

Chapter 8

Maxwell Equations and Conservation Laws

Topics Maxwell's equations. Conservation laws: energy, momentum and angular momentum of the electromagnetic field. Poynting's theorem. Radiation pressure.

Basic equations of this chapter:

(Note: Gaussian cgs units are used in this chapter unless otherwise specified.)

Maxwell's equations

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (8.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (8.2)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\partial_t \mathbf{B}, \quad (8.3)$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\partial_t \mathbf{E}. \quad (8.4)$$

Energy conservation (Poynting's) theorem

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \quad (8.5)$$

where

$$u = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) \quad (8.6)$$

is the energy density of the EM field, and

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \quad (8.7)$$

is the Poynting (also named Poynting-Umov) vector.

Momentum conservation theorem:

$$\partial_t \mathbf{g} + \nabla \cdot \mathbf{T} = - \left(\rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right), \quad (8.8)$$

where

$$\mathbf{g} = \frac{1}{4\pi c} (\mathbf{E} \times \mathbf{B}) = \frac{\mathbf{S}}{c^2} \quad (8.9)$$

is the momentum density of the EM field, and \mathbf{T} is Maxwell's stress tensor with components

$$T_{ij} = \frac{1}{4\pi} \left[\frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) \delta_{ij} - E_i E_j - B_i B_j \right]. \quad (8.10)$$

Thus, $\nabla \cdot \mathbf{T}$ is a vector with components

$$(\nabla \cdot \mathbf{T})_i = \sum_{j=1}^{j=3} \partial_j T_{ij}. \quad (8.11)$$

Angular momentum density of an EM field

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \mathbf{r} \times \frac{\mathbf{S}}{c}. \quad (8.12)$$

8.1 Poynting Vector(s) in an Ohmic Wire

A constant and uniformly distributed current density $\mathbf{J} = \sigma\mathbf{E}$ flows inside an infinite straight wire of radius a and conductivity σ .

a) Calculate the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ and discuss the energy conservation in the wire.

b) The Poynting vector occurs in Poynting’s theorem only through its divergence, since the theorem only requires that the flux of the Poynting vector through any a closed surface describes the net flow of electromagnetic energy. Show that, consequently, $\mathbf{S}' = \varphi\mathbf{J}$, where φ is the electrostatic potential, is also a suitable choice for \mathbf{S} (hint: substitute $\mathbf{E} = -\nabla\varphi$ into (8.7) and manipulate the result).

8.2 Poynting Vector(s) in a Capacitor

A plane capacitor consists of two parallel circular plates of radius a , at a distance $h \ll a$ from each other. The electric field inside the capacitor is slowly varying in time, $\mathbf{E} = E(t)\hat{\mathbf{z}}$, for instance, assume $E = E_0 t/\tau$. Boundary effects are negligible (Fig. 8.1).

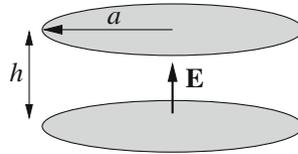


Fig. 8.1

a) Evaluate the magnetic field \mathbf{B} inside the capacitor.

b) Calculate the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$, and show that the flux of \mathbf{S} though any surface enclosing the capacitor equals the time variation of the energy associated to the electromagnetic field.

c) Show that an alternative Poynting vector is

$$\mathbf{S}' = \frac{1}{4\pi} \varphi \partial_t \mathbf{E}, \tag{8.13}$$

where φ is the electric potential ($\mathbf{E} = -\nabla\varphi$). Verify that also the flux of \mathbf{S}' through the closed surface of point **b)** equals the variation of the energy in the volume inside the surface [hint: proceed as in point **b)** of Problem 8.1].

8.3 Poynting’s Theorem in a Solenoid

A time-dependent current, $I = I(t) = I_0 t/\tau$, flows through the coils of an infinitely long, cylindrical solenoid. The solenoid has radius a and n turns per unit length.

a) Find the magnetic and electric fields, \mathbf{B} and \mathbf{E} , inside the solenoid.

- b) Verify the law of energy conservation (Poynting's theorem), for a closed internal cylindrical surface, coaxial to the solenoid.
- c) Now verify Poynting's theorem for an *external*, coaxial cylindrical surface (remember that $\mathbf{B} = 0$ outside an infinite solenoid).

