Chapter 3
Electrostatics of Dielectric Media

Topics. Polarization charges. Dielectrics. Permanent and induced polarization. The auxiliary vector **D**. Boundary conditions at the surface of dielectrics. Relative dielectric permittivity ε_r.

Basic equations We **P** denote the *electric polarization* (electric dipole moment per unit volume) of a material. Some special materials have a permanent non-zero electric polarization, but in most cases a polarization appears only in the presence of an electric field **E**. We consider linear dielectric materials, for which **P** is parallel and proportional to **E**, thus

$$\mathbf{P} = \begin{cases} \varepsilon_0 \chi \mathbf{E}\,, & \text{where} \quad \chi = \varepsilon_r - 1\,, \quad \text{SI} \\ \chi \mathbf{E}\,, & \text{where} \quad \chi = \dfrac{\varepsilon_r - 1}{4\pi}\,, \quad \text{Gaussian,} \end{cases} \tag{3.1}$$

where χ is called the *electric susceptibility* and ε_r the *relative permittivity* of the material.[1] Notice that ε_r is a dimensionless quantity with the same numerical value both in SI and Gaussian units.

We shall denote by ϱ_b and ϱ_f the volume densities of bound electric charge and of free electric charge, respectively, and by σ_b and σ_f the surface densities of bound charge. Quantities ϱ_b and σ_b are related to the electric polarization **P** by

$$\varrho_b = -\nabla \cdot \mathbf{P}\,, \quad \text{and} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}\,, \tag{3.2}$$

[1] In anisotropic media (such as non-cubic crystals) **P** and **E** may be not parallel to each other, in this case χ and ε_r are actually second rank tensors. Here, however, we are interested only in isotropic and homogeneous media, for which χ and ε_r are scalar quantities.

A. Macchi et al., *Problems in Classical Electromagnetism*,
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where $\hat{\mathbf{n}}$ is the unit vector pointing outwards from the boundary surface of the polarized material. We may thus rewrite (1.4) as

$$\nabla \cdot \mathbf{E} = \begin{cases} \dfrac{\varrho_f + \varrho_b}{\varepsilon_0} = \dfrac{\varrho_f}{\varepsilon_0} - \dfrac{1}{\varepsilon_0} \nabla \cdot \mathbf{P} \,, & \text{SI} \\ 4\pi(\varrho_f + \varrho_b) = 4\pi\varrho_f - 4\pi \nabla \cdot \mathbf{P} \,, & \text{Gaussian.} \end{cases} \tag{3.3}$$

We can also introduce the auxiliary vector **D** (also called *electrical displacement*) defined as

$$\mathbf{D} = \begin{cases} \varepsilon_0 \mathbf{E} + \mathbf{P} \,, & \text{SI,} \\ \mathbf{E} + 4\pi \mathbf{P} \,, & \text{Gaussian,} \end{cases} \tag{3.4}$$

so that

$$\nabla \cdot \mathbf{D} = \begin{cases} \varrho_f \,, & \text{SI,} \\ 4\pi\varrho_f \,, & \text{Gaussian.} \end{cases} \tag{3.5}$$

In addition, $\nabla \times \mathbf{E} = 0$ holds in static conditions. Thus, at the interface between two different dielectric materials, the component of **E** parallel to the interface surface, and the perpendicular component of **D** are continuous. In a material of electric permittivity ε_r

$$\mathbf{D} = \begin{cases} \varepsilon_0 \varepsilon_r \mathbf{E} \,, & \text{SI} \\ \varepsilon_r \mathbf{E} \,, & \text{Gaussian.} \end{cases} \tag{3.6}$$

To facilitate the use of the basic equations in this chapter also with the system independent units, we summarize some of them in the following table:

Table 3.1 Basic equations for electrostatics in dielectrics

Quantity	SI	Gaussian	System independent
Polarization **P** of an isotropic dielectric medium of relative permittivity ε_r	$\varepsilon_0(\varepsilon_r - 1)\,\mathbf{E}$	$\dfrac{\varepsilon_r - 1}{4\pi}\,\mathbf{E}$	$\dfrac{\varepsilon_r - 1}{4\pi k_e}\,\mathbf{E}$
$\nabla \cdot \mathbf{E}$	$\dfrac{\varrho_f + \varrho_b}{\varepsilon_0}$	$4\pi(\varrho_f + \varrho_b)$	$4\pi k_e\ (\varrho_f + \varrho_b)$
$\nabla \cdot (\varepsilon_r \mathbf{E})$	$\dfrac{\varrho_f}{\varepsilon_0}$	$4\pi\varrho_f$	$4\pi k_e\ \varrho_f$
$\nabla \times \mathbf{E}$	0	0	0

3.1 An Artificial Dielectric

We have a tenuous suspension of conducting spheres, each of radius a, in a liquid dielectric material of relative dielectric permittivity $\varepsilon_{\mathrm{r}} = 1$. The number of spheres per unit volume is n.

a) Evaluate the dielectric susceptibility χ of the system as a function of the fraction of the volume filled by the conducting spheres. Use the mean field approximation (MFA), according to which the electric field may be assumed to be uniform throughout the medium.

b) The MFA requires the field generated by a single sphere on its nearest neighbor to be much smaller than the mean field due to the collective contribution of all the spheres. Derive a condition on n and a for the validity of the MFA.

