Chapter 1
Basics of Electrostatics

Topics. The electric charge. The electric field. The superposition principle. Gauss's law. Symmetry considerations. The electric field of simple charge distributions (plane layer, straight wire, sphere). Point charges and Coulomb's law. The equations of electrostatics. Potential energy and electric potential. The equations of Poisson and Laplace. Electrostatic energy. Multipole expansions. The field of an electric dipole.

Units. An aim of this book is to provide formulas compatible with both SI (French: *Système International d'Unités*) units and Gaussian units in Chapters 1–6, while only Gaussian units will be used in Chapters 7–13. This is achieved by introducing some system-of-units-dependent constants.

The first constant we need is *Coulomb's constant*, k_e, which for instance appears in the expression for the force between two electric point charges q_1 and q_2 in vacuum, with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. The Coulomb force acting, for instance, on q_1 is

$$\mathbf{f}_1 = k_e \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\,\hat{\mathbf{r}}_{12}\,, \tag{1.1}$$

where k_e is Coulomb's constant, dependent on the units used for force, electric charge, and length. The vector $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ is the distance from q_2 to q_1, pointing towards q_1, and $\hat{\mathbf{r}}_{12}$ the corresponding unit vector. Coulomb's constant is

$$k_e = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} 8.987\cdots\times 10^9\ \mathrm{N\cdot m^2\cdot C^{-2}} \simeq 9\times 10^9\ \mathrm{m/F} & \text{SI} \\ 1 & \text{Gaussian.} \end{cases} \tag{1.2}$$

Constant $\varepsilon_0 \simeq 8.854\,187\,817\,620\cdots\times 10^{-12}$ F/m is the so-called "dielectric permittivity of free space", and is defined by the formula

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_1

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}\,, \tag{1.3}$$

where $\mu_0 = 4\pi\times 10^{-7}$ H/m (by definition) is the vacuum magnetic permeability, and c is the speed of light in vacuum, $c = 299\,792\,458$ m/s (this is a precise value, since the length of the meter is defined from this constant and the international standard for time).

Basic equations The two basic equations of this Chapter are, in differential and integral form,

$$\nabla\cdot\mathbf{E} = 4\pi k_\mathrm{e}\,\varrho\,, \qquad \oint_S \mathbf{E}\cdot\mathrm{d}\mathbf{S} = 4\pi k_\mathrm{e}\int_V \varrho\,\mathrm{d}^3r \tag{1.4}$$

$$\nabla\times\mathbf{E} = 0\,, \qquad \oint_C \mathbf{E}\cdot\mathrm{d}\ell = 0\,. \tag{1.5}$$

where $\mathbf{E}(\mathbf{r},t)$ is the electric field, and $\varrho(\mathbf{r},t)$ is the volume charge density, at a point of location vector $\mathbf{r}$ at time t. The infinitesimal volume element is $\mathrm{d}^3r = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. In (1.4) the functions to be integrated are evaluated over an arbitrary volume V, or over the surface S enclosing the volume V. The function to be integrated in (1.5) is evaluated over an arbitrary closed path C. Since $\nabla\times\mathbf{E} = 0$, it is possible to define an electric potential $\varphi = \varphi(\mathbf{r})$ such that

$$\mathbf{E} = -\nabla\varphi\,. \tag{1.6}$$

The general expression of the potential generated by a given charge distribution $\varrho(\mathbf{r})$ is

$$\varphi(\mathbf{r}) = k_\mathrm{e}\int_V \frac{\varrho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}^3r'\,. \tag{1.7}$$

The force acting on a volume charge distribution $\varrho(\mathbf{r})$ is

$$\mathbf{f} = \int_V \varrho(\mathbf{r}')\,\mathbf{E}(\mathbf{r}')\,\mathrm{d}^3r'\,. \tag{1.8}$$

As a consequence, the force acting on a point charge q located at $\mathbf{r}$ (which corresponds to a charge distribution $\varrho(\mathbf{r}') = q\delta(\mathbf{r}-\mathbf{r}')$, with $\delta(\mathbf{r})$ the Dirac-delta function) is

$$\mathbf{f} = q\,\mathbf{E}(\mathbf{r})\,. \tag{1.9}$$

The electrostatic energy U_es associated with a given distribution of electric charges and fields is given by the following expressions

$$U_\mathrm{es} = \int_V \frac{\mathbf{E}^2}{8\pi k_\mathrm{e}}\,\mathrm{d}^3r\,. \tag{1.10}$$

$$U_{\mathrm{es}} = \frac{1}{2}\int_V \varrho\varphi\,\mathrm{d}^3r\,, \tag{1.11}$$

Equations (1.10–1.11) are valid provided that the volume integrals are finite and that all involved quantities are well defined.

