Chapter S-10
Solutions for Chapter 10

S-10.1 Cyclotron Radiation

a) The electric dipole moment $\mathbf{p} = -e\mathbf{r}$ rotates in the xy plane with frequency ω_L, which is also the frequency of the emitted radiation. The dipole approximation is valid if the dimensions of the radiating source are much smaller than the emitted wavelength λ. Here this corresponds to the condition $2r_L = 2v/\omega_L \ll \lambda = 2\pi c/\omega_L$, always true for non-relativistic velocities.

The rotating dipole can be written as $\mathbf{p} = p_0 (\hat{\mathbf{x}}\cos\omega_L t + \hat{\mathbf{y}}\sin\omega_L t)$. For the electric field of the dipole radiation observed in a direction of unit vector $\hat{\mathbf{n}}$, we have $\mathbf{E} \propto -(\mathbf{p}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}$. If $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, then $\mathbf{E} \propto \hat{\mathbf{x}}\cos\omega_L t + \hat{\mathbf{y}}\sin\omega_L t$ (circular polarization); if $\hat{\mathbf{n}} = \hat{\mathbf{x}}$ or $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, we vave $\mathbf{E} \propto -\hat{\mathbf{y}}\sin\omega_L t$ and $\mathbf{E} \propto -\hat{\mathbf{x}}\cos\omega_L t$, respectively (linear polarization).

Since $\ddot{\mathbf{r}} = v\times\boldsymbol{\omega}_L$ (where $\boldsymbol{\omega}_L = \hat{\mathbf{z}}\omega_L$), the radiated power can be written as

$$P_{\text{rad}} = \frac{2}{3}\frac{|e\ddot{\mathbf{r}}|^2}{c^3} = \frac{2}{3}\frac{e^2 v^2 \omega_L^2}{c^3} . \tag{S-10.1}$$

b) We assume that the energy loss due to radiation is small enough to cause a variation of the orbit radius $\Delta r_c \ll r_c$ during a single period, so that, during a single period, the motion is still approximately circular. Thus the magnitude of the electron velocity $v = v(t)$ can be written as $v \simeq \omega_L r$, where $r = r(t)$ is the radius of the orbit at time t. The electron energy is

$$U = \frac{m_e v^2}{2} = \frac{m_e \omega_L^2 r^2}{2} , \tag{S-10.2}$$

and the equation for the energy loss, $dU/dt = -P_{\text{rad}}$, becomes

$$\frac{d}{dt}\left(\frac{m_e \omega_L^2 r^2}{2}\right) = -\frac{2}{3c^3}\left(e^2 \omega_L^4 r^2\right) = -\frac{2r_e m_e \omega_L^4}{3c}r^2 , \tag{S-10.3}$$

where $r_{\mathrm{e}} = e^2/(m_{\mathrm{e}} c^2)$ is the classical electron radius. Substituting the relation $\mathrm{d}(r^2)/\mathrm{d}t = 2r\,\mathrm{d}r/\mathrm{d}t$ into (S-10.3) we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2r_{\mathrm{e}}\omega_{\mathrm{L}}^2}{3c}r \equiv -\frac{r}{\tau}, \quad \text{with} \quad \tau = \frac{3c}{2r_{\mathrm{e}}\omega_{\mathrm{L}}^2} = \frac{3m_{\mathrm{e}}c^3}{2e^2\omega_{\mathrm{L}}^2}, \tag{S-10.4}$$

whose solution is

$$r(t) = r(0)\,\mathrm{e}^{-t/\tau}, \tag{S-10.5}$$

and the trajectory of the electron is a spiral with a decay time τ. Inserting the expressions for r_{e} and ω_{L} we have

$$\tau = \frac{3}{2}\frac{m_{\mathrm{e}}^3 c^5}{e^4 B_0^2} = \frac{5.2\times10^5}{B_0^2}\ \mathrm{s} \tag{S-10.6}$$

where the magnetic field B_0 is in G. The condition $\tau \gg \omega_{\mathrm{L}}^{-1}$ implies

$$\frac{3}{2}\frac{m_{\mathrm{e}}^3 c^5}{e^4 B_0^2} \gg \frac{m_{\mathrm{e}}c}{eB_0}, \quad \text{or} \quad B_0 \ll \frac{3}{2}\frac{m_{\mathrm{e}}^2 c^4}{e^3} = 9.2\times10^{13}\,\mathrm{G}, \tag{S-10.7}$$

a condition well verified in all experimental conditions: such high fields can be found only on neutron stars! (see Problem 10.5)

c) We insert a frictional force $\mathbf{f}_{\mathrm{fr}} = -m_{\mathrm{e}}\eta\mathbf{v}$ into the equation of motion, obtaining

$$m_{\mathrm{e}}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\frac{e}{c}\mathbf{v}\times\mathbf{B}_0 - m_{\mathrm{e}}\eta\mathbf{v}. \tag{S-10.8}$$

This corresponds to the following two coupled equations for the the x and y components the electron velocity

$$\dot{v}_x = -\omega_{\mathrm{L}}v_y - \eta v_x, \qquad \dot{v}_y = \omega_{\mathrm{L}}v_x - \eta v_y. \tag{S-10.9}$$

An elegant method to solve these equations is to combine the x and y coordinates of the electron into a single complex variable $R = x + \mathrm{i}y$, and the velocity components into the complex variable $V = v_x + \mathrm{i}v_y$. The two equations (S-10.9) are thus combined into the single complex equation

$$\dot{V} = (\mathrm{i}\omega_{\mathrm{L}} - \eta)V, \quad \text{with solution} \quad V = V(0)\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}t - \eta t} = v_0\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}t - \eta t}. \tag{S-10.10}$$

For the electron position we have

$$R = \int V\,\mathrm{d}t + C = \frac{v_0}{\mathrm{i}\omega_{\mathrm{L}} - \eta}\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}t - \eta t} + C = -\frac{(\eta + \mathrm{i}\omega_{\mathrm{L}})v_0}{\omega_{\mathrm{L}}^2 + \eta^2}\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}t - \eta t} + C, \tag{S-10.11}$$

where C is a complex constant depending on our choice of the origin of the coordinates. We choose $C = 0$, and rewrite (S-10.11) as

$$R = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}(\cos\phi + i\sin\phi)\,e^{i\omega_L t - \eta t} = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\,e^{i(\omega_L t + \phi) - \eta t}\,, \qquad \text{(S-10.12)}$$

where

$$\cos\phi = \frac{\eta}{\sqrt{\omega_L^2 + \eta^2}}\,, \quad \sin\phi = \frac{\omega_L}{\sqrt{\omega_L^2 + \eta^2}}\,, \quad \phi = \arctan\left(\frac{\omega_L}{\eta}\right). \qquad \text{(S-10.13)}$$

Going back to the real quantities we have

$$v_x = \text{Re}(V) = v_0\,e^{-\eta t}\cos\omega_L t\,, \qquad\qquad\qquad \text{(S-10.14)}$$

$$v_y = \text{Im}(V) = v_0\,e^{-\eta t}\sin\omega_L t\,, \qquad\qquad\qquad \text{(S-10.15)}$$

$$x = \text{Re}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\,e^{-\eta t}\cos(\omega_L t + \phi)\,, \qquad \text{(S-10.16)}$$

$$y = \text{Im}(R) = -\frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\,e^{-\eta t}\sin(\omega_L t + \phi)\,. \qquad \text{(S-10.17)}$$

