Chapter S-13
Solutions for Chapter 13

S-13.1 Electrically and Magnetically Polarized Cylinders

(a) Long cylinders. In the "magnetic case", the parallel component of the auxiliary field, $\mathbf{H} = \mathbf{B}/[(\mu_0)\mu_r]$ (here, and the following, the parentheses mean that μ_0 appears in SI units only, not in Gaussian units) is continuous at the lateral surface of the cylinder. Thus the magnetic field inside the cylinder, $\mathbf{B}_i$, is

$$\mathbf{B}_i = \mu_r \mathbf{B}_0 \,. \tag{S-13.1}$$

The interface condition for the electric field is that the parallel component of **E** must be continuous at the lateral surface, thus we have for the internal field

$$\mathbf{E}_i = \mathbf{E}_0 \,. \tag{S-13.2}$$

These results are consistent with the analogy between the equations for **E** in electrostatics and **H** in magnetostatics and in the absence of free currents, i.e., $\nabla \times \mathbf{E} = 0$ and $\nabla \times \mathbf{H} = 0$.

(b) Flat cylinders. In the "magnetic case", the perpendicular component of **B** is continuous at the bases, thus we have

$$\mathbf{B}_i = \mathbf{B}_0 \,. \tag{S-13.3}$$

In the "electric case", the perpendicular component of the auxiliary vector **D** must be continuous at the interface, thus internal field is

$$\mathbf{E}_i = \frac{1}{\varepsilon_r}\mathbf{E}_0 \,. \tag{S-13.4}$$

These results are consistent with the analogy between the equations for **B** and for **D** in electrostatics and in the absence of free charges, i.e., $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{D} = 0$.

A. Macchi et al., *Problems in Classical Electromagnetism*,
DOI 10.1007/978-3-319-63133-2_26

(c) Let us assume (S-13.2) as zero-order solution for the case of the "long" dielectric cylinder. According to (3.1) the cylinder acquires a uniform electric polarization

$$\mathbf{P} = \frac{\varepsilon_r - 1}{4\pi k_e}\,\mathbf{E}_i = \frac{\varepsilon_r - 1}{4\pi k_e}\,\mathbf{E}_0\,, \tag{S-13.5}$$

corresponding to two bound surface charge densities $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \pm P$ at the cylinder bases. When evaluating the field at the cylinder center, due to the condition $a \ll h$ the total bound charges on the two bases can be approximated by two point charges $\pm Q$, with

$$Q = \pi a^2 P = \frac{a^2(\varepsilon_r - 1)}{4k_e}\,E_0 = \begin{cases} \pi a^2 \varepsilon_0 (\varepsilon_r - 1)\,E_0\,, & \text{SI,} \\ \dfrac{a^2(\varepsilon_r - 1)}{4}\varepsilon_r\,E_0\,, & \text{Gaussian,} \end{cases} \tag{S-13.6}$$

located at distances $\pm h/2$. Thus, at the cylinder center we have an additional field

$$E_b \simeq -2k_e\,\frac{Q}{(h/2)^2} = -2\,(\varepsilon_r - 1)\,E_0\left(\frac{a}{h}\right)^2, \tag{S-13.7}$$

corresponding to a second-order correction. The electric field up to the second order in (a/h) is thus

$$\mathbf{E}_i^{(2)} = \mathbf{E}_i + \mathbf{E}_b = \mathbf{E}_0\left[1 - 2(\varepsilon_r - 1)\left(\frac{a}{h}\right)^2\right]. \tag{S-13.8}$$

In the corresponding "magnetic case", the formal analogy between **H** and **E** leads to a second-order correction to the auxiliary field $\mathbf{H}_i$ at the cylinder center

$$\mathbf{H}_b = -2(\mu_r - 1)\,\mathbf{H}_0\left(\frac{a}{h}\right)^2, \tag{S-13.9}$$

where $H_0 = B_0/(\mu_0)$. Because of the formal analogy between **H** and **E**, the correction to **H** at the center of the cylinder can be interpreted as due to the presence of *fictitious equivalent magnetic charges* $Q_m = \pm\pi a^2 M$ on the two cylinder bases. The fictitious magnetic charge densities $\sigma_m = \pm M$ at the two bases are associated to the magnetization $\mathbf{M} = \chi_m \mathbf{H}_0$, where χ_m is given by (5.22) in terms of μ_r. Each magnetic charge gives origin to an auxiliary field

$$\mathbf{H} = \begin{cases} \dfrac{1}{4\pi}\,\dfrac{Q_m}{r^2}\,\hat{\mathbf{r}}, & \text{SI,} \\ \dfrac{1}{c}\,\dfrac{Q_m}{r^2}\,\hat{\mathbf{r}}, & \text{Gaussian.} \end{cases} \tag{S-13.10}$$

Recalling that, in SI units, $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, we obtain for the magnetic field at the cylinder center

$$\mathbf{B} = \mu_0(\mathbf{H}_0 + \mathbf{M}) + \mu_0\mathbf{H}_\mathrm{b} = \mu_r H_0 + \mu_0 H_\mathrm{b} \equiv \mathbf{B}_\mathrm{i} + \mathbf{B}_\mathrm{b}\,, \tag{S-13.11}$$

and the second-order correction is

$$\mathbf{B}_\mathrm{b} = -2(\mu_\mathrm{r} - 1)\,\mathbf{B}_0\left(\frac{a}{h}\right)^2 . \tag{S-13.12}$$

In Gaussian units we have $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, the second order correction remaining the same as in (S-13.12).

Notice that it would have been *wrong* to write

$$\mathbf{B}_\mathrm{b} = (\mu_0)\mu_r\mathbf{H}_\mathrm{b} \quad \text{(wrong!)}\,, \tag{S-13.13}$$

as it would have been wrong to write

$$E_\mathrm{b} \simeq -2\frac{k_\mathrm{e}}{\varepsilon_\mathrm{r}}\frac{Q}{(h/2)^2} = -2\,\frac{(\varepsilon_\mathrm{r} - 1)}{\varepsilon_\mathrm{r}}\,E_0\left(\frac{a}{h}\right)^2 \quad \text{(wrong!)}\,, \tag{S-13.14}$$

instead of (S-13.7), because we are considering the fields generated by polarization charges, and inserting μ_r or ε_r would mean taking the effects of the medium polarization into account twice.

Alternatively, we can recall that the zero-order approximation of the cylinder magnetization is

$$\mathbf{M} = \chi_\mathrm{m}\mathbf{H}_\mathrm{i} = \chi_\mathrm{m}\frac{\mathbf{B}_\mathrm{i}}{(\mu_0)\mu_\mathrm{r}} = \chi_\mathrm{m}\frac{\mathbf{B}_0}{(\mu_0)}\,, \tag{S-13.15}$$

again, μ_0 appearing in SI units only. The magnetization is associated to a surface magnetization current density $\mathbf{K}_\mathrm{m} = \mathbf{M}\times\hat{\mathbf{n}}/b_\mathrm{m}$ on the lateral surface of the cylinder

$$\mathbf{K}_\mathrm{m} = \frac{\chi_\mathrm{m}}{b_\mathrm{m}}\frac{B_0}{(\mu_0)}\,\hat{\boldsymbol{\phi}}\,, \tag{S-13.16}$$

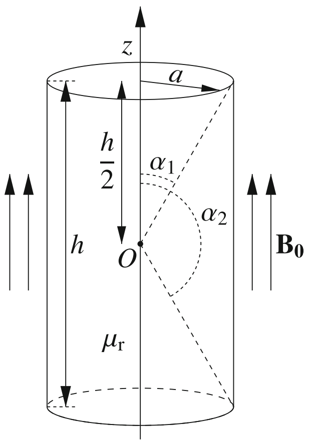

Fig. S-13.1

where $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector of the cylindrical coordinates with the cylinder axis as longitudinal axis. Thus, the cylinder is equivalent to a finite solenoid of height h and radius a, with the product nI equal to a K_m. The magnetic field of a finite solenoid on its axis is

$$\begin{aligned} B_M &= 2\pi k_\mathrm{m} nI\,(\cos\alpha_1 - \cos\alpha_2) = 2\pi k_\mathrm{m} K_\mathrm{m}\,(\cos\alpha_1 - \cos\alpha_2) \\ &= 2\pi k_\mathrm{m}\frac{\chi_\mathrm{m}}{b_\mathrm{m}}\frac{B_0}{(\mu_0)}(\cos\alpha_1 - \cos\alpha_2) = (\mu_\mathrm{r} - 1)\,\frac{B_0}{2}\,(\cos\alpha_1 - \cos\alpha_2)\,, \end{aligned} \tag{S-13.17}$$

where the angles α_1 and α_2 are shown in Fig. S-13.1. At the solenoid center we have

$$\cos\alpha_1 = -\cos\alpha_2 = \frac{h/2}{\sqrt{a^2+(h/2)^2}} \simeq 1 - \frac{1}{2}\left(\frac{2a}{h}\right)^2 = 1 - 2\left(\frac{a}{h}\right)^2 , \tag{S-13.18}$$

thus

$$B_M \simeq (\mu_\mathrm{r}-1)\,B_0\left[1-2\left(\frac{a}{h}\right)^2\right]. \tag{S-13.19}$$

The total field at the cylinder center equals the external field B_0 plus the field due to the cylinder magnetization

$$\begin{aligned} B(0) = B_0 + B_M &= \mu_\mathrm{r}B_0 - 2\mu_\mathrm{r}B_0\left(\frac{a}{h}\right)^2 + 2B_0\left(\frac{a}{h}\right)^2 \\ &= \mu_\mathrm{r}B_0 - 2(\mu_\mathrm{r}-1)B_0\left(\frac{a}{h}\right)^2 , \end{aligned} \tag{S-13.20}$$

in agreement with (S-13.19).

