

Chapter S-11

Solutions for Chapter 11

S-11.1 Wave Propagation in a Conductor at High and Low Frequencies

a) We determine the conductivity of the metal by searching for a steady-state solution in complex form, $\mathbf{v} = \tilde{\mathbf{v}} e^{-i\omega t}$, of (11.5) in the presence of an oscillating electric field $\mathbf{E}(\mathbf{r}, t) = \tilde{\mathbf{E}} e^{-i\omega t}$. We find

$$\tilde{\mathbf{v}} = -\frac{ie}{m_e(\omega + i\eta)} \tilde{\mathbf{E}}, \tag{S-11.1}$$

corresponding to a current density

$$\tilde{\mathbf{J}} = -en_e \tilde{\mathbf{v}} = \frac{ie^2 n_e}{m_e \omega(\omega + i\eta)} \tilde{\mathbf{E}} = \frac{i\omega_p^2}{4\pi(\omega + i\eta)} \tilde{\mathbf{E}} \equiv \sigma(\omega) \tilde{\mathbf{E}}, \tag{S-11.2}$$

where ω_p is the plasma frequency of the metal. At the limits of high frequencies $\omega \gg \eta$, and of low frequencies $\omega \ll \eta$, we have

$$\sigma(\omega) \simeq \begin{cases} \frac{i\omega_p^2}{4\pi\omega}, & \text{for } \omega \gg \eta, \\ \frac{\omega_p^2}{4\pi\eta}, & \text{for } \omega \ll \eta. \end{cases} \tag{S-11.3}$$

The DC conductivity is thus $\sigma_{\text{DC}} = \sigma(0) = \omega_p^2/4\pi\eta$. In a metal, typically we have $\omega_p \sim 10^{16} \text{ s}^{-1}$, since $n_e \sim 10^{23} \text{ cm}^{-3}$ and $\eta \sim 10^{13} \text{ s}^{-1}$. It is thus a very good approximation to assume σ to be purely imaginary for optical frequencies, i.e., for $\omega \sim 10^{15} \text{ s}^{-1}$, and to be purely real and equal to σ_{DC} (i.e. independent of frequency) for microwaves and longer wavelengths.

b) Assuming plane geometry and monochromatic waves, in the absence of sources at $x = +\infty$, the electric field of the wave for $x > 0$ can be written as (in complex notation)

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{ikx - i\omega t}, \quad (\text{S-11.4})$$

where the wave vector k is determined by the general dispersion relation (11.4) in a medium where the refractive index $n = n(\omega)$ or, equivalently, the permittivity $\varepsilon = \varepsilon(\omega) = n^2$ are known. For an incident wave of amplitude \mathbf{E}_i , the electric field at the surface is given by the Fresnel formula

$$\mathbf{E}_t = \frac{2}{1+n} \mathbf{E}_i. \quad (\text{S-11.5})$$

The permittivity $\varepsilon(\omega)$ is related to the complex conductivity of the medium by (11.3). Inserting (S-11.3) for $\sigma(\omega)$, if $\omega \gg \eta$ we have $\varepsilon \approx 1 - \omega_p^2/\omega^2$, and k^2 is real, so that the wave is propagating. For $k^2 < 0$, i.e., for $\omega_p > \omega$, we have $ikx = -|k|x = -x/\ell_p$, with $\ell_p = c/\sqrt{\omega_p^2 - \omega^2}$, and the wave is evanescent:

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{-x/\ell_p - i\omega t} \quad (\text{S-11.6})$$

(the solution $\propto e^{x/\ell_p}$ has been disregarded as unphysical because it is divergent for $x \rightarrow \infty$). For a metal, the condition $\omega < \omega_p$ implies that the metal is reflecting for frequencies in the optical range, while it becomes transparent for ultraviolet frequencies.

If $\omega \ll \eta$, we have that also $\sigma_{\text{DC}} \ll \eta$, so that $\varepsilon \approx 4\pi i \sigma_{\text{DC}}/\omega$ is an imaginary number. In this case, since $k = \pm(1+i)/\ell_c$ with $\ell_c = \sqrt{\omega \sigma_{\text{DC}}/2c}$, the evanescent solution is

$$\mathbf{E}(x, t) = \mathbf{E}_t e^{-x/\ell_c - ix/\ell_c - i\omega t}. \quad (\text{S-11.7})$$

c) The net flux of energy through the surface is given by the time average of the x -component of the Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ at $x = 0$. We obtain the magnetic field of the wave from the relation $\partial_t \mathbf{B} = -c\nabla \times \mathbf{E}$. Thus the complex field amplitudes for $x \geq 0$ can be written as

$$\tilde{E}_y = E_t e^{ik_0 n x}, \quad \tilde{B}_z = n E_t e^{ik_0 n x}, \quad (\text{S-11.8})$$

where $k_0 = \omega/c$. Thus we need to evaluate

$$\langle S_x(0) \rangle = \frac{1}{2} \frac{c}{4\pi} \text{Re} \left[\tilde{E}_y(0) \tilde{B}_z^*(0) \right] = \frac{c}{8\pi} |E_t|^2 \text{Re}(n). \quad (\text{S-11.9})$$

At the limit $\omega \gg \eta$, n is purely imaginary, as found above, and $\langle S_x(0) \rangle = 0$, and there is no energy dissipated into the metal (it can be easily shown that the reflection coefficient obtained from the Fresnel formulas has unity modulus, i.e., all the incident energy is reflected). At the limit $\omega \gg \eta$ we obtain

$$\langle S_x(0) \rangle = \frac{c}{8\pi} |E_t|^2 \sqrt{\frac{2\pi\sigma_{\text{DC}}}{\omega}} \simeq \frac{c}{16\pi} |E_t|^2 \sqrt{\frac{\omega}{2\pi\sigma_{\text{DC}}}}, \quad (\text{S-11.10})$$

where in the latter expression $|1 + n|^2 \simeq |n|^2 = 2(2\pi\sigma_{\text{DC}}/\omega)$ has been assumed.

