Chapter 5
Magnetostatics

Topics. Stationary magnetic field in vacuum. Lorentz force. Motion of an electric point charge in a magnetic field. The magnetic force on a current. The magnetic field of steady currents. "Mechanical" energy of a circuit in a magnetic field. Biot-Savart law. Ampères' circuital law. The magnetism of matter. Volume and surface magnetization current densities (bound currents). Magnetic susceptibility. The "auxiliary" vector **H**. Magnetic field boundary conditions. Equivalent magnetic charge method.

Units. In order to write formulas compatible with both SI and Gaussian units, we introduce two new "system dependent" constants, k_m and b_m, defined as

$$k_\mathrm{m} = \begin{cases} \dfrac{\mu_0}{4\pi}, & \text{SI,} \\[2ex] \dfrac{1}{c}, & \text{Gaussian,} \end{cases} \qquad b_\mathrm{m} = \begin{cases} 1, & \text{SI,} \\[2ex] \dfrac{1}{c}, & \text{Gaussian,} \end{cases} \tag{5.1}$$

where, again, $\mu_0 = 4\pi\times 10^{-7}$ T·m/A is the "magnetic permeability of vacuum", and $c = 29\,979\,245\,800$ cm/s is the light speed in vacuum.

Basic equations The two Maxwell equations for the magnetic field **B** relevant to this chapter are

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \tag{5.2}$$

$$\boldsymbol{\nabla}\times\mathbf{B} = 4\pi k_\mathrm{m}\,\mathbf{J}\,. \tag{5.3}$$

Equation (5.2) is always valid (in the absence of *magnetic monopoles*), while (5.3) is valid in the absence of time-dependent electric fields. It is thus possible to introduce a *vector potential* **A**, such that

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r})\,, \tag{5.4}$$

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Imposing the gauge condition $\nabla \cdot \mathbf{A} = 0$, the vector potential satisfies

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -4\pi k_{\mathrm{m}} \mathbf{J}(\mathbf{r}) , \tag{5.5}$$

which is the vector analogous of Poisson's equation (2.1). Thus,

$$\mathbf{A}(\mathbf{r}) = k_{\mathrm{m}} \int_V \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \, \mathrm{d}^3 r' . \tag{5.6}$$

A particular and typical case is that of closed "line" currents, e.g. flowing in a circuit having wires of negligible thickness. In such case one may replace $\mathbf{J}(\mathbf{r}')\,\mathrm{d}^3 r'$ by $I(\mathbf{r}')\mathrm{d}\boldsymbol{\ell}$ and calculate the field via the Biot-Savart formula

$$\mathbf{B}(\mathbf{r}) = k_{\mathrm{m}} \oint \frac{I(\mathbf{r}-\mathbf{r}')\mathrm{d}\boldsymbol{\ell} \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} , \tag{5.7}$$

where the integral is extended to the closed path of the current.

The force exerted by a magnetic field over a distribution of currents is

$$\mathbf{f} = b_{\mathrm{m}} \int_v \mathbf{J}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}') \, \mathrm{d}^3 r' . \tag{5.8}$$

A single point charge q located at $\mathbf{r}$ and moving with velocity $\mathbf{v}$ is equivalent to a current density $\mathbf{j}(\mathbf{r}') = q\delta(\mathbf{r}-\mathbf{r}')\mathbf{v}$, so that the magnetic force on the point charge is

$$\mathbf{f} = b_{\mathrm{m}} q \mathbf{v} \times \mathbf{B} . \tag{5.9}$$

The energy associated to a magnetic field distribution is given by the expression

$$U_{\mathrm{m}} = \int_V \frac{b_{\mathrm{m}} B^2}{8\pi k_{\mathrm{m}} \mu_{\mathrm{r}}} \, \mathrm{d}^3 r . \tag{5.10}$$

In the absence of magnetic monopoles, the first non-vanishing term of the multipole expansion is the magnetic dipole $\mathbf{m}$

$$\mathbf{m} = \frac{1}{2} \int_V \mathbf{r}' \times \mathbf{J}(\mathbf{r}') \mathrm{d}^3 \mathbf{r}' . \tag{5.11}$$

