Chapter 7
Electromagnetic Oscillators and Wave Propagation

Topics Harmonic oscillators. Resonances. Coupled oscillators, normal modes and eigenfrequencies. Basics of the Fourier transform. Electric circuits: impedances, simple LC and RLC circuits. Waves. The wave equation. Monochromatic waves. Dispersion. Wavepackets. Phase velocity and group velocity. Transmission lines.

Useful formulas for this chapter:

Fourier transform of the Gaussian function

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-(\alpha k)^2} \mathrm{e}^{\mathrm{i}kx} dk = \frac{\sqrt{\pi}}{\alpha} \mathrm{e}^{-x^2/4\alpha^2} , \tag{7.1}$$

where in general α is a complex number with $\mathrm{Re}(\alpha) > 0$.

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_7

7.1 Coupled *RLC* Oscillators (1)

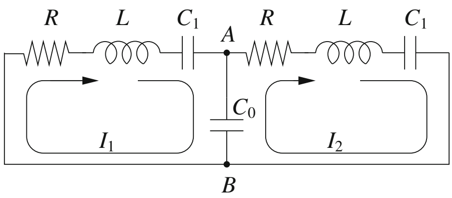

Fig. 7.1

Consider an electrical circuit consisting of two identical resistors R, two identical inductors L, two identical capacitors C_1, and a capacitor C_0, all arranged in two meshes as in Fig. 7.1. Let I_1 and I_2 be the current intensities flowing in the left and right mesh of the circuit, respectively, as shown in the figure. Initially, assume that I_1 and I_2 are flowing in the absence of voltage sources, and assume $R = 0$.

a) Find the equations for the time evolution of I_1 and I_2. Describe the normal modes of the system, i.e., look for steady-state solutions of the form

$$I_1(t) = A_1 \mathrm{e}^{-\mathrm{i}\omega t}\,, \qquad I_2(t) = A_2 \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{7.2}$$

determining the possible values for ω. Find a mechanical equivalent of the circuit.

b) Now consider the effect of the nonzero resistances R in series with each of the two inductances L. Find the solutions for I_1 and I_2 in this case.

c), Evaluate I_1 and I_2 as functions of ω if a voltage source $V = V_0\,\mathrm{e}^{-\mathrm{i}\omega t}$ is inserted into the left mesh of the circuit.

7.2 Coupled *RLC* Oscillators (2)

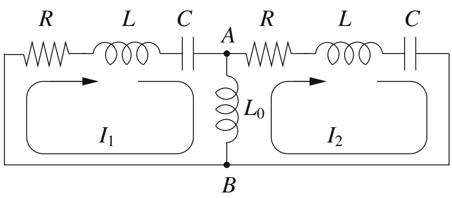

Fig. 7.2

An electrical circuit consists of two identical resistors R, two identical inductors L, two identical capacitors C, and an inductor L_0, all arranged in two meshes as in Fig. 7.2. Let I_1 and I_2 be the currents flowing in the left and right mesh of the circuit, respectively, as shown in the figure.

a) Initially, assume that the currents are flowing in the absence of sources, and assume $R = 0$. Find the equations for the time evolution of $I_1 = I_1(t)$ and $I_2 = I_2(t)$. Determine the normal modes of the circuit.

b) Now assume $R \neq 0$. Show that now the modes of the system are damped, and determine the damping rates.

7.3 Coupled *RLC* Oscillators (3)

An electrical circuit consists of three identical resistors R, three identical inductors L, and two identical capacitors C, arranged in three meshes as in Fig. 7.3. Let I_1, I_2, and I_3 be the currents flowing in the three meshes, as in the figure. Initially, assume $R = 0$.

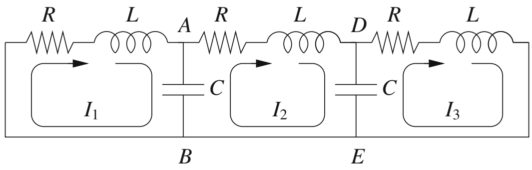

Fig. 7.3

a) Write the equations for the time evolution of $I_n(t)$. Find a mechanical system with three degrees of freedom and the same equations of motion as those for $I_n(t)$.
b) Determine the normal oscillation modes of the system and their frequencies.
c) Now assume $R \neq 0$, and determine the decay rate of the normal modes.

7.4 The *LC* Ladder Network

An LC ladder network is formed by N inductors L, and N capacitors C, arranged as shown in Fig. 7.4. We denote by $I_n = I_n(t)$ the current in the nth inductor. Resistance effects are assumed to be negligible. The distance between two neighboring nodes is a.

