Chapter 12
Transmission Lines, Waveguides, Resonant Cavities

Topics. Guided propagation of EM waves. Transmission lines, TEM mode. Waveguides, TE and TM modes. Resonant cavities and discretization of frequencies.

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12.1 The Coaxial Cable

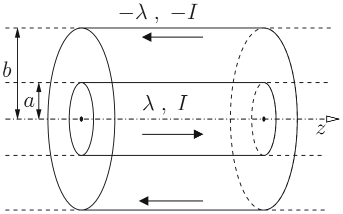

Fig. 12.1

A coaxial cable consists of two coaxial, infinitely long conductors: an inner cylinder of radius a, and an outer cylindrical shell of internal radius $b > a$. In general, if there is a charge per unit length λ on the inner conductor, there is an opposite charge $-\lambda$ on the outer conductor. Similarly, if a total current I flows through the inner conductor, an opposite "return" current $-I$ flows in the outer one.

We use a cylindrical coordinate system (r, ϕ, z) with the cable axis as z axis, and, at first, we assume that the region $a < r < b$ is filled by an insulating medium of dielectric permittivity $\varepsilon = 1$ and magnetic permeability $\mu = 1$.

a) Evaluate the capacitance and inductance per unit length of the cable.

b) Describe the propagation of a current signal $I(z,t)$ and of an associated linear charge signal $\lambda(z,t)$ along the cable, remembering the results of Problem 7.4. How are $I(z,t)$ and $\lambda(z,t)$ related to each other?

c) For given $I(z,t)$ and $\lambda(z,t)$, find the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ in the space between the conductors, assuming that both $\mathbf{E}$ and $\mathbf{B}$ are *transverse*, i.e. perpendicular to the direction of propagation (such configuration is called TEM mode).

d) Now consider a semi-infinite cable with an ideal source imposing the voltage $V(t)$ between the inner and outer conductors at the end of the cable. Show that the work done by the generator equals the flux of the Poynting vector through the cable (far enough from the end, so that we may neglect boundary effects).

e) How do the preceding answers change if the medium between the internal and external conductors has real and positive values for ε and μ, but different from unity?

12.2 Electric Power Transmission Line

Consider a thin, infinite straight wire along the z axis of a cylindrical coordinate system (r, ϕ, z). The wire is located in a medium of relative electric permittivity $\varepsilon_\mathrm{r} = 1$ and relative magnetic permeability $\mu_r = 1$. Assume a current $I = I(z,t)$ to flow in the wire, with

$$I = I(z,t) = I_0\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\,. \tag{12.1}$$

a) Calculate the linear charge density $\lambda = \lambda(z,t)$ on the wire.

b) Assume that the electric and magnetic fields have only their radial and azimuthal components, respectively,

$$E_\phi = E_z = 0,\quad E_r = E_r(r)\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t},\quad B_r = B_z = 0,\quad B_\varphi = B_\varphi(r)\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}. \tag{12.2}$$

Calculate E_r and B_φ as functions of I_0 and ω, and use Maxwell's equations to evaluate the phase velocity of the signal $v_\varphi = \omega/k$.

c) A high voltage transmission line comprises two straight parallel wires, at a constant distance $d = 5$ m and typical height over the ground $h = 30$ m. The two wires have opposite current intensities $\pm I(z,t)$ given by (12.1), where typically $I_0 = 10^3$ A and $\omega = 2\pi\times 50$ s^{-1}. Calculate the electric and magnetic fields on the symmetry plane between the two wires, and evaluate their magnitude on the ground.

