Chapter 2
Electrostatics of Conductors

Topics. The electrostatic potential in vacuum. The uniqueness theorem for Poisson's equation. Laplace's equation, harmonic functions and their properties. Boundary conditions at the surfaces of conductors: Dirichlet, Neumann and mixed boundary conditions. The capacity of a conductor. Plane, cylindrical and spherical capacitors. Electrostatic field and electrostatic pressure at the surface of a conductor. The method of image charges: point charges in front of plane and spherical conductors.

Basic equations Poisson's equation is

$$\nabla^2 \varphi(\mathbf{r}) = -4\pi k_e \varrho(\mathbf{r}), \tag{2.1}$$

where $\varphi(\mathbf{r})$ is the electrostatic potential, and $\varrho(\mathbf{r})$ is the electric charge density, at the point of vector position $\mathbf{r}$. The solution of Poisson's equation is unique if one of the following boundary conditions is true

1. **Dirichlet boundary condition**: φ is known and well defined on all of the boundary surfaces.
2. **Neumann boundary condition**: $\mathbf{E} = -\boldsymbol{\nabla}\varphi$ is known and well defined on all of the boundary surfaces.
3. **Modified Neumann boundary condition** (also called Robin boundary condition): conditions where boundaries are specified as conductors with known charges.
4. **Mixed boundary conditions**: a combination of Dirichlet, Neumann, and modified Neumann boundary conditions:

Laplace's equation is the special case of Poisson's equation

$$\nabla^2 \varphi(\mathbf{r}) = 0, \tag{2.2}$$

which is valid in vacuum.

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_2

2.1 Metal Sphere in an External Field

A a metal sphere of radius R consists of a “rigid” lattice of ions, each of charge $+Ze$, and valence electrons each of charge $-e$. We denote by n_i the ion density, and by n_e the electron density. The net charge of the sphere is zero, therefore $n_\mathrm{e} = Zn_\mathrm{i}$. The sphere is located in an external, constant, and uniform electric field $\mathbf{E}_0$. The field causes a displacement $\boldsymbol{\delta}$ of the “electron sea” with respect to the ion lattice, so that the total field inside the sphere, $\mathbf{E}$, is zero. Using Problem 1.1 as a model, evaluate

a) the displacement $\boldsymbol{\delta}$, giving a numerical estimate for $E_0 = 10^3$ V/m;

b) the field generated by the sphere at its exterior, as a function of $\mathbf{E}_0$;

c) the surface charge density on the sphere.

2.2 Electrostatic Energy with Image Charges

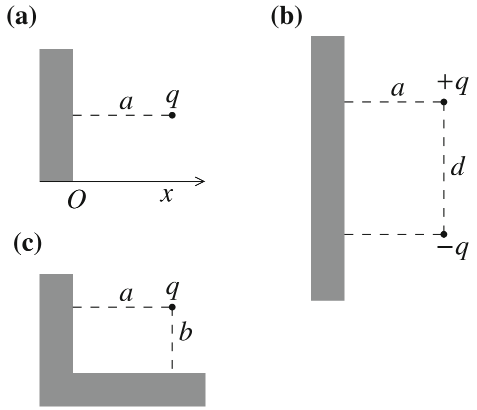

Fig. 2.1

Consider the configurations of point charges in the presence of conducting planes shown in Fig. 2.1. For each case, find the solution for the electrostatic potential over the whole space and evaluate the electrostatic energy of the system. Use the method of image charges.

a) A charge q is located at a distance a from an infinite conducting plane.

b) Two opposite charges $+q$ and $-q$ are at a distance d from each other, both at the same distance a from an infinite conducting plane.

c) A charge q is at distances a and b, respectively, from two infinite conducting half planes forming a right dihedral angle.

2.3 Fields Generated by Surface Charge Densities

Consider the case **a)** of Problem 2.2: we have a point charge q at a distance a from an infinite conducting plane.

a) Evaluate the surface charge density σ, and the total induced charge q_ind, on the plane.

b) Now assume to have a *nonconducting* plane with the same surface charge distribution as in point **a)**. Find the electric field in the whole space.

c) A non conducting spherical surface of radius a has the same charge distribution as the conducting sphere of Problem 2.4. Evaluate the electric field in the whole space.

2.4 A Point Charge in Front of a Conducting Sphere

A point charge q is located at a distance d from the center of a conducting grounded sphere of radius $a < d$. Evaluate

a) the electric potential φ over the whole space;

b) the force on the point charge;

c) the electrostatic energy of the system.

Answer the above questions also in the case of an isolated, uncharged conducting sphere.

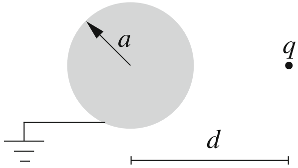

Fig. 2.2

2.5 Dipoles and Spheres

An electric dipole **p** is located at a distance d from the center of a conducting sphere of radius a. Evaluate the electrostatic potential φ over the whole space assuming that

a) p is perpendicular to the direction from **p** to the center of the sphere,

b) p is directed towards the center of the sphere.

c) p forms an arbitrary angle θ with respect to the straight line passing through the center of the sphere and the dipole location.

