Chapter S-12
Solutions for Chapter 12

S-12.1 The Coaxial Cable

a) Since the capacitance has been defined assuming *static* conditions and boundary effects are negligible for an "infinite" wire, we evaluate the capacitance per unit length of the cable, C, as for a cylindrical capacitor assuming the charge density to be constant in time and uniformly distributed. For symmetry reasons the electrostatic field between the two conductors is radial and independent of z and ϕ, and it is obtained easily from Gauss's law as

$$\mathbf{E} = \frac{2\lambda}{r}\hat{\mathbf{r}}, \qquad a < r < b. \tag{S-12.1}$$

Thus, the potential drop between the two conductors is

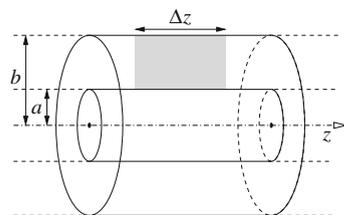$$V = -\int_a^b E_r(r)\,\mathrm{d}r = -2\lambda\ln\left(\frac{b}{a}\right), \tag{S-12.2}$$

so that we obtain

$$C = \frac{\lambda}{|V|} = \frac{1}{2\ln(b/a)}. \tag{S-12.3}$$

Similarly, a static current I uniformly distributed on the inner conductor generates a magnetic field

$$\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}} = \frac{2I}{cr}\hat{\boldsymbol{\phi}}. \tag{S-12.4}$$

The inductance per unit length of the cable can be obtained by evaluating the flux of $\mathbf{B}$

Fig. S-12.1

through a rectangle of width Δz, lying on a plane containing the z axis, and extending from $r = a$ to $r = b$, as highlighted in Fig. S-12.1. The flux is

$$\Phi(\mathbf{B}) = \int_a^b B_\phi(r)\,\Delta z\,\mathrm{d}r = \frac{2I}{c}\ln\left(\frac{b}{a}\right)\Delta z\,, \qquad\text{(S-12.5)}$$

corresponding to an inductance per unit length $\mathcal{L}$

$$\mathcal{L} = \frac{\Phi(\mathbf{B})}{Ic\Delta z} = \frac{2}{c^2}\ln\left(\frac{b}{a}\right). \qquad\text{(S-12.6)}$$

The same result can be obtained by calculating the magnetic energy in a cable section of length Δz, and inductance $\Delta z\mathcal{L}$,

$$\frac{1}{2}\Delta z\mathcal{L}I^2 \equiv \Delta z\int_a^b \frac{B^2}{8\pi}2\pi r\,\mathrm{d}r = \frac{I^2}{c^2}\ln\left(\frac{b}{a}\right)\Delta z\,. \qquad\text{(S-12.7)}$$

b) The coaxial cable is a continuous system with finite capacitance and inductance per unit length, thus we know from Problem 7.4 that a current signal propagates along the wire according to the wave equation (S-7.49), with velocity

$$v = \frac{1}{\sqrt{\mathcal{L}C}} = c\,. \qquad\text{(S-12.8)}$$

The general solution for the propagating current signal is thus $I(z,t) = I(z - vt)$, and propagation occurs with no dispersion. The associated charge signal $\lambda(z,t)$ is related to $I(z,t)$ by the continuity equation,

$$\partial_t\lambda(z,t) = -\partial_z I(z,t) = -I'(z - ct)\,, \qquad\text{(S-12.9)}$$

where I' denotes the derivative of I with respect to its argument. Since $\partial_t\lambda(z - ct) = -c\lambda'(z - ct)$, we obtain

$$\lambda(z,t) = \lambda(z - ct) = \frac{1}{c}I(z - ct)\,. \qquad\text{(S-12.10)}$$

c) A transverse electric field $\mathbf{E}$ must be radial for symmetry reasons, $\mathbf{E} = E_r(r,z,t)\hat{\mathbf{r}}$. Applying Gauss's law to a cylindrical surface of radius $a < r < b$, infinitesimal height Δz, and coaxial to the cable, we find $E_r = 2\lambda(z,t)/r$. Again for symmetry reasons, a transverse magnetic field must be azimuthal, $\mathbf{B} = B_\phi(r,z,t)\hat{\boldsymbol{\phi}}$. Applying Stokes' law to a circle of radius $a < r < b$, coaxial to the cable, we obtain $B_\phi = 2I(z,t)/rc$. The displacement current does not contribute to the flux through the circle, since $\mathbf{E}$ is radial. Thus, the fields of have the same dependence on λ and I as the static fields, the only difference being that here both $\lambda = \lambda(z,t)$ and $I = I(z,t)$ depend on z and t. Notice that it is such peculiar character of the TEM configuration which

allows to use the capacitance and inductance calculated for static fields to obtain the propagation velocity of electromagnetic signals along the cable, a result also true for any transmission line in TEM mode.