Thus, the velocity rotates with frequency ω_L, while its magnitude decays exponentially, $|\boldsymbol{v}(t)| = v_0\,e^{-\eta t}$. For the radius of the trajectory we have

$$r(t) = |R(t)| = \frac{v_0}{\sqrt{\omega_L^2 + \eta^2}}\,e^{-\eta t}\,. \qquad \text{(S-10.18)}$$

Thus, choosing $\eta = 1/\tau$, the motion with frictional force is identical to the motion with radiative power loss, and

$$\mathbf{f}_{\text{fr}}\cdot\boldsymbol{v} = -m_e\eta v^2 = -\frac{m_e v^2}{\tau} = -m_e v^2\frac{2e^2\omega_L^2}{3m_e c^3} = -\frac{2e^2 v^2\omega_L^2}{3c^3} = -P_{\text{rad}}\,. \qquad \text{(S-10.19)}$$

A drawback of this approach is that the frictional coefficient inserted here is not universal but is dependent on the force on the electron (in this case, via the dependence on ω_L). See Problem 10.12 for a more general approach to radiation friction.

S-10.2 Atomic Collapse

a) An electron describing a circular orbit of radius a_0 (Bohr radius) around a proton corresponds to a counterrotating electric dipole $\mathbf{p}(t)$ of magnitude $p_0 = ea_0$. The angular velocity of the orbit ω can be evaluated by considering that the centripetal acceleration is due to the Coulomb force,

$$\omega^2 a_0 = \frac{1}{m_e} \frac{e^2}{a_0^2} , \tag{S-10.20}$$

from which we obtain

$$\omega = \sqrt{\frac{e^2}{m_e a_0^3}} = 4.1 \times 10^{16} \, \text{rad/s} . \tag{S-10.21}$$

Actually, the strongest emission from the hydrogen atom occurs at a frequency smaller by about one order of magnitude.

Since $\mathbf{p}$ is perpendicular to ω, we have $\ddot{\mathbf{p}} = (\mathbf{p} \times \omega) \times \omega$ and $|\ddot{\mathbf{p}}|^2 = \left(\omega^2 p_0\right)^2$ (the same result can be obtained by considering the rotating dipole as the superposition of two perpendicularly oscillating dipoles). Thus the radiated power is

$$P_{\text{rad}} = \frac{2}{3c^3} |\ddot{\mathbf{p}}|^2 = \frac{2}{3} \frac{\omega^4 e^2 a_0^2}{c^3} = \frac{2}{3} \frac{e^2 r_e^2 c}{a_0^4} , \tag{S-10.22}$$

where r_e is the classical electron radius.

b) We assume that, due to the emission of radiation, the electron loses its energy according to $dU/dt = -P_{\text{rad}}$, where $U = K + V$ is the total electron energy, K and V being the kinetic and potential energy, respectively. If the energy lost per period is small with respect to the total energy, we may assume that the electron the orbit is almost circular during a period, with the radius *slowly* decreasing with time, $r = r(t)$ with $\dot{r}/r \ll \omega$.

Since the velocity is $v = r\omega$, the total energy can be written as a function of a:

$$U = K + V = \frac{m_e v^2}{2} - \frac{e^2}{r} = -\frac{e^2}{2r} . \tag{S-10.23}$$

Therefore

$$\frac{dU}{dt} \simeq -\frac{e^2}{2} \frac{d}{d} \left(\frac{1}{r}\right) = \frac{e^2}{2r^2} \frac{dr}{dt} . \tag{S-10.24}$$

Since

$$P_{\text{rad}} = \frac{2}{3} \frac{e^2 r_e^2 c}{r^4} \tag{S-10.25}$$

the equation $dU/dt = -P_{\text{rad}}$ can be written as

$$\frac{e^2}{2r^2} \frac{dr}{dt} = -\frac{2}{3} \frac{e^2 r_e^2 c}{r^4} \quad \Rightarrow \quad r^2 \frac{dr}{dt} = -\frac{4}{3} r_e^2 c \quad \Rightarrow \quad \frac{1}{3} \frac{dr^3}{dt} = \frac{4}{3} r_e^2 c \tag{S-10.26}$$

The solution, assuming $r(0) = a_0$, is

$$r^3 = a_0^3 - 4 r_e^2 c t, \tag{S-10.27}$$

giving for the time need by the electron to fall on the nucleus

$$\tau = \frac{a_0^3}{4 r_e^2 c} \simeq 1.6 \times 10^{-11} \text{ s} . \tag{S-10.28}$$

This is a well-known result, showing that a classical "Keplerian" atom is not stable. It is however interesting to notice that the value of τ is of the same order of magnitude of the lifetime of the first excited state, i.e., of the time by which the excited state decays to the ground state emitting radiation.

S-10.3 Radiative Damping of the Elastically Bound Electron

a) The solution of (10.5) with the given initial conditions and $\eta = 0$ is

$$\mathbf{r} = \mathbf{s}_0 \cos \omega_0 t . \tag{S-10.29}$$

The corresponding average radiated power in the dipole approximation is

$$P_{\text{rad}} = \frac{2}{3c^3} \langle -e|\ddot{\mathbf{r}}|^2 \rangle = \frac{2e^2}{3c^3} \omega_0^4 s_0^2 \langle \cos^2 \omega_0 t \rangle = \frac{e^2}{3c^3} \omega_0^4 s_0^2 . \tag{S-10.30}$$

The radiated power is emitted at the expense of the energy of the oscillating electron. Thus, the total mechanical energy of electron must decrease in time, and the harmonic-oscillator solution of (S-10.29) cannot be exact. Assuming that the energy of the oscillator decays very slowly, i.e., with a decay constant $\tau \gg \omega_0^{-1}$, we can approximate (S-10.29) as

$$\mathbf{r} \simeq \mathbf{s}(t) \cos \omega_0 t . \tag{S-10.31}$$

where $s(t)$ is a decreasing function of time to be determined. Consequently, we must replace s_0 by $s(t)$ also in equation (S-10.30) for the actual average radiated power.

b) At time t, the total energy of the oscillating electron is $U(t) = m_e \omega_0^2 s^2(t)$. The time decay constant τ is defined as

$$\tau = \frac{U(t)}{P_{\text{rad}}(t)} = \frac{3m_e c^3}{2e^2\omega_0^2} = \frac{3c}{2r_e\omega_0^2} \,, \tag{S-10.32}$$

and is thus independent of t. Since the classical electron radius is $r_e \simeq 2.82 \times 10^{-15}$ m, the condition $\tau > 2\pi/\omega_0$ leads to

$$\omega_0 < \frac{3}{4\pi} \frac{c}{r_e} \simeq 3 \times 10^{22} \text{ rad/s} \,. \tag{S-10.33}$$

For a comparison, estimating ω_0 as the frequency of the 1S←2P Lyman-alpha emission line of the hydrogen atom, we have $\omega_0 \simeq 3 \times 10^{16}$ rad/s.