3.2 Charge in Front of a Dielectric Half-Space

A plane divides the whole space into two halves, one of which is empty and the other filled by a dielectric medium of relative permittivity ε_{r}. A point charge q is located in vacuum at a distance d from the medium as shown in Fig. 3.1.

a) Find the electric potential and electric field in the whole space, using the method of image charges.

b) Evaluate the surface polarization charge density on the interface plane, and the total polarization charge of the plane.

c) Find the field generated by the polarization charge in the whole space.

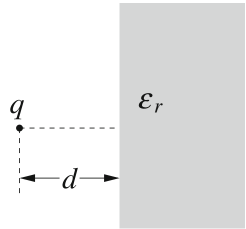

Fig. 3.1

3.3 An Electrically Polarized Sphere

Ferroelectricity is the property of some materials like Rochelle salt, carnauba wax, barium titanate, lead titanate, ..., that possess a spontaneous electric polarization in the absence of external fields.

a) Consider a ferroelectric sphere of radius a and uniform polarization $\mathbf{P}$, in the absence of external fields, and evaluate the electric field in the whole space (hint: see Problem 1.1).

b) Now consider again a ferroelectric sphere of radius a and uniform polarization $\mathbf{P}$, but with a concentrical spherical hole of radius $b < a$. Evaluate the electric field and the displacement field in the whole space.

3.4 Dielectric Sphere in an External Field

A dielectric sphere of relative permittivity ε_r and radius a is placed in vacuum, in an initially uniform external electric field $\mathbf{E}_0$, as shown in Fig. 3.2.
a) Find the electric field in the whole space (hint: use the results of Problem 3.3 and the superposition principle).

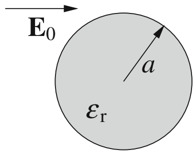

Fig. 3.2

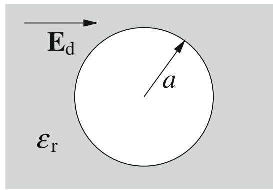

Fig. 3.3

A spherical cavity of radius a is located inside an infinite dielectric medium of relative permittivity ε_r, as in Fig. 3.3. The system is in the presence of an external electric field which, far from the cavity (i.e., at a distance $\gg a$), is uniform and equal to $\mathbf{E}_d$.
b) Find the electric field in the whole space.

3.5 Refraction of the Electric Field at a Dielectric Boundary

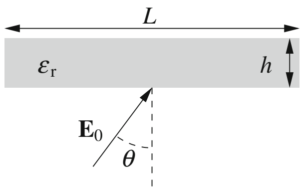

Fig. 3.4

A dielectric slab of thickness h, length $L \gg h$, and dielectric permittivity ε_r, is placed in an external uniform electric field $\mathbf{E}_0$. The angle between $\mathbf{E}_0$ and the normal to the slab surface is θ, as in Fig. 3.4.
a) Find the electric field $\mathbf{E}'$ inside the slab and the angle θ' between $\mathbf{E}'$ and the normal to the slab surface.
b) Find the polarization charge densities in the dielectric medium.
c) Evaluate the torque exerted by the external field on the slab, if any.
Neglect all boundary effects.

3.6 Contact Force between a Conducting Slab and a Dielectric Half-Space

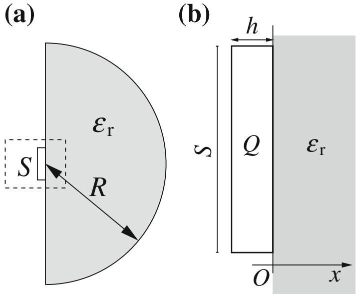

Fig. 3.5

A conducting square slab of surface $S = a^2$ and thickness $h \ll a$ is in contact with a dielectric medium of relative permittivity ε_r. The dielectric medium is much larger than the slab, thus, we can consider it as a hemisphere of radius $R \gg a$, with the slab in contact with its base, as shown in Fig. 3.5.a. Part b) of Fig. 3.5 is an enlargement of the area enclosed in the dashed rectangle of part a). With this assumption, we can assume the slab to be in contact with a semi-infinite medium filling the half-space $x > 0$, while we have vacuum in the half space $x < 0$. The conducting slab carries a total charge Q, and we assume that the boundary effects at its edges are negligible.

a) Considering both the cases in which the slab is in contact with the dielectric, and in which it is displaced by an amount $\xi \ll a$ to the left, find the free charge densities on the left (σ_1) and right (σ_2) surfaces of the slab, the polarization charge density (σ_b) at the surface of the dielectric, and the electric field in the whole space.

b) Calculate the electrostatic force acting on the slab.

c) How do these results change if the dielectric medium is assumed to be an infinite (in the y and z directions) layer of *finite* thickness w in the x direction?