The correction to the field at the center of the magnetic “flat” cylinder can be evaluated as due to a circular loop of radius a carrying an electric current $I_s = K_\mathrm{m}h$:

$$B_\mathrm{b} = \frac{2\pi k_\mathrm{m} I_s}{a} = 2\pi k_\mathrm{m} K_m \frac{h}{a} = 2\pi \frac{k_\mathrm{m}}{b_\mathrm{m}} M \frac{h}{a}. \tag{S-13.21}$$

At zeroth order we have

$$\mathbf{H}_\mathrm{i} \simeq \frac{\mathbf{B}_0}{(\mu_0)\mu_\mathrm{r}}, \quad \text{thus} \quad \mathbf{M} \simeq \chi_\mathrm{m}\frac{\mathbf{B}_0}{(\mu_0)\mu_\mathrm{r}}, \tag{S-13.22}$$

and we get

$$B_\mathrm{b} = \frac{\mu_\mathrm{r}-1}{2\mu_r}\frac{a}{h}B_0 . \tag{S-13.23}$$

The auxiliary field **H** is given by (5.19), thus we have, up to the second order

$$H_\mathrm{i} + H_\mathrm{b} = \frac{B_0 + B_\mathrm{b}}{(\mu_0)\mu_\mathrm{r}} = \begin{cases} \dfrac{B_0 + B_\mathrm{b}}{\mu_0} - M = H_\mathrm{i} + \dfrac{B_\mathrm{b}}{\mu_0} & \text{SI,} \\ B_0 + B_\mathrm{b} - 4\pi M = H_\mathrm{i} + B_\mathrm{b} & \text{Gaussian.} \end{cases} \tag{S-13.24}$$

Thus we have

$$H_\mathrm{b} = \frac{B_\mathrm{b}}{(\mu_0)} = \frac{\mu_\mathrm{r}-1}{2\mu_r}\frac{a}{h}H_0 . \tag{S-13.25}$$

Due to the formal analogy between **H** and **E** we have for the flat dielectric cylinder

$$E_b = \frac{\varepsilon_r - 1}{2\varepsilon_r} \frac{a}{h} E_0 . \tag{S-13.26}$$

S-13.2 Oscillations of a Triatomic Molecule

(a) The equations of motion for the two lateral masses are

$$m\ddot{x}_1 = -k(x_1 - x_c + \ell), \qquad m\ddot{x}_2 = -k(x_2 - x_c - \ell); \tag{S-13.27}$$

from (13.1) we obtain for the position of the central mass

$$x_c = -\frac{m}{M}(x_1 + x_2), \tag{S-13.28}$$

which, substituted into (S-13.27) after dividing by m, leads to a system of two equations of motion involving x_1 and x_2 only

$$\ddot{x}_1 = -k\left(\frac{1}{m} + \frac{1}{M}\right)x_1 - \frac{k}{M}x_2 - \frac{k}{m}\ell, \tag{S-13.29}$$

$$\ddot{x}_2 = -k\left(\frac{1}{m} + \frac{1}{M}\right)x_2 - \frac{k}{M}x_1 + \frac{k}{m}\ell. \tag{S-13.30}$$

Adding and subtracting these equations we obtain

$$\ddot{x}_1 + \ddot{x}_2 = -k\left(\frac{1}{m} + \frac{2}{M}\right)(x_1 + x_2) = -k\frac{M_{tot}}{mM}(x_1 + x_2) \tag{S-13.31}$$

$$\ddot{x}_1 - \ddot{x}_2 = -\frac{k}{m}(x_1 - x_2 + 2\ell), \tag{S-13.32}$$

where $M_{tot} = M + 2m$ is the total mass of the molecule. Thus, introducing the new variables

$$x_+ = x_1 + x_2 \quad \text{and} \quad x_- = x_1 - x_2 + 2\ell, \tag{S-13.33}$$

we obtain the following equations for the normal longitudinal modes of the molecule

$$\ddot{x}_\pm = -\omega_\pm^2 x_\pm, \quad \text{where} \quad \omega_+ = \sqrt{\frac{kM_{tot}}{mM}} \quad \text{and} \quad \omega_- = \sqrt{\frac{k}{m}}. \tag{S-13.34}$$

Frequency ω_+ corresponds an antisymmetric (!) motion of the masses: while the lateral masses move, for instance, to the right by the same amount, the central mass moves to the left, and vice versa, so that $x_{cm} = 0$. Frequency ω_- corresponds to a

symmetric motion: the lateral masses perform opposite oscillations, while the central mass does not move.
(b) The electric dipole moment of the molecule is parallel to the molecular axis and its magnitude is

$$p = -qx_1 + 2qx_c - qx_2 = -q\left(1 + \frac{2m}{M}\right)(x_1 + x_2) = -q\,\frac{M_{tot}}{M}\,x_+ \,. \tag{S-13.35}$$

Thus, the dipole oscillates in the antisymetric mode at frequency ω_+. The dipole moment is zero when the molecule oscillates in the symmetric mode, and radiation at frequency ω_- is due only to quadrupole emission, which is weaker than dipole emission.
(c) The initial conditions for x_+ are

$$x_+(0) = x_1(0) + x_2(0) = d_1 + d_2, \qquad \dot{x}_+(0) = 0\,, \tag{S-13.36}$$

thus for $t > 0$

$$x_+(t) = (d_1 + d_2)\cos\omega_+ t\,. \tag{S-13.37}$$

The symmetric mode is also excited, but does not contribute to the dipole radiation. The instantaneous radiated power is

$$P = \frac{2}{3c^3}\,|\ddot{p}|^2 = \frac{2q^2}{3c^3}\left(\frac{M_{tot}}{M}\right)^2 \omega_+^2\,(d_1 + d_2)^2\cos^2\omega_+ t\,. \tag{S-13.38}$$

S-13.3 Impedance of an Infinite Ladder Network

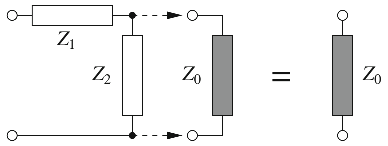

Fig. S-13.2

(a) Our infinite network is a sequence of identical sections. As we did for Problem 4.10, we note that adding a further L-section to the left of of Fig. 13.3 does not change the impedance of the ladder network. Thus we must have (see Fig. S-13.2).

$$Z_0 = Z_1 + \frac{Z_2 Z_0}{Z_2 + Z_0}\,, \tag{S-13.39}$$

from which $Z_0^2 - Z_1 Z_0 - Z_1 Z_2 = 0$ follows. The solution is

$$Z_0 = \frac{Z_1}{2} + \sqrt{\frac{Z_1^2}{4} + Z_1 Z_2}\,, \qquad \text{(S-13.40)}$$

The other solution of the quadratic equation has been discarded because in the case of real, positive impedances (the purely resistive case of Problem 4.10) it would give an unphysical negative value. Thus, a finite ladder of N sections, terminated by an impedance Z_0 as shown in Fig. S-13.3, is equivalent to the infinite ladder.

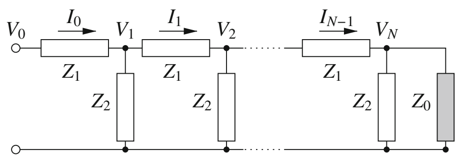

Fig. S-13.3

(b) In Fig. 13.3, current I_n flows through the Z_1 impedance of the $(n+1)$-th section, thus, the voltage drop across the impedance, $V_n - V_{n+1}$, must equal $I_n Z_1$. On the other hand, I_n is input into the semi-infinite ladder network starting at node n, thus we must have $I_n = V_n/Z_0$. The two conditions give

$$V_n - V_{n+1} = \frac{V_n}{Z_0} Z_1\,, \qquad \text{(S-13.41)}$$

so that we obtain for the ratio of the voltages at adjacent nodes

$$\alpha \equiv \frac{V_{n+1}}{V_n} = 1 - \frac{Z_1}{Z_0}\,. \qquad \text{(S-13.42)}$$

If $V_0(t) = V_0\,\mathrm{e}^{-\mathrm{i}\omega t}$ is the input voltage, we have $V_n = \alpha^n V_0\,\mathrm{e}^{-\mathrm{i}\omega t}$ at the n-th node. For a purely resistive network we have

$$Z_0 \equiv R_0 = \frac{R_1}{2} + \sqrt{\frac{R_1^2}{4} + R_1 R_2}\,, \qquad \text{(S-13.43)}$$

which is a real number, and $\alpha = 1 - R_1/R_0 < 1$. At each successive node the signal is damped by a factor α.