c) We look for a solution of the form $\mathbf{r} = \text{Re}(\mathbf{s}_0 e^{-i\omega t})$, with *complex* ω. Substituting this into (10.5), the characteristic equation becomes

$$\omega^2 + i\eta\omega + \omega_0^2 = 0 \,, \tag{S-10.34}$$

whose solution is

$$\omega = -i\frac{\eta}{2} \pm \sqrt{\omega_0^2 - \frac{\eta^2}{4}} \simeq -i\frac{\eta}{2} \pm \omega_0 \,, \tag{S-10.35}$$

where we have neglected the terms of the order $(\eta/\omega_0)^2$ and higher. Thus, the approximated solution for the electron position is

$$\mathbf{r} \simeq \mathbf{s}_0 e^{-\eta t/2} \cos \omega_0 t \,. \tag{S-10.36}$$

Actually, this approximation gives an initial velocity $\dot{\mathbf{r}}(0) = -\eta \mathbf{s}_0/2$ instead of zero. However, this discrepancy can be neglected if $\eta \ll \omega_0$. The maximum speed reached by the electron is $v_{\max} \simeq \omega_0 s_0$, and $\eta s_0/2 \ll \omega_0 s_0$.

The time-dependent total energy of the electron and average radiated power are

$$U(t) \simeq \frac{m_e}{2} \omega_0^2 s_0^2 e^{-\eta t} \,, \quad \text{and} \quad P_{\text{rad}}(t) \simeq \frac{e^2}{3c^3} \omega_0^4 s_0^2 e^{-\eta t} \,. \tag{S-10.37}$$

The condition $dU/dt = -P_{\text{rad}}$ leads to

$$\eta = \frac{2r_e\omega_0^2}{3c} = \frac{1}{\tau} \,. \tag{S-10.38}$$

S-10.4 Radiation Emitted by Orbiting Charges

a) Let us denote by $\mathbf{r}_1$ and $\mathbf{r}_2$ the location vectors of the two charges with respect to the center of their common circular orbit. In polar coordinates we have

$$\mathbf{r}_1 \equiv [R, \phi_1(t)], \quad \text{and} \quad \mathbf{r}_2 \equiv [R, \phi_2(t)] . \tag{S-10.39}$$

Defining $\Delta\phi = \phi_2 - \phi_1$ and choosing an appropriate origin of time, the equations of motion in polar coordinates are

$$\mathbf{r}_1 \equiv \left(R, \omega t - \frac{\Delta\phi}{2}\right), \quad \text{and} \quad \mathbf{r}_2 \equiv \left(R, \omega t + \frac{\Delta\phi}{2}\right). \tag{S-10.40}$$

In Cartesian coordinates we have

$$\mathbf{r}_1 \equiv [x_1(t), y_1(t)], \quad \text{and} \quad \mathbf{r}_2 \equiv [x_2(t), y_2(t)], \tag{S-10.41}$$

with, respectively,

$$x_1(t) = R\cos\left(\omega t - \frac{\Delta\phi}{2}\right), \qquad y_1(t) = R\sin\left(\omega t - \frac{\Delta\phi}{2}\right), \tag{S-10.42}$$

$$x_2(t) = R\cos\left(\omega t + \frac{\Delta\phi}{2}\right), \qquad y_2(t) = R\sin\left(\omega t + \frac{\Delta\phi}{2}\right). \tag{S-10.43}$$

The dipole moment of the system is $\mathbf{p} = q(\mathbf{r}_1 + \mathbf{r}_2)$, with Cartesian components

$$p_x = qR\left[\cos\left(\omega t - \frac{\Delta\phi}{2}\right) + \cos\left(\omega t + \frac{\Delta\phi}{2}\right)\right] = 2qR\cos\left(\frac{\Delta\phi}{2}\right)\cos\omega t, \tag{S-10.44}$$

$$p_y = qR\left[\sin\left(\omega t - \frac{\Delta\phi}{2}\right) + \sin\left(\omega t + \frac{\Delta\phi}{2}\right)\right] = 2qR\cos\left(\frac{\Delta\phi}{2}\right)\sin\omega t, \tag{S-10.45}$$

i.e., $\mathbf{p}$ has constant magnitude $p = 2qR\cos(\Delta\phi/2)$, and rotates in the $z = 0$ plane with angular frequency ω.
b) In the dipole approximation, the electric field of the radiation emitted along a direction of unit vector the $\hat{\mathbf{n}}$ is parallel to the vector

$$(\mathbf{p} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}} = \mathbf{p}_\perp . \tag{S-10.46}$$

Since for a dipole rotating in the $z = 0$ plane

$$(\mathbf{p} \times \hat{\mathbf{x}}) \times \hat{\mathbf{x}} \text{ is parallel to } \hat{\mathbf{y}}, \quad \text{and} \quad (\mathbf{p} \times \hat{\mathbf{y}}) \times \hat{\mathbf{y}} \text{ is parallel to } \hat{\mathbf{x}}, \tag{S-10.47}$$

the polarization of the radiation observed in the $\hat{\mathbf{x}}$ ($\hat{\mathbf{y}}$) direction is linear and along $\hat{\mathbf{y}}$ ($\hat{\mathbf{x}}$). For radiation observed the $\hat{\mathbf{z}}$ direction

$$(\mathbf{p} \times \hat{\mathbf{z}}) \times \hat{\mathbf{z}} \text{ is parallel to } \mathbf{p}\,, \tag{S-10.48}$$

and the observed polarization is circular.

The total radiated power is

$$P_{\mathrm{rad}} = \frac{2}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{4q^2R^2\omega^4}{3c^3}\cos^2\!\left(\frac{\Delta\phi}{2}\right), \tag{S-10.49}$$

which obviously vanishes when $\mathbf{p} = 0$, i.e., for $\Delta\phi = \pi$ (charges on opposite ends of a rotating diameter), and has a maximum for $\Delta\phi = 0$ (superposed charges).

c) In this case charges are superposed to each other every half turn. We choose the coordinates and the time origin so that the charges are superposed at $t = 0$ we have $\mathbf{r}_1 = \mathbf{r}_2 = (R,0)$. Thus the trajectories can be written as

$$r_1 = r_2 = R\,, \qquad \phi_1(t) = \omega t\,, \qquad \phi_2(t) = -\omega t\,, \tag{S-10.50}$$

in polar coordinates, and as

$$\begin{aligned}
x_1(t) &= R\cos\omega t\,, & y_1(t) &= R\sin\omega t\,, \\
x_2(t) &= R\cos\omega t\,, & y_2(t) &= -R\sin\omega t\,,
\end{aligned} \tag{S-10.51}$$

in Cartesian coordinates. The total dipole moment is thus $\mathbf{p} = (2qR\cos\omega t)\hat{\mathbf{x}}$. No radiation is emitted along x, while the radiation emitted along all other directions is linearly polarized. The total average radiated power is

$$P_{\mathrm{rad}} = \frac{2}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{4q^2R^2\omega^4}{3c^3}\,. \tag{S-10.52}$$

d) With an appropriate choice of the time origin the equations of motion of the three charges can be written, in polar coordinates, as

$$\begin{aligned}
r_1 = r_2 = r_3 &= R\,, & \phi_1(t) &= \omega t\,, \\
\phi_2(t) &= \omega t + \Delta\phi_2\,, & \phi_3(t) &= \omega t + \Delta\phi_3\,,
\end{aligned} \tag{S-10.53}$$

and, in Cartesian coordinates,

$$x_i = R\cos\phi_i(t)\,, \qquad y_i = R\sin\phi_i(t)\,, \qquad (i = 1,2,3)\,. \tag{S-10.54}$$