The energy dissipated per unit volume is

$$\begin{aligned} \langle \mathbf{J} \cdot \mathbf{E} \rangle &= \frac{1}{2} \text{Re}(\sigma \tilde{E}_y \tilde{E}_y^*) = \frac{|E_t|^2}{2} \text{Re}(\sigma e^{ik_0 n x} e^{-ik_0 n^* x}) \\ &= \frac{|E_t|^2}{2} \text{Re}(\sigma) \exp[-2k_0 \text{Im}(n)x]. \end{aligned} \quad (\text{S-11.11})$$

If σ is imaginary then there is no dissipation, consistently with what found above. In the $\omega \ll \eta$ regime, the total energy dissipated per unit surface is given by the integral

$$\begin{aligned} \int_0^\infty \langle \mathbf{J} \cdot \mathbf{E} \rangle dx &= \frac{|E_t|^2}{2} \frac{\sigma_{\text{DC}}}{2k_0 \text{Im}(n)} = \frac{|E_t|^2}{2} \frac{\sigma_{\text{DC}}}{2\omega_c \sqrt{2\pi\sigma_{\text{DC}}/\omega}} \\ &= \frac{c}{8\pi} |E_t|^2 \sqrt{\frac{2\pi\sigma_{\text{DC}}}{\omega}}, \end{aligned} \quad (\text{S-11.12})$$

which is equal to the EM energy flux of (S-11.10).

S-11.2 Energy Densities in a Free Electron Gas

a) We use the complex representation for all fields, $A(x, t) = \text{Re}(\tilde{A}e^{ikx - i\omega t})$, where A is the considered field. For the electric field of the wave we have $\tilde{\mathbf{E}} = \mathbf{E}_0$, where E_0 can be considered as a real quantity. The equation of motion for an electron, neglecting the nonlinear magnetic term, is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = m_e \frac{d\mathbf{v}}{dt} = -e\mathbf{E}, \quad (\text{S-11.13})$$

which has the steady-state solution for the electron velocity and position

$$\tilde{\mathbf{v}} = -\frac{ie}{m_e \omega} \mathbf{E}_0, \quad \tilde{\mathbf{r}} = \frac{e}{m_e \omega^2} \mathbf{E}_0. \quad (\text{S-11.14})$$

The polarization density is

$$\tilde{\mathbf{P}} = -en_e \tilde{\mathbf{r}} = -\frac{n_e e^2}{m_e \omega^2} \mathbf{E}_0 = -\frac{1}{4\pi} \frac{\omega_p^2}{\omega^2} \mathbf{E}_0 \equiv \chi(\omega) \mathbf{E}_0, \quad (\text{S-11.15})$$

corresponding to a dielectric permittivity of the medium

$$\varepsilon(\omega) = 1 + 4\pi\chi(\omega) = 1 - \frac{\omega_p^2}{\omega^2}. \quad (\text{S-11.16})$$

Using (11.4), the dispersion relation is obtained as

$$\omega^2 = \frac{k^2 c^2}{\varepsilon(\omega)} = \omega_p^2 + k^2 c^2. \quad (\text{S-11.17})$$

The phase and group velocities are

$$v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}, \quad v_g = \frac{\partial \omega}{\partial k} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \quad (\text{S-11.18})$$

so that both v_φ and v_g are real if $\omega > \omega_p$, and $v_\varphi v_g = c^2$. Finally, using the equation $c \nabla \times \mathbf{E} = -\partial_t \mathbf{B}$. i.e., $ikc \tilde{\mathbf{E}} = i\omega \tilde{\mathbf{B}}$, we obtain $E_0 = (v_\varphi/c) B_0$.

b) From the definition of the EM energy density

$$\begin{aligned} u_{\text{EM}} &= \left\langle \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \right\rangle = \frac{1}{16\pi} (E_0^2 + B_0^2) = \frac{1}{16\pi} E_0^2 \left(1 + \frac{c^2}{v_\varphi^2} \right) \\ &= \frac{1}{16\pi} E_0^2 \left(2 - \frac{\omega_p^2}{\omega^2} \right). \end{aligned} \quad (\text{S-11.19})$$

c) From the definition of the kinetic energy density

$$\begin{aligned} u_{\text{K}} &= \left\langle n_e \frac{m_e}{2} \mathbf{v}^2 \right\rangle = n_e \frac{m_e}{2} \frac{1}{2} \left| \frac{eE_0}{m_e \omega} \right|^2 = \frac{1}{4} \frac{n_e e^2}{m_e \omega^2} E_0^2 \\ &= \frac{1}{16\pi} E_0^2 \frac{\omega_p^2}{\omega^2}. \end{aligned} \quad (\text{S-11.20})$$

Thus

$$u = u_{\text{EM}} + u_{\text{K}} = \frac{1}{8\pi} E_0^2, \quad (\text{S-11.21})$$

independently of n_e .

d) In our case (11.6) can be rewritten

$$v_g E_t^2 = c(E_0^2 - E_r^2). \quad (\text{S-11.22})$$

Using Fresnel formulas as functions of the phase velocity $v_\varphi = c/n$, with $n = \sqrt{\varepsilon_r}$, we obtain

$$E_r = \frac{v_\varphi - c}{v_\varphi + c} E_0, \quad E_t = \frac{2v_\varphi}{v_\varphi + c} E_0, \quad (\text{S-11.23})$$

leading to

$$4v_g v_\varphi^2 = 4c^2 v_\varphi, \quad (\text{S-11.24})$$

which is equivalent to $v_g v_\varphi = c^2$.

S-11.3 Longitudinal Waves

a) We obtain from Maxwell's equations, assuming $\mathbf{B} = 0$,

$$0 = \nabla \times \mathbf{B} = \frac{1}{c} (4\pi \mathbf{J} + \partial_t \mathbf{E}) = \frac{1}{c} (4\pi \partial_t \mathbf{P} + \partial_t \mathbf{E}) = \frac{1}{c} \partial_t (4\pi \mathbf{P} + \mathbf{E}). \quad (\text{S-11.25})$$

where \mathbf{P} is the polarization density of the medium and $\mathbf{J} = \partial_t \mathbf{P}$ the associated polarization current. Assuming all fields to have an harmonic dependence $\sim e^{-i\omega t}$, we have $\mathbf{P} = \chi(\omega) \mathbf{E}$ with $\chi = [\varepsilon_r(\omega) - 1]/(4\pi)$. Now, using (11.7), we can write

$$0 = -i\omega(4\pi \mathbf{P} + \mathbf{E}) = -i\omega\{[\varepsilon_r(\omega) - 1]\mathbf{E} + \mathbf{E}\} = -i\omega \varepsilon_r(\omega) \mathbf{E}, \quad (\text{S-11.26})$$

implying $\varepsilon_r(\omega) = 0$.