(c) For the *LC* network we have

$$Z_0 = -\frac{i\omega L}{2} + \sqrt{-\frac{\omega^2 L^2}{4} + \frac{i\omega L}{i\omega C}} = -\frac{i\omega L}{2} + \sqrt{\frac{L}{C} - \frac{\omega^2 L^2}{4}}$$
$$= \sqrt{\frac{4L^2}{4LC} - \frac{\omega^2 L^2}{4}} - \frac{i\omega L}{2} = \frac{L}{2}\sqrt{\frac{4}{LC} - \omega^2} - \frac{i\omega L}{2}$$
$$= \frac{L}{2}\left(\sqrt{\omega_{co}^2 - \omega^2} - i\omega\right), \tag{S-13.44}$$

where $\omega_{co} \equiv 2/\sqrt{LC}$. Thus

$$\alpha = 1 - \frac{Z_1}{Z_0} = 1 + \frac{2i\omega L}{L\left(\sqrt{\omega_{co}^2 - \omega^2} - i\omega\right)} = \frac{\sqrt{\omega_{co}^2 - \omega^2} + i\omega}{\sqrt{\omega_{co}^2 - \omega^2} - i\omega}. \tag{S-13.45}$$

If $\omega < \omega_{co}$, the square roots are real and α is the ratio of a complex number to its own complex conjugate, therefore $|\alpha| = 1$, and we can write $\alpha = e^{i\phi}$ with

$$\tan\left(\frac{\phi}{2}\right) = \frac{\omega}{\sqrt{\omega_{co}^2 - \omega^2}}. \tag{S-13.46}$$

Thus the voltage at node n is $V_n = V_0\, e^{in\phi - i\omega t}$, and the signal propagates along the network without damping. The above equation also gives the dispersion relation

$$\omega = \omega_{co}\left|\sin\left(\frac{\phi}{2}\right)\right|. \tag{S-13.47}$$

This is analogous to the dispersion relation (S-7.42) found in Problem 7.4, when we substitute ϕ for ka.

If $\omega > \omega_{co}$, Z_0 is a purely imaginary number,

$$Z_0 = \pm i\sqrt{\omega^2 - \omega_{co}}, \tag{S-13.48}$$

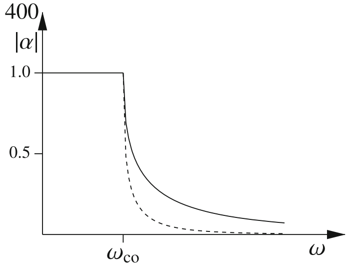

Fig. S-13.4

and α is real

$$\alpha = \frac{\pm\sqrt{\omega^2 - \omega_{co}^2} + \omega}{\pm\sqrt{\omega^2 - \omega_{co}^2} - \omega}. \tag{S-13.49}$$

Inserting the negative root into (S-13.49) leads to $|\alpha| < 1$, and the signal is damped. The positive root would lead to an unphysical $|\alpha| > 1$, implying an amplification of the signal along the network, without an external energy source.

Thus the LC network behaves as a low-pass filter, since signals at frequencies $\omega > \omega_{\rm co}$ are attenuated by a factor $|\alpha|^N$ after N nodes. The dependence of the network transmission on frequency approaches an ideal low-pass filter, for which transmission is zero for $\omega > \omega_{\rm co}$, at high numbers of circuit sections N. Figure S-13.4 shows $|\alpha|$ (solid line) and $|\alpha|^2$ (dashed line) as a functions of the signal frequency ω.

(d) For the CL network (Problem 7.5) we proceed analogously to point **(c)** for the LC network, and obtain

$$Z_0 = \frac{\mathrm{i}}{2\omega C} + \sqrt{-\frac{1}{4\omega^2 C^2} + \frac{\omega L}{\omega C}} = \frac{1}{2C}\left[\sqrt{\frac{1}{\omega_{\rm co}^2} - \frac{1}{\omega^2}} + \frac{\mathrm{i}}{\omega}\right], \qquad \text{(S-13.50)}$$

and

$$\alpha = \frac{\sqrt{\omega_{\rm co}^{-2} - \omega^{-2}} - \mathrm{i}/\omega}{\sqrt{\omega_{\rm co}^{-2} - \omega^{-2}} + \mathrm{i}/\omega}. \qquad \text{(S-13.51)}$$

We have undamped propagation for $|\alpha| = 1$, i.e., when $\omega > \omega_{\rm co}$. For $\omega < \omega_{\rm co}$ the signals are damped, and the network acts as a high-pass filter.

S-13.4 Discharge of a Cylindrical Capacitor

(a) We use cylindrical coordinates (r, ϕ, z). For symmetry reasons, assuming $h \gg b$, the electric field between the capacitor plates is radial, and easily evaluated from Gauss's law as

$$E_r = E_r(r) = \frac{2Q_0}{hr}, \quad \text{(Gaussian units).} \qquad \text{(S-13.52)}$$

The potential difference V across the plates is

$$V = \left|\int_a^b \mathbf{E}\cdot \mathrm{d}\mathbf{s}\right| = \frac{2Q_0}{h}\int_a^b \frac{\mathrm{d}r}{r} = \frac{2Q_0}{h}\ln(b/a), \qquad \text{(S-13.53)}$$

and the capacity of our cylindrical capacitor is

$$C = \frac{Q_0}{V} = \frac{h}{2\ln(b/a)} \qquad \text{(S-13.54)}$$

The initial electrostatic energy is $U_{es}(0) = Q_0^2/2C$.

After the plates are connected through the resistor at $t = 0$, the system is an RC circuit, and the capacitor charge at time t is

$$Q(t) = Q_0\,\mathrm{e}^{-t/\tau}, \quad \text{where} \quad \tau = RC = \frac{Rh}{2\ln(b/a)} \tag{S-13.55}$$

Assuming that the charge densities remain uniform over the plates during the discharge, the absolute value of the charge of each plate between its bottom, $z = 0$, and any height $z < h$ (see Fig. S-13.5 for the case of the inner plate of Fig. 13.4) is

$$\Delta Q(z,t) = Q(t)\,\frac{z}{h}. \tag{S-13.56}$$

The decay of the charge implies a current flowing over each plate, along the $\hat{\mathbf{z}}$ direction. Let $I_a(z,t)$ and $I_b(z,t)$ be the currents in the inner and outer plate, respectively, which can obtained from the continuity equation: for the inner plate

$$I_a(z,t) = -\frac{\mathrm{d}[\Delta Q(z,t)]}{\mathrm{d}t} = \frac{Q(t)}{\tau}\frac{z}{h} = \frac{Q_0}{\tau}\frac{z}{h}\,\mathrm{e}^{-t/\tau}. \tag{S-13.57}$$

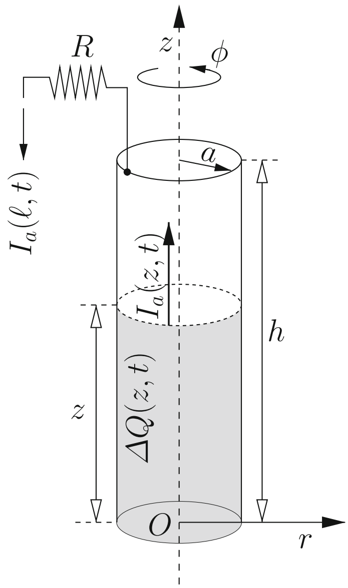

Fig. S-13.5

Since, in the assumption of uniform charge densities, the charge on the outer plate is $-\Delta Q(z,t)$, then $I_b(z,t) = -I_a(z,t)$.

We can evaluate $\mathbf{B}$ in the $a < r < b$ region from Maxwell's equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\,\mathbf{J} + \frac{1}{c}\,\partial_t\mathbf{E}. \tag{S-13.58}$$

The only nonzero component of $\mathbf{J}$ is along z and the only nonzero component of $\mathbf{E}$ is along r, given by

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{2Q(t)}{hr}, \tag{S-13.59}$$

while $\mathbf{B}$ must be independent of ϕ because of the symmetry of our problem. Thus, according to the curl components in cylindrical coordinates of Table A.1 of the Appendix we have

$$(\boldsymbol{\nabla}\times\mathbf{B})_r = -\partial_z B_\phi = \frac{1}{c}\,\partial_t E_r\,,$$
$$(\boldsymbol{\nabla}\times\mathbf{B})_z = \frac{1}{r}\,\partial_r(rB_\phi) = \frac{4\pi}{c}\,J_z\,, \qquad \text{(S-13.60)}$$

and we see that the only nonzero component of **B** is B_ϕ, which can be evaluated from either of (S-13.60). We choose the second of (S-13.60), and apply Stokes' theorem to a circle C of radius $a < r < b$, coaxial to the capacitor and located at height $0 < z < h$,

$$\oint_C \mathbf{B}(r,z,t)\cdot \mathrm{d}\boldsymbol{\ell} = 2\pi r B_\phi(r,z,t) = \frac{4\pi}{c}\,I_a(z,t)\,, \qquad \text{(S-13.61)}$$

$$B_\phi(r,z,t) = \frac{2}{c}\,\frac{I_a(z,t)}{r} = \frac{2}{ch\tau}\,\frac{z}{r}\,Q_0\,\mathrm{e}^{-t/\tau}\,. \qquad \text{(S-13.62)}$$