In the simple case of a small plane coil of area A and electric current I this reduces to the line integral over the coil path C

$$\mathbf{m} = \frac{I}{2} \oint_C \mathbf{r}' \times \mathrm{d}\boldsymbol{\ell} = AI\hat{\mathbf{n}} , \tag{5.12}$$

where $\hat{\mathbf{n}}$ is perpendicular to the coil surface. A magnetic dipole term located at $\mathbf{r} = 0$ generates a magnetic field

$$\mathbf{B}(\mathbf{r}) = k_\mathrm{m} \frac{3\hat{\mathbf{r}}(\mathbf{m} \cdot \hat{\mathbf{r}}) - \mathbf{m}}{r^3} \,. \tag{5.13}$$

In an external magnetic field $\mathbf{B}_\mathrm{ext}$, the magnetic force on a magnetic dipole is

$$\mathbf{f} = (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B}_\mathrm{ext} \,. \tag{5.14}$$

The magnetization density $\mathbf{M}$ of a material is defined as the dipole moment per unit volue,

$$\mathbf{M} = \frac{\mathrm{d}\mathbf{m}}{\mathrm{d}V} \,. \tag{5.15}$$

Ampère equivalence theorem states that a magnetization density $\mathbf{M} = \mathbf{M}(\mathbf{r})$ is always equivalent to a distribution of volume current density $\mathbf{J}_\mathrm{m}$ and surface current density $\mathbf{K}_\mathrm{m}$ *bound* to the material, and given by

$$\mathbf{J}_\mathrm{m} = \frac{1}{b_\mathrm{m}} \boldsymbol{\nabla} \times \mathbf{M} \,, \tag{5.16}$$

$$\mathbf{K}_\mathrm{m} = \frac{1}{b_\mathrm{m}} \mathbf{M} \cdot \hat{\mathbf{n}} \,, \tag{5.17}$$

where $\hat{\mathbf{n}}$ is the unit normal vector pointing outwards from the boundary surface of the material. The total volume and surface current densities are thus

$$\mathbf{J} = \mathbf{J}_\mathrm{f} + \mathbf{J}_\mathrm{m} \,, \qquad \mathbf{K} = \mathbf{K}_\mathrm{f} + \mathbf{K}_\mathrm{m} \,, \tag{5.18}$$

the subscript f denoting the *free* (e.g., conduction) current densities.

The auxiliary field $\mathbf{H}$ is defined as

$$\mathbf{H} = \begin{cases} \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M} \,, & \text{SI,} \\ \mathbf{B} - 4\pi\mathbf{M} \,, & \text{Gaussian,} \end{cases} \tag{5.19}$$

so that Equation (5.3) becomes

$$\boldsymbol{\nabla} \times \mathbf{H} = 4\pi k_\mathrm{m} \mathbf{J}_\mathrm{f} \,, \tag{5.20}$$

A material may have either a permanent magnetization, or a magnetization induced by a magnetic field. In linear, isotropic diamagnetic and paramagnetic materials $\mathbf{M}$ is parallel and proportional to $\mathbf{H}$,

$$\mathbf{M} = \chi_\mathrm{m} \mathbf{H} \,, \tag{5.21}$$

where χ_m is the *magnetic susceptibility* of the material, with $\chi_m < 0$ for diamagnetic materials and $\chi_m > 0$ for paramagnetic materials.[1]. The (*relative*) *magnetic permeability* μ_r is defined as

$$\mu_r = \begin{cases} 1 + \chi_m \,, & \text{SI,} \\ 1 + 4\pi\chi_m \,, & \text{Gaussian.} \end{cases} \tag{5.22}$$

We have $\mu_r < 1$ for diamagnetic materials and $\mu_r > 1$ for paramagnetic materials. Inserting (5.21) and (5.22) into (5.19) we obtain

$$\mathbf{B} = \begin{cases} \mu_0 \mu_r \mathbf{H} \,, & \text{SI,} \\ \mu_r \mathbf{H} \,, & \text{Gaussian,} \end{cases} \tag{5.23}$$

valid for isotropic, non-ferromagnetic, materials.