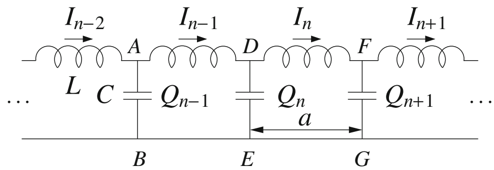

Fig. 7.4

a) Find the equations for the time evolution of I_n. Which is a mechanical equivalent of the system?
b) Show that solutions exist in the form of propagating monochromatic waves

$$I_n = C\,\mathrm{e}^{\mathrm{i}(kna-\omega t)} \tag{7.3}$$

and find the dispersion relation between k and ω.
c) For a given value of ω, find the allowed values of k with the boundary conditions $I_0 = I_N = 0$.
d) Discuss the limit to a continuum system, $N \to \infty$, $n \to \infty$, $a \to 0$, with $na \to x$. In this case inductance and capacity are continuously distributed, i.e., defined per unit length.

7.5 The *CL* Ladder Network

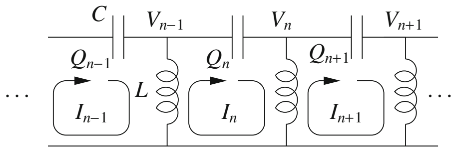

Fig. 7.5

Consider an infinite ladder network of identical capacitors C and inductors L, arranged as shown in Fig. 7.5. Let $Q_n = Q_n(t)$ be the charge on the nth capacitor, $V_n = V_n(t)$ the voltage drop on the nth inductor, and $I_n = I_n(t) = dQ_n/dt$ is the current flowing in the nth mesh, across the nth capacitor, i.e., between the network nodes at V_{n-1} and V_n.

a) Show that the currents I_n satisfy the coupled equations

$$L \frac{\mathrm{d}^2}{\mathrm{d}t^2}(I_{n+1} - 2I_n + I_{n-1}) = \frac{I_n}{C} . \tag{7.4}$$

b) Show that the solutions of (7.4) have the form

$$I_n = A\,\mathrm{e}^{\mathrm{i}(kna-\omega t)} , \tag{7.5}$$

with a the distance between two adjacent network elements, and determine the dispersion relation $\omega = \omega(k)$.

7.6 Non-Dispersive Transmission Line

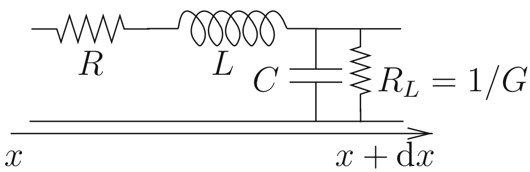

Fig. 7.6

The "elementary cell" scheme of a transmission line is sketched in the figure. In addition to the inductance L and capacitance C typical of the ideal "LC" transmission line, there is a resistance R in series with L, which accounts for the finite resistivity of the two conductors which form the line. In addition, we assume a finite leakage of current between the two conductors (i.e., in the direction "transverse" to the propagation) which is modeled by a second resistance R_L in parallel to C. The corresponding *conductance* is $G = 1/R_L$. In the limit of a continuous system with homogeneous, distributed properties, we define all quantities per unit length by replacing R with $R_\ell \mathrm{d}x$, L with $L_\ell \mathrm{d}x$, C with $C_\ell \mathrm{d}x$ and G with $G_\ell \mathrm{d}x$ (it is proper to use G as a quantity defined per unit length instead of R_L because the latter is proportional to the *inverse* of the length of the line).

a) Show that the current intensity $I = I(x,t)$ satisfies the equation

$$(\partial_x^2 - L_\ell C_\ell \partial_t^2)I = (R_\ell C_\ell + L_\ell G_\ell)\partial_t I + R_\ell G_\ell I = 0\,. \tag{7.6}$$

b) Study the propagation of a monochromatic current signal of frequency ω, i.e., search for solutions

$$I = I_0 \mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}\,, \tag{7.7}$$

for $x > 0$ with the boundary condition $I(0,t) = I_0 \mathrm{e}^{-\mathrm{i}\omega t}$, and determine the dispersion relation $k = k(\omega)$.
c) Find the condition on the line parameters for which a wavepacket traveling along the lines undergoes attenuation of the amplitude but *no* dispersion. This condition corresponds to solutions having the general form

$$I(x,t) = \mathrm{e}^{-\kappa x} f(x - \mathbf{v}t)\,, \tag{7.8}$$

where $f(x)$ is an arbitrary differentiable function. Find the expression for $\mathbf{v}$ and κ.