We can check that the above fields constitute a solution to Maxwell's equations by verifying that

$$\nabla \times \mathbf{E} = \partial_z E_r \hat{\boldsymbol{\phi}} = \frac{2}{r} \partial_z \lambda(z - ct) \hat{\boldsymbol{\phi}} = \frac{2}{r} \lambda'(z - ct) \hat{\boldsymbol{\phi}}$$

$$= \frac{2}{rc} I'(z - ct) \hat{\boldsymbol{\phi}} = -\frac{2}{rc^2} \partial_t I(z - ct) \hat{\boldsymbol{\phi}}$$

$$= -\frac{1}{c} \partial_t \mathbf{B} . \tag{S-12.11}$$

d) The source at $z = 0$ must do a work $W(t)$ in order to drive the current between the inner and outer conductors,

$$W(t) = V(0, t) I(0, t) = -2c\lambda^2(0, t) \ln\left(\frac{b}{a}\right) . \tag{S-12.12}$$

The local flux of energy at any point (r, ϕ, z), with $a < r < b$ and $z > 0$, is

$$\mathbf{S}(r, z, t) = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \hat{\mathbf{z}} \frac{c}{4\pi} \frac{2\lambda(z - ct)}{r} \frac{2I(z - ct)}{rc}$$

$$= \hat{\mathbf{z}} \frac{c}{\pi r^2} \lambda^2(z - ct) , \tag{S-12.13}$$

corresponding to a total flow of energy at z

$$\Phi(z, t) = \int_a^b S_z \, 2\pi r \, dr = 2c\lambda^2(z - ct) \ln\left(\frac{b}{a}\right) = -W(z - ct) . \tag{S-12.14}$$

This shows that the energy flow is sustained by the source.
e) The expressions for the fields, and for the capacitance and inductance per unit length, are, in the presence of generic values of ε and μ,

$$E_r = 2\frac{\lambda}{\varepsilon r} , \qquad\qquad B_\phi = \frac{2\mu I}{rc} , \tag{S-12.15}$$

$$C = \frac{\varepsilon}{2\ln(b/a)} , \qquad\qquad \mathcal{L} = \frac{2}{c^2} \ln\left(\frac{b}{a}\right) , \tag{S-12.16}$$

corresponding to a wave velocity $v = c/\sqrt{\varepsilon\mu} < c$. In general, however, both ε and μ can depend on frequency, and the cable becomes a dispersive transmission line with phase velocity $v_\phi(\omega) = c/\sqrt{\varepsilon(\omega)\mu(\omega)}$.

S-12.2 Electric Power Transmission Line

a) The continuity equation is $\partial_t \lambda = -\partial_z I$. Writing λ in the form $\lambda = \lambda_0 \, \mathrm{e}^{\mathrm{i}kz - \mathrm{i}\omega t}$, we obtain

$$-\mathrm{i}\omega \lambda_0 = -\mathrm{i}k I_0 , \quad \text{or} \quad \lambda_0 = \frac{k}{\omega} I_0 = \frac{I_0}{v_\varphi} , \tag{S-12.17}$$

where v_φ is the phase velocity of the signal.
b) The electric field $\mathbf{E}$ can be calculated by applying Gauss's law to a cylindrical surface coaxial to the wire, obtaining

$$E_r(r, z, t) = \frac{2\lambda(z, t)}{r} . \tag{S-12.18}$$

The magnetic field $\mathbf{B}$ can be obtained from the equation $c\nabla \times \mathbf{B} = 4\pi \mathbf{J} + \partial_t \mathbf{E}$. If we choose a circle of radius r coaxial to the wire and apply Stokes' theorem we have

$$\oint \mathbf{B} \cdot \mathrm{d}\boldsymbol{\ell} = \frac{1}{c} \int (4\pi \mathbf{J} + \partial_t \mathbf{E}) \cdot \mathrm{d}\mathbf{S} . \tag{S-12.19}$$

The $\partial_t \mathbf{E}$ term is radial and thus does not contribute to the flux at the right-hand side, so that

$$2\pi r B_\phi = 4\pi \frac{I}{c} , \quad \text{and} \quad B_\phi(r, z, t) = \frac{2I(z, t)}{rc} . \tag{S-12.20}$$

The equations for $E_r(r, \phi, z)$ and $B_\phi(r, \phi, z)$ have the same form as in the static case of a wire with constant and uniform charge density and current, respectively. We also have $|E_r|/|B_\phi| = c/v_\varphi$. These are a typical properties of the TEM (transverse electromagnetic) mode for the transmission lines. Maxwell's equation $c\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$ gives $c\partial_z E_r = -\partial_t B_\varphi$ leads to

$$\mathrm{i}k\lambda_0 = \mathrm{i}\omega \frac{I_0}{c^2} \quad \Rightarrow \quad k\frac{I_0}{v_\varphi} = \omega \frac{I_0}{c^2} \quad \Rightarrow \quad \frac{\omega}{v_\varphi^2} = \frac{\omega}{c^2} \quad \Rightarrow \quad v_\varphi = c , \tag{S-12.21}$$

where we have used (S-12.17) and $k = \omega/v_\varphi$.
   In SI units we have

$$E_r = \frac{\lambda}{2\pi\varepsilon_0} , \qquad B_\phi = \frac{\mu_0 I}{2\pi r} , \qquad \frac{|E_r|}{|B_\phi|} = \frac{c^2}{v_\varphi} . \tag{S-12.22}$$

c) Consider a line on the midplane, at a distance h from the plane containing the two wires, as in Figs. S-12.2 and S-12.3. The distance of the line from each wire is $r = \sqrt{h^2 + d^2/4}$. The electric and magnetic fields generated by the two wires sum up to

$$\mathbf{E} = 2\hat{\mathbf{x}} \frac{2I_0}{rc} \sin\theta \, e^{-i\omega t}, \qquad \text{(S-12.23)}$$

$$\mathbf{B} = 2\hat{\mathbf{y}} \frac{2I_0}{rc} \sin\theta \, e^{-i\omega t}, \qquad \text{(S-12.24)}$$

with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ the unit vectors parallel and perpendicular to the plane containing the wires, respectively.