b) The *total* charge and current densities in the medium can be obtained from \mathbf{E} using the equations

$$\varrho = \frac{1}{4\pi} \nabla \cdot \mathbf{E}, \quad \mathbf{J} = -\frac{1}{4\pi} \partial_t \mathbf{E}, \quad (\text{S-11.27})$$

which also imply the continuity equation $4\pi \partial_t \varrho = -\nabla \cdot \mathbf{J}$. For \mathbf{E} given by (11.7) we obtain

$$\varrho = \frac{ik}{4\pi} E_0 e^{ikx - i\omega t}, \quad \mathbf{J} = \hat{\mathbf{x}} \frac{i\omega}{4\pi} E_0 e^{ikx - i\omega t}. \quad (\text{S-11.28})$$

c) Assuming electrons moving with negligible friction, the equation of motion for the single electron is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = -m_e \omega_0^2 \mathbf{r} - e \mathbf{E}, \quad (\text{S-11.29})$$

where m_e is the electron mass, and \mathbf{r} is the distance of the electron from its equilibrium position. For a monochromatic field $\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$ the stationary solution is

$$\mathbf{r} = \frac{e \mathbf{E}}{m_e (\omega^2 - \omega_0^2)}. \quad (\text{S-11.30})$$

The polarization density of the medium is

$$\mathbf{P} = -en_e \mathbf{r} = -\frac{n_e e^2}{m_e (\omega^2 - \omega_0^2)} \mathbf{E} \equiv \chi(\omega) \mathbf{E}, \quad (\text{S-11.31})$$

where n_e is the number of electrons per unit volume, and

$$\chi(\omega) = -\frac{n_e e^2}{m_e (\omega^2 - \omega_0^2)} = -\frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad (\text{S-11.32})$$

is the dielectric susceptibility of the medium, and $\omega_p = \sqrt{4\pi n_e e^2 / m_e}$ is its plasma frequency. The dielectric permittivity is thus

$$\varepsilon_r(\omega) = 1 + 4\pi\chi(\omega) = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2}, \quad (\text{S-11.33})$$

and the longitudinal-wave condition $\varepsilon_r(\omega) = 0$ leads to

$$\omega = \sqrt{\omega_p^2 + \omega_0^2}. \quad (\text{S-11.34})$$

It is important to notice that the wavevector \mathbf{k} is *not* determined by this equation; it may have any value, and the phase velocity may thus be arbitrary (lower or greater than c). Longitudinal waves in condensed matter physics are also called *polaritons*. In a free electron medium where $\omega_0 = 0$ (a simple metal, a ionized gas or a plasma), we have $\omega = \omega_p$; in this case the waves are called plasma waves or *plasmons*.

S-11.4 Transmission and Reflection by a Thin Conducting Foil

a) Since the problem of determining the transmission and reflection coefficient is linear, and the medium is isotropic, the choice of polarization is arbitrary. For definiteness, we assume linear polarization, with the electric field \mathbf{E} of the incoming wave parallel to the y axis, and the magnetic \mathbf{B} parallel to the z axis.

We apply Stokes's theorem to a closed rectangular path C , delimiting a surface area A , twice: once for \mathbf{E} and once for \mathbf{B} . In both cases the base of the path extends from $x = -h/2$ to $x = +h/2$, while the height, of length ℓ , is parallel to the y axis for the electric field \mathbf{E} , as shown in Fig. S-11.1, and to the z axis for the magnetic field \mathbf{B} . For the electric field we have

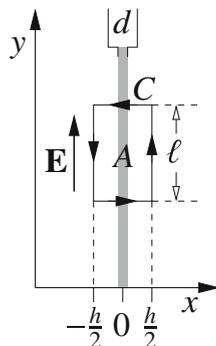


Fig. S-11.1

$$\begin{aligned} \oint_C \mathbf{E} \cdot d\mathbf{l} &= \left[E\left(+\frac{h}{2}\right) - E\left(-\frac{h}{2}\right) \right] \ell \\ &= +i \frac{\omega}{c} \int_A \mathbf{B} \cdot d\mathbf{A} = i \frac{\omega}{c} \bar{B} \ell h, \end{aligned} \quad (\text{S-11.35})$$

where \bar{B} is the amplitude of \mathbf{B} at some point of the surface A , according to the mean value theorem. Since \mathbf{B} is limited, at the limit $h \rightarrow 0$ we have $\bar{B}h \rightarrow 0$, and the first of (11.8) is proved. For the magnetic field we have

$$\begin{aligned} \oint_C \mathbf{B} \cdot d\mathbf{l} &= \left[B\left(+\frac{h}{2}\right) - B\left(-\frac{h}{2}\right) \right] \ell = \int_A \left(\frac{4\pi}{c} \mathbf{J} - i \frac{\omega}{c} \mathbf{E} \right) \cdot d\mathbf{A} \\ &= \int_A \frac{4\pi}{c} K \delta(x) dx dz - i \int_A \frac{\omega}{c} \mathbf{E} \cdot d\mathbf{A} \\ &= \frac{4\pi}{c} \ell \bar{K} + -i \frac{\omega}{c} \bar{E} \ell h, \end{aligned} \quad (\text{S-11.36})$$

where, in the second line, we have replaced \mathbf{J} by $\mathbf{K}\delta(x)$, and, in the third line, \bar{K} is a value assumed by K somewhere on the segment of length ℓ . Since, again, \bar{E} is limited, the product $\bar{E}\ell h \rightarrow 0$ as $h \rightarrow 0$, and the second of (11.8) is proved.

b) The most general expression for the field is the sum of the incident and the reflected wave for $x < 0$, and the transmitted wave only for $x > 0$:

$$E(x, t) = \begin{cases} E_i e^{ikx - i\omega t} + E_r e^{-ikx - i\omega t}, & x < 0, \\ E_t e^{ikx - i\omega t}, & x > 0. \end{cases} \quad (\text{S-11.37})$$

The amplitudes E_r and E_t must be determined as functions of E_i and other parameters, by imposing (11.8) as boundary conditions. Noticing that $K = \sigma d E(0) = \sigma d E_t$ and that $c \partial_x E(x, t) = -\partial_t B(x, t)$, we have

$$E_t - E_i - E_r = 0, \quad E_t - E_i + E_r = -4\pi \frac{\sigma d}{c} E_t, \quad (\text{S-11.38})$$

so that, writing $2\pi\sigma d/c = \eta$ as a shorthand, we have

$$E_r = -\frac{\eta}{1+\eta} E_i, \quad E_t = \frac{1}{1+\eta} E_i. \quad (\text{S-11.39})$$

c) At the limit $\eta \gg \omega$ the conductivity is given by $\sigma = n_e e^2 / m_e \eta$ and is a real number (Ohmic conductor). The mechanical power P is the cycle average of $\mathbf{J} \cdot \mathbf{E}$ integrated over the volume of the foil, thus we obtain (per unit surface)

$$P = \frac{1}{2} |E(0)|^2 d = \frac{1}{2} \frac{\sigma d}{(1+\eta)^2} E_i^2 = \frac{c}{4\pi} \frac{\eta}{(1+\eta)^2} E_i^2, \quad (\text{S-11.40})$$

(notice that E_i can be taken as a real quantity).

