Chapter 6
Magnetic Induction and Time-Varying Fields

Topics. Magnetic induction. Faraday's law. Electromotive force. The slowly varying current approximation. Mutual inductance and self-inductance. Energy stored in an inductor. Magnetically coupled circuits. Magnetic energy. Displacement current and the complete Maxwell's equations.

Basic equations In the presence of a time-varying magnetic field, Equation (1.5) is modified into the exact equation

$$\boldsymbol{\nabla}\times\mathbf{E} = -b_{\mathrm{m}}\partial_t\mathbf{B}\,, \tag{6.1}$$

so that the line integral of $\boldsymbol{\nabla}\times\mathbf{E}$ around a closed path C is

$$\oint_C \mathbf{E}\cdot\mathrm{d}\ell = -b_{\mathrm{m}}\int_S \partial_t\mathbf{B}\cdot\mathrm{d}\mathbf{S} \tag{6.2}$$

Thus, for a *fixed* path, the line integral of **E** equals the time derivative of the flux of the time-varying field **B** through a surface delimited by the contour C.

The *electromotive force* (emf) $\mathcal{E}$ in a *real* circuit having *moving* parts is the work done by the Lorentz force on a unit charge over the circuit path,

$$\mathcal{E} = \oint_{\mathrm{circ}} (\mathbf{E} + b_{\mathrm{m}}\mathbf{V}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{\ell} \equiv -b_{\mathrm{m}}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\mathrm{circ}}(\mathbf{B})\,, \tag{6.3}$$

where **V** is the velocity of the circuit element; now in (6.3) the flux $\Phi_{\mathrm{circ}}(\mathbf{B})$ of **B** through the circuit may vary because of *both* the temporal variation of **B** and of the circuit geometry. Equation (6.3) is the general Faraday's law of induction.

For a system of two electric circuits, the magnetic flux through each circuit can be written as a function of the currents flowing in each circuit,

$$\Phi_2 = L_1 I_1 + M_{21} I_2\,, \qquad \Phi_1 = L_2 I_2 + M_{12} I_1\,, \tag{6.4}$$

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_6

where the terms containing the (self-)inductance coefficients L_i are the contribution to flux generated by the circuit itself, and the terms containing the mutual inductance coefficients $M_{21} = M_{12}$ give the flux generated by one circuit over the other.

Finally, for time-varying fields the complete Maxwell's equation replacing (5.3) is

$$\nabla \times \mathbf{B} = 4\pi k_\mathrm{m}\,\mathbf{J} + \frac{k_\mathrm{m}}{k_\mathrm{e}}\,\partial_t \mathbf{E} = \begin{cases} \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\partial_t \mathbf{E} & \text{(Gaussian)}, \\ \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial_t \mathbf{E} & \text{(SI.)} \end{cases} \tag{6.5}$$

6.1 A Square Wave Generator

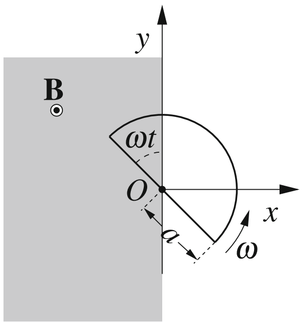

Fig. 6.1

We have a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ in the half space $x < 0$ of a Cartesian coordinate system, while the field is zero for $x > 0$. A semicircular loop of radius a and resistance R lyes in the xy plane, with the center of the full circumference at the origin O of our coordinate system, as in Fig. 6.1. The loop rotates around the z axis at constant angular velocity ω.

First, assume that the self-inductance of the coil is negligible and evaluate

a) the current circulating in the coil;

b) the torque exerted by the magnetic forces on the coil, and the mechanical power needed to keep the coil in rotation. Compare this to the electric power dissipated in the coil.

c) Now consider the presence of the self-inductance of the coil, and discuss how it affects the answer to point **a)**.