Fig. S-12.2

Since $\sin\theta = d/(2r)$, we obtain

$$|E_x| = |B_y| = \frac{2I_0 d}{r^2 c} . \qquad \text{(S-12.25)}$$

The corresponding expressions in SI units are

$$|E_x| = |B_y|c = \frac{I_0 d}{2\pi c\varepsilon_0 r^2} = \frac{\mu_0 I_0 dc}{2\pi r^2} . \qquad \text{(S-12.26)}$$

Thus

$$|B_y| = \frac{4\pi \times 10^{-7} \times 10^3 \times 5}{2\pi \times (30^2 + 5^2/2^2)} \, \text{T} \simeq 10^{-6} \, \text{T} , \qquad \text{(S-12.27)}$$

and

$$|E_x| \simeq 3 \times 10^2 \, \text{V/m} . \qquad \text{(S-12.28)}$$

Fig. S-12.3

For a comparison, the average magnetic field at the Earth surface is $\sim 5 \times 10^{-5}$ T, while the electric field is $\sim 1.5 \times 10^2$ V/m. Possible screening effects by the Earth's surface have been neglected.

S-12.3 TEM and TM Modes in an "Open" Waveguide

a) Inserting (12.23) into the wave equation for $\mathbf{B}$

$$\left(\nabla^2 - \frac{1}{c^2} \partial_t^2\right) B_z = 0 , \qquad \text{(S-12.29)}$$
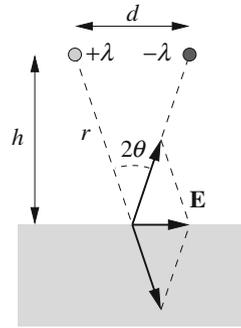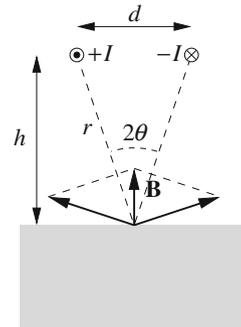
and recalling that $\partial_z B_z = 0$, we obtain the following relation between k_x, k_y and ω

$$k_x^2 + k_y^2 - \frac{\omega^2}{c^2} = 0 \, . \tag{S-12.30}$$

b) The electric field of the wave can be obtained from

$$\partial_t \mathbf{E} = c\boldsymbol{\nabla} \times \mathbf{B} = c\,(\hat{\mathbf{x}}\partial_y - \hat{\mathbf{y}}\partial_x)\,B_z,$$

$$-\mathrm{i}\omega\mathbf{E} = cB_0 \left[-\hat{\mathbf{x}}\,k_y\sin(k_y y) - \hat{\mathbf{y}}\,\mathrm{i}k_x\cos(k_y y) \right] \mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}\omega t} \, , \tag{S-12.31}$$

which leads to

$$E_x = -\mathrm{i}\frac{k_y c}{\omega} B_0 \sin(k_y y)\,\mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}\omega t} \, , \tag{S-12.32}$$

$$E_y = \frac{k_x c}{\omega} B_0 \cos(k_y y)\,\mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}\omega t} \, . \tag{S-12.33}$$

c) The parallel component $E_\parallel$ of the electric field $\mathbf{E}$ must vanish at the boundary with a perfectly conducting surface, thus we must have $E_x(y = \pm a/2) = 0$. This implies that $\sin(k_y a/2) = 0$, and $k_y a = 2m\pi$, with $m \in \mathbb{N}$. By substitution into (S-12.30) we obtain

$$\omega^2 = k_x^2 c^2 + \left(\frac{\pi c}{a}\right)^2 (2m)^2 \, . \tag{S-12.34}$$

The $m = 0$ mode corresponds to $E_x = 0$ and to E_y and B_z independent of y. The fields are thus uniform over any cross-section of the waveguide parallel to the yz plane, and we have $\omega = k_x c$. This is the TEM mode typical of transmission lines. The $m = 1$ mode has frequency

$$\omega = \sqrt{k_x^2 c^2 + \left(\frac{2\pi c}{a}\right)^2} > \frac{2\pi c}{a} \equiv \omega_{\mathrm{co}} \, , \tag{S-12.35}$$

where $\omega_{\mathrm{co}} \equiv 2\pi c/a$ is the cut-off frequency.

d) The energy flux is given by Poynting's vector, parallel to the $z = 0$ plane,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} = \frac{c}{4\pi}(E_y B_z \hat{\mathbf{x}} - E_x B_z \hat{\mathbf{y}}) \, . \tag{S-12.36}$$

By averaging over one full cycle we find $\langle S_y \rangle = 0$, i.e., there is no net energy flux along y. Averaging S_x over one cycle we obtain

$$\langle S_x \rangle = \frac{c^2}{8\pi}\frac{k_x}{\omega} B_0^2 \cos^2(k_y y) \, . \tag{S-12.37}$$

The group velocity of the wave is

$$v_g = \partial_k \omega = \frac{k_x c^2}{\sqrt{k_x^2 c^2 + \omega_{co}^2}} = \frac{k_x c^2}{\omega} \,, \tag{S-12.38}$$

thus we can also write

$$\langle S_x \rangle = v_g \frac{B_0^2}{8\pi} \cos^2(k_y y) \,. \tag{S-12.39}$$

S-12.4 Square and Triangular Waveguides

a) The electric field must satisfy the wave equation in vacuum

$$\left(\boldsymbol{\nabla}^2 - \frac{1}{c^2} \partial_t^2 \right) \mathbf{E} = 0 \,, \tag{S-12.40}$$

and, substituting (12.4) for **E**, we obtain the time-independent Helmoltz's equation for the only nonzero component of the electric field, $\tilde{E}_x$,

$$\left(\partial_x^2 + \partial_y^2 - k_z^2 + \frac{\omega^2}{c^2} \right) \tilde{E}_x = 0 \,. \tag{S-12.41}$$