At the limit $\eta \ll \omega$ the conductivity is $\sigma = i n_e e^2 / m_e \omega = i \omega_p^2 / 4\pi \omega$ and is thus imaginary, corresponding to a real permittivity $\epsilon = 1 - \omega_p^2 / \omega^2$. Accordingly, \mathbf{J} and \mathbf{E} have opposite phase, and $\langle \mathbf{J} \cdot \mathbf{E} \rangle = 0$, as can be directly verified.

d) The energy flux through the foil is given by the difference between the values of the Poynting flux at the two surfaces (here we switch back to real fields for simplicity),

$$S(0^+) - S(0^-) = \frac{c}{4\pi} [E(0^+)B(0^+) - E(0^-)B(0^-)]. \quad (\text{S-11.41})$$

Inserting the boundary conditions we may write

$$E(0^+)B(0^+) - E(0^-)B(0^-) = E(0)[B(0^+) - B(0^-)] = -E(0) \frac{4\pi}{c} J d, \quad (\text{S-11.42})$$

so that

$$S(0^+) - S(0^-) = -J E(0) d = -K E(0), \quad (\text{S-11.43})$$

i.e., the energy flux through the foil equals the mechanical power dissipated in the foil (all quantities have been defined per unit surface).

Alternatively, we may compute the energy flux directly and compare it to the mechanical power. For the cycle-averaged Poynting vector at the two surfaces we have

$$\langle S(0^+) \rangle = \frac{c}{4\pi} \langle E^2(0^+) \rangle = \frac{2\pi}{c} |E_t|^2 = \frac{2\pi}{c} \frac{1}{|1+\eta|^2} E_i^2, \quad (\text{S-11.44})$$

$$\begin{aligned} \langle S(0^-) \rangle &= \varepsilon_0 c^2 \langle E(0^-) B(0^-) \rangle = \frac{2\pi}{c} \text{Re} \left[(E_i + E_r)(E_i^* - E_r^*) \right] \\ &= \frac{2\pi}{c} \frac{1}{|1+\eta|^2} \text{Re}(2\eta^* + 1) E_i^2. \end{aligned} \quad (\text{S-11.45})$$

If $\eta \ll \omega$, then η is purely imaginary and $S(0^-) = S(0^+)$: there is no net energy flux inside the foil, consistently with the vanishing of the mechanical power.

If $\eta \gg \omega$, then η is real and the net flux of energy is

$$\langle S(0^+) \rangle - \langle S(0^-) \rangle = \frac{2\pi}{c} \frac{1 - (2\eta + 1)}{(1 + \eta)^2} E_i^2 = -\frac{c}{4\pi} \frac{\eta}{(1 + \eta)^2} E_i^2, \quad (\text{S-11.46})$$

which is equal to minus the absorbed power (S-11.40).

S-11.5 Anti-Reflection Coating

a) In the absence of sources at $x = +\infty$, the general solution can be written as (omitting the common time dependence $e^{-i\omega t}$)

$$E = \begin{cases} E_i e^{ikx} + E_r e^{-ikx} & (x < 0), \\ E_+ e^{in_1 kx} + E_- e^{-in_1 kx} & (0 < x < d), \\ E_t e^{in_2 kx} & (x > d), \end{cases} \quad (\text{S-11.47})$$

where $k = \omega/c$, E_i is the amplitude of the incident wave, E_r the amplitude of the wave reflected at $x = 0$, E_+ and E_- the amplitudes of the waves propagating along $+\hat{x}$ and $-\hat{x}$, respectively, in the $0 < x < d$ layer, and E_t the amplitude of the wave propagating along $+\hat{x}$ in the $x > d$ half-space. The subscripts of the electric fields E in (S-11.47) are in agreement with the subscripts of the wave vectors \mathbf{k} in Fig. 11.2.

b) The matching conditions require the electric field and its derivative with respect to x (which is proportional to the magnetic field) to be continuous at the planes $x = 0$ and $x = d$. We thus obtain

$$E_i + E_r = E_+ + E_-, \quad (\text{S-11.48})$$

$$E_i - E_r = n_1(E_+ - E_-), \quad (\text{S-11.49})$$

$$E_+ e^{+in_1 kd} + E_- e^{-in_1 kd} = E_t e^{in_2 kd}, \quad (\text{S-11.50})$$

$$n_1(E_+ e^{+in_1 kd} - E_- e^{-in_1 kd}) = n_2 E_t e^{in_2 kd}. \quad (\text{S-11.51})$$

c) Since we require that there is no reflected wave in vacuum, E_r must be zero. Posing $E_r = 0$ in (S-11.48)-(S-11.51), the latter can be regarded as an homogeneous linear system in E_i , E_+ , E_- and E_t . Such system has non-trivial solutions only if its determinant is zero, i.e. if

$$e^{2in_1kd} = \frac{n_1 + n_2}{n_1 - n_2} \frac{n_1 - 1}{n_1 + 1}. \quad (\text{S-11.52})$$

In the case of a layer of thickness d with vacuum at both sides, $n_2 = 1$ and the right-hand side of (S-11.52) equals unity, thus $e^{2in_1kd} = 1$. This implies $2n_1kd = 2m\pi$, with m any integer. Thus, there is no reflected wave when the layer thickness is $d = m\lambda/2n_1$ (since $k = 2\pi/\lambda$), i.e. when the “optical depth” nd equals an half-integer number of wavelengths.

d) In the general case, the left-hand side of (S-11.52) is a complex number of modulus 1, while the right-hand side is always real number if n_1 and n_2 are real as we assumed. Thus, we have solutions only if $e^{2in_1kd} = \pm 1$. The case $e^{2in_1kd} = +1$ is the case of $n_2 = 1$, considered above at the end of point c). In the second case $e^{2in_1kd} = -1$ we have the condition

$$2n_1kd = (2m + 1)\pi, \quad \frac{n_1 + n_2}{n_1 - n_2} \frac{n_1 - 1}{n_1 + 1} = -1, \quad (\text{S-11.53})$$

the second equation implying $n_2 = \sqrt{n_1}$. The thickness of the layer must be

$$d = (2m + 1) \frac{\lambda}{4n_1}, \quad (\text{S-11.54})$$

with m , again, any integer. The smallest possible thickness is $d = \lambda/(4n_1)$, corresponding to $m = 0$. This shows that, with a suitable choice of materials and of layer thickness, we can produce an “anti-reflection” coating on an optical element (such as a window or lens) from which we do not want any reflection to occur.

