Chapter 4
Electric Currents

Topics. Electric current density. Continuity equation. Stationary electric currents. Drude model for a conductor. Ohm's law. Joule heating.

Basic equations The electric current density $\mathbf{J} = \mathbf{J}(\mathbf{r}, t)$ is the local flow of charge per unit area and surface, which appears in the continuity equation

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 , \tag{4.1}$$

that states the conservation of the total electric charge. In integral form

$$\frac{\mathrm{d}Q}{\mathrm{d}t} \equiv \int_V \partial_t \rho \, \mathrm{d}^3 r = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \equiv I . \tag{4.2}$$

where Q is the total charge contained into the volume V bounded by the closed surface S. Usually the flux (or electric current) I is defined also for an open surface, as the total charge crossing the surface per unit time.

The quantity

$$w = \mathbf{J} \cdot \mathbf{E} \tag{4.3}$$

is interpreted as the work per unit time and volume done by the EM fields on a distribution of currents.

In a model of matter where there are several species of charged particles (labeled with the index s) each having a charge q_s, a density of particles $n_s = n_s(\mathbf{r}, t)$ and flowing with velocity $\boldsymbol{v}_s = \boldsymbol{v}_s(\mathbf{r}, t)$, the current density is given by

$$\mathbf{J} = \sum_s q_s n_s \boldsymbol{v}_s . \tag{4.4}$$

In a metal where electrons are the only charge carrier, $\mathbf{J} = -e n_e \boldsymbol{v}_e$.

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_4

Drude's model for electrons in a metal assumes the classical equation of motion

$$m_e \frac{\mathrm{d}\boldsymbol{v}_e}{\mathrm{d}t} = -e\mathbf{E} - m_e \nu \boldsymbol{v}_e \, , \tag{4.5}$$

where ν is a phenomenological friction coefficient. In a steady state ($\mathrm{d}\boldsymbol{v}_e/\mathrm{d}t = 0$) this leads to Ohm's law for a conductor

$$\mathbf{J} = \frac{n_e e^2}{m_e \nu}\mathbf{E} \equiv \sigma \mathbf{E} \equiv \frac{\mathbf{E}}{\rho} \, , \tag{4.6}$$

where σ is the conductivity and $\rho = 1/\sigma$ the resistivity of the material.[1]

In a material satisfying (4.6), the latter implies that the current I flowing between two points (or layers) at different values of the electric potential, the potential drop V is proportional to I, leading to the definition of the resistance R:

$$V = RI \, . \tag{4.7}$$

In the common (but particular) example of a straight conductor of length ℓ and cross-section area A, such that the electric field is uniform inside the conductor, one obtains $R = \ell/(\sigma A) = \rho\ell/A$. The equations (Kirchoff's laws) describing DC electric circuits, i.e. networks of interconnected conductors each satisfying (4.7), can be found in any textbook and will not be repeated here.

Equation (4.7) is known as Ohm's law, but it is appropriate to use this name also for the underlying and more general Equation (4.6) due to G. Kirchoff. An Ohmic conductor is defined as any material which satisfies (4.6). For such materials, Equation (4.3) gives the power per unit volume dissipated into the material as a consequence of the friction term,

$$\mathbf{J} \cdot \mathbf{E} = \sigma E^2 = \frac{E^2}{\rho} \, , \tag{4.8}$$

which causes the heating of the material (Joule effect). For the above mentioned example, this is equivalent to state that the power dissipated into the whole conductor is $W = RI^2$.

Notice that all the above equations have the same form both in the SI and in the Gaussian system. However, the units of measure are different. For example, the current I is measure in C/s or Ampère (1 A = 1 C/S) in SI, and in statCoulomb/s or "statAmpère" in Gaussian units, while the resistance is measured in Ohms (Ω) in SI and in s/cm in Gaussian units. For the latter, σ has the dimensions of the inverse of a time, and is thus measured in s^{-1}, while ρ can be measured in s.

[1] Unfortunately the lower-case Greek letters commonly used as symbols for resistivity and conductivity are the same used for volume and surface charge densities, respectively. However, the meaning of the symbols used in the formulas throughout the book should be clear from the context.

4.1 The Tolman-Stewart Experiment

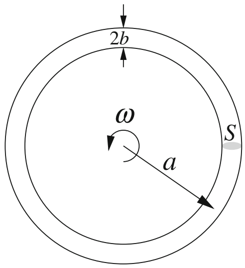

Fig. 4.1

The experiment of Tolman and Stewart [1] was conceived in order to show that conduction in metals is due to electrons. A metallic torus (ring) of major radius a and minor radius b is spun at a very high angular velocity ω around its axis. We assume that $b \ll a$, so that the radial motion of the charge carriers can be neglected. The cross section of the ring is $S = \pi b^2$.