To facilitate the use of the basic equations in this chapter also with the system independent units, we summarize some of them in the following table (Table 5.1):

Table 5.1 Basic equations for magnetostatics

Quantity	SI	Gaussian	System independent
$\nabla \times \mathbf{B}$	$\mu_0 \mathbf{J}$	$\frac{4\pi}{c}\mathbf{J}$	$4\pi k_m \mathbf{J}$
Magnetic force on a point charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$	$q\,\mathbf{v} \times \mathbf{B}$	$q\,\frac{\mathbf{v}}{c} \times \mathbf{B}$	$b_m\, q\,\mathbf{v} \times \mathbf{B}$
Magnetic field $\mathrm{d}\mathbf{B}$ generated by a wire element $\mathrm{d}\boldsymbol{\ell}$ carrying a current I at a distance $\mathbf{r}$ (Biot-Savart's law)	$\frac{\mu_0}{4\pi}\frac{I\mathrm{d}\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$	$\frac{1}{c}\frac{I\mathrm{d}\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$	$k_m \frac{I\mathrm{d}\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}$
Magnetic moment $\mathbf{m}$ of a ring circuit carrying an electric current I, and enclosing a surface $\mathbf{S}$	$I\mathbf{S}$	$\frac{1}{c} I\mathbf{S}$	$b_m\, I\mathbf{S}$
Volumetric magnetic energy density u_m	$\frac{B^2}{2\mu_0 \mu_r}$	$\frac{B^2}{8\pi\mu_r}$	$\frac{b_m B^2}{8\pi k_m \mu_r}$

[1]The magnetization is expressed in terms of the auxiliary field **H**, rather than in terms of the magnetic field **B**, for historical reasons. In ferromagnetic materials there is no one-to-one correspondence between **M** and **H** (between **M** and **B**) because of *magnetic hysteresis*

5.1 The Rowland Experiment

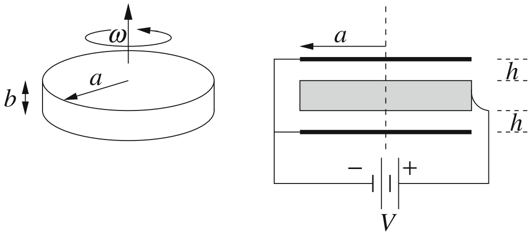

Fig. 5.1

This experiment by Henry A. Rowland (1876) aimed at showing that moving charges generate magnetic fields. A metallic disk or radius a and thickness $b \ll a$ is electrically charged and kept in rotation with a constant angular velocity ω.

a) The disk rotates between two conducting plates, one at a distance $h \simeq 0.5$ cm above its upper surface, and the other at h below its lower surface, as in Fig. 5.1. The two plates are connected to the same terminal of a voltage source maintaining a potential difference $V_0 = 10^4$ V, while the other terminal is connected to the disk by a sliding contact. Evaluate the surface charge density on the disk surfaces.

b) Calculate the magnetic field $\mathbf{B}_c$ near the center of the disk and the magnetic field component B_r parallel and close to the disk surfaces, as a function of the distance r from the axis. Typical experimental values were $a = 10$ cm, and $\omega \simeq 2\pi \times 10^2$ rad/s (period $T = 2\pi/\omega = 10^{-2}$ s).

c) The field component B_r generated by the disk at $r = a$ can be measured by orienting the apparatus so that $\hat{\mathbf{r}}$ is perpendicular to the Earth's magnetic field $\mathbf{B}_\oplus$, of strength $B_\oplus \simeq 5 \times 10^{-5}$ T, and measuring the deviation of a magnetic needle when the disk rotates. Find the deviation angle of the needle.