S-11.6 Birefringence and Waveplates

a) The incident wave can be considered as the superposition of two waves having, respectively, P and S polarization, i.e., one having the electric field lying in the xy plane, and the other parallel to z . The difference between the refractive indices for P and S polarization, n_p and n_s , gives origin to two different refraction angles, $\theta_{t,p}$ and $\theta_{t,s}$, according to Snell’s law. With our assumptions, the refraction angles are

$$\sin \theta_{t,p} = \frac{\sin \theta_i}{n_p} = \sin(\theta_t - \alpha), \quad \sin \theta_{t,s} = \frac{\sin \theta_i}{n_s} = \sin(\theta_t + \alpha), \quad (\text{S-11.55})$$

at the limit $\alpha \ll 1$ we can approximate $\sin \alpha \simeq \alpha$ and $\cos \alpha \simeq 1$, obtaining

$$\sin(\theta_t \pm \alpha) = \sin \theta_t \cos \alpha \pm \cos \theta_t \sin \alpha \simeq \sin \theta_t \pm \alpha \cos \theta_t. \quad (\text{S-11.56})$$

The refractive indices are $n_p = \bar{n} + \delta n$, and $n_s = \bar{n} - \delta n$, respectively, with $\delta n \ll \bar{n}$. For P polarization we have, up to the first order in $\delta n/\bar{n}$,

$$\sin \theta_t - \alpha \cos \theta_t = \frac{\sin \theta_i}{\bar{n} + \delta n} \simeq \frac{\sin \theta_i}{\bar{n}} \left(1 - \frac{\delta n}{\bar{n}}\right), \quad (\text{S-11.57})$$

and, analogously, for S polarization we have

$$\sin \theta_{t,s} \simeq \frac{\sin \theta_i}{\bar{n}} \left(1 + \frac{\delta n}{\bar{n}}\right). \quad (\text{S-11.58})$$

The above results lead to

$$\bar{n} = \frac{\sin \theta_i}{\sin \theta_t}, \quad \text{and} \quad \alpha = 2 \frac{\sin \theta_i}{\cos \theta_t} \frac{\delta n}{\bar{n}^2} = 2 \delta n \frac{\sin^2 \theta_t}{\cos \theta_t \sin \theta_i}. \quad (\text{S-11.59})$$

b) In order to have exiting circularly polarized light, the exiting P - and S -polarized components must be phase-shifted by $\delta\phi = \pi/2$. This can be obtained making use of the difference between the two optical path lengths, $n_p d$ and $n_s d$. The condition for circularly polarized light is thus

$$\delta\phi = k 2\delta n d = \frac{4\pi \delta n d}{\lambda} \doteq \frac{\pi}{2}, \quad (\text{S-11.60})$$

i.e. $d = \lambda/(8\delta n)$. This is called a *quarter-wave plate*. If $\delta\phi = \pi$ instead, i.e., if $d = \lambda/(4\delta n)$, there is a relative change of sign between the two components, which leads to a polarization rotation of $\pi/2$; this is an *half-wave plate*.

S-11.7 Magnetic Birefringence and Faraday Effect

a) Neglecting the effect of the magnetic field of the wave, much smaller than the external field \mathbf{B}_0 , the equation of motion for the electrons is

$$m_e \frac{d^2 \mathbf{r}}{dt^2} = -e\mathbf{E} - e \frac{\mathbf{v}}{c} \times \mathbf{B}_0 - m_e \omega_0^2 \mathbf{r}. \quad (\text{S-11.61})$$

The electric field of the circularly polarized EM wave can be written, in complex notation, as

$$\mathbf{E}_{\pm} = E(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{ikz - i\omega t}, \quad (\text{S-11.62})$$

where the plus and minus signs correspond to left-handed (clockwise) and right-handed (counter-clockwise) circular polarizations, respectively. We look for solutions of (S-11.61) of the form

$$\mathbf{r}_{\pm} = r_{\pm}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{ikz - i\omega t}, \quad \mathbf{v}_{\pm} = v_{\pm}(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{ikz - i\omega t}, \quad (\text{S-11.63})$$

with $v_{\pm} = -i\omega r_{\pm}$. The vector product $\mathbf{v}_{\pm} \times \mathbf{B}_0$ is

$$\mathbf{v}_{\pm} \times \mathbf{B}_0 = v_{\pm} B_0 (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times \hat{\mathbf{z}} = v_{\pm} B_0 (-\hat{\mathbf{y}} \pm i\hat{\mathbf{x}}) = \pm i v_{\pm} B_0 (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \quad (\text{S-11.64})$$

thus (S-11.61) leads to the equation for r_{\pm}

$$(\omega_0^2 - \omega^2)r_{\pm} = -\frac{e}{m_e} E \mp i e \frac{v_{\pm}}{m_e c} B_0 = -\frac{e}{m_e} E \mp \omega \omega_c r_{\pm}, \quad (\text{S-11.65})$$

where $\omega_c = eB_0/m_e c$ is the cyclotron frequency. The solution for r_{\pm} is

$$r_{\pm} = \frac{eE}{m_e(\omega^2 - \omega_0^2 \mp \omega \omega_c)}. \quad (\text{S-11.66})$$

Thus, we have a different polarization of the medium \mathbf{P}_{\pm} , and a corresponding different dielectric susceptibility χ_{\pm} , for each circular-polarization state of the EM wave,

$$\mathbf{P}_{\pm} = -en_e \mathbf{r}_{\pm} \equiv \chi_{\pm} \mathbf{E}_{\pm}. \quad (\text{S-11.67})$$