6.2 A Coil Moving in an Inhomogeneous Magnetic Field

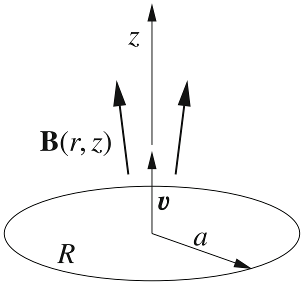

Fig. 6.2

A magnetic field has rotational symmetry around a straight line, that we choose as the longitudinal axis, z, of a cylindrical reference frame (r, ϕ, z). The z component of the field on the z axis, $B_z(0,z)$, is known and equals $B_z(0,z) = B_0\, z/L$, where L is a constant. A circular coil has radius a, resistance R, and axis coinciding with the z axis of our reference frame. The coil performs a translational motion at constant velocity $\boldsymbol{v} = v\,\hat{\mathbf{z}}$, and its radius a is assumed to be small enough that the magnetic field is always approximately uniform over the surface limited by the coil.

a) Find the current **I** flowing in the coil.
b) Find the power P dissipated by the coil due to Joule heating, and the corresponding frictional force **f** on the coil.
c) Calculate **f** as the resultant magnetic force on the loop carrying the current I.

6.3 A Circuit with "Free-Falling" Parts

In the presence of the Earth's gravitational field **g**, two high-conducting bars are located vertically, at a distance a from each other. A uniform, horizontal magnetic field **B** is perpendicular to the plane defined by the vertical bars. Two horizontal bars, both of mass m, resistance $R/2$ and length a, are constrained to move, without friction, with their ends steadily in contact with the two vertical bars. The resistance of the two fixed vertical bars is assumed to be much smaller than $R/2$, so that the net resistance of the resulting rectangular circuit is, with very good approximation, always R, independently of the positions of the two horizontal bars.

First, assume that the upper horizontal bar is fixed, while the lower bar starts a "free" fall at $t = 0$. Let's denote by $v = v(t)$ the velocity of the falling bar at time t, with $v(0) = 0$.

a) Write the equation of motion for the falling bar, find the solution for $v(t)$ and show that, asymptotically, the bar approaches a terminal velocity v_t.

b) Evaluate the power dissipated in the circuit by Joule heating when $v(t) = v_t$, and the mechanical work done per unit time by gravity in these conditions.

Now consider the case in which, at $t = 0$, the upper bar already has a velocity $v_0 \neq 0$ directed downwards, while the lower bar starts a "free" fall.

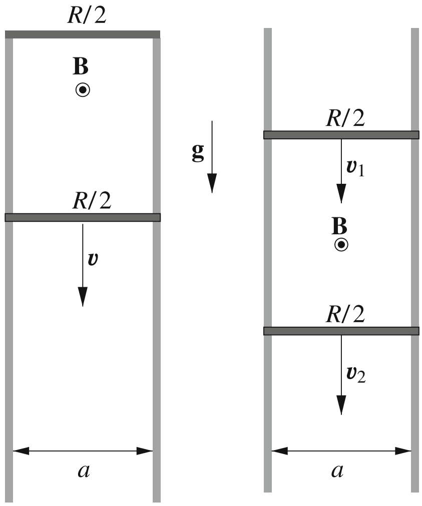

Fig. 6.3

c) Write the equations of motion for both falling bars, and discuss the asymptotic behavior of their velocities $v_1(t)$ and $v_2(t)$, and of the current in the circuit $I(t)$.

6.4 The Tethered Satellite

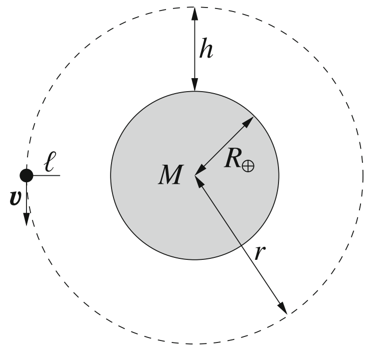

Fig. 6.4

The Earth's magnetic field at the Earth's surface roughly approximates the field of a magnetic dipole placed at the Earth's center. Its magnitude ranges from 2.5×10^{-5} to 6.5×10^{-5} T (0.25 to 0.65 G in Gaussian units), with a value $B_{\mathrm{eq}} \simeq 3.2\times10^{-5}$ T at the equator. A satellite moves on the magnetic equatorial plane with a velocity $v \simeq 8$ km/s at a constant height $h \simeq 100$ km over the Earth's surface, as shown in the figure (not to scale!). A tether (leash, or lead line), consisting in a metal cable of length $\ell = 1$ km, hangs from the satellite, pointing to the Earth's center.

a) Find the electromotive force on the wire.

b) The satellite is traveling through the ionosphere, where charge carriers in outer space are available to close the circuit, thus a current can flow along the wire. Assume that the ionosphere is rigidly rotating at the same angular velocity as the Earth. Find the power dissipated by Joule heating in the wire and the mechanical force on the wire as a function of its resistance R.