In vacuum we must also have $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$, this condition is automatically satisfied if we assume that $\tilde{E}_x$ is independent of x, $\tilde{E}_x = \tilde{E}_x(y)$, and (S-12.41) reduces to

$$\left(\partial_y^2 - k_z^2 + \frac{\omega^2}{c^2} \right) \tilde{E}_x(y) = 0 \,. \tag{S-12.42}$$

According to the boundary conditions, the parallel component of **E** must be zero at the perfectly reflecting walls of the waveguide $y = 0$ and $y = a$. This condition is satisfied if we assume

$$\tilde{E}_x(y) = E_{0x} \sin(k_y y) \,, \quad \text{with} \quad k_y = n\frac{\pi}{a} \,, \quad n = 1, 2, 3, \ldots \,, \tag{S-12.43}$$

where E_{0x} is an arbitrary, constant amplitude. The electric field of our $\hat{\mathbf{x}}$ polarized wave can thus be written

$$\mathbf{E} = \hat{\mathbf{x}} \tilde{E}_x(y) e^{ik_z z - i\omega t} = \hat{\mathbf{x}} E_{0x} \sin\left(\frac{n\pi}{a} y\right) e^{ik_z z - i\omega t} \,. \tag{S-12.44}$$

Substituting (S-12.43) for $\tilde{E}_x$ into (S-12.42) leads to

$$\left(-k_y^2 - k_z^2 + \frac{\omega^2}{c^2}\right)E_{0x} = 0 \,, \tag{S-12.45}$$

which, diregarding the trivial case $E_{0x} = 0$, is true only if

$$k_y^2 + k_z^2 - \frac{\omega^2}{c^2} = 0 \,, \quad \text{or} \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_y^2} = \sqrt{\frac{\omega^2}{c^2} - n^2\frac{\pi^2}{a^2}} \,. \tag{S-12.46}$$

The wave can propagate only if k_z is real, thus we must have

$$\omega > n\frac{\pi c}{a} \,. \tag{S-12.47}$$

The cutoff frequency ω_a is the lowest value of ω at which wave propagation occurs. Since we must have $n \geqslant 1$, we have $\omega_a = \pi c/a$. If we choose a frequency such that $\pi c/a < \omega < 2\pi c/a$, only the $n = 1$ mode can propagate in the guide.

The cross-section of the waveguide being square, the conditions for a $\hat{\mathbf{y}}$ polarized TE wave are obtained by interchanging the roles of x and y in all the above formulae, and the electric field is

$$\mathbf{E} = \hat{\mathbf{y}}\,E_{0y}\sin(k_x x)\,\mathrm{e}^{\mathrm{i}k_z z - \mathrm{i}\omega t} = \hat{\mathbf{y}}\,E_{0y}\sin\left(\frac{m\pi}{a}x\right)\mathrm{e}^{\mathrm{i}k_z z - \mathrm{i}\omega t} \,, \tag{S-12.48}$$

with, again, E_{0y} an arbitrary amplitude, $m = 1, 2, 3, \ldots$, and the same dispersion relation as between ω and k_z as above. Modes with $m = n$ are degenerate, sharing the same wavevector k_z.

In general, a monochromatic TE wave propagating in the guide will be a superposition of the two polarizations. The electric field will be

$$\mathbf{E} = \left[\hat{\mathbf{x}}\,E_{0x}\sin\left(\frac{n\pi}{a}y\right) + \hat{\mathbf{y}}\,E_{0y}\sin\left(\frac{n\pi}{a}x\right)\right]\mathrm{e}^{\mathrm{i}k_z z - \mathrm{i}\omega t} \,. \tag{S-12.49}$$

b) In the case of the triangular waveguide, the parallel component of the electric field $\mathbf{E}$ must be zero on the three $x = 0$, $y = 0$, and $y = x$ planes. A field of the form (S-12.49) already satisfies the boundary conditions at the $x = 0$ and $y = 0$ planes. The additional condition at the $y = x$ plane is $\mathbf{E}(x, x) \cdot \hat{\mathbf{n}} = 0$, where $\hat{\mathbf{n}} = (-1, 1, 0)/\sqrt{2}$ is the unit vector perpendicular to the $y = x$ plane. Thus

$$\mathbf{E} \cdot \mathbf{h} = E_{0x}\sin\left(\frac{n\pi}{a}x\right) - E_{0y}\sin\left(\frac{n\pi}{a}x\right) = 0 \,, \tag{S-12.50}$$

which is satisfied if $E_{0y} = -E_{0x} \equiv E_0$, so that we eventually obtain

$$\mathbf{E} = E_0\left[\hat{\mathbf{x}}\sin\left(\frac{n\pi}{a}y\right) - \hat{\mathbf{y}}\sin\left(\frac{n\pi}{a}x\right)\right]\mathrm{e}^{\mathrm{i}k_z z - \mathrm{i}\omega t} \,. \tag{S-12.51}$$