(b) The Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{B} = \hat{\mathbf{z}}\,\frac{Q_0^2}{\pi h^2\tau}\,\frac{z}{r^2}\,\mathrm{e}^{-2t/\tau}\,, \qquad a < r < b\,, \qquad \text{(S-13.63)}$$

and $\mathbf{S} = 0$ if $r < a$ or $r > b$. The flux of **S** through a plane perpendicular to z at height $0 < z < h$ is thus

$$\Phi_\mathrm{S}(z,t) = \frac{Q_0^2 z}{\pi h^2\tau}\,\mathrm{e}^{-2t/\tau}\int_a^b \frac{1}{r^2}\,2\pi r\,\mathrm{d}r = \frac{2Q_0^2 z\ln(b/a)}{h^2\tau}\,\mathrm{e}^{-2t/\tau}\,. \qquad \text{(S-13.64)}$$

The electrostatic energy associated to the volume between the bottom of the capacitor ($z = 0$) and height z at time t is

$$\Delta U_\mathrm{es}(z,t) = \frac{z}{h}\,\frac{Q^2(t)}{2C} = \frac{z}{h}\,\frac{Q_0^2\,\mathrm{e}^{-2t/\tau}}{2C} = \frac{z}{h}\,\frac{Q_0^2\ln(b/a)}{h}\,\mathrm{e}^{-2t/\tau}\,, \qquad \text{(S-13.65)}$$

because the electric field does not depend on z. Thus we have

$$\frac{\mathrm{d}\,[\Delta U_\mathrm{es}(z,t)]}{\mathrm{d}t} = -\frac{2\Delta U_\mathrm{es}(z,t)}{\tau} = -\Phi_\mathrm{S}(z,t)\,. \qquad \text{(S-13.66)}$$

(c) The assumptions of slowly varying currents and of uniform charge density are closely related. In fact, the capacitor can be viewed as a portion of a coaxial cable along which charge and current signals are propagating in TEM mode, at velocity c. In these conditions, the charge density can be assumed as uniform if the propagation of the signals is "instantaneous" with respect to the duration of the discharge, i.e., if the propagation time $h/c \ll \tau$. This is equivalent to assuming that the wavelengths corresponding to the frequency spectrum of the signal are much larger than h, so that the field can be considered as uniform along z.

We can reach the same conclusion by checking that the electric field $\mathbf{E}_1$, generated by the magnetic induction, is much smaller than the electrostatic field $\mathbf{E}_0$. From Maxwell's equation

$$\boldsymbol{\nabla}\times\mathbf{E}_1 \simeq -\frac{1}{c}\partial_t\mathbf{B}, \tag{S-13.67}$$

where the only nonzero component of $\mathbf{B}$ is B_ϕ, we obtain

$$\begin{aligned}\partial_z E_{1r} &= \frac{1}{c}\,\frac{2}{ch\tau^2}\,\frac{z}{r}\,Q_0\,\mathrm{e}^{-t/\tau}\\ E_{1r} &= \frac{Q_0}{c^2h\tau^2}\,\frac{z^2}{r}\,\mathrm{e}^{-t/\tau} = \frac{1}{2}\left(\frac{z}{c\tau}\right)^2 E_{0r}\,.\end{aligned} \tag{S-13.68}$$

where E_{0r} is from the second of (S-13.59). Thus $E_{1r} \ll E_{0r}$ if $h \ll c\tau$.

S-13.5 Fields Generated by Spatially Periodic Surface Sources

(a) In this case fields and potential are electrostatic. The potential $\varphi = \varphi(x,y)$ is a solution of the 2D Laplace's equation

$$(\partial_x^2+\partial_y^2)\,\varphi = 0 \quad \text{for} \quad y \neq 0, \tag{S-13.69}$$

and, due to the symmetry of the source, must be an even function of y. We attempt to find a solution by the method of separation of variables, i.e., we look for a solution of the form $\varphi = X(x)Y(y)$, where X depends only on x and Y only on y. Equation (S-13.69) becomes

$$X''(x)Y(y)+X(x)Y''(y) = 0, \tag{S-13.70}$$

where the double primes denote the second derivatives. Dividing by $X(x)Y(y)$ we obtain

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)}, \tag{S-13.71}$$

which must hold for every x,y, implying that both sides of the equation must equal some constant value, which, for convenience, we denote by α^2,

$$\frac{Y''(y)}{Y(y)} = \alpha^2, \qquad \frac{X''(x)}{X(x)} = -\alpha^2, \tag{S-13.72}$$

whose solutions are

$$Y(y) = A_y\,\mathrm{e}^{+\alpha y}+B_y\,\mathrm{e}^{-\alpha y}, \quad \text{and} \quad X(x) = A_x\,\mathrm{e}^{+\mathrm{i}\alpha x}+B_x\,\mathrm{e}^{-\mathrm{i}\alpha x}, \tag{S-13.73}$$

where A_x, A_y, B_x, and B_y are constants to be determined. Discarding the solutions that diverge for $|y| \to \infty$, and fitting the x dependence to the dependence of σ, which implies $\alpha = k$, we obtain

$$\varphi = \varphi_0 \, \mathrm{e}^{-k|y|} \cos(kx) \,, \tag{S-13.74}$$

where φ_0 is a constant to be determined. The nonzero components of the electric field are

$$\begin{aligned} E_x &= -\partial_x \varphi = k\varphi_0 \, \mathrm{e}^{-k|y|} \sin(kx) \,, \\ E_y &= -\partial_y \varphi = \mathrm{sgn}(y) \, k\varphi_0 \, \mathrm{e}^{-k|y|} \cos(kx) \,. \end{aligned} \tag{S-13.75}$$

The component E_x is continuous at the $y = 0$ plane, as expected, since $\boldsymbol{\nabla} \times \mathbf{E} = 0$. We can obtain the relation between E_y at $y = 0$ and the surface charge density by using Gauss's law,

$$E_y(x, y = 0^+) - E_y(x, y = 0^-) = 4\pi\sigma(x) \,, \tag{S-13.76}$$

from which we obtain the value of φ_0, namely $\varphi_0 = 2\pi\sigma_0/k$, and, finally

$$\varphi = \frac{2\pi\sigma_0}{k} \, \mathrm{e}^{-k|y|} \cos(kx) \,. \tag{S-13.77}$$

(b) Here we have magnetostatic fields. Due to the analogy between the Poisson equations for the vector potential $\nabla^2 \mathbf{A} = -4\pi\mathbf{J}/c$, and for the scalar potential $\nabla^2 \varphi = -4\pi\rho$, we can use (S-13.77) for obtaining the vector potential $\mathbf{A}$ as

$$\mathbf{A} = \hat{\mathbf{z}} A_0 \, \mathrm{e}^{-k|y|} \cos(kx) \,, \quad \text{where} \quad A_0 = \frac{2\pi K_0}{kc} \,. \tag{S-13.78}$$

The nonzero components of the magnetic field are

$$\begin{aligned} B_x &= \partial_y A_z = -\,\mathrm{sgn}(y) \, kA_0 \, \mathrm{e}^{-k|y|} \cos(kx) = -\,\mathrm{sgn}(y) \, \frac{2\pi K_0}{c} \, \mathrm{e}^{-k|y|} \cos(kx) \,, \\ B_y &= -\partial_x A_z = kA_0 \, \mathrm{e}^{-k|y|} \sin(kx) = \frac{2\pi K_0}{c} \, \mathrm{e}^{-k|y|} \sin(kx) \,. \end{aligned} \tag{S-13.79}$$

Thus, B_y is continuous at $y = 0$, as expected from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Further we have

$$B_x(x, y = 0^+) - B_x(x, y = 0^-) = -\frac{4\pi}{c} K_0 \cos(kx) \,, \tag{S-13.80}$$

in agreement with Ampère's law.

(c) Since $\sigma = 0$, also the scalar potential is zero, $\varphi = 0$. The inhomogeneous wave equation for the vector potential $\mathbf{A}$ is, in the Lorentz gauge condition,

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} = -\hat{\mathbf{z}}\,\frac{4\pi}{c}\,\delta(y)\,K_0\,\mathrm{e}^{-\mathrm{i}\omega t}\cos(kx)\,. \tag{S-13.81}$$

As an educated guess, we search for a solution of the form

$$\mathbf{A} = \hat{\mathbf{z}} A_0\,\mathrm{e}^{-q|y|-\mathrm{i}\omega t}\cos(kx)\,, \tag{S-13.82}$$

which, for $y \neq 0$, leads to

$$\left(-k^2 + q^2 + \frac{\omega^2}{c^2}\right)\mathbf{A} = 0\,, \quad \text{or} \quad q^2 = k^2 - \left(\frac{\omega}{c}\right)^2 . \tag{S-13.83}$$

Thus, if $\omega < kc$, q is real and $\mathbf{A}$ decays exponentially with $|y|$. If $\omega > kc$, q is imaginary and the waves propagates, $\mathbf{A}$ being proportional to $\mathrm{e}^{\mathrm{i}|q||y|-\mathrm{i}\omega t}$. If we integrate (S-13.81) in dy from $-h$ to $+h$ we obtain

$$\lim_{h\to 0}\int_{-h}^{+h}\left(\frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right)\mathrm{d}y = -\hat{\mathbf{z}}\,\frac{4\pi}{c}\,K_0\,\mathrm{e}^{-\mathrm{i}\omega t}\cos(kx)\,. \tag{S-13.84}$$

Now, both $\partial^2\mathbf{A}/\partial x^2$ and $\partial^2\mathbf{A}/\partial t^2$ are continuous at $y = 0$ and don't contribute to the integral at the limit $h \to 0$. Thus, the left-hand side of (S-13.84) is

$$\begin{aligned}\lim_{h\to 0}\int_{-h}^{+h}\frac{\partial^2 \mathbf{A}}{\partial y^2}\,\mathrm{d}y &= \lim_{h\to 0}\Big[\partial_y \mathbf{A}\Big]_{-h}^{+h} = -\hat{\mathbf{z}} A_0\,q\cos(kx)\,\lim_{h\to 0}\Big[\mathrm{sgn}(y)\,\mathrm{e}^{-q|y|-\mathrm{i}\omega t}\Big]_{-h}^{+h} \\ &= -\hat{\mathbf{z}}\,2A_0\mathrm{e}^{-\mathrm{i}\omega t}\,q\cos(kx)\,,\end{aligned} \tag{S-13.85}$$

which must equal the right-hand side of (S-13.84), leading to

$$A_0 = \frac{2\pi}{qc}K_0 \tag{S-13.86}$$

which, at the static limit $\omega \to 0$, $q \to k$, equals (S-13.78).