The electric dipole moment vanishes if the three charges are on the vertices of a rotating equilateral triangle ($\Delta\phi_2 = -\Delta\phi_3 = 2\pi/3$), and has its maximum value when the three charges are overlapped ($\Delta\phi_2 = \Delta\phi_3 = 0$).

e) The magnetic dipole moment for a point charge q, traveling at angular velocity ω on a circular orbit of radius R, is defined by

$$\mathbf{m} = \frac{1}{2c} \int \mathbf{r} \times \mathbf{J} \, \mathrm{d}^3 x = \frac{qR^2\omega}{2c} \, , \qquad (\text{S-10.55})$$

and is constant (notice that $\mathbf{m}$ is proportional to the angular momentum of the orbiting charge). Thus the magnetic dipole does not contribute to radiation, because the radiation fields are proportional to $\ddot{\mathbf{m}}$.

This problem explains why a circular coil carrying a constant current does not radiate, although we may consider the current as produced by charges moving on circular orbits, and thus subject to acceleration.

S-10.5 Spin-Down Rate and Magnetic Field of a Pulsar

a) Due to the nonzero angle α between the magnetic moment and the rotation axis of the pulsar, the component of $\mathbf{m}$ perpendicular to ω rotates with frequency ω. Thus the Pulsar emits magnetic dipole radiation of frequency ω. The total power is

$$P = \frac{2}{3c^3} |\ddot{\mathbf{m}}_\perp|^2 = \frac{2}{3} \frac{m_\perp^2 \omega^4}{c^3} \, , \qquad (\text{S-10.56})$$

where $m_\perp = m\sin\alpha$.

b) The mechanical energy is $U = I\omega^2/2$, where $I = 2MR^2/5 \simeq 1.1 \times 10^{43} \, \mathrm{g \, cm}^2$ is the moment of inertia of the pulsar, assuming a uniform mass distribution over the volume of a sphere of radius R. Assuming that the energy loss is due to radiation emission only, we can write

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{I\omega^2}{2}\right) = I\omega\dot{\omega} = -P \, , \qquad (\text{S-10.57})$$

and, substituting (S-10.56), we have

$$I\omega\dot{\omega} = -\frac{2}{3}\frac{m_\perp^2\omega^4}{c^3} \quad \Rightarrow \quad \frac{\dot{\omega}}{\omega^3} = -\frac{2m_\perp^2}{3Ic^3} \, . \qquad (\text{S-10.58})$$

By integrating over time from 0 to t we obtain

$$\frac{1}{2\omega^2(t)} - \frac{1}{2\omega^2(0)} = \frac{2m_\perp^2}{3Ic^3}t \, , \qquad (\text{S-10.59})$$

and thus

$$\omega(t) = \frac{\omega(0)}{\sqrt{1+\dfrac{t}{\tau}}} \, , \quad \text{where} \quad \tau = \frac{3Ic^3}{4m_\perp^2\omega^2(0)} \, . \qquad (\text{S-10.60})$$

c) We can rewrite $\dot\omega/\omega^3$ as $\dot\omega/\omega^3 = -T\dot{T}/4\pi^2$, where $T = 2\pi/\omega$ is the rotation period of the pulsar, and $\dot{T} = -2\pi\dot\omega/\omega^2$. Thus we can obtain the magnetic dipole moment $m = m_\perp$ of the pulsar as a function of the experimentally measured parameters from (S-10.58):

$$m = \sqrt{\frac{3Ic^3}{8\pi^2}T\dot{T}} \simeq 3.3\times 10^{36}\sqrt{T\dot{T}}\,\text{erg/G}\,, \tag{S-10.61}$$

where T is in seconds. The magnetic field immediately outside of the pulsar surface is the field of a magnetic dipole located at the pulsar center:

$$\mathbf{B} = \frac{3(\hat{\mathbf{r}}\cdot\mathbf{m})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,, \tag{S-10.62}$$

and thus $B_{\text{max}} = 2m/R^3$. Thus we obtain the practical formula

$$B_{\text{max}} \simeq 6.6\times 10^{21}\sqrt{T\dot{T}}\,\text{G}\,, \tag{S-10.63}$$

Inserting the experimental values for T and $\dot{T}$ we obtain

$$B_{\text{max}} \approx (9.6\pm 0.25)\times 10^{16}\,\text{G}\,. \tag{S-10.64}$$

S-10.6 A Bent Dipole Antenna

a) If we divide the antenna into a series of infinitesimal resistors, each of length $\mathrm{d}z$ and resistance $\mathrm{d}R = (R/a)\,\mathrm{d}z$, we can write the dissipated power as

$$P_{\text{diss}} = \int\langle I^2\rangle\,\mathrm{d}R = \int_{-a}^{+a}\frac{I_0^2}{2}\left(1 - \frac{|z|}{a}\right)^2\frac{R}{a}\,\mathrm{d}z = \frac{I_0^2 R}{3}\,. \tag{S-10.65}$$

b) The linear charge density on the antenna q_ℓ can be obtained from the continuity equation $\partial_t q_\ell = -\partial_z I$, obtaining

$$q_\ell = \pm\frac{\mathrm{i}I_0}{a\omega}\mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{S-10.66}$$

where the signs $+$ and $-$ apply to $z > 0$ and $z < 0$, respectively. The linear charge density is uniform (independent of z) on each half of the antenna. For symmetry reasons, the only non-vanishing component of the electric dipole $\mathbf{p}$ is along z and it is given by

$$p_z = \int_{-a}^{+a} z\,q_\ell\,\mathrm{d}z = 2\int_0^{+a}\frac{\mathrm{i}I_0}{a\omega}\mathrm{e}^{-\mathrm{i}\omega t}z\,\mathrm{d}z = \frac{\mathrm{i}I_0 a}{\omega}\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-10.67}$$

c) The average radiated power, in the dipole approximation, is

$$P_{\text{rad}} = \frac{1}{3c^3}\left\langle|\ddot{p}_z|^2\right\rangle = \frac{I_0^2 a^2 \omega^2}{6c^3} = \frac{2\pi^2 a^2 I_0^2}{3c\lambda^2} \ , \tag{S-10.68}$$

where $\lambda = 2\pi c/\omega$ is the radiation wavelength. Thus

$$\frac{P_{\text{rad}}}{P_{\text{diss}}} = \frac{2\pi^2 a^2}{c\lambda^2 R} \ , \tag{S-10.69}$$

where we recall that R has the dimensions of the inverse of a velocity in Gaussian units.

d) The angular distribution of the radiated power is proportional to $\sin^2\theta$, where θ is the angle between the observation direction and $\mathbf{p}$. Thus the emitted radiation intensity is zero along the z axis and maximum for observation in the xy plane.

e) The bent antenna has a linear charge density $\pm(iI_0/a\omega)\mathrm{e}^{-i\omega t}$ on its horizontal and vertical arms, respectively. Thus the electric dipole moment has two components

$$p_x = \int_0^a \frac{iI_0}{a\omega}\mathrm{e}^{-i\omega t}x\,\mathrm{d}x = \frac{iI_0 a}{2\omega}\mathrm{e}^{-i\omega t} \ , \tag{S-10.70}$$

$$p_z = -\int_{-a}^0 \frac{iI_0}{a\omega}\mathrm{e}^{-i\omega t}z\,\mathrm{d}z = \frac{iI_0 a}{2\omega}\mathrm{e}^{-i\omega t} \ . \tag{S-10.71}$$

Since the components are perpendicular to each other, the cycle-averaged radiated power can be calculated as the sum of the powers from each dipole:

$$P_{\text{rad}} = \frac{1}{3c^3}\left\langle|\ddot{p}_x|^2 + |\ddot{p}_z|^2\right\rangle = \frac{1}{12c^3}(I_0\omega a)^2 \ , \tag{S-10.72}$$

which is one half of the value for the linear antenna, while the dissipated power P_{diss} does not change.