In all three cases consider the two possibilities of i) a grounded sphere, and ii) an electrically uncharged isolated sphere.

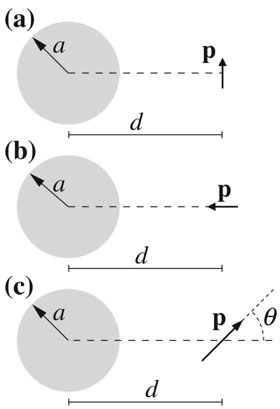

Fig. 2.3

2.6 Coulomb's Experiment

Coulomb, in his original experiment, measured the force between two charged metal spheres, rather than the force between two "point charges". We know that the field of a sphere whose surface is uniformly charged equals the field of a point charge, and that the force between two charge distributions, each of spherical symmetry, equals the force between two point charges

$$\mathbf{F} = k_e \frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}}, \tag{2.3}$$

where q_1 and q_2 are the charges on the spheres, and $\mathbf{r} = r\hat{\mathbf{r}}$ is the distance between the two centers of symmetry. But we also know that electric induction modifies the surface charge densities of conductors, so that a correction to (2.3) is needed. We expect the induction effects to be important if the radius a of the spheres is not negligibly small with respect to r.

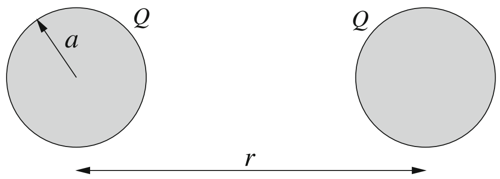

Fig. 2.4

a) Using the method of image charges, find the solution for the electrical potential outside the spheres as a series expansion, and identify the expansion parameter. For simplicity, assume the spheres to be identical and to have the same charge Q, as in the figure.

b) Evaluate the lowest order correction to the force between the spheres with respect to Coulomb's law (2.3).

2.7 A Solution Looking for a Problem

An electric dipole $\mathbf{p}$ is located at the origin of a Cartesian frame, parallel to the z axis, in the presence of a uniform electric field $\mathbf{E}$, also parallel to the z axis.

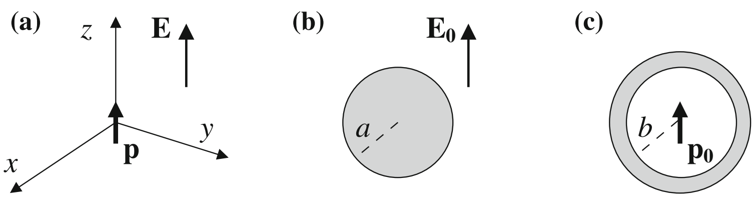

Fig. 2.5

a) Find the total electrostatic potential $\varphi = \varphi(\mathbf{r})$, with the condition $\varphi = 0$ on the xy plane. Show that, in addition to the xy plane, there is another equipotential surface with $\varphi = 0$, that this surface is spherical, and calculate its radius R.

Now use the result from point **a)** to find the electric potential in the whole space for the following problems:

b) A conducting sphere of radius a is placed in a uniform electric field $\mathbf{E}_0$;

c) a dipole $\mathbf{p}_0$ is placed in the center of a conducting spherical shell of radius b.

d) Find the solution to problem **c)** using the method of image charges.

2.8 Electrically Connected Spheres

Two conducting spheres of radii a and $b < a$, respectively, are connected by a thin metal wire of negligible capacitance. The centers of the two spheres are at a distance $d \gg a > b$ from each other. A total net charge Q is located on the system.

Evaluate to zeroth order approximation, neglecting the induction effects on the surfaces of the two spheres,

a) how the charge Q is partitioned between the two spheres,

b) the value V of the electrostatic potential of the system (assuming zero potential at infinity) and the capacitance $C = Q/V$,

c) the electric field at the surface of each sphere, comparing the intensities and discussing the limit $b \to 0$.

d) Now take the electrostatic induction effects into account and improve the preceding results to the first order in a/d and b/d.

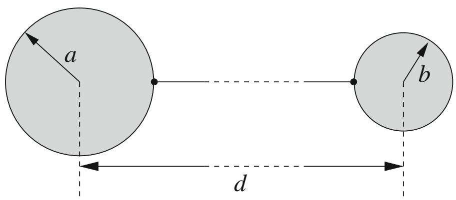

Fig. 2.6

2.9 A Charge Inside a Conducting Shell

A point charge q is located at a distance d from the center of a spherical conducting shell of internal radius $R > d$, and external radius $R' > R$. The shell is grounded, so that its electric potential is zero.

a) Find the electric potential and the electric field in the whole space.

b) Evaluate the force acting on the charge.

c) Show that the total charge induced on the surface of the internal sphere is $-q$.

d) How does the answer to **a)** change if the shell is not grounded, but electrically isolated with a total charge equal to zero?