The nonzero components of the magnetic field are

$$\begin{aligned}B_x &= \quad \partial_y A_z = -\,\mathrm{sgn}(y)\,\frac{2\pi}{c}\,K_0\,\mathrm{e}^{-q|y|-\mathrm{i}\omega t}\cos(kx)\,, \\ B_y &= -\partial_x A_z = \frac{2\pi k}{qc}\,K_0\,\mathrm{e}^{-q|y|-\mathrm{i}\omega t}\sin\omega t\,,\end{aligned} \tag{S-13.87}$$

which, at the static limit $\omega \to 0$, $q \to k$, equal (S-13.79). The electric field is obtained from $\mathbf{E} = -\partial_t \mathbf{A}/c = \mathrm{i}\omega\mathbf{A}/c$, and its only nonzero component is

$$E_z = -\frac{2\pi\mathrm{i}\omega K_0}{qc^2}\,\mathrm{e}^{-q|y|-\mathrm{i}\omega t}\cos(kx)\,. \tag{S-13.88}$$

(d) In this context, given a function $f = f(x,t)$, we denote its time average by angle brackets, and its space average by a bar, as follows

$$\langle f \rangle = \frac{\omega}{2\pi} \int_{-\pi/\omega}^{+\pi/\omega} f \, \mathrm{d}t, \qquad \overline{f} = \frac{k}{2\pi} \int_{-\pi/k}^{+\pi/k} f \, \mathrm{d}x. \tag{S-13.89}$$

Thus we write the average power dissipated per unit time and unite surface on the $y = 0$ plane as

$$\left\langle \overline{K_z E_z} \right\rangle = \frac{1}{2} \, \mathrm{Re}\left[K_0 \left(\frac{2\pi \mathrm{i}\omega}{qc} K_0 \right)^* \right] \overline{\cos^2(kx)} = \frac{\pi\omega}{4c} |K_0|^2 \, \mathrm{Re}\left(\frac{-\mathrm{i}}{q} \right). \tag{S-13.90}$$

If q is real we have $\left\langle \overline{K_z E_z} \right\rangle = 0$, consistently with the fields being evanescent for $|y| \to \infty$. There is no energy flow out of the $y = 0$ plane, and the work done by the currents is zero on average. On the other hand, if q is imaginary, we have

$$\left\langle \overline{K_z E_z} \right\rangle = -\frac{\pi\omega |K_0|^2}{2|q|c}, \tag{S-13.91}$$

which equals minus the flux of electromagnetic energy out of the $y = 0$ plane. In fact, the averaged Poynting vector is

$$\begin{aligned} \left\langle \overline{S_y} \right\rangle &= \frac{c}{4\pi} \left\langle \overline{E_z B_x} \right\rangle = \frac{1}{2} \frac{c}{4\pi} \, \mathrm{Re}\left[\frac{2\pi \mathrm{i}\omega K_0}{2qc} \left(\mathrm{sgn}(y)\, q^* \frac{2\pi K_0^*}{q^* c} \right) \right] \overline{\cos^2(kx)} \\ &= \mathrm{sgn}(y) \frac{\pi\omega |K_0|^2}{4|q|c}, \end{aligned} \tag{S-13.92}$$

where we have used $\mathrm{Re}(\mathrm{i}/q) = 1/|q|$ (for imaginary q). The flux of energy out of the $y = 0$ plane is thus $2\left| \left\langle \overline{S_y} \right\rangle \right| = -\left\langle \overline{K_z E_z} \right\rangle$.

S-13.6 Energy and Momentum Flow Close to a Perfect Mirror

(a) The total electric field in front of the mirror is the sum of the fields of the incident ($\mathbf{E}_\mathrm{i}$) and of the reflected ($\mathbf{E}_\mathrm{r}$) waves, which have equal amplitude and frequency, but opposite polarizations and wavevectors,

$$
\begin{aligned}
\mathbf{E} = \mathbf{E}_\mathrm{i} + \mathbf{E}_\mathrm{r} &= \hat{\mathbf{y}}\, E_\epsilon \left[\cos(kx-\omega t) - \cos(-kx-\omega t)\right] \\
&\quad - \hat{\mathbf{z}}\,\epsilon E_\epsilon \left[\sin(kx-\omega t) - \sin(-kx-\omega t)\right] \\
&= \hat{\mathbf{y}}\, E_\epsilon [\cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t) - \cos(kx)\cos(\omega t) + \sin(kx \sin(\omega t)] \\
&\quad - \hat{\mathbf{z}}\,\epsilon E_\epsilon [\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t) + \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)] \\
&= \hat{\mathbf{y}}\, 2E_\epsilon \sin(kx)\sin(\omega t) - \hat{\mathbf{z}}\, 2\epsilon E_\epsilon \sin(kx)\cos(\omega t)\,,
\end{aligned}
\qquad \text{(S-13.93)}
$$

where $E_\epsilon \equiv E_0/\sqrt{1+\epsilon^2}$. We can obtain the magnetic field from Maxwell's equation

$$
\begin{aligned}
\partial_t \mathbf{B} &= -c\boldsymbol{\nabla}\times\mathbf{E} = \hat{\mathbf{y}}\, c\,\partial_x E_z - \hat{\mathbf{z}}\, c\,\partial_x E_y \\
&= -\hat{\mathbf{y}}\, 2\epsilon E_\epsilon\, ck \cos(kx)\cos(\omega t) - \hat{\mathbf{z}}\, 2E_\epsilon\, ck \cos(kx)\sin(\omega t)\,,
\end{aligned}
\qquad \text{(S-13.94)}
$$

which yields, after integration in dt,

$$
\begin{aligned}
\mathbf{B} &= -\hat{\mathbf{y}}\, 2\epsilon E_\epsilon \frac{ck}{\omega} \cos(kx)\sin(\omega t) + \hat{\mathbf{z}}\, 2E_\epsilon \frac{ck}{\omega}\cos(kx)\cos(\omega t) \\
&= -\hat{\mathbf{y}}\, 2\epsilon E_\epsilon \cos(kx)\sin(\omega t) + \hat{\mathbf{z}}\, 2E_\epsilon \cos(kx)\cos(\omega t)\,,
\end{aligned}
\qquad \text{(S-13.95)}
$$

where we have used $k = \omega/c$. The Poynting vector is

$$
\begin{aligned}
\mathbf{S} &= \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = \hat{\mathbf{x}}\,\frac{c}{4\pi}\left(E_y B_z - E_z B_y\right) \\
&= \hat{\mathbf{x}}\,\frac{cE_\epsilon^2}{\pi}\left[\sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t) - \epsilon^2 \sin(kx)\cos(kx)\cos(\omega t)\sin(\omega t)\right] \\
&= \hat{\mathbf{x}}\,\frac{cE_\epsilon^2}{\pi}\sin(kx)\cos(kx)\sin(\omega t)\cos(\omega t)\left(1-\epsilon^2\right) \\
&= \hat{\mathbf{x}}\,\frac{c}{4\pi}E_\epsilon^2\left(1-\epsilon^2\right)\sin(2kx)\sin(2\omega t)\,.
\end{aligned}
\qquad \text{(S-13.96)}
$$

Thus $\mathbf{S} = 0$ if $\epsilon = 1$, corresponding to circular polarization. In such a case, $\mathbf{E}$ is parallel to $\mathbf{B}$. In general, also when $\mathbf{S} \neq 0$, we have $\langle \mathbf{S} \rangle = 0$, and there is no net energy flow.

(b) From the definition of T_{ij} we find

$$
\begin{aligned}
F_x = T_{xx} &= \frac{1}{8\pi}\mathbf{B}^2(0^-) = \frac{1}{4\pi}E_\epsilon^2(\cos^2\omega t + \epsilon^2\sin^2\omega t) \\
&= \frac{2I}{c}\left[1 + \frac{1-\epsilon^2}{1+\epsilon^2}\cos 2\omega t\right].
\end{aligned}
\qquad \text{(S-13.97)}
$$

The oscillating (at 2ω) component vanishes for circular polarization. The average of F_x is the radiation pressure on the mirror (Problem 9.8), which does not depend on polarization.