The electric dipole of the bent antenna lies along the diagonal direction, which thus corresponds to the direction of zero emitted intensity. The intensity is maximum in the plane perpendicular to the dipole.

S-10.7 A Receiving Circular Antenna

a) We choose a Cartesian reference frame such that the wave is propagating in the z direction, its electric field $\mathbf{E}$ is along the x axis, and its magnetic field $\mathbf{B}$ is along the y axis. The current I flowing in the antenna is $I = \mathcal{E}_{\text{circ}}/R$, where $\mathcal{E}_{\text{circ}} = -(1/c)\,\mathrm{d}\Phi(\mathbf{B})/\mathrm{d}t$ is the electromotive force, and $\Phi(\mathbf{B})$ is the flux of $\mathbf{B}$ through the circle delimited by the antenna. Since we have assumed $\lambda \gg a$, $\mathbf{B}$ is practically uniform over the whole surface of the circle, and $\Phi(\mathbf{B}) \simeq \pi a^2 \mathbf{B}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is unit

vector perpendicular to the circle surface. Thus the circular antenna must lie on the xz plane in order to maximize $\Phi(\mathbf{B})$. With a proper choice of the time origin the magnetic field on the circle surface can be written as $\mathbf{B} \simeq \hat{\mathbf{y}} B_0 \cos \omega t$, and

$$\mathcal{E}_{\text{circ}} = \pi a^2 \frac{\omega}{c} E_0 \sin(\omega t) , \tag{S-10.73}$$

since $B_0 = E_0$ in Gaussian units.

b) The electromotive force on a linear antenna parallel to the x axis is practically $\mathcal{E}_{\text{lin}} = \ell E_0 \cos(\omega t + \phi)$, where ℓ is rhe length of the antenna and ϕ is a phase angle. The ratio of the average electromotive force of the circular antenna to the average electromotive force of the linear antenna is thus

$$\frac{\langle \mathcal{E}_{\text{circ}} \rangle}{\langle \mathcal{E}_{\text{lin}} \rangle} \simeq \frac{\langle \mathcal{E}_{\text{circ}} \rangle}{E_0 \ell} = \frac{\pi a^2 \omega}{\ell c} = 2\pi^2 \frac{a^2}{\ell \lambda} . \tag{S-10.74}$$

In the range $10^2 \, \text{cm} < \lambda < 10^3 \, \text{cm}$, and with our assumptions $\ell \simeq 50 \, \text{cm}$ and $a \simeq 25 \, \text{cm}$, this ratio varies between 2.5 and 0.25. The circular antenna is more convenient for shorter wavelengths.

c) The radiation emission from the circular antenna is dominated by the magnetic dipole term. The dipole moment of the antenna is

$$\mathbf{m} = \frac{1}{c} \mathcal{I} \pi a^2 \, \hat{\mathbf{n}} , \tag{S-10.75}$$

where $\mathcal{I}$ is the current circulating in the antenna due to the electromotive force induced by the incident wave. The corresponding time-averaged radiated power is

$$\begin{aligned} P_{\text{rad}} &= \frac{2}{3c^3} \left\langle |\ddot{\mathbf{m}}|^2 \right\rangle = \frac{2}{3c^5} (\pi a^2)^2 \omega^4 \left\langle \mathcal{I}^2 \right\rangle \\ &= \frac{(\pi a^2)^4 \omega^6}{3c^7 R^2} E_0^2 . \end{aligned} \tag{S-10.76}$$

In Gaussian units, the intensity of the incoming wave is $I = cE_0^2/4\pi$, and (S-10.76) can be rewritten

$$P_{\text{rad}} = \frac{4\pi(\pi a^2)^4 \omega^6}{3c^8 R^2} I = \frac{2(2\pi)^7 (\pi a^2)^4}{3c^2 R^2 \lambda^6} I . \tag{S-10.77}$$

The factor multiplying I,

$$\sigma_{\text{scatt}} = \frac{2(2\pi)^7 (\pi a^2)^4}{3c^2 R^2 \lambda^6} , \tag{S-10.78}$$

has the dimensions of a surface (R has the dimensions of an inverse velocity in Gaussian units), and is the radiative scattering cross section for our circular antenna, in the magnetic dipole approximation.

The time-averaged power dissipated by Joule heating is $P_{\text{diss}} = R\langle I^2\rangle$, so that

$$\frac{P_{\text{rad}}}{P_{\text{diss}}} = \frac{2}{3c^5 R}(\pi a^2)^2\omega^4 = \frac{2(2\pi)^4(\pi a^2)^2}{3Rc\lambda^4}\,. \tag{S-10.79}$$

S-10.8 Polarization of Scattered Radiation

a) We choose a Cartesian reference frame with the origin located on the scattering particle, and the z axis parallel to the wave vector of $\mathbf{k}$ the incident wave. In order to have complete rotational symmetry around the z axis it is convenient to assume that the incident wave is circularly polarized. The electric field of the incoming wave can thus be written

$$\mathbf{E}_{\text{i}} = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\,e^{ikz - i\omega t}\,. \tag{S-10.80}$$

Thus, the dipole moment of the scatterer is $\mathbf{p} = \alpha\mathbf{E}_{\text{i}} = \alpha E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\,e^{ikz - i\omega t}$.