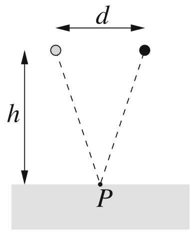

Fig. 12.2

12.3 TEM and TM Modes in an "Open" Waveguide

An "open" waveguide comprises two parallel, perfectly conducting planes, between which the waves propagate. Let us choose a Cartesian coordinate system (x,y,z) such that the two conducting planes are at $y = \pm a/2$, respectively, as in Fig. 12.3. An EM wave of frequency ω propagates in the waveguide along $\hat{\mathbf{x}}$. The magnetic field of the wave is directed along $\hat{\mathbf{z}}$ and has the form

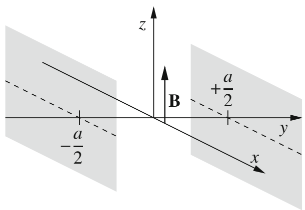

Fig. 12.3

$$B_z(x,y,t) = B_0 \cos(k_y y)\, \mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}\omega t} . \tag{12.3}$$

a) Find the relations between ω, k_x and k_y.
b) Find the expression for the electric field $\mathbf{E} = \mathbf{E}(x,y,t)$ of the EM wave.
c) Find how the possible values for k_y are determined by the boundary conditions on $\mathbf{E}$, and discuss the existence of cut-off frequencies.
d) Find the flux of energy along the direction of propagation $\hat{\mathbf{x}}$, showing that it is proportional to the group velocity of the wave.

12.4 Square and Triangular Waveguides

A waveguide has perfectly conducting walls and a square section of side a, as shown in Fig. 12.4. We choose a Cartesian coordinate system (x,y,z) where the interior of the waveguide is delimited by the four planes $x = 0$, $x = a$, $y = 0$ and $y = a$. Consider the

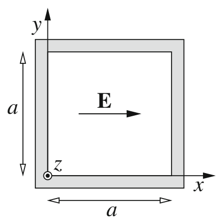

Fig. 12.4

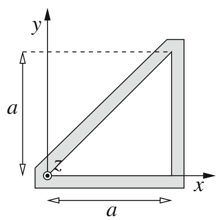

Fig. 12.5

propagation along $\hat{\mathbf{z}}$ of a wave of frequency ω, whose electric field $\mathbf{E}(x, y, z, t)$ is perpendicular to $\hat{\mathbf{z}}$ (a TE mode). Assume that the electric field can be written as

$$\mathbf{E}(x, y, z, t) = \tilde{\mathbf{E}}(x, y)\, e^{ik_z z - i\omega t}, \tag{12.4}$$

where $\tilde{\mathbf{E}}(x, y)$ on x and y only.
a) Assume that $\mathbf{E}$ is parallel to $\hat{\mathbf{x}}$, i.e. $\mathbf{E} = \hat{\mathbf{x}} E_x$, and determine the lowest value of ω for which the TE mode can propagate in the waveguide, and the corresponding expressions for the electric and magnetic fields.
b) Determine the lowest frequency and the EM fields for a waveguide delimited by the conducting planes $x = 0$, $y = 0$, and $y = x$, whose cross section is the right isosceles triangle shown in Fig. 12.5.

12.5 Waveguide Modes as an Interference Effect

An electric dipole $\mathbf{p} = p\hat{\mathbf{y}}$ is located at the origin of a Cartesian coordinate system (x, y, z), between two infinite, perfectly conducting planes located at $y = \pm a$, respectively, as shown in Fig. 12.6.

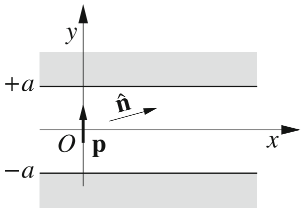

Fig. 12.6

a) Find the the electrostatic potential between the two conducting planes, using the method of images.

Now assume that the dipole is oscillating, in complex notation $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$, and consider the emitted radiation in the region between the two conducting planes, at large distances from the dipole, i.e., with both $|x| \gg \lambda$ and $|x| \gg a$.
b) Find in which directions $\hat{\mathbf{n}}$, lying in the $z = 0$ plane, we observe constructive interference between the waves emitted by the dipole and its images, and the corresponding constraints on the possible values of the oscillation frequency ω.
Now consider two types of waves, labeled "0" and "1", respectively, propagating between the two conducting planes with their wavevectors $\mathbf{k}_{0,1}$ lying in the $z = 0$ plane. Assume that the only nonzero component of the magnetic field of both waves is parallel to $\hat{\mathbf{z}}$ (TM waves), and that the magnetic fields have the form

$$\mathbf{B}_0 = \hat{\mathbf{z}} B_0\, e^{ik_{0x}x - i\omega t}, \qquad \mathbf{B}_1 = \hat{\mathbf{z}} B_1 \sin(k_{1y} y)\, e^{ik_{1x}x - i\omega t}. \tag{12.5}$$

c) Find the relation between the components of the wavevectors and ω for both waves.
d) Find the expressions for the electric fields $\mathbf{E}_{0,1}$ of the waves corresponding to the magnetic fields (12.5).