6.5 Eddy Currents in a Solenoid

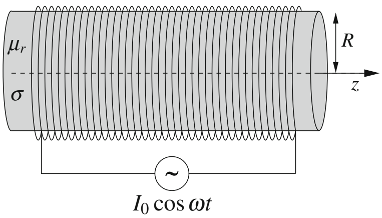

Fig. 6.5

A long solenoid consists of a helical coil of n turns per unit length wound around a soft ferromagnetic cylinder of radius R and length $\ell \gg R$. The ferromagnetic material has a relative magnetic permittivity μ_{r}, and an electrical conductivity σ. An AC current $I = I_0 \cos\omega t$ flows in the coil.

a) Find the *electric* field induced in the solenoid.

b) Explain why the cylinder warms up and evaluate the dissipated power.

c) Evaluate how the induced currents affect the magnetic field in the solenoid. (Boundary effects and the displacement current are assumed to be negligible).

6.6 Feynman's "Paradox"

A non-conducting ring of radius R is at rest on the xy plane, with its center at the origin of the coordinate system. The ring has mass m, negligible thickness, and an electric charge Q distributed uniformly on it, so that the ring has a linear charge density $\lambda = Q/(2\pi a)$. The ring is free to rotate around its axis without friction.

A superconducting circular ring of radius $a \ll R$, coaxial to the charged ring and carrying an electric current I_0, also lies on the xy plane, as in Fig. 6.6. At time $t = 0$ the superconducting loop is heated above its critical temperature, and switches to normal conductivity. Consequently, its current decays to zero according to a law $I = I(t)$.

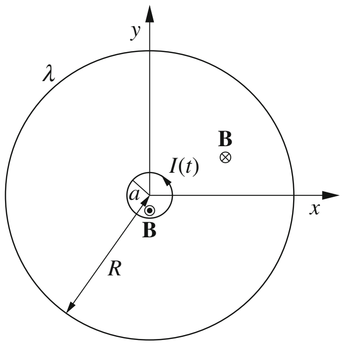

Fig. 6.6

a) Neglecting self-induction effects, evaluate the angular velocity $\omega = \omega(t)$ of the charged ring as a function of the current $I(t)$ in the smaller ring. Evaluate the final angular velocity ω_f, and the final angular momentum L_f, of the charged ring.
b) Evaluate the magnetic field at the ring center, $\mathbf{B}_c$, generated by the rotation of the ring.
c) Discuss how the results of **a)** are modified by taking the "self-inductance" $\mathcal{L}$ of the charged ring into account.

This is one of the possible versions of the so-called *Feynman's disc paradox* [2], presented in Vol. II, Section 17-4, of *The Feynman's Lectures on Physics*. The apparent paradox arises because the initial total *mechanical* angular momentum of the system is zero, no external torque is applied, and one could (wrongly) expect the final total angular momentum to be zero, i.e., no rotation of the ring. This conclusion is wrong, of course, for reasons further discussed in Prob. 8.8.

6.7 Induced Electric Currents in the Ocean

A fluid flows with uniform velocity $\boldsymbol{v}$ in the presence of a constant and uniform magnetic field $\mathbf{B}$ perpendicular to $\boldsymbol{v}$. The fluid has an electrical conductivity σ and volumetric mass density ϱ.
a) Evaluate the electric current density $\mathbf{J}$ induced in the fluid.
b) Give a numerical estimate of $|\mathbf{J}|$ for the terrestrial oceans, knowing that the Earth's magnetic field has an average value $B \simeq 0.5\,\mathrm{G} = 5\times 10^{-5}$ T, the conductivity of sea water is $\sigma \simeq 4\,\Omega^{-1}\mathrm{m}^{-1}$ ($\sigma \simeq 3.6\times 10^{10}\,\mathrm{s}^{-1}\mathrm{cm}^{-1}$ in Gaussian units), and a typical value of the flow velocity is $v = 1\,\mathrm{m/s}$.
c) Due to the presence of the induced current, the magnetic force tends to slow down the fluid. Estimate the order of magnitude for the time constant of this effect.