8.4 Poynting Vector in a Capacitor with Moving Plates

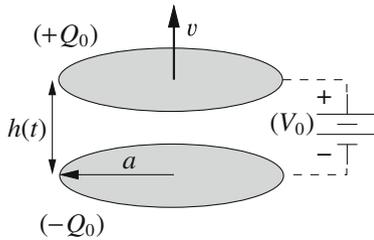


Fig. 8.2

A plane capacitor consists of two circular metallic plates of radius a , parallel to each other. One plate is kept at rest while the other moves at constant velocity v , so that the distance between the plates is $h = h(t) = h_0 + vt$. In the following we consider only the case in which $h \ll a$ at any time t , so that boundary effects are negligible. We also assume that v is small enough to ensure the validity of the slowly varying current approximation.^{7.11} A Quasi-Gaussian Wave Packet (Fig. 8.2).

Considering both the case of electrically isolated plates having opposite charges $\pm Q_0$, and the case of plates connected through a voltage source keeping a constant electric potential drop V_0 between them, calculate

- the force F needed to keep v constant,
- the rate of change of the electrostatic energy U ,
- the magnetic field between the plates,
- the Poynting vector \mathbf{S} and its flux through a cylindrical surface enclosing, and coaxial with, the capacitor; use this last result to discuss energy conservation in the system.

8.5 Radiation Pressure on a Perfect Mirror

A perfect mirror is defined as a medium inside which $\mathbf{E} = 0$ and $\mathbf{B} = 0$. Thus, an EM wave cannot penetrate the mirror surface and will be reflected by it.

Find the radiation pressure P_{rad} , i.e., the cycle-averaged force per unit surface exerted by a plane wave incident on the surface of a perfect plane mirror, as a function of the intensity I of the wave by each of the following three methods:

- Consider the reflection of a square wave packet of arbitrary, but finite, duration. Determine P_{rad} from the difference between the total momentum of the incident wave packet and the momentum of the reflected wave packet.

- b) Calculate the force on the mirror directly, from the knowledge of the EM fields and of the charge and current densities on the mirror surface.
 c) Determine P_{rad} from Maxwell's stress tensor.

8.6 A Gaussian Beam

In optics, a Gaussian beam is a beam of monochromatic electromagnetic radiation whose transverse magnetic and electric field amplitude profiles are given by the Gaussian function. Gaussian beams are important because they are a very good approximation of the radiation emitted by most laser sources. Here we consider a linearly-polarized Gaussian beam propagating along the z -axis and whose transverse profile is symmetrical around such axis. The origin of the coordinate systems is chosen so that the beam has minimum width on the $z = 0$ plane. We assume that, *close to the $z = 0$ plane*, the *transverse* components of the EM fields can be written as

$$\begin{aligned} E_x &= E_0(r) \cos(kz - \omega t) = E_0 e^{-r^2/r_0^2} \cos(kz - \omega t), \\ B_y &= B_0(r) \cos(kz - \omega t) = B_0 e^{-r^2/r_0^2} \cos(kz - \omega t), \end{aligned} \quad (8.14)$$

where $r = \sqrt{x^2 + y^2} < r_0$ and $k = \omega/c$. The parameter r_0 is called the *waist* of the beam.

- a) Show that, in addition to the transverse components (8.14), *longitudinal* components E_z and B_z must exist, and give their expression.
 b) Compute the Poynting vector of the beam \mathbf{S} , and its average over a period $\langle \mathbf{S} \rangle$, showing which components are vanishing.
 c) Verify that the fields (8.14) do *not* satisfy the wave equation in vacuum, hence they are only an approximate expression, as mentioned above. Explain in which range of z , depending on the value of kr_0 , the approximate expressions are accurate.

8.7 Intensity and Angular Momentum of a Light Beam

A circularly polarized monochromatic light beam of frequency ω propagates along the z direction. The beam has a finite width in the plane perpendicular to z . We assume that in a region of space, close to the “waist” (i.e., to the plane where the beam has minimal width), the *transverse* components of the EM fields can be written approximately as

$$\begin{aligned} E_x &= +E_0(r) \cos(kz - \omega t), & E_y &= -E_0(r) \sin(kz - \omega t), \\ B_x &= E_0(r) \sin(kz - \omega t), & B_y &= E_0(r) \cos(kz - \omega t), \end{aligned} \quad (8.15)$$

where $r = \sqrt{x^2 + y^2}$, $k = \omega/c$, and $E_0(r)$ is a known real function.