S-12.5 Waveguide Modes as an Interference Effect

a) The electrostatic potential ϕ must
be zero on the two conducting planes
at $y = \pm a$, and the electric fields at
$y = a^-$ and $y = -a^+$ must be perpen-
dicular to their surfaces (parallel to
$\hat{\mathbf{y}}$). The real dipole $\mathbf{p}$ is located at the
origin of our coordinate system, thus,
we need an image dipole equal to $\mathbf{p}$,
located at $(0, 2a, 0)$ and represented by
$\mathbf{p}_1$ in Fig. S-12.4, in order to fulfill
these conditions at the generic point A
of the $y = +a$ plane. Analogously, the
real dipole $\mathbf{p}$ requires a further image
dipole $\mathbf{p}$ located at $(0, -2a, 0)$, repre-
sented by $\mathbf{p}_{-1}$ in Fig. S-12.4, in order
to fulfill the conditions at the $y = -a$
conducting plane. But now the three
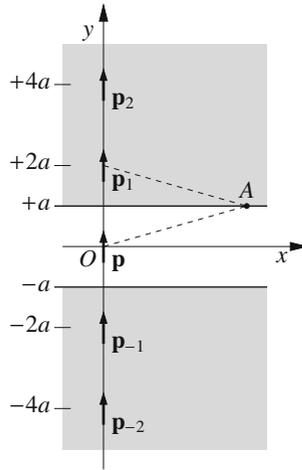dipoles $\mathbf{p}$, $\mathbf{p}_1$, and $\mathbf{p}_{-1}$ together do not

Fig. S-12.4

generate a potential equal to zero on either plane. We can readjust the potential at
$y = +a$ by adding a new image dipole equal to $\mathbf{p}$, symmetrical to $\mathbf{p}_{-1}$, at $(0, 4a, 0)$,
represented by $\mathbf{p}_2$. But this requires adding a further image dipole $\mathbf{p}_{-2}$, and so on.
Thus, the exact solution requires two infinite sets of equal image dipoles, $\mathbf{p}_n$ and $\mathbf{p}_{-n}$,
with $n = 1, 2, 3, \ldots$, located respectively at $(0, 2na, 0)$ and $(0, -2na, 0)$. The resulting
electrostatic potential between the plates is finite because, for high n values, the
contribution of $\pm n$th dipole is proportional to $(2na)^{-2}$.

b) In order to fulfill the boundary conditions, all the image dipoles must oscillate
in phase with the real dipole. Consider the radiation emitted by each dipole in the
$\hat{\mathbf{n}} \equiv (\sin\theta, \cos\theta, 0)$ direction in the $z = 0$ plane, with wavevector $\mathbf{k} = (\omega/c)\,\hat{\mathbf{n}}$. In the
following we consider wavevectors lying in the $z = 0$ plane, but our considerations
apply to wavevectors lying in any plane containing the y axis, due to the rotational
symmetry of the problem. The optical path difference between the waves emitted by
two neighboring dipoles (real or images) is $\Delta\ell = 2a\cos\theta$, as shown in Fig. S-12.5 for
the case of the real dipole $\mathbf{p}$ and the image $\mathbf{p}_1$. This corresponds to a phase difference
$\Delta\varphi = k\Delta\ell$, and the condition for constructive interference is

$$k\Delta\ell = \frac{2\omega a}{c}\cos\theta = 2\pi m, \qquad \theta = \arccos\left(m\frac{\pi c}{\omega a}\right), \qquad \text{(S-12.52)}$$
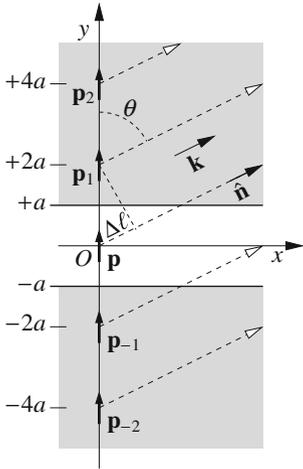
Fig. S-12.5

with $m = 0, 1, 2, \ldots$. Due to the mirror symmetry of the system for reflections through the $y = 0$ plane (actually, *antisymmetry*, since all dipoles are inverted by the reflection), if an angle θ satisfies (S-12.52) for constructive interference, so does $\pi - \theta$. In other words, at large distance from the oscillating dipole, each interference order $m > 0$ corresponds to the superposition of two waves with wavevectors $\mathbf{k}_\pm \equiv (\sin\theta, \pm\cos\theta, 0)\omega/c$, respectively.

The $m = 0$ condition corresponds to $\theta = \pi/2$, and the waves travels along the x axis. For $m > 0$, we can write

$$k_x = \frac{\omega}{c}\sin\theta = \frac{\omega}{c}\sqrt{1-\cos^2\theta} = \frac{\omega}{c}\sqrt{1-\left(m\frac{\pi c}{\omega a}\right)^2}$$

$$= \sqrt{\frac{\omega^2}{c^2}-\left(m\frac{\pi}{a}\right)^2}\,, \tag{S-12.53}$$

and k_x is real only if $\omega > m\pi c/a$. Thus, given a frequency ω, we observe only the modes with $m < \omega a/(\pi c)$. If $\omega < \pi c/a$, corresponding to a wavelength $\lambda > 2a$, only the mode $m = 0$ can propagate.