6.8 A Magnetized Sphere as Unipolar Motor

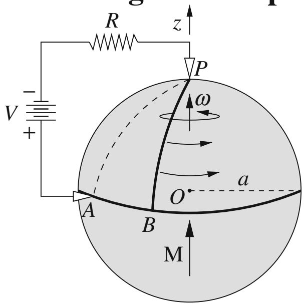

Fig. 6.7

A magnetized, non-conducting sphere has radius a, mass m and permanent, uniform magnetization $\mathbf{M}$ throughout its volume. An electric circuit is formed by pasting a conducting wire along a half meridian, from the pole P to the equator, and another conducting wire around the whole equator of the sphere, as shown in Fig. 6.7. The circuit is closed by two brush contacts (the white arrows in Fig. 6.7) connecting the pole P, and a point A of the wire on the equator of the sphere, to a voltage source of electromotive force V. The resulting circuit has resistance R.

a) Evaluate the torque on the sphere when a current I flows in the circuit.
b) If the sphere is free to rotate without friction around the z axis of a cylindrical coordinate system, parallel to $\mathbf{M}$ and passing through the center O of the sphere, it reaches asymptotically a terminal angular velocity ω_t. Evaluated ω_t and the characteristic time of the system.

6.9 Induction Heating

Consider a homogeneous material of electrical conductivity σ and relative magnetic permeability μ_r, both real, positive and independent of frequency. The electric permittivity is $\varepsilon_r = 1$.
a) Show that, if the displacement current density $\partial_t \mathbf{E}/(4\pi k_e)$ can be neglected, the magnetic field $\mathbf{B}$ inside the material obeys the equation

$$\partial_t \mathbf{B} = \alpha \nabla^2 \mathbf{B} \,, \tag{6.6}$$

and determine the value of the real constant α.

The material fills the half-space $x > 0$ in the presence of a uniform oscillating magnetic field $\mathbf{B}_0 = \hat{\mathbf{y}}\, B_0 \cos(\omega t) = \hat{\mathbf{y}}\, \mathrm{Re}\left(B_0\, \mathrm{e}^{-\mathrm{i}\omega t}\right)$ in the half-space $x < 0$.
b) Evaluate the magnetic field $\mathbf{B}(x,t)$ for $x > 0$, assuming that the displacement current is negligible. Discuss under what conditions the result is a good approximation for the case of a finite slab of the material.
c) Evaluate the power dissipated in the medium by Joule heating.

6.10 A Magnetized Cylinder as DC Generator

A long hard-iron cylinder has height h, radius $a \ll h$, and permanent, uniform magnetization $\mathbf{M}$ throughout its volume. The magnetization is parallel to the cylinder axis, which we choose as the z axis of a cylindrical coordinate system (r, ϕ, z).

a) Show that the magnetic field inside the cylinder, far from the two bases, is $\mathbf{B}_0 \simeq 4\pi(k_m/b_m)\mathbf{M}$, or $\mathbf{B}_0 \simeq \mu_0\mathbf{M}$ in SI units, $\mathbf{B}_0 = 4\pi\mathbf{M}$ in Gaussian units. Show that the magnitude of the z component of the field at the two bases is $B_z \simeq B_0/2$.

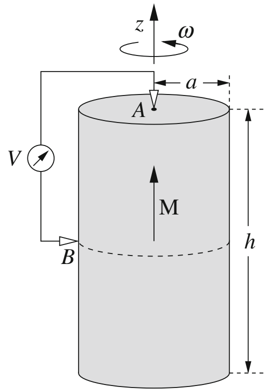

Fig. 6.8

b) Two brush contacts (the white arrows in Fig. 6.8) connect the center of the upper base of the cylinder, A, and a point on the equator of the cylinder, B, to a voltmeter. The cylinder is kept in rotation around the z axis with constant angular velocity $\boldsymbol{\omega}$. Evaluate the electromotive force measured by the voltmeter.

This problem is taken from an example of [1], Section 88, page 379.

6.11 The Faraday Disk and a Self-Sustained Dynamo

A perfectly conducting disk, of radius a and thickness $h \ll a$, rotates at constant angular velocity $\boldsymbol{\omega}$ (parallel to the disk axis), in the presence of a uniform and constant magnetic field $\mathbf{B}$ parallel to $\boldsymbol{\omega}$.

a) Evaluate the electric field $\mathbf{E}$ in the disk in steady state conditions, and the corresponding potential drop between the center and the boundary of the disk (hint: the total force on charge carriers must be zero at equilibrium).