e) Verify (or impose when appropriate) that for the expressions found in **d)** the component of **E** parallel to the planes vanishes at their surface, and the related constraints on $\mathbf{k} = (k_x, k_y)$. What is the relation with the orders of interference found at point **b)**?

12.6 Propagation in an Optical Fiber

Figure 12.7 represents a simple model for an optical fiber. In a Cartesian reference frame (x, y, z) the space between the planes $y = \pm a/2$ is filled by a material of a real and positive refractive index $\mathsf{n} > 1$ (in the frequency range of interest), while we have vacuum ($\mathsf{n} = 1$) in the regions $y > a/2$ and $y < -a/2$. A monochromatic electromagnetic wave of frequency ω propagates parallel to $\hat{\mathbf{x}}$ inside the fiber. We assume that the only nonzero component of the electric field **E** of the wave is parallel to z (i.e. perpendicular to the plane of the figure). Further, we assume that the wave is the superposition of two plane waves with wavevectors $\mathbf{k}_1 \equiv (k_x, k_y, 0) \equiv k(\sin\theta, \cos\theta, 0)$, and $k_2 \equiv (k_x, -k_y, 0) \equiv k(\sin\theta, -\cos\theta, 0)$, where θ is the angle of incidence shown in the figure. We have, in complex notation,

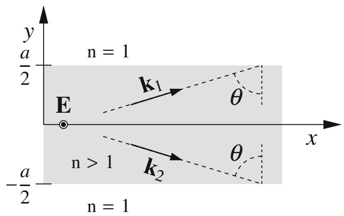

Fig. 12.7

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{y}}\, E_z(x, y, t) = \hat{\mathbf{y}} \left(E_1\, \mathrm{e}^{\mathrm{i}\mathbf{k}_1 \cdot \mathbf{r} - \mathrm{i}\omega t} + E_2\, \mathrm{e}^{\mathrm{i}\mathbf{k}_2 \cdot \mathbf{r} - \mathrm{i}\omega t} \right) \\ &= \hat{\mathbf{y}} \left(E_1\, \mathrm{e}^{\mathrm{i}k_x x + \mathrm{i}k_y y - \mathrm{i}\omega t} + E_2\, \mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}k_y y - \mathrm{i}\omega t} \right) . \end{aligned} \tag{12.6}$$

a) Find the relation between k and ω, and the range of θ for which the wave propagates without energy loss through the boundary surfaces at $y = \pm a/2$.

b) The *amplitude reflection coefficient* $\mathsf{r} = E_\mathrm{r}/E_\mathrm{i}$ is the ratio of the complex amplitude of the reflected wave to the amplitude of the incident wave, at the surface separating two media. In the case of total reflection we have $\mathsf{r} = \mathrm{e}^{\mathrm{i}\delta}$, with δ a real number. Show that, in our case, we have

$$k_y a + \delta = m\pi\,, \quad \text{with} \quad m \in \mathbb{N}\,, \tag{12.7}$$

and write the equation for the cut-off frequencies of the fiber. Find the values of k_y explicitly at the $\mathsf{n}\sin\theta \gg 1$, $\theta \to \pi/2$ limit.

c) How do the results change if **E** lies in the xy plane?

12.7 Wave Propagation in a Filled Waveguide

A waveguide has rectangular cross section and perfectly conducting walls. We choose a Cartesian reference frame where the waves propagate parallel to the x axis, and the conducting walls lie on the $y = \pm a/2$ and $z = \pm a/2$ planes, as in Fig. 12.8. The waveguide is uniformly filled with a medium having refractive index $\mathrm{n}=\mathrm{n}(\omega)$.