In turn, this gives two different dielectric constants $\varepsilon_{\pm} = 1 + 4\pi\chi_{\pm}$

$$\varepsilon_{\pm} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_0^2 \mp \omega \omega_c}, \quad (\text{S-11.68})$$

where $\omega_p = \sqrt{4\pi e^2 n_e / m_e}$ is the plasma frequency of the medium. The propagation of the wave requires $\varepsilon_{\pm} > 0$, i.e., $\omega > \omega_{\text{co}\pm}$, where the two cutoff frequencies $\omega_{\text{co}\pm}$ depend on the polarization of the wave

$$\omega_{\text{co}\pm} = \sqrt{\omega_0^2 + \omega_p^2 + \frac{\omega_c^2}{4}} \pm \frac{\omega_c}{2}. \quad (\text{S-11.69})$$

The magnetized medium is thus birefringent. For waves of frequency in the range $\omega_{\text{co}-} < \omega < \omega_{\text{co}+}$, only one state of circular polarization can propagate in the medium, while we have an evanescent wave for the opposite polarization. The two resonant frequencies $\omega_{\text{res}\pm}$, defined by $\chi(\omega_{\text{res}\pm}) \rightarrow \infty$, also depend on polarization:

$$\omega_{\text{res}\pm} = \sqrt{\omega_0^2 + \frac{\omega_c^2}{4}} \pm \frac{\omega_c}{2}. \quad (\text{S-11.70})$$

Notice that in the case $\omega_0 = 0$, i.e., for a magnetized free-electron medium, there is a single resonance at $\omega = \omega_c$, for *only one* circular polarization (see Problem 7.9).

The knowledge of the permittivity (or, equivalently, of the refraction index) for the two independent states of circular polarization is sufficient to study the propagation of a transverse wave of arbitrary polarization, since the latter can be always expressed as a linear superposition of circularly polarized states. Notice that if we had searched for linearly polarized solutions, we would have found a *mixing* of polarization vectors directed along $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, i.e. the permittivity would have been a matrix instead of a number. It can be shown that such matrix can be diagonalized, with circularly polarized states as eigenvectors and (S-11.68) as eigenvalues.

b) The linearly polarized wave can be considered as a superposition of the two states of circular polarization, so that at $z = 0$ the electric field of the wave can be written

$$\mathbf{E}(z = 0, t) = \hat{\mathbf{x}} E e^{-i\omega t} = \frac{E}{2} [(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})] e^{-i\omega t}. \quad (\text{S-11.71})$$

The two circularly polarized components travel at different phase velocities $v_{\pm} = c/n_{\pm}$, where $n_{\pm} = \sqrt{\epsilon_{\pm}}$ is the refractive index associated to each polarization state. At $z = \ell$, the electric field of the wave is

$$\mathbf{E}(z = \ell, t) = \frac{E}{2} [(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{ik_+\ell} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) e^{-ik_-\ell}] e^{-i\omega t}, \quad (\text{S-11.72})$$

where $k_{\pm} = \omega/v_{\pm} = (\omega/c)n_{\pm}$. To first order in ω_c/ω , we can write $n_{\pm} \simeq n_0 \pm \delta n$, where $n_0 = n(\omega_c = 0)$ and

$$\delta n = \frac{\omega\omega_c\omega_p^2}{2n_0(\omega^2 - \omega_0^2)^2}. \quad (\text{S-11.73})$$

Thus, the wave vectors for the two polarizations can be written $k_{\pm} \simeq k_0 \pm \delta k$, where $k_0 = (\omega/c)n_0$ and $\delta k = (\omega/c)\delta n$. The electric field at $z = \ell$ can be rewritten as

$$\begin{aligned} \mathbf{E}(z = \ell, t) &= \frac{E}{2} [(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) e^{i\delta k \ell} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}}) e^{-i\delta k \ell}] e^{ik_0 \ell - i\omega t} \\ &\propto \hat{\mathbf{x}} \cos(\delta k \ell) - \hat{\mathbf{y}} \sin(\delta k \ell). \end{aligned} \quad (\text{S-11.74})$$

The polarization has thus rotated by an angle $\phi = \delta k \ell$, proportional to the intensity of the magnetic field.

S-11.8 Whistler Waves

The dielectric permittivity of a magnetized free electron gas for circularly polarized transverse waves, propagating along the magnetic field, is (see Problem 11.7)

$$\varepsilon = \varepsilon_{\pm}(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}, \quad (\text{S-11.75})$$

where $\omega_p = \sqrt{4\pi e^2 n_e / m_e}$ is the plasma frequency of the medium, $\omega_c = eB_0 / m_e c$ is the cyclotron (Larmor) frequency, and the plus and minus signs refer to left-handed (counterclockwise) and right-handed (clockwise) circular polarizations, respectively. Since, in general, the dispersion relation is $\omega^2 = k^2 c^2 / \varepsilon(\omega)$, (11.11) implies that $\varepsilon = c^2 / \alpha \omega$. For $\omega \ll \omega_c$ and $\omega \ll \omega_p^2 / \omega_c$, (S-11.75) reduces to

$$\varepsilon_{\pm} \approx \pm \frac{\omega_p^2}{\omega \omega_c}. \quad (\text{S-11.76})$$

Wave propagation requires $\varepsilon > 0$. Thus, only left-handed polarized waves can propagate in the presence of a dispersion relation given by (11.11), with $\alpha = c^2 \omega_c / \omega_p^2$.

Assuming the values of n_e and B_0 given in the text, we estimate $\omega_p \approx 5.6 \times 10^6 \text{ s}^{-1}$ and $\omega_c \approx 8.8 \times 10^6 \text{ s}^{-1}$. A typical frequency for which (S-11.76) holds is $\omega \sim 10^5 \text{ s}^{-1}$.

b) First, we notice that, in general, (11.11) implies $v_g = \partial_k \omega = 2\alpha k = 2\omega/k = 2v_{\varphi}$. Thus, the phase velocity depends on frequency as

$$v_{\varphi} = \frac{\omega}{k} = \sqrt{\alpha \omega} = \sqrt{\frac{\omega_c \omega}{\omega_p^2}} c \ll c. \quad (\text{S-11.77})$$

For $\omega = 10^5$, and the above values of ω_p and ω_c , we obtain $v_{\varphi} \approx 0.03c$.

c) With a spectral range from ω_1 to $2\omega_1$, the frequency components travel with velocities differing by a factor up to 2, so that the wave packet generated by the lightning will spread out and increase its length during its propagation. The higher frequencies travel faster, and are thus received earlier by the observer, than the slower frequencies. This is the origin of name “whistlers”.