3.7 A Conducting Sphere between two Dielectrics

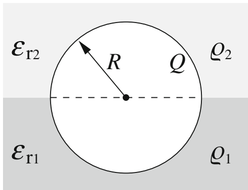

Fig. 3.6

A conducting sphere of mass density ϱ and radius R floats in a liquid of density $\varrho_1 > 2\varrho$ and relative dielectric permittivity ε_{r1} in the presence of the gravitational field. Above the liquid there is a gaseous medium of mass density $\varrho_2 \ll \varrho$ and relative dielectric permittivity $\varepsilon_{r2} < \varepsilon_{r1}$. The sphere is given a charge Q such that exactly one half of its volume is submerged. Evaluate

a) the electric field in the whole space, the surface free charge densities on the sphere, and the surface polarization charge densities of the two dielectrics, as functions of R, ε_{r1}, ε_{r2} and Q;

b) the value of Q.

3.8 Measuring the Dielectric Constant of a Liquid

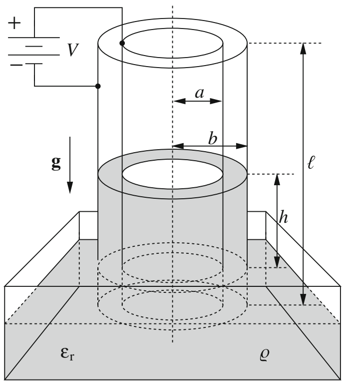

Fig. 3.7

A cylindrical capacitor has internal radius a, external radius $b > a$, and length $\ell \gg b$, so that the boundary effects are negligible. The axis of the capacitor is vertical, and the bottom of the capacitor is immersed in a vessel containing a liquid of mass density ϱ and dielectric permittivity ε_r, in the presence of the gravitational field. If a voltage source maintains a potential difference V between the two cylindrical plates, the liquid rises for a height h in the cylindrical shell between the plates. Show how one can evaluate the value of ε_r from the measurement of h.
(This is a problem from Ref. [1]).

3.9 A Conducting Cylinder in a Dielectric Liquid

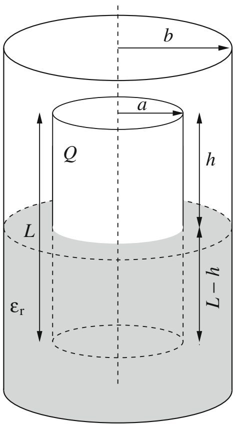

Fig. 3.8

A conducting cylinder of mass M, radius a and height $L \gg a$ is immersed for a depth $L-h$ (with $h \gg a$) in a dielectric liquid having relative permittivity ε_r. The liquid is contained in a cylindrical vessel of radius $b > a$, with conducting lateral surface. A free charge Q is located on the internal cylinder. Boundary effects are assumed to be negligible. The cylinder is free to move vertically preserving its axis. Find

a) the electric field $\mathbf{E}(a)$ at the surface of the internal cylinder, and the surface charge densities;

b) the electric field in the region between the lateral surface of the internal cylinder and the container of the liquid ($a < r < b$);

c) the electrostatic force on the internal cylinder.

d) Assume that the internal cylinder has mass M, and the liquid has mass density $\varrho > M/(\pi a^2 L)$. Discuss the equilibrium conditions.

3.10 A Dielectric Slab in Contact with a Charged Conductor

A dielectric slab of relative permeability ε_r, thickness h and surface $S \gg h$ is in contact with a plane conducting surface, carrying a uniform surface charge density σ, as in Fig. 3.9. Boundary effects are negligible.

a) Evaluate the electric field in the whole space.

b) Evaluate the polarization surface-charge densities on the dielectric surfaces.

c) How do the answers to points **a)** and **b)** change if the slab is moved at a distance $s < h$ from the conducting plane? How does the electrostatic energy of the system depend on s? Is there an interaction force between slab and conductor?

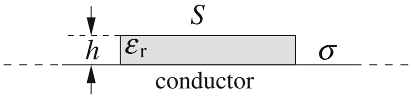

Fig. 3.9

3.11 A Transversally Polarized Cylinder

An infinite cylinder of radius a has an internal uniform electric polarization $\mathbf{P}$, perpendicular to its axis, as shown in Fig. 3.10. Evaluate the electric charge density on the lateral surface of the cylinder, the electric potential and the electric field in the whole space.

Hint: see Problem 1.1.

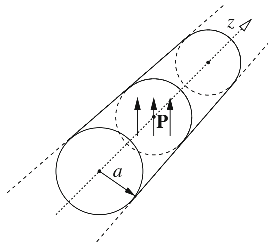

Fig. 3.10

Reference

1. J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York,1975, Problem 4.13