At time $t = 0$ the rotation of the ring is suddenly stopped. A current $I = I(t)$ flowing in the ring and decaying in time is observed for $t > 0$.

a) Using the Drude model for conduction in metals, find $I = I(t)$ and its characteristic decay time τ for a ring of copper (electrical conductivity $\sigma \simeq 10^7\ \Omega^{-1}\text{m}^{-1}$ and electron density $n_\text{e} = 8.5 \times 10^{28}\ \text{m}^{-3}$).

b) Evaluate the charge that flows in the ring from $t = 0$ to $t = \infty$ as a function of σ.

4.2 Charge Relaxation in a Conducting Sphere

A conducting sphere of radius a and conductivity σ has a net charge Q. At time $t = 0$ the charge is uniformly distributed over the volume of the sphere, with a volume charge density $\rho_0 = Q(3/4\pi a^3)$. Since in static conditions the charge in an isolated conductor can only be located on the conductor's surface, for $t > 0$ the charge progressively migrates to the surface of the sphere.

a) Evaluate the time evolution of the charge distribution on the sphere, and of the electric field everywhere in space. Give a numerical value for the time constant τ in the case of a good conductor (e.g., copper).

b) Evaluate the time evolution of the electrostatic energy of the sphere during the charge redistribution.

c) Show that the energy dissipated into Joule heat equals the loss of electrostatic energy.

4.3 A Coaxial Resistor

Two coaxial cylindrical plates of very low resistivity ρ_0 have radii a and b, respectively, with $a < b$. The space between the cylindrical plates is filled up to a height h with a medium of resistivity $\rho \gg \rho_0$, as in Fig. 4.2. A voltage source maintains a constant potential difference V between the plates.

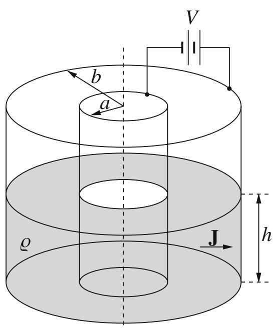

Fig. 4.2

a) Evaluate the resistance R of the system.
b) Discuss the relation between R and the capacitance of a cylindrical capacitor of radii a and b.

4.4 Electrical Resistance between two Submerged Spheres (1)

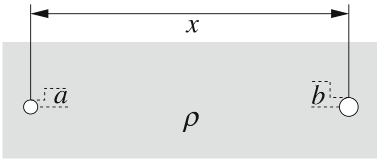

Fig. 4.3

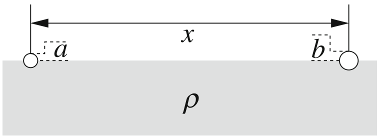

Fig. 4.4

a) Two highly conducting spheres of radii a and b, respectively, are deeply submerged in the water of a lake, at a distance x from each other, with $x \gg a$ and $x \gg b$. The water of the lake has resistivity ρ. Evaluate the approximate resistance between the two spheres, using the results of the answer to point **b**) of problem 4.3.
b) Now suppose that the two spheres are not completely submerged, but just sunk so that their centers are exactly at the level of the surface of the lake, as shown in the figure. Evaluate the resistance between them.

4.5 Electrical Resistance between two Submerged Spheres (2)

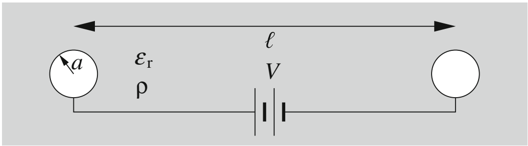

Fig. 4.5

Two identical, perfectly conducting spheres of radius a are immersed in a fluid of resistivity ρ and relative electric permittivity ε_r. The distance between the centers of the two spheres is $\ell \gg a$. A constant potential difference difference V is maintained between the spheres by a suitable voltage source.

As a first approximation, assume the charge to be uniformly distributed over the surface of each sphere, neglecting electrostatic induction effects. Evaluate
a) the charge on each sphere,
b) the resistance R and the current I flowing between the spheres.
c) Find the temporal law and the decay time for the discharge of the spheres when the voltage source is disconnected.
d) Discuss how electrostatic induction modifies the previous answers, to the lowest order in a/ℓ.

4.6 Effects of non-uniform resistivity

Two geometrically identical cylindrical conductors have both height h and radius a, but different resistivities ρ_1 and ρ_2. The two cylinders are connected in series as in Fig. 4.6, forming a single conducting cylinder of height $2h$ and cross section $S = \pi a^2$. The two opposite bases are connected to a voltage source maintaining a potential difference V through the system, as shown in the figure.

a) Evaluate the electric fields, the electric current and the current densities flowing in the two cylinders in stationary conditions.

b) Evaluate the surface charge densities at the surface separating the two materials, and at the base surfaces connected to the voltage source.