Because of the rotational symmetry of the problem around the z axis, it is sufficient to consider the scattered radiation with the wave vector $\mathbf{k}_{\text{d}}$ lying in the yz plane and forming an angle θ with the z axis, as shown in Fig. S-10.1. Disregarding a proportionality factor depending on α and θ, the electric field $\mathbf{E}_{\text{d}}$ of the scattered radiation can be written
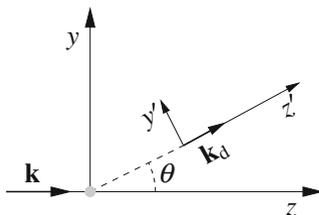

Fig. S-10.1

$$\begin{aligned}\mathbf{E}_{\text{d}} &\propto -(\mathbf{p}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}\\ &\propto -[(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}}\,,\end{aligned} \tag{S-10.81}$$

where $\hat{\mathbf{n}} = (0, \sin\theta, \cos\theta)$ is the unit vector parallel to $\mathbf{k}_{\text{d}}$. Now, recalling that

$$(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\times\hat{\mathbf{n}} = (\pm i\cos\theta, -\cos\theta, \sin\theta), \tag{S-10.82}$$

$$[(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\times\hat{\mathbf{n}}]\times\hat{\mathbf{n}} = (-1, \mp i\cos^2\theta, \pm i\sin\theta\cos\theta), \tag{S-10.83}$$

we fint that

$$\mathbf{E}_{\text{d}} \propto \left(1, \pm i\cos^2\theta, \mp i\sin\theta\cos\theta\right)\,. \tag{S-10.84}$$

Since an observer would measure the polarization of the scattered radiation with respect to the direction $\hat{\mathbf{n}}$, we calculate the components of the field in the rotated

coordinate system (x', y', z'), rotated by an angle θ around the x axis, so that $x' = x$ and z' is along $\hat{\mathbf{n}}$:

$$E'_{dx} = E_{dx} \propto 1, \tag{S-10.85}$$

$$E'_{dy} = E_{dy}\cos\theta - E_{dz}\sin\theta \propto \pm I_i\cos^3\theta \pm i\sin^2\theta\cos\theta$$

$$= \pm i\cos\theta\left(\cos^2\theta + \sin^2\theta\right) = \pm i\cos\theta, \tag{S-10.86}$$

$$E'_{dz} = E_{dy}\sin\theta + E_{dz}\cos\theta \propto \pm i\sin\theta\cos^2\theta \mp i\sin\theta\cos^2\theta = 0. \tag{S-10.87}$$

The last equality is a check that the radiation field is transverse. We thus obtain

$$\mathbf{E}_d \propto \hat{\mathbf{x}}' \pm i\cos\theta\,\hat{\mathbf{y}}', \tag{S-10.88}$$

which gives the dependence of the polarization on the scattering angle θ. In addition, the angular distribution or the radiated power is given by

$$\frac{dP_{\text{scatt}}}{d\Omega} \propto |\mathbf{E}_d|^2 \propto 1 + \cos^2\theta. \tag{S-10.89}$$

b) The radiation from most sources (sunlight is a typical example) is usually incoherent. This means that its phase and electric field direction change randomly at time intervals not much longer than the oscillation period. Thus, the radiation is effectively unpolarized at direct observation, in the sense that it is not possible to measure a definite polarization because of its fast variations. However, (S-10.88) shows that, independently of the source polarization, the radiation scattered at 90° ($\cos\theta = 0$) is always linearly polarized (in the direction perpendicular to both the wave vector of the incoming light and the observation direction). Hence, incoherent radiation that has undergone scattering (as the blue light from the sky) tends to be polarized, even if the radiation from the primary source (in this case the sun) is unpolarized. A measurement of the polarization might help, then, to localize the position of the Sun on a cloudy day.

S-10.9 Polarization Effects on Thomson Scattering

a) Equation (10.7) leads to the following two equations for the velocity components of the electron, v_x and v_y,

$$m_e\dot{v}_x = -eE_0\cos\theta\cos(kz - \omega t), \quad m_e\dot{v}_x = -eE_0\sin\theta\sin(kz - \omega t), \tag{S-10.90}$$

where m_e is the electron mass. We search for a steady-state solution of the form

$$v_x = V_{0x}\sin(kz - \omega t) , \quad v_y = V_{0y}\cos(kz - \omega t) , \tag{S-10.91}$$

with V_{0x} and V_{0y} two real constants to be determined. Substituting into (S-10.90) we obtain

$$V_{0x} = \frac{eE_0\cos\theta}{m_e\omega} , \quad V_{0y} = -\frac{eE_0\sin\theta}{m_e\omega} . \tag{S-10.92}$$

The second derivative of the electric dipole moment of the electron with respect to time is

$$\ddot{\mathbf{p}} = -e\dot{\mathbf{v}} = -\frac{e^2E_0}{m_e}\left[(\hat{\mathbf{x}}\cos\theta\cos(kz - \omega t) + \hat{\mathbf{y}}\sin\theta\sin(kz - \omega t)\right] , \tag{S-10.93}$$

and the electron radiates at frequency ω. The polarization for scattered radiation propagating in a generic direction of unit vector $\hat{\mathbf{n}}$ direction is parallel to the projection of the dipole moment onto the plane perpendicular to $\hat{\mathbf{n}}$, i.e., to $\mathbf{p}_\perp = (\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}$. Thus, we observe linear polarization parallel to $\hat{\mathbf{y}}$ for the radiation emitted along $\hat{\mathbf{x}}$, and linear polarization parallel to $\hat{\mathbf{x}}$ for the radiation emitted along $\hat{\mathbf{y}}$, and elliptical polarization for the radiation emitted along $\hat{\mathbf{z}}$.

If $0 < \theta < \pi/4$, so that $\sin\theta < \cos\theta$, we choose the observation-direction unit vector $\hat{\mathbf{n}} = (\sin\psi, 0, \cos\psi)$, lying in the xz plane, and forming an angle ψ with the z axis, as shown in Fig. S-10.2, where $\mathbf{k}_i$ is the wave vector of the incident wave. Now we choose a Cartesian reference frame x', y', z', with $y' \equiv y$ and z' along $\hat{\mathbf{n}}$, so that the scattered radiation of interest is propagating along z'.

If we perform an orthogonal projection onto the $x'y'$ plane of an ellipse lying on the xy plane, of half axes $\cos\theta$ parallel to x, and $\sin\theta$ parallel to y, we obtain an ellipse of half-axes $\cos\theta\cos\psi$ along x', and $\sin\theta$ along y'. Thus we observe a circular polarization if $\cos\theta\cos\psi = \sin\theta$, i.e., if $\cos\psi = \tan\theta$. Analogously, if $\pi/4 < \theta < \pi/2$, so that $\sin\theta > \cos\theta$, we choose the observation-direction unit vector $\hat{\mathbf{n}} = (0, \sin\psi, \cos\psi)$, lying in the yz plane, and we observe circular polarization if $\sin\theta\cos\psi = \cos\theta$, i.e., if $\cos\psi = \cot\theta$. **b)** The average total scattered power is
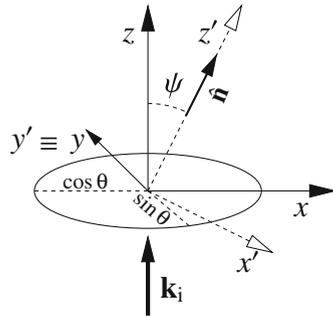

Fig. S-10.2

$$P = \frac{2}{3c^3}\left\langle |\ddot{\mathbf{p}}|^2 \right\rangle = \frac{2e^4}{3m_e^2c^3}\left\langle |\mathbf{E}|^2 \right\rangle , \tag{S-10.94}$$

where

$$\langle|\mathbf{E}|^2\rangle = \langle E_x^2 + E_y^2\rangle = \frac{1}{2}E_0^2\left(\cos^2\theta + \sin^2\theta\right) = \frac{1}{2}E_0^2 . \tag{S-10.95}$$