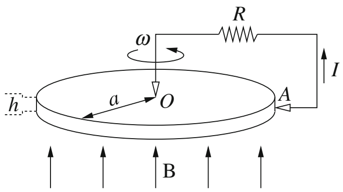

Fig. 6.9

b) We now form a closed circuit by connecting the center of the disk to a point of the circumference by brush contacts (white arrows in the figure), as in Fig. 6.9. Let R be the total resistance of the resulting circuit. Calculate the external torque needed to keep the disk in rotation at constant angular speed.

c) Finally, we place the rotating disk at the center of a long solenoid of radius $b > a$ and n turns per unit length. The disk and the solenoid are coaxial, as shown in Fig. 6.10.

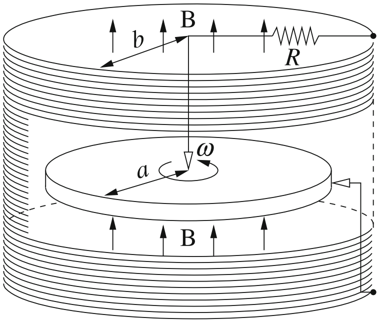

Fig. 6.10

The two brush contacts of point **b)** are now connected to the ends of the solenoid coil, so that the rotating disk provides the current circulating in the turns. The total resistance of the disk-solenoid circuit is R. The circulating current is thus due to the disk rotation and to the presence of the magnetic field **B**, that the current itself generates by circulating in the solenoid (self-sustained dynamo). Find the value of ω for steady-state conditions. This is an elementary model for a dynamo self-sustained by rotation, such as the generation mechanism of the Earth's magnetic field [3].

6.12 Mutual Induction between Circular Loops

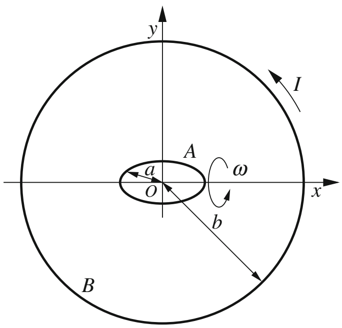

Fig. 6.11

The centers of two circular conducting loops A and B, of radii a and $b \gg a$, respectively, are located at the origin O of a Cartesian reference frame. At time $t = 0$ both loops lie on the xy plane. While the larger loop remains at rest, the smaller loop, of resistance R, rotates about one of its diameters, lying on the x axis, with angular velocity ω, as shown in Fig. 6.11. A constant current I circulates in the larger loop.

a) Evaluate the current I_A induced in loop A, neglecting self-inductance effects.

b) Evaluate the power dissipated in loop A due to Joule heating.

c) Evaluated the torque needed to keep loop A in rotation, and the associated mechanical power.

d) Now consider the case when loop A is at rest on the xy plane, with a constant current I circulating in it, while loop B rotates around the x axis with constant angular velocity ω. Evaluate the electromotive force induced in B, neglecting self-inductance effects.

6.13 Mutual Induction between a Solenoid and a Loop

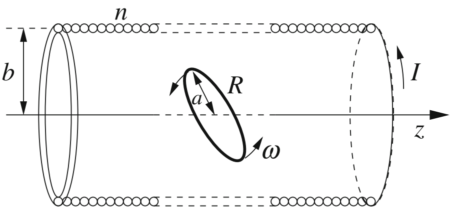

Fig. 6.12

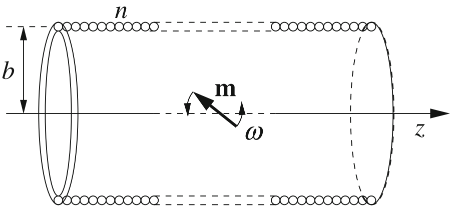

Fig. 6.13

A conducting loop of radius a and resistance R is located with its center at the center of solenoid of radius $b > a$ and n turns per unit length, as in Fig. 6.12. The loop rotates at constant angular velocity ω around a diameter perpendicular to the solenoid axis, while a steady current I flows in the solenoid.

a) Evaluate the flux of the magnetic field through the rotating coil as a function of time.

b) Evaluate the torque exerted by the external forces on the loop in order to keep it rotating at constant angular velocity.

Now assume that the solenoid is disconnected from the current source, and that the rotating loop is replaced by a magnetic dipole $\mathbf{m}$, still rotating at constant angular velocity ω, as in Fig. 6.13.

c) Evaluate the electromotive force induced in the solenoid.

6.14 Skin Effect and Eddy Inductance in an Ohmic Wire

A long, straight cylindrical wire of radius r_0 and conductivity σ (which we assume to be real and constant in the frequency range considered) carries an alternating current of angular frequency ω. The impedance per unit length of the wire, Z_ℓ, can be defined as the ratio of the electric field at the wire surface to the total current through the wire cross section. Evaluate Z_ℓ as a function of ω.