In order to estimate the spread of the packet after a distance $L = 10^9 \text{ cm}$, we assume that the center of the wave packet travels with a group velocity $v_g \approx 0.06c$, reaching a distance L after a time $\tau = L/v_g = 0.56 \text{ s}$. The “extreme” frequencies ω_1 and ω_2 will have group velocities $v_1 \approx 0.04c$ and $v_2 \approx 0.08c$, respectively, and the pulse duration may be roughly estimated as the difference $\Delta\tau = \tau_1 - \tau_2 = L/v_1 - L/v_2 \approx (0.83 - 0.42) \text{ s} = 0.41 \text{ s}$, provided that the duration at the emission is much shorter than $\Delta\tau$. This rough estimate neglects the deformation of the wave packet due to the strong dispersion.

S-11.9 Wave Propagation in a “Pair” Plasma

Actually, it is convenient to calculate the dispersion relation in the presence of an external magnetic field \mathbf{B}_0 first, then, the answer to point **a**) is simply obtained as a special case with $\mathbf{B}_0 = 0$. We assume $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$ and a wave linearly polarized along $\hat{\mathbf{x}}$ in a Cartesian reference frame xyz . The differential equations for the velocities of positrons, v_+ , and electrons, v_- , are respectively

$$\frac{dv_{x\pm}}{dt} = \pm \frac{e}{m_e c} (E_x + v_{y\pm} B_0), \quad \frac{dv_{y\pm}}{dt} = \mp \frac{e}{m_e c} (v_{x\pm} B_0), \quad (\text{S-11.78})$$

where we have assumed $v_{z\pm} = 0$. Differentiating the first of (S-11.78) once more with respect to t , and substituting the second of (S-11.78) for $dv_{y\pm}/dt$, we obtain

$$\frac{d^2 v_{x\pm}}{dt^2} = \mp i\omega \frac{e}{m_e} E_x \pm \frac{e B_0}{m_e c} \frac{dv_{y\pm}}{dt} = \mp i\omega \frac{e}{m_e} E_x + \omega_c^2 v_{x\pm}, \quad (\text{S-11.79})$$

where $\omega_c = \sqrt{e B_0 / m_e c}$ is the cyclotron frequency. Substituting $E_x = E_0 e^{-i\omega t}$ we obtain

$$v_{x\pm} = \mp i\omega \frac{e}{m_e (\omega_c^2 - \omega^2)} E_0. \quad (\text{S-11.80})$$

Analogously, for $v_{y\pm}$ we have

$$v_{y\pm} = \pm \frac{e B_0}{m_e} v_{x\mp} = -i\omega \frac{e}{m_e} E_0, \quad (\text{S-11.81})$$

which has the same value for both electrons and positrons. The components of the current density are thus

$$\begin{aligned} J_x &= n_0 e (v_{x+} - v_{x-}) = -\frac{2i\omega n_0 e^2}{m_e (\omega_c^2 - \omega^2)} E_0, \\ J_y &= n_0 e (v_{y+} - v_{y-}) = 0. \end{aligned} \quad (\text{S-11.82})$$

The dielectric permittivity of the pair plasma, $\varepsilon(\omega)$, is obtained from the usual definitions $\mathbf{J} = \sigma \mathbf{E} = -i\omega \chi \mathbf{E}$ and is

$$\varepsilon(\omega) = 1 - \frac{2\omega_p^2}{\omega^2 - \omega_c^2}. \quad (\text{S-11.83})$$

The same result can be obtained for circular polarization, both for left-handed and right-handed waves, confirming that there is no magnetically induced birefringence

in a pair plasma. This is different from the case of a medium containing free electrons only, considered in Problem 11.7.

For case **a**), where $\mathbf{B}_0 = 0$, we set $\omega_c = 0$, and obtain a cut-off frequency at $\omega = 2\omega_p$.

For case **b**), there is a resonance at $\omega = \omega_c$, while wave propagation is forbidden for frequencies in the range $\omega_c < \omega < \sqrt{\omega_c^2 + 2\omega_p^2}$.

S-11.10 Surface Waves

a) In a dielectric medium described by $\varepsilon = \varepsilon(\omega)$, a monochromatic EM field of frequency ω satisfies the Helmholtz equation. Thus we have for the magnetic field

$$\left(\nabla^2 + \varepsilon \frac{\omega^2}{c^2}\right) B_z = 0. \quad (\text{S-11.84})$$

Substituting (11.12) for B_z into the Helmholtz equation, we obtain

$$q^2 - k^2 + \frac{\omega^2}{c^2} \varepsilon = 0. \quad (\text{S-11.85})$$

b) From the equation $c\nabla \times \mathbf{B} = 4\pi\mathbf{J} + \partial_t \mathbf{E}$ and the definition of ε we obtain (for monochromatic waves in complex notation) $c\nabla \times \mathbf{B} = -i\omega\varepsilon \mathbf{E}$. By substituting (11.12) for \mathbf{B} we obtain

$$-i\omega\varepsilon \mathbf{E} = (\hat{x}\partial_y - \hat{y}\partial_x)B_z c = (ik\hat{x} - q\hat{y})B_z c, \quad (\text{S-11.86})$$

which gives for the electric field

$$\mathbf{E} = -(k\hat{x} + iq\hat{y}) \frac{c}{\varepsilon\omega} B_z. \quad (\text{S-11.87})$$

c) From the definition of $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/(4\pi)$ we find that \mathbf{S} has components both along x and along y , given by

$$S_x = \frac{c}{4\pi} E_y B_z = \frac{qcB_0^2}{4\pi\varepsilon\omega} e^{2qx} \cos(ky - \omega t) \sin(ky - \omega t), \quad (\text{S-11.88})$$

$$S_y = -\frac{c}{4\pi} E_x B_z = \frac{kcB_0^2}{4\pi\varepsilon\omega} e^{2qx} \cos^2(ky - \omega t). \quad (\text{S-11.89})$$

However, averaging over one oscillation period we obtain $\langle S_x \rangle = 0$, thus the net energy flux is in the y -direction only, since $\langle S_y \rangle \neq 0$.

d) The tangential component of the magnetic field at the interface between two media must be continuous. Thus, from $B_z(0^-) = B_z(0^+)$ we get $B_1 = B_2$.