c) Both magnetic fields must satisfy the wave equation

$$(c^2\nabla^2 - \partial_t^2)\mathbf{B}_i = (c^2\partial_x^2 + c^2\partial_y^2 - \partial_t^2)\mathbf{B}_i = 0\,, \quad i = 0, 1\,, \tag{S-12.54}$$

from which we obtain, denoting by $\mathbf{k}_0$ and $\mathbf{k}_1$ the respective wavevectors,

$$k_{0x}^2 c^2 = \omega^2\,, \qquad k_{1x}^2 c^2 + k_{1y}^2 c^2 = \omega^2\,. \tag{S-12.55}$$

d) Assuming electric fields of the form $\mathbf{E} = \tilde{\mathbf{E}}\,e^{-i\omega t}$, where $\tilde{\mathbf{E}}$ depends on the space coordinates only, Maxwell's equation in vacuum, $\partial_t\mathbf{E} = c\nabla\times\mathbf{B}$, gives

$$-i\omega\mathbf{E} = c(\hat{\mathbf{x}}\,\partial_y B_z - \hat{\mathbf{y}}\,\partial_x B_z)\,. \tag{S-12.56}$$

For the wave of type "0" we obtain

$$\mathbf{E}_0 = \hat{\mathbf{y}}\,\frac{k_{0x}c}{\omega}\,B_0\,e^{ik_{0x}x-i\omega t} = \hat{\mathbf{y}}\,B_0\,e^{ik_{0x}x-i\omega t}\,. \tag{S-12.57}$$

For the wave of type "1" we obtain

$$\mathbf{E}_1 = \frac{ic}{\omega}B_1\left[\hat{\mathbf{y}}\,k_{1x}\cos(k_{1y}y) - \hat{\mathbf{x}}\,ik_{1y}\sin(k_{1y}y)\right]e^{ik_{1x}x-i\omega t}\,. \tag{S-12.58}$$

e) The "type-0" wave has the three vectors $\mathbf{E}_0$, $\mathbf{B}$ and $\mathbf{k}$ perpendicular to one another, analogously to a plane wave in the free space (TEM mode). Further, $\mathbf{E}_0$ is perpendicular to the two conducting surfaces, automatically satisfying the boundary conditions. Thus, the frequency ω and the wavevector $\mathbf{k} = \hat{\mathbf{x}}k_{0x}$, with $k_{0x} = \omega/c$, are subject to no constraint.

On the other hand, the electric field of the "type-1" wave has a component E_x parallel to the two conducting surfaces, in addition to the transverse E_y component (the mode is TM rather than TEM). The boundary conditions at $y = \pm a$ require that $E_x(y = \pm a) = 0$. Thus we must have $\sin(\pm k_y a) = 0$, or $k_y = m\pi/a$, with $m = 1, 2, 3, \ldots$, leading to

$$k_x = \sqrt{\frac{\omega^2}{c^2} - \left(m\frac{\pi}{a}\right)^2}. \tag{S-12.59}$$

The m-th mode can propagate only if the corresponding k_x is real, and has a lower cut-off frequency $\omega_{\text{co}}(m) = 2\pi mc/a$-

A comparison to point (b) shows that the type-0 wave (TEM mode) corresponds to the $m = 0$ interference order, while the type-1 waves (TM modes) correspond to the > 0 interference orders. Actually, more precisely, we need not single dipoles, but "dipole layers", spread parallel to the z axis, in order to generate waves with fields independent of z. If the real dipole of points (a) and (b) is parallel, rather than perpendicular, to the conducting planes, the different boundary conditions would lead to TE, rather than TM modes [1].

S-12.6 Propagation in an Optical Fiber

a) The electric field (12.6) corresponds to the sum of two plane waves of the same frequency and different wavevectors, $\mathbf{k}_1$ and $\mathbf{k}_2$, propagating in the medium. For both waves the dispersion relation is $\omega = kc/\mathsf{n}$, where $\mathsf{n} = \mathsf{n}(\omega)$ is the refractive index of the medium. Both waves impinge on the medium-vacuum interface at the angle θ, and the condition for total reflection is, according to Snell's law,

$$\sin\theta > \frac{1}{\mathsf{n}}. \tag{S-12.60}$$

b) The internal reflections at the $y = \pm a/2$ planes turn the wave of type "1" into a wave of type "2", and vice versa. Thus the field amplitudes of the two waves at the interface are related by the amplitude reflection coefficient r

$$E_2(x, y = +a/2, t) = \mathsf{r}E_1(x, y = +a/2, t),$$
$$E_1(x, y = -a/2, t) = \mathsf{r}E_2(x, y = -a/2, t). \tag{S-12.61}$$

For total reflection there is no transmission of energy through the $y = \pm a/2$ planes, thus the amplitudes of the incident and the reflected fields must be equal but for a change of phase. For S-polarization ($\mathbf{E}$ parallel to the interface, as in our case) r is written, according to the Fresnel equations,

$$r = \frac{n\cos\theta - i\sqrt{n^2\sin^2\theta - 1}}{n\cos\theta + i\sqrt{n^2\sin^2\theta - 1}} \,, \tag{S-12.62}$$

and, if $n\sin\theta > 1$, the square roots are real and $|r| = 1$. Thus we can write

$$r = e^{i\delta} = \cos\delta + i\sin\delta \,, \qquad \tan\frac{\delta}{2} = -\frac{\sqrt{n^2\sin^2\theta - 1}}{n\cos\theta} \,. \tag{S-12.63}$$

Substituting $r = e^{i\delta}$ into (S-12.61) we obtain the following conditions at the $y = \pm a/2$ planes

$$E_2\,e^{-ik_ya/2} = E_1\,e^{+ik_ya/2}\,e^{i\delta} \,, \qquad E_1\,e^{-ik_ya/2} = E_2\,e^{+ik_ya/2}\,e^{i\delta} \,. \tag{S-12.64}$$