S-13.7 Laser Cooling of a Mirror

(a) A plane wave of intensity I exerts a radiation pressure $2I/c$ on a perfectly reflecting surface. Thus the total force on the mirror, directed along the x axis of Fig. 13.5, is

$$F = \frac{2A}{c}(I_1 - I_2). \tag{S-13.98}$$

If $I_1 > I_2$ we have $F > 0$.
(b) The amplitudes of the electric fields of the two waves, in the mirror rest frame S', are

$$E_1' = \gamma(E_1 - \beta B_1) = \gamma(1-\beta)E_1 = \sqrt{\frac{1-\beta}{1+\beta}}\,E_1\,, \tag{S-13.99}$$

$$E_2' = \gamma(E_2 + \beta B_2) = \gamma(1+\beta)E_2 = \sqrt{\frac{1+\beta}{1-\beta}}\,E_2\,, \tag{S-13.100}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$, and $E_1 = B_1$, $E_2 = B_2$ in Gaussian units. The intensity of a plane wave is $I = (c/4\pi)|\mathbf{E}\times\mathbf{B}| = cE^2/4\pi$, thus we have

$$I_1' = \frac{1-\beta}{1+\beta}\,I_1\,, \qquad I_2' = \frac{1+\beta}{1-\beta}\,I_2\,. \tag{S-13.101}$$

Since we have assumed $I_1 = I_2$, the total force is

$$F' = \frac{2A}{c}(I_1' - I_2') = \frac{2A}{c}\,\frac{(1-\beta)^2 - (1+\beta)^2}{1-\beta^2} = -8A\beta\gamma^2\,\frac{I}{c}\,. \tag{S-13.102}$$

(c) From the answer to point **(b)** we have $F' < 0$, the direction of the force is opposite to the direction of $\boldsymbol{v}$. At the limit $v \ll c$, the force in the laboratory frame is equal to the force in the mirror frame, and we have

$$\mathbf{F} \simeq \mathbf{F}' \simeq -8A\,\frac{I}{c^2}\,\boldsymbol{v}\,, \tag{S-13.103}$$

which is a viscous force. Under the action of this force, the mirror velocity will decrease exponentially in time

$$v(t) = v(0)\,\mathrm{e}^{-t/\tau}\,, \quad \text{where} \quad \tau = \frac{Mc^2}{8AI}\,. \tag{S-13.104}$$

This effect has some analogies with the “laser-cooling” techniques, used in order to cool atoms down to temperatures of the order of 10^{-6} K. These include, for instance, Doppler cooling and Sisyphus cooling. The cooling of a macroscopic mirror by radiation pressure has also been studied [1] for possible applications in experiments of optical interferometry of ultra-high precision, e.g., for the detection of gravitational waves.

S-13.8 Radiation Pressure on a Thin Foil

(a) It is instructive to solve this problem by three different methods. For definiteness we assume a linearly polarized incident wave, with electric field $\mathbf{E}_\mathrm{i} = \hat{\mathbf{y}}\, E_\mathrm{i}\, \mathrm{e}^{\mathrm{i}k_i x - \mathrm{i}\omega t}$, where $k_i = \omega/c$; generalization to arbitrary polarization is straightforward.

First method (heuristic): we assume the incident plane wave to be a square pulse of arbitrary but finite duration τ, and thus length $c\tau$. The momentum of the wave packet impinging on the surface A of the foil is, neglecting boundary effects,

$$\mathbf{p}_\mathrm{i} = \frac{\langle \mathbf{S}_\mathrm{i} \rangle}{c^2}\, c\tau A = \hat{\mathbf{x}}\, \frac{\left\langle |\mathbf{E}_\mathrm{i}|^2 \right\rangle}{4\pi c}\, \tau A = \hat{\mathbf{x}}\, \frac{E_i^2}{8\pi c}\, \tau A = \hat{\mathbf{x}}\, \frac{I}{c}\, \tau A\,, \tag{S-13.105}$$

where $I = \langle |\mathbf{S}_\mathrm{i}| \rangle = c\left\langle |\mathbf{E}_\mathrm{i}|^2 \right\rangle/(4\pi) = cE_i^2/(8\pi)$ is the intensity of the incident wave. The reflected and transmitted wave packets have momenta

$$\mathbf{p}_\mathrm{r} = \frac{\langle \mathbf{S}_\mathrm{r} \rangle}{c^2}\, c\tau A = -\hat{\mathbf{x}} R\, \frac{E_i^2}{8\pi c}\, \tau A = -\hat{\mathbf{x}} R\, \frac{I}{c}\, \tau A\,, \tag{S-13.106}$$

$$\mathbf{p}_\mathrm{t} = \frac{\langle \mathbf{S}_\mathrm{t} \rangle}{c^2}\, c\tau A = +\hat{\mathbf{x}} T\, \frac{E_i^2}{8\pi c}\, \tau A = +\hat{\mathbf{x}} T\, \frac{I}{c}\, \tau A\,, \tag{S-13.107}$$

respectively, where $R = |\mathsf{r}|^2$, $T = |\mathsf{t}|^2$, and $R + T = 1$ because of energy conservation. The amount of momentum transfered from the incident wave packet to the foil is

$$\Delta\mathbf{p} = \mathbf{p}_\mathrm{i} - (\mathbf{p}_\mathrm{r} + \mathbf{p}_\mathrm{t})\,, \tag{S-13.108}$$

resulting in a pressure pushing the foil toward positive x values (because $\Delta\mathbf{p} > 0$)

$$P_\mathrm{rad} = \frac{|\Delta\mathbf{p}|}{\tau A} = [1 - (-R + T)]\, \frac{I}{c} = 2R\, \frac{I}{c}\,. \tag{S-13.109}$$

Second method: we calculate the average force on the foil, parallel to $\hat{\mathbf{x}}$, directly as

$$\langle \mathbf{F} \rangle = \int_0^d \langle \mathbf{J} \times \mathbf{B} \rangle\, A\, \mathrm{d}x\,, \tag{S-13.110}$$

where we have assumed the left surface of the foil located at $x = 0$, and the right surface located at $x = d$. For a very small thickness d we can write

$$A\int_0^d \langle \mathbf{J}\times\mathbf{B}\rangle\,\mathrm{d}x = \frac{1}{2}Ad\langle J(t)\,[B(0^+)+B(0^-)]\rangle$$
$$= -\frac{Ac}{8\pi}\left\langle\left[B^2(0^+)-B^2(0^-)\right]\right\rangle, \qquad \text{(S-13.111)}$$

where we have substituted $J(t) = -[B(0^+)-B(0^-)]\,c/(4\pi d)$. Since we have $|B(0^+)| = |E_\mathrm{t}| = |\mathsf{t}\,E_\mathrm{i}|$, and $|B(0^-)| = \left|E_\mathrm{i}-E_\mathrm{r}\right| = \left|(1-\mathsf{r})\,E_\mathrm{i}\right|$, we can write the radiation pressure on the foil as

$$P_\mathrm{rad} = \frac{\langle|\mathbf{F}|\rangle}{A} = -\frac{E_\mathrm{i}^2}{16\pi c}\left(|\mathsf{t}|^2-|1-\mathsf{r}|^2\right) = -\frac{I}{2c}\left(|\mathsf{t}|^2-|1-\mathsf{r}|^2\right). \qquad \text{(S-13.112)}$$

Introducing the shorthand $\alpha = (\omega_\mathrm{p}^2 d)/(2\omega c)$ in (13.5), so that $\eta = \mathrm{i}\alpha$, we have

$$\mathsf{t} = \frac{1}{1+\mathrm{i}\alpha}, \qquad T = |\mathsf{t}|^2 = \frac{1}{1+\alpha^2}, \qquad \text{(S-13.113)}$$
$$\mathsf{r} = -\frac{\mathrm{i}\alpha}{1+\mathrm{i}\alpha}, \qquad R = |\mathsf{r}|^2 = \frac{\alpha^2}{1+\alpha^2}, \qquad \text{(S-13.114)}$$
$$|1-\mathsf{r}|^2 = \frac{1+5\alpha^2+4\alpha^4}{(1+\alpha^2)^2}, \qquad |\mathsf{t}|^2-|1-\mathsf{r}|^2 = -\frac{4\alpha^2}{1+\alpha^2}, \qquad \text{(S-13.115)}$$

and finally

$$P_\mathrm{rad} = \frac{I}{2c^2}\,\frac{4\alpha^2}{1+\alpha^2} = 2R\,\frac{I}{c}. \qquad \text{(S-13.116)}$$

Third method: we calculate the flow of EM momentum directly using Maxwell's stress tensor T_{ij}. The theorem of EM momentum conservation states that

$$\frac{\mathrm{d}\mathbf{p}_\mathrm{i}}{\mathrm{d}t} = \oint_S T_{ij}n_j\,\mathrm{d}S \qquad \text{(S-13.117)}$$

(summation over the repeated index is implied), where $\hat{\mathbf{n}}$ is the unit vector normal to the surface S which envelops the thin foil, and $\mathbf{p}$ is the total momentum (EM and mechanical) of the foil. Since in a steady state the EM contribution is constant, the RHS of (S-13.117) equals the variation of mechanical momentum, i.e., the force.