- a) Write the intensity $I = I(r)$, defined as the “energy flow along z ”, i.e., $I(r) = S_z = \mathbf{S} \cdot \hat{\mathbf{z}}$ where \mathbf{S} is the Poynting vector.
- b) Show that, in addition to the transverse components of the fields, also *longitudinal* components (E_z, B_z) must exist, and give their expression.
- c) Evaluate the S_x and S_y component of \mathbf{S} , and discuss the result.
- d) Show that the density of angular momentum (8.12) of the beam can be written as

$$\ell_z = \ell_z(r) = -\frac{r}{2c\omega} drI, \tag{8.16}$$

and compute the quantity

$$L_z = \int_0^\infty \ell_z(r) 2\pi r dr \tag{8.17}$$

as a function of the total power of the beam $W = \int_0^\infty I(r) 2\pi r dr$.

8.8 Feynman’s Paradox solved

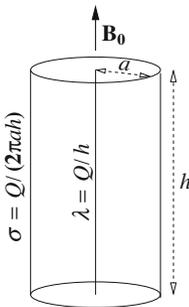


Fig. 8.3

The system in Fig. 8.3 is composed by a non-conducting cylindrical surface of height h and radius a , over which there is a net charge Q uniformly distributed with surface density $\sigma = Q/(2\pi ah)$, and a wire of same length oriented along the cylinder axis and having charge $-Q$ distributed with uniform linear density $\lambda = -Q/h$, so that the system is globally neutral. The cylindrical surface is free to rotate around its axis without friction, and has moment of inertia \mathcal{I} per unit length. The system is at rest in the presence of an external

uniform magnetic field \mathbf{B}_{ext} , parallel to the system axis. Assume that boundary effects can be neglected.

Starting at time $t = 0$, the external magnetic field is reduced from its initial value $\mathbf{B}_{\text{ext}} = \mathbf{B}_0$ to zero at a time $t_f \gg a/c$, according to some temporal law $\mathbf{B}_{\text{ext}} = \mathbf{B}_{\text{ext}}(t)$.

- a) Initially assuming that the field generated by the motion of the charges on the cylinder is negligible, evaluate the angular velocity $\omega = \omega(t)$ of the cylinder as a function of time during the decay of \mathbf{B}_{ext} , and the corresponding mechanical angular momentum \mathbf{L}_c of the cylinder.
- b) Now take the field generated by rotating charges into account, and evaluate how the results of a) change.
- c) Consistently with Eqs. (8.8–8.9), we introduce the *angular momentum* of a given distribution of electromagnetic fields as

$$\mathbf{L}_{\text{EM}} = \int \mathbf{r} \times \mathbf{g} d^3r, \quad (8.18)$$

where $\mathbf{g} = \mathbf{E} \times \mathbf{B}/4\pi$ is the electromagnetic momentum density. Use Eq. (8.18) to check the conservation of the *total* angular momentum for the system (thus solving the “paradox” as outlined in Problem 6.6).

8.9 Magnetic Monopoles

Assume that an experiment gives evidence of the existence of “magnetic monopoles”, i.e., of point-like particles with a *net magnetic charge* q_m , such that the magnetic field \mathbf{B}_m generated by such charge is

$$\mathbf{B}_m = \alpha \frac{q_m}{r^2} \hat{\mathbf{r}}, \quad (8.19)$$

while in the presence of an “external” magnetic field \mathbf{B}_{ext} the force on the particle is $\mathbf{f} = q_m \mathbf{B}_{\text{ext}}$. Thus, for example, the interaction force between two particles with magnetic charges q_{m1} and q_{m2} is given by

$$\mathbf{f}_{1 \rightarrow 2} = \alpha \frac{q_{m1} q_{m2}}{r_{12}^2} \hat{\mathbf{r}}_{12}, \quad \mathbf{f}_{2 \rightarrow 1} = -\mathbf{f}_{1 \rightarrow 2}. \quad (8.20)$$

where \mathbf{r}_{12} is the distance vector directed from charge 1 to charge 2. We also assume that conservation of the total magnetic charge holds.

a) Determine, both in SI and Gaussian units, the expressions for the coefficient α and the dimensions of the magnetic charge q_m with respect to the electric charge q_e . (Hint: we may assume that the field generated by two magnetic charges $+q_m$ and $-q_m$, separated by a distance \mathbf{h} , is equivalent to the field of a magnetic dipole $\mathbf{m} = q_m \mathbf{h}$ at distances $r \gg |\mathbf{h}|$.)

b) Complete Maxwell's equations in order to take the presence of magnetic monopoles into account.

c) Now consider a beam of magnetic monopoles of radius a , of uniform density and infinite length. The number density of the particles of the beam is n , and all particles have the same magnetic charge q_m and the same velocity \mathbf{v} . Find the electric and magnetic fields generated by the beam.