By calculating the determinant of the homogeneous system for E_1 and E_2 we obtain the condition

$$1 = e^{2i(k_ya+\delta)} \,, \tag{S-12.65}$$

true if

$$2k_ya + 2\delta = 2m\pi \,, \qquad m = 0, 1, 2,\dots \,. \tag{S-12.66}$$

The implicit relation determining the allowed frequencies is

$$k_x^2 = \frac{\omega^2}{c^2}n^2 - k_y^2 > 0 \,. \tag{S-12.67}$$

If $n\sin\theta \gg 1$ then $\delta \simeq -2\theta$, and if $\theta \to \pi/2$ then

$$k_y \to (m+1)\frac{\pi}{a} \,. \tag{S-12.68}$$

c) All the above results are valid also for P-polarization, where the electric field of the wave lies in the xy plane. Only (S-12.62) must be replaced by

$$r_\| = e^{i\delta_\|} = \frac{-n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}}{n^2\cos\theta + i\sqrt{\sin^2\theta - n^2}} \,, \tag{S-12.69}$$

corresponding to a different dependence of r and δ on θ.

S-12.7 Wave Propagation in a Filled Waveguide

a) The electric field $\mathbf{E}$ of a monochromatic EM wave of frequency ω propagating in a medium of refractive index $n = n(\omega)$ satisfies Helmholtz's equation

$$\left(\nabla^2 + n^2(\omega)\frac{\omega^2}{c^2}\right)\mathbf{E} = 0 \,. \tag{S-12.70}$$

We are considering a TE mode with $\mathbf{E} = \hat{\mathbf{z}}\,E_z(y)\,e^{ikx - i\omega t}$, thus we have

$$\left(\partial_y^2 - k^2 + n^2(\omega)\frac{\omega^2}{c^2}\right)E_z(y) = 0 \,, \tag{S-12.71}$$

whose general solution has the form s $E_z(y) = A\cos(qy) + B\sin(qy)$, with A and B two arbitrary constants. The electric field being parallel to the conducting walls at $y = \pm a/2$, the boundary conditions are $E_z(y = \pm a/2) = 0$, from which we obtain

$$E_z(y) = E_0 \begin{cases} \cos(q_n y), & n = 1,3,5\ldots \\ \sin(q_n y), & n = 2,4,6\ldots \end{cases}, \qquad q_n = n\frac{\pi}{a}, \tag{S-12.72}$$

and (S-12.71) turns into

$$q_n^2 + k^2 - n^2(\omega)\frac{\omega^2}{c^2} = 0 \,. \tag{S-12.73}$$

The wave can propagate only if k is real, i.e., if $\omega > q_n c \equiv \omega_n$.
  In the case of a plasma

$$q_n^2 + k^2 - \frac{\omega^2 - \omega_p^2}{c^2} = 0 \,, \tag{S-12.74}$$

and the cut-off frequencies are

$$\omega_n' = \sqrt{q_n^2 c^2 + \omega_p^2} \,. \tag{S-12.75}$$

b) The incident wave must be in the $n = 1$ mode, and its electric field is

$$\mathbf{E}_i = \hat{\mathbf{z}}\,E_0\cos(q_1 y)\,e^{ik_1 x - i\omega t} \,, \tag{S-12.76}$$

where $k_1 = \sqrt{\omega^2/c^2 - q_1^2}$. The total electric field is the sum of the incident field $\mathbf{E}_i$ and the reflected field $\mathbf{E}_r$ for $x < 0$, while only the transmitted field $\mathbf{E}_t$ is present in the $x > 0$ region. The boundary conditions at $x = 0$ is $(E_{iz} + E_{rz})|_{x=0} = E_{tz}|_{x=0}$, thus all the waves must have the same dependence on t and y. The total field must thus be

$$E_z = \begin{cases} \left(E_0\, \mathrm{e}^{\mathrm{i}k_1 x} + E_\mathrm{r}\, \mathrm{e}^{-\mathrm{i}k_1 x}\right)\cos(q_1 y)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, & x < 0\,, \\ E_\mathrm{t}\cos(q_1 y)\, \mathrm{e}^{\mathrm{i}k_\mathrm{t} x - \mathrm{i}\omega t}\,, & x > 0\,, \end{cases} \qquad \text{(S-12.77)}$$

where $k_\mathrm{t} = \sqrt{\mathsf{n}(\omega)\omega^2/c^2 - q_1^2}$. The boundary condition on the electric field yields

$$E_0 + E_\mathrm{r} = E_\mathrm{t}\,. \qquad \text{(S-12.78)}$$

In addition, the magnetic field must be also continuous at $x = 0$. From $\partial_t \mathbf{B} = -c\nabla \times \mathbf{E}$ we obtain

$$B_x = \frac{\mathrm{i}\pi c}{\omega a} \times \begin{cases} \left(E_0\, \mathrm{e}^{\mathrm{i}k_1 x} + E_\mathrm{r}\, \mathrm{e}^{-\mathrm{i}k_1 x}\right)\sin(q_1 y)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, & x < 0\,, \\ E_\mathrm{t}\sin(q_1 y)\, \mathrm{e}^{\mathrm{i}k_\mathrm{t} x - \mathrm{i}\omega t}\,, & x > 0\,, \end{cases} \qquad \text{(S-12.79)}$$

$$B_y = -\frac{c}{\omega} \times \begin{cases} k\left(E_0\, \mathrm{e}^{\mathrm{i}k_1 x} - E_\mathrm{r}\, \mathrm{e}^{-\mathrm{i}k_1 x}\right)\cos(q_1 y)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, & x < 0\,, \\ k_\mathrm{t} E_\mathrm{t}\cos(q_1 y)\, \mathrm{e}^{\mathrm{i}k_\mathrm{t} x - \mathrm{i}\omega t}\,, & x > 0\,, \end{cases} \qquad \text{(S-12.80)}$$