Thus, the total scattered power is independent of θ and can be written as

$$P = \frac{e^4 E_0^2}{3m_e^2 c^3} = \frac{cE_0^2}{3}r_e^2 = \frac{4\pi}{3}r_e^2 I , \tag{S-10.96}$$

where

$$r_e = \frac{e^2}{m_e c^2} , \quad \text{and} \quad I = \frac{cE_0^2}{4\pi} ,$$

are the classical electron radius and the intensity of the incident wave, respectively.
c) The magnetic field of the wave is

$$\mathbf{B} = E_0\left[-\hat{\mathbf{x}}\sin\theta\sin(kz - \omega t) + \hat{\mathbf{y}}\cos\theta\cos(kz - \omega t)\right] . \tag{S-10.97}$$

The only non-vanishing component of $\mathbf{v}\times\mathbf{B}$ is in the $\hat{\mathbf{z}}$ direction, and the magnetic force on the electron can be written as

$$\begin{aligned}
F_z &= -\frac{e}{c}(\mathbf{v}\times\mathbf{B})_z = -\frac{e}{c}(v_x B_y - v_y B_x) \\
&= -\frac{e^2 E_0^2}{2cm_e\omega}\left(\cos^2\theta - \sin^2\theta\right)\sin(2kz - 2\omega t) ,
\end{aligned} \tag{S-10.98}$$

this quantity vanishes for $\theta = \pi/4$, when $\cos\theta = \sin\theta$, i.e., for circular polarization.
d) The magnetic force F_z drives dipole oscillations along the z axis at frequency 2ω. Thus, in addition to the scattered radiation of frequency ω discussed at points a) and b), we observe also scattered radiation of frequency 2ω, angularly distributed as $\sin^2\psi$ around the z axis. Since the dipole oscillating at 2ω is perpendicular to the dipole oscillating at ω, we can simply add the corresponding scattered powers. Now we want to evaluate the power emitted at frequency 2ω.

The equation of motion for the electron along the z axis is (we put $\cos^2\theta - \sin^2\theta = \cos 2\theta$)

$$m_e \dot{v}_z = F_z = -\frac{e^2 E_0^2}{2cm_e\omega}\cos 2\theta\sin(2kz - 2\omega t) . \tag{S-10.99}$$

Once more, we search for a steady-state solution of the form

$$v_z = V_{0z}\cos(2kz - 2\omega t) , \tag{S-10.100}$$

with V_{0z} a constant. Substituting into (S-10.99) we obtain

$$V_{0z} = -\frac{e^2 E_0^2}{4cm_e^2\omega^2}\cos(2\theta) \tag{S-10.101}$$

and

$$\dot{v}_z = -\frac{e^2 E_0^2}{2cm_e^2\omega}\cos(2\theta)\sin(2kz - 2\omega t) . \tag{S-10.102}$$

The total average power emitted by the dipole oscillating at 2ω is

$$
\begin{aligned}
P_{2\omega} &= \frac{2}{3c^3}\left\langle|\ddot{\mathbf{p}}_{2\omega}|^2\right\rangle = \frac{2}{3c^3}\left\langle|e\dot{v}_z|^2\right\rangle = \frac{e^6 E_0^4}{12c^5 m_e^4 \omega^2}\cos^2(2\theta) \\
&= \frac{4\pi}{3}\frac{e^2 E_0^2}{4c^2 m_e^2\omega^2}\cos^2(2\theta)r_e^2 I \\
&= \frac{4\pi}{3}\frac{V_{0z}}{c}\cos^2(2\theta)r_e^2 I .
\end{aligned}
\tag{S-10.103}
$$

S-10.10 Scattering and Interference

a) With a proper choice of the time origin, the electric field of the incident plane wave at $x = \pm d/2$ can be written as

$$\mathbf{E}_i\left(\pm\frac{d}{2}, t\right) = E_0\, e^{\pm ikd/2 - i\omega t}\,\hat{\mathbf{z}} , \tag{S-10.104}$$

and the phase difference between the two scatterers is

$$\phi_+ - \phi_- = kd . \tag{S-10.105}$$

We denote by $r_\pm$ the optical paths between the observation point P and the scatterers located at $(\pm d/2, 0, 0)$, as shown in Fig. 10.5. The difference between the two optical paths is

$$\Delta r = (r_+ - r_-) \simeq -d\sin\theta , \tag{S-10.106}$$

where θ is the angle between the y axis and the line joining the origin to P, as shown in Fig. 10.5. The approximation is valid for $L \gg d$. The phase difference between the two scattered waves in P is obtained by combining (S-10.105) and (S-10.106),

$$\Delta\phi = kd(1 - \sin\theta) . \tag{S-10.107}$$

b) If we neglect the difference between the magnitudes of the scattered fields $\mathbf{E}_+$ and $\mathbf{E}_-$ in P, $\mathbf{E}_\pm$ being the field of the wave scattered at $(\pm d/2, 0, 0)$, the total scattered intensity I_s in P is proportional to

$$I_s \propto |\mathbf{E}_+ + \mathbf{E}_-|^2 \propto |\mathbf{E}_+|^2 \left| e^{ikd(1-\sin\theta)/2} + e^{-ikd(1-\sin\theta)/2} \right|^2$$

$$\propto \frac{1}{r^2} \cos^2\left[\frac{kd}{2}(1-\sin\theta) \right]. \qquad \text{(S-10.108)}$$

Since $r\cos\theta = L$, we can also write

$$I_s \propto \frac{\cos^2\theta}{L^2} \cos^2\left[\frac{kd}{2}(1-\sin\theta) \right]. \qquad \text{(S-10.109)}$$

We denote by $u = (kd/2)(1-\sin\theta)$ the argument of the second $\cos^2$ appearing in (S-10.109). For $-\pi/2 \leqslant \theta \leqslant \pi/2$ the variable u varies continuously and monotonically from kd to 0. If $kd \ll 1$ (i.e., if $d \ll \lambda/2\pi$), then $\cos^2 u \simeq 1$ e $I_s(\theta) \sim \cos^2\theta$, as if a single scatterer was present. If $kd < \pi/2$ the function $\cos^2 u$ has no zeros, meaning that interference fringes are not observed if the distance between the scatterers is less than $\lambda/4$. If

$$\frac{\pi}{2} < \frac{kd}{2} < (n+1)\frac{\pi}{2} \,,$$

with n an integer number and $n \geqslant 1$, the function $\cos^2 u$ has n zeros, and one observes n scattered-intensity minima and $n+1$ maxima as θ varies from $-\pi/2$ to $+\pi/2$. The intensity of the maxima is modulated by the function $\cos^2\theta$.