The multipole expansion allows us to obtain simple expressions for the leading terms of the potential and field generated by a charge distribution at a distance much larger than its extension. In the following we will need only the expansion up to the dipole term,

$$\varphi(\mathbf{r}) \simeq k_{\mathrm{e}}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \ldots\right), \tag{1.12}$$

where Q is the total charge of the distribution and the electric dipole moment is

$$\mathbf{p} \equiv \int_V \mathbf{r}'\rho(\mathbf{r}')\mathrm{d}^3\mathbf{r}'\,. \tag{1.13}$$

If $Q = 0$, then $\mathbf{p}$ is independent on the choice of the origin of the reference frame. The field generated by a dipolar distribution centered at $\mathbf{r} = 0$ is

$$\mathbf{E} = k_{\mathrm{e}}\frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3}\,. \tag{1.14}$$

We will briefly refer to a localized charge distribution having a dipole moment as "an electric dipole" (the simplest case being two opposite point charges $\pm q$ with a spatial separation δ, so that $\mathbf{p} = q\delta$). A dipole placed in an external field $\mathbf{E}_{\mathrm{ext}}$ has a potential energy

$$U_{\mathrm{p}} = -\mathbf{p}\cdot\mathbf{E}_{\mathrm{ext}}\,. \tag{1.15}$$

1.1 Overlapping Charged Spheres

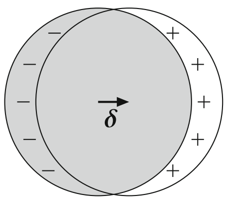

Fig. 1.1

We assume that a neutral sphere of radius R can be regarded as the superposition of two "rigid" spheres: one of uniform positive charge density $+\varrho_0$, comprising the nuclei of the atoms, and a second sphere of the same radius, but of negative uniform charge density $-\varrho_0$, comprising the electrons. We further assume that its is possible to shift the two spheres relative to each other by a quantity δ, as shown in Fig. 1.1, without perturbing the internal structure of either sphere.

Find the electrostatic field generated by the global charge distribution

a) in the “inner” region, where the two spheres overlap,

b) in the “outer” region, i.e., outside both spheres, discussing the limit of small displacements $\delta \ll R$.

1.2 Charged Sphere with Internal Spherical Cavity

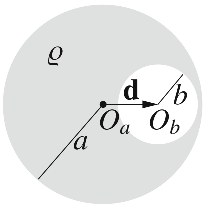

Fig. 1.2

A sphere of radius a has uniform charge density ϱ over all its volume, excluding a spherical cavity of radius $b < a$, where $\varrho = 0$. The center of the cavity, O_b is located at a distance $\mathbf{d}$, with $|\mathbf{d}| < (a - b)$, from the center of the sphere, O_a. The mass distribution of the sphere is proportional to its charge distribution.

a) Find the electric field inside the cavity.

Now we apply an external, uniform electric field $\mathbf{E}_0$. Find

b) the force on the sphere,

c) the torque with respect to the center of the sphere, and the torque with respect to the center of mass.

1.3 Energy of a Charged Sphere

A total charge Q is distributed uniformly over the volume of a sphere of radius R. Evaluate the electrostatic energy of this charge configuration in the following three alternative ways:

a) Evaluate the work needed to assemble the charged sphere by moving successive infinitesimals shells of charge from infinity to their final location.

b) Evaluate the volume integral of $u_{\mathrm{E}} = |\mathbf{E}|^2/(8\pi k_{\mathrm{e}})$ where $\mathbf{E}$ is the electric field [Eq. (1.10)].

c) Evaluate the volume integral of $\varrho\phi/2$ where ϱ is the charge density and ϕ is the electrostatic potential [Eq. (1.11)]. Discuss the differences with the calculation made in **b)**.

1.4 Plasma Oscillations

A square metal slab of side L has thickness h, with $h \ll L$. The conduction-electron and ion densities in the slab are n_e and $n_i = n_e/Z$, respectively, Z being the ion charge.

An external electric field shifts all conduction electrons by the same amount δ, such that $|\delta| \ll h$, perpendicularly to the base of the slab. We assume that both n_e and n_i are constant, that the ion lattice is unperturbed by the external field, and that boundary effects are negligible.

a) Evaluate the electrostatic field generated by the displacement of the electrons.

b) Evaluate the electrostatic energy of the system.

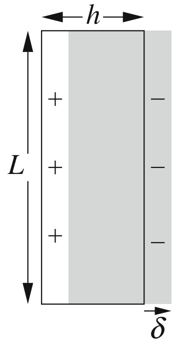

Fig. 1.3

Now the external field is removed, and the "electron slab" starts oscillating around its equilibrium position.

c) Find the oscillation frequency, at the small displacement limit ($\delta \ll h$).

1.5 Mie Oscillations

Now, instead of a the metal slab of Problem 1.4, consider a metal sphere of radius R. Initially, all the conduction electrons (n_e per unit volume) are displaced by $-\delta$ (with $\delta \ll R$) by an external electric field, analogously to Problem 1.1.

a) At time $t = 0$ the external field is suddenly removed. Describe the subsequent motion of the conduction electrons under the action of the self-consistent electrostatic field, neglecting the boundary effects on the electrons close to the surface of the sphere.

b) At the limit $\delta \to 0$ (but assuming $en_e\delta = \sigma_0$ to remain finite, i.e., the charge distribution is a surface density), find the electrostatic energy of the sphere as a function of δ and use the result to discuss the electron motion as in point **a)**.