5.2 Pinch Effect in a Cylindrical Wire

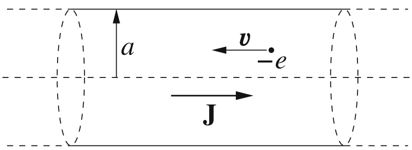

Fig. 5.2

A uniform current density $\mathbf{J}$ flows in an infinite cylindrical conductor of radius a. The current carriers are electrons (charge $-e$) of number volume density n_e and drift velocity $\mathbf{v}$, parallel to the axis of the cylinder. Ions can be considered as fixed in space, with uniform number density n_i and charge Ze. The system is globally neutral.

a) Evaluate the magnetic field generated by the current, and the resulting magnetic force on the electrons.

The magnetic force modifies the volume distribution of the electrons and this, in turn, gives origin to a static electric field. At equilibrium the magnetic force on the electrons is compensated by the electrostatic force.
b) Evaluate the electric field that compensates the magnetic force on the electrons, and the corresponding charge distribution.
c) Evaluate the effect in "standard" conditions for a good Ohmic conductor.

5.3 A Magnetic Dipole in Front of a Magnetic Half-Space

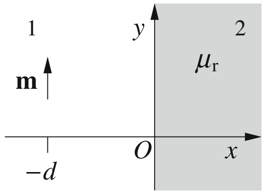

Fig. 5.3

The plane $x = 0$ divides the space into two half-spaces, labeled 1 and 2, respectively. We have vacuum in half-space 1, while half-space 2 is filled by a medium of magnetic permittivity μ_r. A magnetic dipole **m**, parallel to the y axis, is located in vacuum at position $x = -d$. Find
a) the magnetic field **B** in the whole space,
b) the force acting on the magnetic dipole.

5.4 Magnetic Levitation

In a given region of space we have a static magnetic field, which, in a cylindrical reference frame (r, ϕ, z), is symmetric around the z axis, i.e., is independent of ϕ, and can be written $\mathbf{B} = \mathbf{B}(r, z)$. The field component along z is $B_z(z) = B_0 z/L$, where B_0 and L are constant parameters.
a) Find the radial component B_r close to the z axis.
A particle of magnetic polarizability α (such that it acquires an induced magnetic dipole moment $\mathbf{m} = \alpha\mathbf{B}$ in a magnetic field **B**), is located close to the z axis.
b) Find the potential energy of the particle in the magnetic field.
c) Discuss the existence of equilibrium positions for the particle, and find the frequency of oscillations for small displacements from equilibrium either along z or r (let M be the mass of the particle).

5.5 Uniformly Magnetized Cylinder

A magnetically "hard" cylinder of radius R and height h, with $R \ll h$, carries a uniform magnetization **M** parallel to its axis.

a) Show that the volume magnetization current density $\mathbf{J}_{\mathrm{m}}$ is zero inside the cylinder, while the lateral surface of the cylinder carries a surface magnetization current density $\mathbf{K}_{\mathrm{m}}$, with $|\mathbf{K}_{\mathrm{m}}| = |\mathbf{M}|$.
b) Find the magnetic field $\mathbf{B}$ inside and outside the cylinder, at the limit $h \to \infty$.
c) Now consider the opposite case of a "flat" cylinder, i.e., $h \ll R$, and evaluate the magnetic field $\mathbf{B}_0$ at the center of the cylinder.
d) According to the result of **c)**, $\lim_{R/h\to\infty} \mathbf{B}_0 = 0$. Obtain the same result using the equivalent magnetic charge method.