S-10.11 Optical Beats Generating a "Lighthouse Effect"

a) On the $z = 0$ plane the electric fields $\mathbf{E}_\pm$ emitted by the two dipoles are parallel to $\hat{\mathbf{z}}$ (perpendicular to the plane), and their amplitudes are independent of ϕ. Since for each dipole $\mathbf{E}_\pm \propto -\omega_\pm^2 \mathbf{p}_0$, the field amplitudes are $E_+ \simeq E_-$, equal to each other up to the first order in $\delta\omega/\omega_0$. The difference between the optical paths from the two dipoles to P is $\delta r \simeq d\sin\phi = (\pi c/\omega_0)\sin\phi$, which yields a phase difference of $\pi\sin\phi$. The total field may be thus written as

$$\begin{aligned} E &= E_0 \cos[(\omega_0 + \delta\omega/2)t + \pi\sin\phi/2] + E_0 \cos[(\omega_0 - \delta\omega/2)t - \pi\sin\phi/2] \\ &= 2E_0 \cos(\omega_0 t)\cos(\delta\omega t + \pi\sin\phi) \,. \qquad \text{(S-10.110)} \end{aligned}$$

b) The EM energy flux in the radiation zone is given by Poynting's vector $\mathbf{S}$, which is proportional to the square modulus of the electric field. Thus

$$S \propto 4\cos^2(\omega_0 t)\cos^2(\delta\omega t + \pi\sin\phi). \qquad \text{(S-10.111)}$$

Using the "fast", or "instantaneous", detector, only the factor $\cos^2(\omega_0 t)$ is averaged, and the measured signal is proportional to

$$\langle S \rangle \propto 2\cos^2(\delta\omega t + \pi\sin\phi)\,, \quad \text{since} \quad \left\langle \cos^2(\omega_0 t) \right\rangle = \frac{1}{2}\,. \qquad \text{(S-10.112)}$$

At time t, the direction of maximum flux intensity is determined by the condition

$$\delta\omega t + \pi \sin\phi = \begin{cases} 0 \\ \pi \end{cases},$$ (S-10.113)

which means that the direction of maximum flux ϕ_{max} *rotates* in the $z = 0$ plane, similarly to a lighthouse beam, according to

$$\phi_{\mathrm{max}}(t) = \arcsin\left(-\frac{\delta\omega}{\pi} t\right).$$ (S-10.114)

If the EM flux is measured with the "slow" detector, i.e., averaging over times longer than $2\pi/\delta\omega$, both $\cos^2$ terms of (S-10.111) are averaged to $1/2$, and the total flux is the sum of the two independent fluxes from the two dipoles.

c) Now the observation point P is on the $x = 0$ plane, at a distance $\mathbf{r}$ from the origin, as in Fig. S-10.3. The angle between the z axis and $\mathbf{r}$ is θ. Within our approximations, the intensities of the two electric fields $\mathbf{E}_+$ and $\mathbf{E}_-$ in P are equal and proportional to $\sin\theta$. Thus the two separate intensities are dependent

Fig. S-10.3

on θ, while they are independent of ϕ on the $z = 0$ plane. The amplitude of the Poynting vector is proportional to
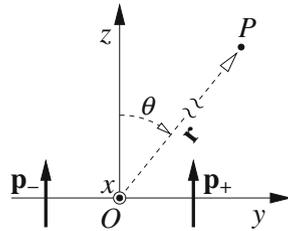
$$S \propto 2\sin^2\theta \cos^2(\omega_0 t)\cos^2(\delta\omega t + \pi \sin\theta).$$ (S-10.115)

Thus the "fast" detector still measures a rotation of the direction of maximum emission, but the intensity is modulated by a $\sin^2\theta$ factor.

S-10.12 Radiation Friction Force

a) We insert (10.11) for $\mathbf{F}_{\mathrm{rad}}$ into (10.9), obtaining

$$\int_t^{t+T} \mathbf{F}_{\mathrm{rad}}(t) \cdot \mathbf{v}(t)\,\mathrm{d}t = m_{\mathrm{e}}\tau \int_t^{t+T} \frac{\mathrm{d}^2\mathbf{v}(t)}{\mathrm{d}t^2} \cdot \mathbf{v}(t)\mathrm{d}t$$

$$= m_{\mathrm{e}}\tau \left[\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t} \cdot \mathbf{v}(t)\right]_t^{t+T} - m_{\mathrm{e}}\tau \int_t^{t+T} \left|\frac{\mathrm{d}\mathbf{v}(t)}{\mathrm{d}t}\right|^2 \mathrm{d}t,$$ (S-10.116)

where we have used integration by parts in the second line. The first term vanishes since the motion is periodic[1]:

$$\left[\frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t}\cdot\boldsymbol{v}(t)\right]_t^{t+T} = \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}v^2(t+T) - \frac{\mathrm{d}}{\mathrm{d}t}v^2(t)\right] = 0 . \qquad \text{(S-10.117)}$$

We thus obtain

$$\int_t^{t+T} \mathbf{F}_{\mathrm{rad}}(t)\cdot\boldsymbol{v}(t)\,\mathrm{d}t = -m_{\mathrm{e}}\tau\int_t^{t+T}\left|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|^2\,\mathrm{d}t . \qquad \text{(S-10.118)}$$

Substituting Larmor's formula (10.10) into the right-hand side of (10.9) we obtain

$$-\int_0^t P_{\mathrm{rad}}(t')\,\mathrm{d}t' = -\int_0^t \frac{2e^2}{3c^2}\left|\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t'}\right|^2\,\mathrm{d}t' , \qquad \text{(S-10.119)}$$

and (10.9) is verified if we choose

$$\tau = \frac{2e^2}{3m_{\mathrm{e}}c^2} . \qquad \text{(S-10.120)}$$

Apart from the 2/3 factor, τ is the time needed by light to travel a distance equal to the classical electron radius $r_{\mathrm{e}} = 2.82\times10^{-13}$ cm, and we have $\tau \sim 10^{-23}$ s.
b) After substituting (10.12) into (10.8), we search for a steady-state solution of the form $\boldsymbol{v}(t) = \boldsymbol{v}_0\,\mathrm{e}^{-\mathrm{i}\omega t}$, and find

$$\boldsymbol{v}_0 = -\frac{\mathrm{i}\,e\mathbf{E}_0}{m_{\mathrm{e}}\omega(1+\mathrm{i}\omega\tau)} . \qquad \text{(S-10.121)}$$

Analogously, the steady-state solution of (10.13) is

$$\boldsymbol{v}_0 = -\frac{\mathrm{i}\,e\mathbf{E}_0}{m_{\mathrm{e}}\omega\left(1+\mathrm{i}\dfrac{\eta}{\omega}\right)} . \qquad \text{(S-10.122)}$$

The two solutions are identical if we choose $\eta = \omega^2\tau$. The same result can be obtained by a direct comparison of F_{rad} to the frictional force $-m_{\mathrm{e}}\eta\boldsymbol{v}$.

   Equation (10.8) represents the first attempt to derive an expression for the "radiation friction" or "radiation reaction" force which is deeply related to the back-action of the electron on itself, since the electron interacts with the electric field it generates (self-force9. However. (10.8) is considered unsatisfactory for for two reasons: (i) it increases the order of the equation of motion, and, consequently, one needs a further initial condition for the acceleration; and ii) it has unphysical "runaway"

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[1]Actually it is not strictly necessary for the motion to be periodic, it is sufficient that $\mathrm{d}v^2(t)\mathrm{d}t$ vanishes at the initial and final instants of the time interval considered.

solutions in the absence of an external field, such as $\mathbf{a}(t) = \mathbf{a}_0 \, \mathrm{e}^{t/\tau}$ with $\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t$. This problem has a long and still open history. Additional discussion may be found in textbooks and in the literature, also in very recent works related to highly relativistic electrons in ultraintense laser fields (for which the radiation friction effect becomes important).