We notice that the continuity of B_x is ensured by the condition $E_0 + E_\mathrm{r} = E_\mathrm{t}$, while the continuity of B_y yields

$$k_1(E_0 - E_\mathrm{r}) = k_\mathrm{t} E_\mathrm{t}\,. \qquad \text{(S-12.81)}$$

Eventually, we obtain

$$E_\mathrm{r} = \frac{k_1 - k_\mathrm{t}}{k_1 + k_\mathrm{t}} E_0\,, \qquad E_\mathrm{t} = \frac{2k_1}{k_1 + k_\mathrm{t}} E_0\,, \qquad \text{(S-12.82)}$$

which are identical to Fresnel's formulas for S-polarization. In fact, the field of the incoming wave (S-12.76) can be written as

$$\begin{aligned} \mathbf{E}_\mathrm{i} &= \hat{\mathbf{z}} E_0 \cos(q_1 y)\, \mathrm{e}^{\mathrm{i}k_1 x - \mathrm{i}\omega t} = \hat{\mathbf{z}}\frac{E_0}{2}\left(\mathrm{e}^{\mathrm{i}q_1 y} + \mathrm{e}^{-\mathrm{i}q_1 y}\right)\mathrm{e}^{\mathrm{i}k_1 x - \mathrm{i}\omega t} \\ &= \hat{\mathbf{z}}\frac{E_0}{2}\mathrm{e}^{\mathrm{i}k_1 x + \mathrm{i}q_1 y - \mathrm{i}\omega t} + \hat{\mathbf{z}}\frac{E_0}{2}\mathrm{e}^{\mathrm{i}k_1 x - \mathrm{i}q_1 y - \mathrm{i}\omega t}\,, \end{aligned} \qquad \text{(S-12.83)}$$

which is the superposition of two z-polarized plane waves of equal amplitude, and wavevectors of equal magnitude, but opposite y component, $\mathbf{k} = \hat{\mathbf{x}}k_1 \pm \hat{\mathbf{y}}q_1$. Thus both plane waves impinge on the vacuum-medium interface at the same incidence angle $|\theta| = \arctan(q_1/k_1)$.

S-12.8 Schumann Resonances

a) Substituting the electric field (12.9) into the periodic boundary conditions (12.8) we obtain

$$e^{ik_x L} = 1 \,, \qquad e^{ik_y L} = 1 \,, \tag{S-12.84}$$

solved by

$$k_x = m\frac{2\pi}{L} \,, \qquad k_y = n\frac{2\pi}{L} \,, \qquad m,n = 0,1,2,\dots \tag{S-12.85}$$

where m and n are not allowed to be zero simultaneously, and $L = 2\pi R_\oplus$. Since the wave equations gives us $\omega^2 = k^2 c^2$, we have

$$\omega^2 = \left(\frac{c}{R_\oplus}\right)^2 (m^2 + n^2) \,. \tag{S-12.86}$$

The lowest frequency corresponds to $m = 1, n = 0$ or $m = 0, n = 1$, and its value is

$$\nu_{\min} = \frac{\omega_{\min}}{2\pi} = \frac{\omega_{10}}{2\pi} = \frac{c}{2\pi R_\oplus} \simeq 7.5\,\mathrm{s}^{-1} \,, \tag{S-12.87}$$

corresponding to a wavelength $\lambda_{\max} = 2\pi R_\oplus \simeq 40\,000\,\mathrm{km}$, the length of a great circle of the Earth. The experimentally observed value is $\nu_{\min} \simeq 8\,\mathrm{s}^{-1}$.

b) An ohmic conductor can be considered as perfectly reflecting at a frequency ω if its conductivity $\sigma(\omega)$, assumed to be real, fulfills the condition $\sigma(\omega) \gg \omega/4\pi k_e$, where $k_e = 1$ in Gaussian units, and $k_e = 1/(4\pi\varepsilon_0)$ in SI units. Heuristically, the condition corresponds to the conduction current $\mathbf{J}$ being much larger than Maxwell's displacement current. Since $\varepsilon_0 = 8.854 \times 10^{-12}$ SI units, and $\sigma/\omega \approx 0.6\,\mathrm{s}\,\Omega^{-1}\mathrm{m}^{-1}$, sea water can be considered as a perfect conductor in the frequency range of the Schumann resonances. In Gaussian units, the low-frequency conductivity of sea water is $\sigma \simeq 4 \times 10^{10}\,\mathrm{s}^{-1}$.

A discussion of Schumann resonances based on a "realistic" spherical geometry can be found in Reference [2], Section 8.9 and Problem 8.7. Nevertheless, our simplified approach reveals the essential point that the characteristic length L of the system, which determines the maximum wavelength for a standing wave ($\lambda \approx L$), is the Earth's circumference, rather than the height of the ionosphere above the the Earth's surface.

References

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2. J.D. Jackson, *Classical Electrodynamics*, § 9.2 and 9.4, 3rd Ed., Wiley, New York, London, Sidney (1998)