6.15 Magnetic Pressure and Pinch effect for a Surface Current

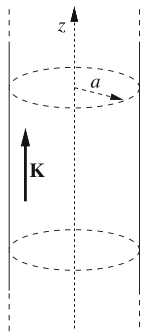

Fig. 6.14

A current I flows on the surface of a cylinder of radius a and infinite length, in the direction parallel to the axis $\hat{\mathbf{z}}$. The current layer has negligible thickness, so that we can write $I = 2\pi a K$, with $\mathbf{K} = K\hat{\mathbf{z}}$ the surface current density. Calculate
a) the magnetic field $\mathbf{B}$ in the whole space,
b) the force per unit surface P on the cylinder surface
c) the variation of magnetic field energy (per unit length) $\mathrm{d}U_\mathrm{m}$ associated to an infinitesimal variation of the radius $\mathrm{d}a$. Explain why $P \neq -(2\pi a)^{-1}\mathrm{d}U_\mathrm{m}/\mathrm{d}a$ and how to calculate P correctly from the energy variation.

Notice: for point **b)** it might be useful to show first that for a magnetostatic field we have

$$\mathbf{J}\times\mathbf{B} = \frac{1}{4\pi k_\mathrm{m}}\left[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} - \tfrac{1}{2}\boldsymbol{\nabla}B^2\right]. \tag{6.7}$$

6.16 Magnetic Pressure on a Solenoid

A current source supplies a constant current I to a solenoid of radius a, length $h \gg a$, so that boundary effects are negligible, and n coils per unit length.
a) Evaluate the magnetic pressure on the solenoid surface directly, by computing the magnetic force on the coils.
b) Now evaluate the magnetic pressure on the solenoid surface by evaluating the variation of the magnetic energy of the system for an infinitesimal increase $\mathrm{d}a$ of the radius of the solenoid, and the corresponding work done by the current source in order to keep I constant.

6.17 A Homopolar Motor

A *homopolar motor* is a direct current electric motor consisting of a circuit carrying a direct current I in the presence of a static magnetic field. The circuit is free to rotate around a fixed axis, so that the angle between the current and the magnetic field remains constant in time in each part of the circuit. The resulting electromotive force is continuous, and the homopolar motor needs no device, like a commutator, to switch the current flow. But it still requires slip rings (or brush contacts) to operate. "Homopolar" means that the electrical polarity of the conductor (the direction of the current flow at each point of the circuit) and the magnetic field do not change in time, and the motor does not require commutation. A simple practical realization of a homopolar motor is shown in Figs. 6.15 and 6.16, based on a Wikipedia entry.

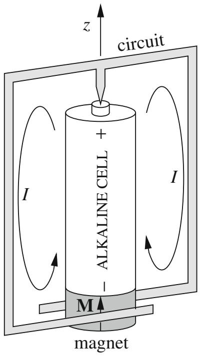

Fig. 6.15

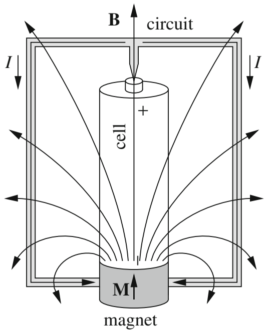

Fig. 6.16

The idea is the following: an electrochemical cell drives a DC current into the double circuit shown in the figures, while a magnetic field is generated generated by the permanent magnet cylindrical located at the bottom of the cell, in electrical contact with its negative pole, as shown in Fig. 6.16. The magnetic field has rotational symmetry around the z axis and is constant in time, in spite of the magnet rotation, and the circuit is free to rotate around the z axis. The magnetic forces on the current-carrying circuit exert a torque, and the circuit starts to rotate.

The dimensions, mass and resistance of the circuit (the mass includes battery and magnet), the voltage of the battery and the magnetic field strength generated by the magnet at each point of the circuit are known. Find the torque acting on the circuit, and the angular velocity of the system as a function of time,

References

1. R. Becker, *Electromagnetic Fields and Interactions*, vol. I (Electromagnetic Theory and Relativity, Blackie, London and Glasgow, 1964)
2. R. P. Feynman, R. B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, Reading, MA, 2006. Volume II, Section 17-4
3. R. T. Merrill, *Our magnetic Earth: the Science of Geomagnetism*, Chapter 3, Chicago University Press, 2010