5.6 Charged Particle in Crossed Electric and Magnetic Fields

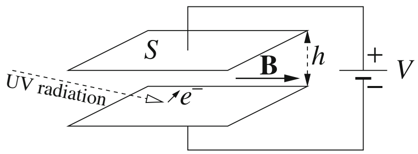

Fig. 5.4

A particle of electric charge q and mass m is initially at rest in the presence of a uniform electric field $\mathbf{E}$ and a uniform magnetic field $\mathbf{B}$, perpendicular to $\mathbf{E}$.
a) Describe the subsequent motion of the particle.
b) Use the above result to discuss the following problem. We have a parallel-plate capacitor with surface S, plate separation h and voltage V, as in Fig. 5.4. A uniform magnetic field $\mathbf{B}$ is applied to the capacitor, perpendicular to the capacitor electric field, i.e., parallel to the plates. Ultraviolet radiation causes the negative plate to emit electrons with zero initial velocity. Evaluate the minimum value of $\mathbf{B}$ for which the electrons cannot reach the positive plate.

5.7 Cylindrical Conductor with an Off-Center Cavity

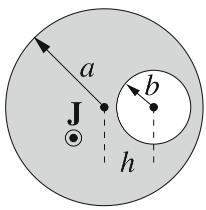

Fig. 5.5

An infinite cylindrical conductor of radius a has a cylindrical cavity of radius b bored parallel to, and centered at a distance $h < a - b$ from the cylinder axis as in Fig. 5.5, which shows a section of the conductor. The current density $\mathbf{J}$ is perpendicular to, and uniform over the section of the conductor (i.e., excluding the cavity!). The figure shows a section of the conductor. Evaluate the magnetic field $\mathbf{B}$, showing that it is *uniform* inside the cavity.

5.8 Conducting Cylinder in a Magnetic Field

A conducting cylinder of radius a and height $h \gg a$ rotates around its axis at constant angular velocity ω in a uniform magnetic field $\mathbf{B}_0$, parallel to the cylinder axis.

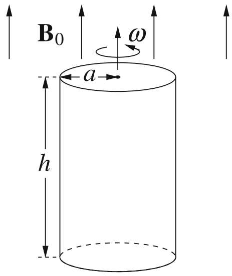

Fig. 5.6

a) Evaluate the magnetic force acting on the conduction electrons, assuming $\omega = 2\pi \times 10^2\,\mathrm{s}^{-1}$ and $B = 5 \times 10^{-5}\,\mathrm{T}$ (the Earth's magnetic field), and the ratio of the magnetic force to the centrifugal force.
Assume that the cylinder is rotating in stationary conditions. Evaluate
b) the electric field inside the cylinder, and the volume and surface charge densities;
c) the magnetic field $\mathbf{B}_1$ generated by the rotation currents inside the cylinder, and the order of magnitude of B_1/B_0 (assume $a \approx 0.1\,\mathrm{m}$).

5.9 Rotating Cylindrical Capacitor

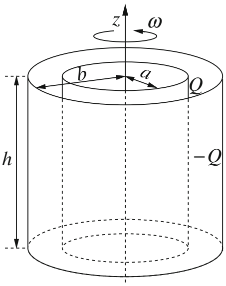

Fig. 5.7

The concentric cylindrical shells of a cylindrical capacitor have radii a and $b > a$, respectively, and height $h \gg b$. The capacitor charge is Q, with $+Q$ on the inner shell of radius a, and $-Q$ on the outer shell of radius b, as in Fig. 5.7. The whole capacitor rotates about its axis with angular velocity $\omega = 2\pi/T$. Boundary effects are negligible.

a) Evaluate the magnetic field $\mathbf{B}$ generated by the rotating capacitor over the whole space.

b) Evaluate the magnetic forces on the charges of the two rotating cylindrical shells, and compare them to the electrostatic forces.

5.10 Magnetized Spheres

a) A sphere of radius R has a uniform and permanent magnetization $\mathbf{M}$. Calculate the magnetic field inside and outside the sphere. (Hint: see Problem 3.3.)
b) A sphere of radius R has a total charge Q uniformly distributed on its surface. The sphere rotates with angular velocity ω. Calculate the magnetic field inside and outside the sphere.

c) A sphere of radius R has a magnetic permeability μ_{r} and is located in an external, uniform magnetic field $\mathbf{B}_0$. Calculate the total magnetic field inside and outside the sphere, discussing the limit of a perfectly diamagnetic material ($\mu_r = 0$), as a superconductor.