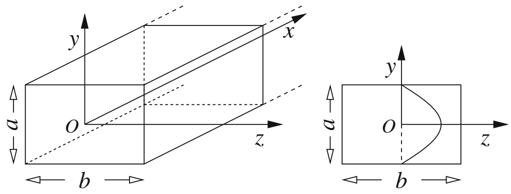

Fig. 12.8

a) Consider the propagation of a TE mode of frequency ω, for which the electric field is $\mathbf{E} = \hat{\mathbf{z}}\,E_z(y)\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t}$. Find the general expression for $E_z(y)$ and the dispersion relation $\omega = \omega(k)$. Determine the cut-off frequencies for the particular case in which the filling medium is a gas of free electrons, i.e., a plasma, with plasma frequency ω_p. In this case we have for the refractive index $\mathrm{n}^2(\omega) = 1 - \omega_p^2/\omega^2$.

b) Now assume that the medium fills only the $x > 0$ region of the waveguide. A monochromatic wave of the lowest frequency that can propagate in both regions ($x < 0$ and $x > 0$) travels in the guide from $x = -\infty$. Find the amplitudes of the reflected and transmitted waves at the $x = 0$ interface.

12.8 Schumann Resonances

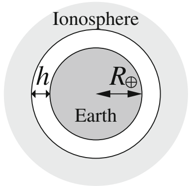

Fig. 12.9

The system formed by the Earth and the ionosphere can be considered as a resonant cavity. The cavity is delimited by two conducting, concentrical spherical surfaces: the Earth's surface (radius $R_\oplus \simeq 6400$ km) and to the lower border of the ionosphere, located at an altitude $h \simeq 100$ km above, as shown in Fig. 12.9, obviously out of scale. Inside this "cavity" there are standing electromagnetic waves of particular frequencies, called *Schumann resonances*.

We want to estimate the typical frequency ω of these resonances, assuming that both the Earth and the ionosphere are perfect conductors, and thus completely reflect the electromagnetic waves in the resonant frequency range.

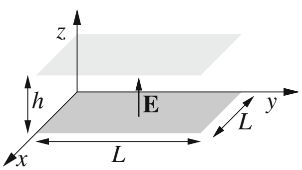

Fig. 12.10

In order to avoid mathematical complications due to the spherical geometry of the problem, we choose a simplified, flat model consisting in a rectangular parallelepiped with two square, conducting bases of side L, and height h. In a Cartesian reference frame, the base standing for the Earth surface lies on the $z = 0$ plane, while the base standing for the surface at the bottom of the ionosphere lies on the $z = h$ plane, as shown in Fig. 12.10. We choose $L = 2\pi R_\oplus$, and, in order to reproduce somehow spherical geometry, we impose periodic boundary conditions on the lateral surface of the parallelepiped, namely

$$\mathbf{E}(0, y, z, t) = \mathbf{E}(L, y, z, t)\,, \quad \mathbf{E}(x, 0, z, t) = \mathbf{E}(x, L, z, t)\,, \tag{12.8}$$

where $\mathbf{E}$ is the field of the wave, the same conditions are assumed for the magnetic field of the wave. We assume $\varepsilon_r = 1$ and $\mu_r = 1$ in the interior of our parallelepiped. Further, we assume a TE mode with an electric field of the form

$$\mathbf{E} = \hat{\mathbf{z}}\, E_0\, \mathrm{e}^{\mathrm{i}k_x x + \mathrm{i}k_y y - \mathrm{i}\omega t}\,. \tag{12.9}$$

a) Find the possible values of k_x, k_y, ω and give a numerical estimate of ω and the corresponding wavelength for the lowest frequency mode.
b) The low-frequency conductivity of sea water is $\sigma \simeq 4.4\ \Omega^{-1}\mathrm{m}^{-1}$. Discuss if approximating the surface of the oceans as a perfect conductor is reasonable at the frequency of the Schumann resonances .