate the pulse length after propagation over a distance $L \simeq 10^4$ km. Try to explain why these signals are called *whistlers*.

(Refer to [1], Sect. 7.6, and to Problem 11.7 for the propagation of EM waves along a magnetic field).

11.9 Wave Propagation in a "Pair" Plasma

A "pair" plasma is composed by electrons and positrons with equal density n_0 (pair annihilation is neglected).
a) In the absence of external fields, find the dispersion relation for transverse EM waves, determining cut-off and/or resonance frequencies, if any.
b) Find and discuss the dispersion relation as in **a)**, but for waves propagating along the direction of an external, static magnetic field $\mathbf{B}_0$ (see also Problem 11.7).

11.10 Surface Waves

A homogeneous medium fills the $x > 0$ half-space of a Cartesian reference frame, while we have vacuum for $x < 0$. The dielectric permittivity of the medium, $\varepsilon = \varepsilon(\omega)$, assumes real values in the frequency range of interest. A monochromatic EM wave propagates along the y-direction, parallel to the interface between the medium and vacuum. Inside the medium, the magnetic field of the wave has the z-component only, given by

$$B_z = B_0\,\mathrm{e}^{-qx}\cos(ky-\omega t) = \mathrm{Re}\left(B_0\,\mathrm{e}^{-qx}\mathrm{e}^{\mathrm{i}ky-\mathrm{i}\omega t}\right) \qquad (x>0)\,, \tag{11.12}$$

where q is a real and positive quantity.
a) Using the wave equation for $\mathbf{B}$ inside the dielectric medium, find a relation between q, k and ω.
b) Write the expression for the electric field $\mathbf{E}$ inside the medium.
c) Calculate the Poynting vector $\mathbf{S}$ and specify the direction of the time-averaged EM energy flow.

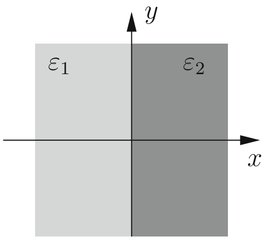

Fig. 11.5

Now consider two different homogeneous media of dielectric permittivities ε_1 and ε_2, respectively, filling the $x < 0$ and $x > 0$ half-spaces. A linearly-polarized EM wave propagates along the y-axis on the $x = 0$ interface, with the magnetic field given by

$$\mathbf{B} = \mathrm{Re}\left[\hat{\mathbf{z}}\,B_z(x)\,\mathrm{e}^{\mathrm{i}ky-\mathrm{i}\omega t}\right]\,, \tag{11.13}$$

where

$$B_z(x) = \begin{cases} B_1\,\mathrm{e}^{+q_1 x}\,, & x<0 \\ B_2\,\mathrm{e}^{-q_2 x}\,, & x>0 \end{cases}\,. \tag{11.14}$$

d) Using the boundary conditions for B_z at the $x = 0$ surface, find the relation between B_1 and B_2.

e) Using the continuity of E_y at the $x = 0$ surface, find the relation between q_1 and q_2. Show that ε_1 and ε_2 must have *opposite* sign in order to have $q_{1,2} > 0$, i.e., vanishing fields for $|x| \to \infty$.
f) From the results of points **a)** and **e)** find the dispersion relation $\omega = \omega(k)$ as a function of ε_1 and ε_2, showing that wave propagation requires $\varepsilon_1 + \varepsilon_2 < 0$.
g) If medium 1 is vacuum ($\varepsilon_1 = 1$), how should medium 2 and the wave frequency be chosen in order to fulfill the condition found at point **f)**?

11.11 Mie Resonance and a "Plasmonic Metamaterial"

A plane, monochromatic wave of frequency ω impinges on a a small sphere of radius $a \ll \lambda = 2\pi c/\omega$. The sphere is made of a material whose dielectric function $\varepsilon = \varepsilon(\omega)$ can be written as

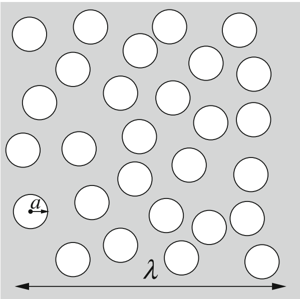

Fig. 11.6

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 + i\omega\eta} \tag{11.15}$$

where, according to the model of the elastically bound electron, ω_p is the plasma frequency, ω_0 is the resonance frequency of bound electrons, and η is a damping constant.
a) Find the induced field and polarization inside the sphere, and discuss any resonant behavior. (Hint: have a look back at Problem 3.4)
b) Assume that the EM wave is propagating inside a material where there are n_s metallic ($\omega_0 = 0$) nanospheres per unit volume, with $n_s\lambda^3 \gg 1 \gg \lambda/a$. Find the *macroscopic* polarization of the material and discuss the propagation of the wave as a function of the frequency ω.

Reference

1. J.D. Jackson, *Classical Electrodynamics*, §9.2 and 9.4, 3rd edn. (Wiley, New York, London, Sydney, 1998)