Taking into account that the electric field has only the component E_y and the magnetic field only the component B_z, and that $\hat{\mathbf{n}} = \mp\hat{\mathbf{x}}$ on the left ($x = 0^-$) and right ($x = 0^+$) surfaces, respectively, the only relevant component of T_{ij} is T_{xx}, and

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = [T_{xx}(0^+)-T_{xx}(0^-)]\,A. \qquad \text{(S-13.118)}$$

For $T_{xx}(0^+)$ and $T_{xx}(0^-)$ we have

$$
\begin{aligned}
T_{xx}(0^+) &= -\frac{1}{8\pi}\left\langle E^2(0^+) + B^2(0^+)\right\rangle = -\frac{1}{4\pi}\left\langle |E_\mathrm{t}|^2(0^+)\right\rangle \\
&= -T\,\frac{E_\mathrm{i}^2}{8\pi}\,, \\
T_{xx}(0^-) &= -\frac{1}{8\pi}\left\langle E^2(0^-) + B^2(0^-)\right\rangle = -\frac{\left\langle E_\mathrm{i}^2\right\rangle}{8\pi}\left[|(1+\mathrm{r})|^2 + |(1-\mathrm{r})|^2\right] \\
&= -\frac{E_\mathrm{i}^2}{16\pi}\left(1 + |\mathrm{r}|^2 + \mathrm{r}\mathrm{r}^* + 1 + |\mathrm{r}|^2 - \mathrm{r}\mathrm{r}^*\right) \\
&= -(1+R)\,\frac{E_\mathrm{i}^2}{8\pi}\,.
\end{aligned}
\qquad \text{(S-13.119)}
$$

Thus

$$
\frac{\mathrm{d}p_x}{\mathrm{d}t} = \frac{E_\mathrm{i}^2}{8\pi}(-T + 1 + R)A = 2R\,\frac{I}{c}\,A\,, \qquad \text{(S-13.120)}
$$

which yields (13.6) again.
(b) From the Lorentz transformation of the fields we obtain the intensity of the incident wave in the S' frame, where the foil is at rest,

$$
I' = \frac{1-\beta}{1+\beta}\,I\,, \qquad \text{(S-13.121)}
$$

and the force on the foil in S' is $F' = 2AI'/c$. Since for a force parallel to $\boldsymbol{v}$ we have $F = F'$, in the laboratory frame S we can write

$$
F = F' = 2\,\frac{1-\beta}{1+\beta}\,\frac{I}{c}\,A\,. \qquad \text{(S-13.122)}
$$

(c) The radiation pressure must be multiplied by a factor $R = R(\omega')$ in the frame S', where the frequency is $\omega' = \sqrt{(1-\beta)/(1+\beta)}\,\omega$. Thus

$$
F = 2\,\frac{1-\beta}{1+\beta}\,R(\omega')\,\frac{I}{c}\,A\,, \qquad \omega' = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega\,. \qquad \text{(S-13.123)}
$$

S-13.9 Thomson Scattering in the Presence of a Magnetic Field

(a) We write the fields in the complex notation. Within our assumptions, the equation of motion for the electron is

$$m_e \frac{d\boldsymbol{v}}{dt} = -e\left(\mathbf{E} + \frac{\boldsymbol{v}}{c} \times \mathbf{B}_0\right), \tag{S-13.124}$$

where $-e$ and m_e are the charge and mass of the electron, respectively. The solution has already been evaluated in Problem 7.10, and is

$$v_x = \frac{\omega_c}{\omega_c^2 - \omega^2} \frac{e}{m_e} E_i\, e^{-i\omega t}, \quad v_y = \frac{i\omega}{\omega_c^2 - \omega^2} \frac{e}{m_e} E_i\, e^{-i\omega t}, \tag{S-13.125}$$

and $v_z = 0$.
(b) The cycle-averaged radiated power is

$$\langle P \rangle = \frac{e^4}{3m_e^2 c^3} |E_i|^2 \frac{\omega^2}{\left(\omega_c^2 - \omega^2\right)^2} \left(\omega_c^2 + \omega^2\right), \tag{S-13.126}$$

which is maximum at the cyclotron resonance, $\omega = \omega_c$. At the low-frequency limit $\omega/\omega_c \ll 1$ we have $\langle P \rangle \propto \omega^2/\omega_c^2$, while at the high-frequency limit $\omega/\omega_c \gg 1$ the power is independent of frequency ("white" spectrum).

The orbit of the electron is elliptical, consequently the angular distribution and polarization of the scattered radiation are analogous to what found for an electron in the presence of an elliptically polarized wave, in the absence of external magnetic fields, as discussed in Problem 10.9. According to (S-13.125) we have $v_x/v_y = -i\omega_c/\omega$. At the limit $\omega \ll \omega_c$ we have $\left\langle |v_x| \right\rangle \gg \left\langle |v_y| \right\rangle$, the major axis of the elliptical orbit of the electron is thus parallel to $\hat{\mathbf{x}}$, and the strongest radiation intensity is observed on the yz plane. At the opposite limit, $\omega \gg \omega_c$, we have $\left\langle |v_x| \right\rangle \ll \left\langle |v_y| \right\rangle$, the major axis of the orbit is parallel to $\hat{\mathbf{y}}$, and the strongest radiation intensity is observed on the xz plane.

S-13.10 Undulator Radiation

(a) According to Maxwell's equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, we must have

$$\partial_x B_x = -\partial_y B_y = -(\partial_y b) \cos(kx), \tag{S-13.127}$$

which, after integration in dx, leads to

$$B_x = -(\partial_y b)\,\frac{\sin(kx)}{k}\,, \tag{S-13.128}$$

where we have set to zero the integration constant. In static conditions, and in the absence of electric currents, we have $\boldsymbol{\nabla}\times\mathbf{B} = 0$, thus we must also have

$$0 = \partial_x B_y - \partial_y B_x = -kb(y)\,\sin(kx) + (\partial_y^2 b)\,\frac{\sin(kx)}{k}\,, \tag{S-13.129}$$

which, divided by $\sin(kx)$, reduces to

$$\partial_y^2 b(y) = k^2 b(y)\,. \tag{S-13.130}$$

The even solution (S-13.130) is

$$b(y) = B_0 \cosh(ky)\,, \tag{S-13.131}$$

where B_0 is a constant to be determined. Thus the two nonzero components of **B** are

$$B_x = -B_0 \sin(kx)\sinh(ky)\,, \qquad B_y = B_0 \cos(kx)\cosh(ky)\,, \tag{S-13.132}$$

and on the z axis, where $x = 0$ and $y = 0$, we have

$$\mathbf{B}(0,0,z) = \hat{\mathbf{y}} B_0\,. \tag{S-13.133}$$

(b) The Lorentz transformations from the laboratory frame S to S' give for the fields in S'

$$B'_x = B_x[x(x',t'),y'] = -B_0 \sinh(ky')\sin[k\gamma(x'+vt')]\,, \tag{S-13.134}$$

$$B'_y = \gamma B_y[x(x',t'),y'] = \gamma B_0 \cosh(ky')\cos[k\gamma(x'+vt')]\,, \tag{S-13.135}$$

$$E'_z = \gamma v B_y[x(x',t'),y'] = \gamma v B_0 \cosh(ky')\cos[k\gamma(x'+vt')]. \tag{S-13.136}$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$. Since the boost is parallel to the x axis, we have $y' = y$.

Disregarding the magnetic force in S', the electron oscillates along $\hat{\mathbf{z}}'$ under the action of the electric field $E' = E'_z(0,0,t') = \gamma v B_0 \cos(\omega' t')$, where $\omega' = k\gamma v$. Thus, in S', we observe a Thomson scattering, and the electron emits electric-dipole radiation of frequency ω'.

(d) Transforming back to S, the frequencies of the radiation emitted in the forward (+) and backward (−) directions are

$$\omega_\pm = \gamma(1\pm\beta)\,\omega' = \gamma(1\pm\beta)k\gamma v = kc\gamma^2\beta(1\pm\beta)\,, \tag{S-13.137}$$

where $\beta = v/c$.

In S', the electron does not emit radiation along its direction of oscillation, i.e., along $\hat{\mathbf{z}}'$. This corresponds to two "forbidden" wavevectors $\mathbf{k}' \equiv (0,0,\omega'/c)$ and $\mathbf{k}' \equiv (0,0,-\omega'/c)$. By a back transformation to S we obtain

$$k_x = \gamma\left(k'_x \pm \frac{\omega'}{c}\beta\right) = \pm\gamma\beta\,\frac{\omega'}{c}, \quad k_y = 0, \quad k_z = k'_z = \frac{\omega'}{c}, \tag{S-13.138}$$

thus, in S, we have no radiation emission at the angles $\pm\theta$ in the xz plane such that

$$\tan\theta = \frac{k_z}{k_x} = \frac{1}{\gamma\beta}. \tag{S-13.139}$$

The "undulator radiation", emitted by high-energy electrons injected along a periodically modulated magnetic field, is at the basis of free-electron lasers emitting coherent radiation in the X-ray frequency range.