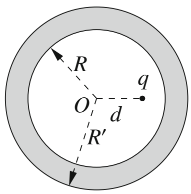

Fig. 2.7

2.10 A Charged Wire in Front of a Cylindrical Conductor

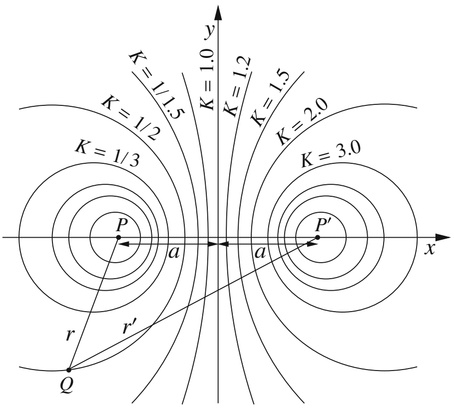

Fig. 2.8

We have two fixed points $P \equiv (-a, 0)$ and $P' \equiv (+a, 0)$ on the xy plane, and a third, generic point $Q \equiv (x, y)$. Let $r = \overline{QP}$ and $r' = \overline{QP'}$ be the distances of Q from P and P', respectively.

a) Show that the family of curves defined by the equation $r/r' = K$, with $K > 0$ a constant, is the family of circumferences drawn in Fig. 2.8.

b) Now consider the electrostatic field generated by two straight infinite, parallel wires of linear charge densities λ and $-\lambda$, respectively. We choose a Cartesian reference frame such that the z axis is parallel to the wires, and the two wires intersect the xy plane at $(-a, 0)$ and $(+a, 0)$, respectively. Use the geometrical result of point **a)** to show that the equipotential surfaces of the electrostatic field generated by the two wires are infinite cylindrical surfaces whose intersections with the xy plane are the circumferences shown in Fig. 2.8.

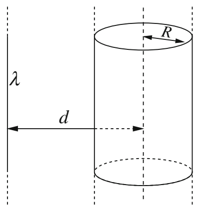

Fig. 2.9

c) Use the results of points **a)** and **b)** to solve the following problem by the method of image charges. An infinite straight wire of linear charge density λ is located in front of an infinite conducting cylindrical surface of radius R. The wire is parallel to the axis of the cylinder, and the distance between the wire and axis of the cylinder is d, with $d > R$, as shown in Fig. 2.9. Find the electrostatic potential in the whole space.

2.11 Hemispherical Conducting Surfaces

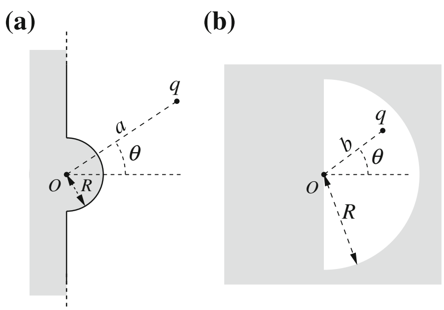

Fig. 2.10

Find the configurations of image charges that solve the problems represented in Fig. 2.10a, 2.10b, and the corresponding induced-charge distributions, remembering that the electric potential of an infinite conductor is zero.

a) The plane infinite surface of a conductor has a hemispherical boss of radius R, with curvature center in O. A point charge q is located at a distance $a > R$ from O, the line segment from O to q forms an angle θ with the symmetry axis of the problem.

b) An infinite conductor has a hemispherical cavity of radius R. A point charge q is located inside the cavity, at a distance $b < R$ from O. Again, the line segment from O to q forms an angle θ with the symmetry axis of the problem.

2.12 The Force Between the Plates of a Capacitor

The plates of a flat, parallel-plate capacitor have surface S and separation $h \ll \sqrt{S}$. Find the force between the plates, both for an isolated capacitor (as a function of the constant charge Q), and for a capacitor connected to an ideal voltage source (as a function of the constant voltage V). In both cases, use *two* different methods, i.e., calculate the force

a) from the electrostatic pressure on the surface of the plates,

b) from the expression of the energy as a function of the distance between the plates.

2.13 Electrostatic Pressure on a Conducting Sphere

A conducting sphere of radius a has a net charge Q and it is electrically isolated. Find the electrostatic pressure at the surface of the sphere

a) directly, from the surface charge density and the electric field on the sphere,

b) by evaluating variation of the electrostatic energy with respect to a.

c) Now calculate again the pressure on the sphere, assuming that the sphere is not isolated, but connected to an ideal voltage source, keeping the sphere at the constant potential V with respect to infinity.

2.14 Conducting Prolate Ellipsoid

a) Show that the equipotential surfaces generated by a uniformly charged line segment are prolate ellipsoids of revolution, with the focal points coinciding with the end points of the segment.

b) Evaluate the electric field generated by a conducting prolate ellipsoid of revolution of major axis $2a$ and minor axis $2b$, carrying a charge Q. Evaluate the electric capacity of the ellipsoid, and the capacity of a confocal ellipsoidal capacitor.

c) Use the above results to evaluate an approximation for the capacity of a straight conducting cylindrical wire of length h, and diameter $2b$.