7.7 An "Alternate" *LC* Ladder Network

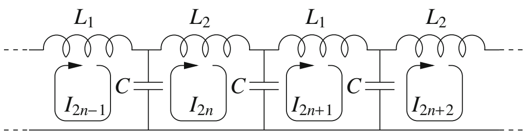

Fig. 7.7

Consider an "alternate" LC ladder network comprising identical capacitors C and inductors of value alternatively L_1 and L_2, as shown in Fig. 7.7. Let I_{2n} be the current flowing in the mesh of the nth inductor of value L_2, and I_{2n+1} the current flowing in the n-th inductor of value L_1.
a) Show that the currents satisfy the equations

$$L_2 \frac{\mathrm{d}^2 I_{2n}}{\mathrm{d}t^2} = \frac{1}{C}(I_{2n-1} - 2I_{2n} + I_{2n+1})\,, \quad L_1 \frac{\mathrm{d}^2 I_{2n+1}}{\mathrm{d}t^2} = \frac{1}{C}(I_{2n} - 2I_{2n+1} + I_{2n+2})\,. \tag{7.9}$$

What is a mechanical equivalent of this network?
b) Search for solutions of (7.9) of the form

$$I_{2n} = I_{\mathrm{e}}\,\mathrm{e}^{\mathrm{i}[2nka - \omega t]}\,, \quad I_{2n+1} = I_{\mathrm{o}}\,\mathrm{e}^{\mathrm{i}[(2n+1)ka - \omega t]}\,, \tag{7.10}$$

where I_e and I_o (the subscripts "e" and "o" stand for *even* and *odd*, respectively) are two constants, and determine the dispersion relation $\omega = \omega(k)$. Determine the allowed frequency range for wave propagation (for simplicity, assume $L_2 \ll L_1$).

7.8 Resonances in an *LC* Ladder Network

Consider the semi-infinite *LC* ladder network shown in Fig. 7.8. Let $I_n = I_n(t)$ be the current flowing in the n-th mesh of the circuit. An ideal current source provides the input current

$$I(t) = I_s e^{-i\omega t}, \tag{7.11}$$

where

a) Assuming $\omega < 2\omega_0$, evaluate $I_n(t)$ as a function of I_s and ω.
b) Now find $I_n(t)$ assuming $\omega > 2\omega_0$. Hint: search for a solution of the form

$$I_n(t) = A\alpha^n e^{-i\omega t}, \tag{7.12}$$

determining the dependence of α on ω and ω_0.

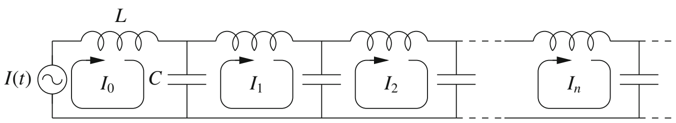

Fig. 7.8

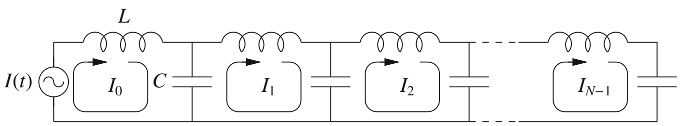

Fig. 7.9

Now assume that our LC ladder is finite, comprising N meshes numbered from 0 to $N-1$, as in Fig. 7.9. Evaluate $I_n(t)$ both for the case $\omega > 2\omega_0$ and for the case $\omega < 2\omega_0$, determining for which values of ω resonances are observed.

7.9 Cyclotron Resonances (1)

Consider a particle of charge q and mass m in the presence of a constant, uniform magnetic field $\mathbf{B} = B_0\,\hat{\mathbf{z}}$, and of a uniform electric field of amplitude E_0, *rotating* with frequency ω in the (x, y) plane, either in clockwise or in counterclockwise direction (Fig. 7.10 shows the counterclockwise case).

a) Describe the motion of the particle as a function of B, E, and ω, and show that, given B, a resonance is observed for the appropriate sign and value of ω.
b) Evaluate the solution of the equations of motion at resonance in the absence of friction.
c) Now assume the presence of a frictional force $\mathbf{f} = -m\gamma\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity of the particle. Find the steady-state solution of the equations of motion, and calculate the power dissipated by friction as a function of ω.

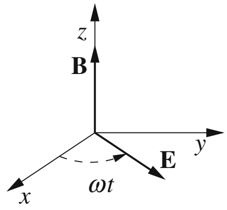

Fig. 7.10

7.10 Cyclotron Resonances (2)

Consider a particle of charge q and mass m in the presence of a constant uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, and of an oscillating uniform electric field $\mathbf{E} = E_0\hat{\mathbf{x}}\cos\omega t$.
a) Write the equations of motion (assuming no friction) and determine the resonance frequency of the system (hint: show that the equations for the velocity components v_x and v_y can be separated into two uncoupled equations of the forced harmonic oscillator type)
b) Now assume the presence of a frictional force $\mathbf{f} = -m\gamma\boldsymbol{v}$ where $\gamma \ll \omega$ and $\gamma \ll qB_0/m$. Find the steady state solution of the equations of motion and the spectrum of the absorbed power (hint: the equations for v_x and v_y cannot be separated in this case, but seeking a solution in the form $\boldsymbol{v} = \boldsymbol{v}_0\mathrm{e}^{-\mathrm{i}\omega t}$, with $\boldsymbol{v}_0$ a complex vector, will work).