1.6 Coulomb explosions

At $t = 0$ we have a spherical cloud of radius R and total charge Q, comprising N point-like particles. Each particle has charge $q = Q/N$ and mass m. The particle density is uniform, and all particles are at rest.

a) Evaluate the electrostatic potential energy of a charge located at a distance $r < R$ from the center at $t = 0$.

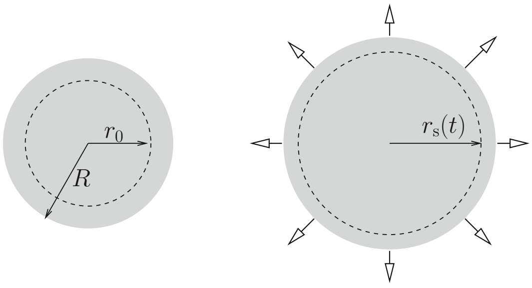

Fig. 1.4

b) Due to the Coulomb repulsion, the cloud begins to expand radially, keeping its spherical symmetry. Assume that the particles do not overtake one another, i.e., that if two particles were initially located at $r_1(0)$ and $r_2(0)$, with $r_2(0) > r_1(0)$, then $r_2(t) > r_1(t)$ at any subsequent time $t > 0$. Consider the particles located in the infinitesimal spherical shell $r_0 < r_s < r_0 + dr$, with $r_0 + dr < R$, at $t = 0$. Show that the equation of motion of the layer is

$$m\frac{d^2 r_s}{dt^2} = k_e \frac{qQ}{r_s^2}\left(\frac{r_0}{R}\right)^3 \tag{1.16}$$

c) Find the initial position of the particles that acquire the maximum kinetic energy during the cloud expansion, and determinate the value of such maximum energy.

d) Find the energy spectrum, i.e., the distribution of the particles as a function of their final kinetic energy. Compare the total kinetic energy with the potential energy initially stored in the electrostatic field.

e) Show that the particle density remains spatially uniform during the expansion.

1.7 Plane and Cylindrical Coulomb Explosions

Particles of identical mass m and charge q are distributed with zero initial velocity and uniform density n_0 in the infinite slab $|x| < a/2$ at $t = 0$. For $t > 0$ the slab expands because of the electrostatic repulsion between the pairs of particles.

a) Find the equation of motion for the particles, its solution, and the kinetic energy acquired by the particles.

b) Consider the analogous problem of the explosion of a uniform distribution having cylindrical symmetry.

1.8 Collision of two Charged Spheres

Two rigid spheres have the same radius R and the same mass M, and opposite charges $\pm Q$. Both charges are uniformly and rigidly distributed over the volumes of the two spheres. The two spheres are initially at rest, at a distance $x_0 \gg R$ between their centers, such that their interaction energy is negligible compared to the sum of their "internal" (construction) energies.

a) Evaluate the initial energy of the system.

The two spheres, having opposite charges, attract each other, and start moving at $t = 0$.

b) Evaluate the velocity of the spheres when they touch each other (i.e. when the distance between their centers is $x = 2R$).

c) Assume that, after touching, the two spheres penetrate each other without friction. Evaluate the velocity of the spheres when the two centers overlap ($x = 0$).

1.9 Oscillations in a Positively Charged Conducting Sphere

An electrically neutral metal sphere of radius a contains N conduction electrons. A fraction f of the conduction electrons ($0 < f < 1$) is removed from the sphere, and the remaining $(1 - f)N$ conduction electrons redistribute themselves to an equilibrium configurations, while the N lattice ions remain fixed.

a) Evaluate the conduction-electron density and the radius of their distribution in the sphere.

Now the conduction-electron sphere is rigidly displaced by $\boldsymbol{\delta}$ relatively to the ion lattice, with $|\boldsymbol{\delta}|$ small enough for the conduction-electron sphere to remain inside the ion sphere.

b) Evaluate the electric field inside the conduction-electron sphere.

c) Evaluate the oscillation frequency of the conduction-electron sphere when it is released.

1.10 Interaction between a Point Charge and an Electric Dipole

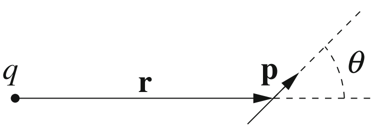

Fig. 1.5

An electric dipole $\mathbf{p}$ is located at a distance $\mathbf{r}$ from a point charge q, as in Fig. 1.5. The angle between $\mathbf{p}$ and $\mathbf{r}$ is θ.

a) Evaluate the electrostatic force on the dipole.

b) Evaluate the torque acting on the dipole.

1.11 Electric Field of a Charged Hemispherical Surface

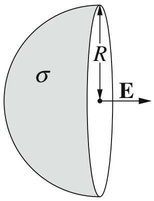

Fig. 1.6

A hemispherical surface of radius R is uniformly charged with surface charge density σ. Evaluate the electric field and potential at the center of curvature (hint: start from the electric field of a uniformly charged ring along its axis).