e) Also the tangential component of the electric must be continuous at the interface, thus $E_y(0^-) = E_y(0^+)$. Using the results of points **b)** and **d)** we obtain

$$\frac{q_1}{\varepsilon_1} = -\frac{q_2}{\varepsilon_2}. \quad (\text{S-11.90})$$

Since both $q_1 > 0$ and $q_2 > 0$, ε_1 and ε_2 must have opposite signs.

f) Using the relationship $(q_1/\varepsilon_1)^2 = (q_2/\varepsilon_2)^2$ and the result of point **a)** we obtain

$$\varepsilon_2^2 \left(k^2 - \frac{\omega^2}{c^2} \varepsilon_1 \right) = \varepsilon_1^2 \left(k^2 - \frac{\omega^2}{c^2} \varepsilon_2 \right), \quad (\text{S-11.91})$$

from which it follows that

$$\omega^2 = k^2 c^2 \frac{\varepsilon_2^2 - \varepsilon_1^2}{\varepsilon_2^2 \varepsilon_1 - \varepsilon_1^2 \varepsilon_2} = k^2 c^2 \frac{\varepsilon_2 + \varepsilon_1}{\varepsilon_2 \varepsilon_1}. \quad (\text{S-11.92})$$

Since wave can propagate only if $k^2 > 0$, and $\varepsilon_1 \varepsilon_2 < 0$, we get the additional condition $\varepsilon_1 + \varepsilon_2 < 0$.

g) Since $\varepsilon_2 < -\varepsilon_1 = -1$ must hold, we may choose a metal, or a free electron gas, or an ideal plasma . . . , for which $\varepsilon_2 = 1 - \omega_p^2/\omega^2$, and a frequency such that $\omega < \omega_p/\sqrt{2}$.

The above described EM modes are *surface waves* (also named *surface plasmons*). These waves propagate along the surface of a conductor and are evanescent along the perpendicular direction, so that the EM energy is confined in a narrow layer, thinner than the wavelength in vacuum. Surface waves are a building block of *plasmonics*, a discipline oriented to develop optical and electronic devices on a nanometric scale.¹

S-11.11 Mie Resonance and a “Plasmonic Metamaterial”

a) The incident field can be written, in complex notation, as

$$\mathbf{E}_i = \mathbf{E}_i(x, t) = \mathbf{E}_0 e^{ikx - i\omega t}. \quad (\text{S-11.93})$$

Since $a \ll \lambda$, the electric field can be considered as uniform over the volume of the sphere, thus $\mathbf{E}_i \approx \mathbf{E}_0 e^{-i\omega t}$, assuming the center of the sphere to be located at $x = 0$.

¹See e.g. W. L. Barnes et al., “Surface plasmon subwavelength optics”, *Nature* **424**, 824 (2003); E. Ozbay, “Plasmonics: merging photonics and electronics at nanoscale dimensions”, *Science* **311**, 189 (2006).

Now we introduce a spherical coordinate system (r, θ, ϕ) with the origin at the center of the sphere, and the zenith direction parallel to \mathbf{E}_i . At the surface of the sphere, $r = a$, we have the usual boundary conditions at the interface between two media

$$E_{\perp}(a^+, \theta) - E_{\perp}(a^-, \theta) = 4\pi\sigma(\theta), \quad E_{\parallel}(a^+, \theta) - E_{\parallel}(a^-, \theta) = 0, \quad (\text{S-11.94})$$

where $\sigma(\theta)$ is the surface charge density on the sphere, independent of ϕ within our approximations. The problem is thus analogous to the case of a dielectric sphere in a static uniform external field, treated in Problem 3.4. We can extend the results for the internal field and polarization to the case of an oscillating field as follows

$$\mathbf{E}_{\text{int}} = \frac{3\mathbf{E}_0}{\varepsilon_r(\omega) + 2}, \quad \mathbf{P} = \chi\mathbf{E}_{\text{int}} = \frac{3(\varepsilon_r(\omega) - 1)}{4\pi(\varepsilon_r(\omega) + 2)}\mathbf{E}_0. \quad (\text{S-11.95})$$

The difference with the electrostatic case is that now ε_r depends on frequency, and is not necessarily positive and greater than one, so that the internal field \mathbf{E}_{int} can be greater than the external applied field \mathbf{E}_0 . A resonance appears when the real part of the denominator vanishes. Setting $\eta = 0$ for simplicity, the resonance condition is

$$\varepsilon_r(\omega) + 2 = 3 - \frac{\omega_p^2}{\omega^2 - \omega_0^2} = 0, \quad (\text{S-11.96})$$

which yields

$$\omega^2 = \omega_0^2 + \frac{\omega_p^2}{3}. \quad (\text{S-11.97})$$

The physical meaning of the resonance is particularly clear for $\omega_0 = 0$, e.g., for a metallic (nano)sphere in a high-frequency (optical) field. In this case the resonance frequency is

$$\omega = \frac{\omega_p}{\sqrt{3}}, \quad (\text{S-11.98})$$

that is the natural frequency of the collective ‘‘Mie oscillations’’ of the electron sphere treated in Problem 1.5, also known as the lowest-order surface plasmon of the sphere. The resonance thus corresponds to the excitation of this oscillation mode.

b) The *macroscopic* polarization is given by the dipole moment of each nanosphere, $\mathbf{p}_{\text{sphere}} = \mathbf{P}V$, with $V = (4\pi/3)a^3$ the volume of the sphere, times the number of nanospheres per unit volume, n_s :

$$\mathbf{P}_{\text{macro}} = n_s\mathbf{p}_{\text{sphere}} = -\frac{3n_sV\omega_p^2}{3\omega^2 - \omega_p^2}\mathbf{E}_0. \quad (\text{S-11.99})$$

This is equivalent to a *macroscopic* dielectric function

$$\varepsilon_r(\omega) = 1 - \frac{3n_s V \omega_p^2}{3\omega^2 - \omega_p^2}. \quad (\text{S-11.100})$$

Wave propagation requires ε_r to be positive, i.e.,

$$\omega < \frac{\omega_p}{\sqrt{3}}, \quad \omega > \frac{\omega_p}{\sqrt{3}} \sqrt{1 + 3n_s V}. \quad (\text{S-11.101})$$

This is a simple example of an artificial “metamaterial”, where the plasmonic properties of the nanostructures composing the material determine the optical response.