7.11 A Quasi-Gaussian Wave Packet

Let us consider a wave packet of Gaussian profile propagating with velocity v along the x axis in a non-dispersive medium, with dispersion relation $\omega(k) = kv$. In these conditions, the wave packet's profile remains constant, and the packet is described by the function $g(x - vt)$ (Fig. 7.11)

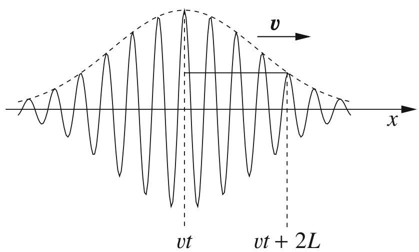

Fig. 7.11

$$\begin{aligned} g(x-vt) = & \sqrt{\pi}\frac{A}{L}\,\mathrm{e}^{\mathrm{i}k_0(x-vt)}\mathrm{e}^{-(x-vt)^2/4L^2} \\ = & \int_{-\infty}^{+\infty} \tilde{g}(k)\,\mathrm{e}^{\mathrm{i}k(x-vt)}\mathrm{d}k, \end{aligned} \tag{7.13}$$

where L, A and k_0 are constant parameters, and $\tilde{g}(k) = A\mathrm{e}^{-(k-k_0)^2L^2}$ is the Fourier transform of g. Now consider a second wave packet described by a function f, whose Fourier transform is

$$\tilde{f}(k) = \tilde{g}(k)\,\mathrm{e}^{\mathrm{i}\phi(k)} = A\mathrm{e}^{-(k-k_0)^2L^2}\mathrm{e}^{\mathrm{i}\phi(k)}, \tag{7.14}$$

where the "phase perturbation" $\phi(k)$ is a smooth function, that can be approximated by its Taylor polynomial expansion of degree 2 around $k = k_0$,

$$\phi(k) \simeq \phi(k_0) + \phi'(k_0)(k-k_0) + \frac{1}{2}\phi''(k_0)(k-k_0)^2, \tag{7.15}$$

where ϕ' and ϕ'' are the first and second derivatives of ϕ. The second wave packet can be considered as an "attempt" to build up a Gaussian wave packet from its spectral components, but with some error on the relative phases of the components themselves. Find the width of the wave packet and discuss its shape in order to show its deviations from the Gaussian profile.

7.12 A Wave Packet along a Weakly Dispersive Line

A transmission line extends from $x = 0$ to $x = +\infty$. A generator at $x = 0$ inputs a signal

$$f(t) = A\mathrm{e}^{-\mathrm{i}\omega_0 t}\mathrm{e}^{-t^2/\tau^2}, \tag{7.16}$$

where A and τ are constant and $\omega_0\tau \gg 1$, i.e., the signal is "quasi-monochromatic". The dispersion relation of the transmission line can be written

$$\omega = \omega(k) = kv\,(1+bk), \tag{7.17}$$

where v and b are known constants, and we assume $k > 0$.
a) Find the expression $f(x,t)$ for the propagating signal, i.e., for the wave packet traveling along the line, assuming $b = 0$.

From now on, assume dispersive effects to be small but not negligible, i.e., assume $bk_0 \ll 1$, where $k_0 = k(\omega_0)$ according to (7.17).
b) Within the above approximation, write the phase and group velocities as functions of ω_0 to the lowest order at which dispersive effects are present.

c) Give an estimate of the instant t_x when the "peak" of the signal reaches the position x, and of the corresponding length of the wave packet.
d) Now find the expression of the wave-packet shape as a function of (x,t), by calculating the integral

$$f(x,t) = \int e^{ik(\omega)x - i\omega t} \tilde{f}(\omega)\, d\omega\,, \tag{7.18}$$

where $\tilde{f}(\omega)$ is the Fourier transform of the wave packet. As a reasonable approximation, keep only factors up to the second order in $(k-k_0)^2$, for instance use

$$k(\omega) \simeq k(\omega_0) + k'(\omega_0)(\omega-\omega_0) + \frac{1}{2}k''(\omega_0)(\omega-\omega_0)^2. \tag{7.19}$$