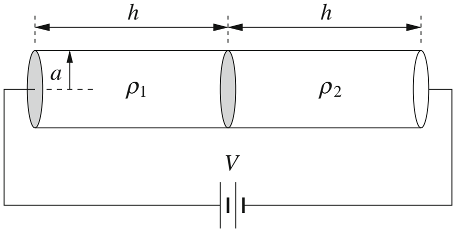

Fig. 4.6

4.7 Charge Decay in a Lossy Spherical Capacitor

A spherical capacitor has internal radius a and external radius b. The spherical shell $a < r < b$ is filled by a lossy dielectric medium of relative dielectric permittivity ε_r and conductivity σ. At time $t = 0$, the charge of the capacitor is Q_0.

a) Evaluate the time constant for the discharge of the capacitor.

b) Evaluate the power dissipated by Joule heating inside the capacitor, and compare it with the temporal variation of the electrostatic energy.

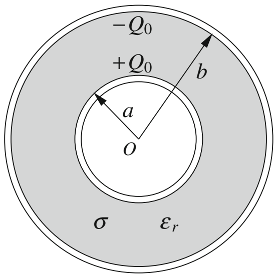

Fig. 4.7

4.8 Dielectric-Barrier Discharge

The plates of a parallel-plate capacitor have surface S and separation d. The space between the plates is divided into two layers, parallel to the plates, of thickness d_1 and d_2, respectively, with $d_1 + d_2 = d$, as in Fig. 4.8. The layer of thickness d_1 is filled with a gas of negligible dielectric susceptibility ($\chi = 0$, $\varepsilon_r \simeq 1$), while the layer of thickness d_2 is filled with a dielectric material of dielectric permittivity $\varepsilon_r > 1$. The electric potential difference between the plates, V, is kept constant by a voltage source. Boundary effects can be neglected.

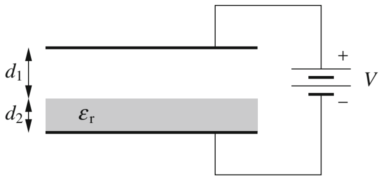

Fig. 4.8

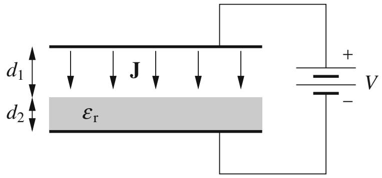

Fig. 4.9

a) Find the electric field inside the capacitor. An ionization discharge is started in the gaseous layer at $t = 0$, and the gas instantaneously becomes conducting. We assume that, for $t > 0$, the ionized gas can be considered as an Ohmic conductor of constant and uniform resistivity ρ.

b) After a sufficiently long time we observe that the current stops flowing in the gas, and the system reaches a steady state (i.e., all physical quantities are constant). Find the electric field in the capacitor in these conditions, and the surface free charge density between the two layers.

c) Find the time dependence of the electric field during the transient phase ($t > 0$), and the relaxation time needed by the system to reach the steady state condition.

4.9 Charge Distribution in a Long Cylindrical Conductor

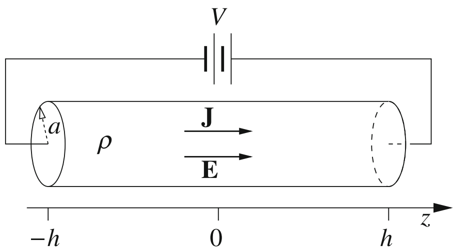

Fig. 4.10

Consider a conducting homogeneous cylindrical wire of radius a and length $2h$, with $a \ll h$, and resistivity ρ. The wire is connected to a voltage source that keeps a constant potential difference V across its ends. We know that the electric field $\mathbf{E}$ and, consequently, the current density $\mathbf{J} = \mathbf{E}/\rho$ must be uniform inside the wire, see Problem 4.6. This implies the presence of charge distributions generating the uniform field. Only surface charge distributions are allowed in a conductor in steady conditions. The charge distributions on the bases of the cylinder are not sufficient for generating an even approximately uniform field in our case of $a \ll 2h$. Thus, a charge density σ_L must be present also on the lateral surface. Verify that a surface charge density $\sigma_\mathrm{L} = \gamma z$, where γ is a constant and z is the coordinate along the cylinder axis, leads to a good approximation for the field inside the conductor far from the ends [2].

4.10 An Infinite Resistor Ladder

An infinite resistor ladder consists of an infinite number of resistors, all of resistance R, arranged as in Fig. 4.11. Evaluate the resistance measured between the terminals A and B. Hint: use an invariance property of the ladder.

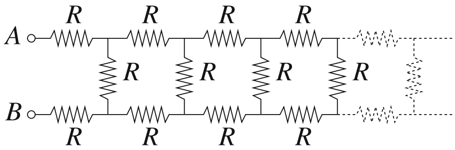

Fig. 4.11

References

1. R.C. Tolman, T.D. Stewart, The electromotive force produced by the acceleration of metals. Phys. Rev. **8**, 97–116 (1916)
2. C.A. Coombes, H. Laue, Electric fields and charge distributions associated with steady currents. Am. J. Phys. **49**, 450–451 (1981)