S-13.11 Electromagnetic Torque on a Conducting Sphere

(a) We can write the electric field of the wave as

$$\begin{aligned}\mathbf{E}(z,t) &= E_0[\hat{\mathbf{x}}\cos(kz-\omega t) - \hat{\mathbf{y}}\sin(kz-\omega t)] \\ &= \mathrm{Re}[E_0(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})\,\mathrm{e}^{\mathrm{i}(kz-\omega t)}],\end{aligned} \tag{S-13.140}$$

where $k = \omega/c = 2\pi/\lambda$. Since $a \ll \lambda$, we can consider the electric and magnetic fields of the wave as uniform over the volume of the sphere, and neglect the magnetic induction effects. Thus, the sphere can be considered as located in a uniform rotating electric field

$$\mathbf{E}_0(t) = \mathrm{Re}\left(\tilde{\mathbf{E}}_0\,\mathrm{e}^{-\mathrm{i}\omega t}\right), \quad \text{where} \quad \tilde{\mathbf{E}}_0 = E_0(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}}). \tag{S-13.141}$$

In the presence of oscillating fields, the complex electric permittivity of a medium of real conductivity σ is defined as

$$\tilde{\varepsilon}(\omega) = 1 + \frac{4\pi\mathrm{i}\sigma}{\omega}. \tag{S-13.142}$$

Thus, our problem is analogous to Problem 3.4, where we considered a dielectric sphere in a uniform external electric field. The internal electric field and the dipole moment of the sphere are

$$\tilde{\mathbf{E}}_{\text{int}} = \frac{3}{\tilde{\varepsilon}+2}\,\tilde{\mathbf{E}}_0 = \frac{3\tilde{\mathbf{E}}_0}{3+4\pi\mathrm{i}\sigma/\omega} = -\frac{3\mathrm{i}\omega t_\mathrm{d}}{1-3\mathrm{i}\omega t_\mathrm{d}}\,\tilde{\mathbf{E}}_0\,, \tag{S-13.143}$$

$$\begin{aligned}\tilde{\mathbf{p}} &= \mathbf{P}V = \chi\tilde{\mathbf{E}}_{\text{int}}V = \frac{3V}{4\pi}\frac{\tilde{\varepsilon}-1}{\tilde{\varepsilon}+2}\,\tilde{\mathbf{E}}_0 = \frac{3V}{4\pi}\frac{4\pi\mathrm{i}\sigma/\omega}{3+4\pi\mathrm{i}\sigma/\omega}\,\tilde{\mathbf{E}}_0\,,\\ &= \frac{3V}{4\pi}\frac{4\pi\mathrm{i}\sigma}{3\omega+4\pi\mathrm{i}\sigma}\,\tilde{\mathbf{E}}_0 = \frac{3V}{4\pi}\frac{\mathrm{i}/t_\mathrm{d}}{3\omega+\mathrm{i}/t_\mathrm{d}}\,\tilde{\mathbf{E}}_0 = \frac{3V}{4\pi}\frac{\mathrm{i}}{3\omega t_\mathrm{d}+\mathrm{i}}\,\tilde{\mathbf{E}}_0\\ &= \frac{3V}{4\pi}\frac{1+3\mathrm{i}\omega t_\mathrm{d}}{(3\omega t_\mathrm{d})^2+1}\,\tilde{\mathbf{E}}_0\end{aligned} \tag{S-13.144}$$

where $V = 4\pi a^3/3$ is the volume of the sphere, and $t_\mathrm{d} = 1/(4\pi\sigma)$. By writing the complex numerator in terms of its modulus and argument we have

$$1+3\mathrm{i}\omega t_\mathrm{d} = \sqrt{1+(3\omega t_\mathrm{d})^2}\,\mathrm{e}^{\mathrm{i}\phi}\,,\quad \text{where}\quad \phi = \arctan(3\omega t_\mathrm{d})\,, \tag{S-13.145}$$

and, substituting into (S-13.144) we obtain

$$\tilde{\mathbf{p}} = \frac{3V}{4\pi}\frac{\tilde{\mathbf{E}}_0}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,\mathrm{e}^{\mathrm{i}\phi}\,, \tag{S-13.146}$$

and, for the real quantity,

$$\begin{aligned}\mathbf{p} &= \mathrm{Re}\left[\frac{3V}{4\pi}\frac{E_0}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})\,\mathrm{e}^{-\mathrm{i}(\omega t-\phi)}\right]\\ &= \frac{3V}{4\pi}\frac{E_0}{\sqrt{1+(3\omega t_\mathrm{d})^2}}[\hat{\mathbf{x}}\cos(\omega t-\phi)+\hat{\mathbf{y}}\sin(\omega t-\phi)]\,.\end{aligned} \tag{S-13.147}$$

Thus the dipole moment of the sphere rotates with a phase delay ϕ relative to the electric field of the wave.

(b) The torque acting on an electric dipole $\mathbf{p}$ in the presence of an electric field $\mathbf{E}$ is $\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}$. In our case, the angle between $\mathbf{p}$ and $\mathbf{E}_0$ is constant in time and equal to ϕ, thus the torque is

$$\boldsymbol{\tau} = \hat{\mathbf{z}}\,|\mathbf{p}||\mathbf{E}_0|\sin\phi = \hat{\mathbf{z}}\,\frac{3V}{4\pi}\frac{E_0^2}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,\sin\phi\,. \tag{S-13.148}$$

The same result can be obtained by evaluating

$$\begin{aligned}\boldsymbol{\tau} &= \frac{1}{2}\,\mathrm{Re}\left(\tilde{\mathbf{p}}\times\tilde{\mathbf{E}}^*\right) = \mathrm{Re}\left[\frac{3V}{8\pi}\frac{E_0^2\,\mathrm{e}^{\mathrm{i}\phi}}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,(\hat{\mathbf{x}}+\mathrm{i}\hat{\mathbf{y}})\times(\hat{\mathbf{x}}-\mathrm{i}\hat{\mathbf{y}})\right]\\ &= \mathrm{Re}\left[\frac{3V}{8\pi}\frac{E_0^2\,(\cos\phi+\mathrm{i}\sin\phi)}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,(-2\mathrm{i}\,\hat{\mathbf{z}})\right] = \hat{\mathbf{z}}\,\frac{3V}{4\pi}\frac{E_0^2}{\sqrt{1+(3\omega t_\mathrm{d})^2}}\,\sin\phi\,.\end{aligned} \tag{S-13.149}$$

S-13.12 Surface Waves in a Thin Foil

(a) As an educated guess, we search for solutions for the unknown quantities E_x, and B_z of the form

$$E_x(x,y,t) = \tilde{E}_x(x)\,\mathrm{e}^{\mathrm{i}ky-\mathrm{i}\omega t}, \qquad B_z(x,y,t) = \tilde{B}_z(x)\,\mathrm{e}^{\mathrm{i}ky-\mathrm{i}\omega t}, \tag{S-13.150}$$

where $\tilde{E}_x(x)$, and $\tilde{B}_z(x)$ are complex functions to be determined. According to (13.10), E_y is symmetric (even) for reflection across the $x=0$ plane. Since in vacuum we have $\nabla\cdot\mathbf{E} = \partial_x E_x + \partial_y E_y = 0$, we obtain

$$\partial_x \tilde{E}_x = -\mathrm{i}kE_0\,\mathrm{e}^{-q|x|}, \tag{S-13.151}$$

which, after integration in dx, leads to

$$\tilde{E}_x(x) = \mathrm{sgn}(x)\,\frac{\mathrm{i}k}{q}\,E_0\,\mathrm{e}^{-q|x|} = \begin{cases} -\dfrac{\mathrm{i}k}{q}\,E_0\,\mathrm{e}^{qx}, & x<0,\\ \dfrac{\mathrm{i}k}{q}\,E_0\,\mathrm{e}^{-qx}, & x>0. \end{cases} \tag{S-13.152}$$

Thus, the E_x component is antisymmetric (odd) for reflection across the $x=0$ plane. Since our fields are independent of z, Maxwell's equation $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}/c$ reduces to

$$-\frac{1}{c}\,\partial_t B_z = \partial_x E_y - \partial_y E_x = \mathrm{sgn}(x)\left(\frac{k^2}{q} - q\right)E_0\,\mathrm{e}^{-q|x|}\,\mathrm{e}^{\mathrm{i}(ky-\omega t)}, \tag{S-13.153}$$

which, after integration in dt and division by $-\mathrm{e}^{\mathrm{i}(ky-\omega t)}/c$, leads to

$$\tilde{B}_z = \mathrm{sgn}(x)\,\frac{\mathrm{i}c}{q\omega}\left(q^2-k^2\right)E_0\,\mathrm{e}^{-q|x|}, \tag{S-13.154}$$

thus $\tilde{B}_z$, like $\tilde{E}_x$, is an odd function of x.

We can obtain the surface charge density $\sigma(y,t)$ and the surface current density $\mathbf{K}(y,t)$ on the foil from the boundary conditions at the $x=0$ plane. Figure S-13.6 shows the surface current $\mathbf{K}$ and the magnetic field close to the foil.

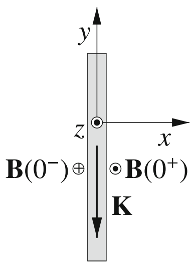

Fig. S-13.6

$$\begin{aligned} \sigma(y,t) &= \frac{1}{4\pi}\left[E_x(x=0^+,y,t) - E_x(x=0^-,y,t)\right] \\ &= \mathrm{i}\,\frac{2k}{q}\,E_0\,\mathrm{e}^{\mathrm{i}(ky-\omega t)}, \end{aligned} \tag{S-13.155}$$

$$K_y(y,t) = -\frac{c}{4\pi}\left[B_z(x=0^+,y,t) - B_z(x=0^-,y,t)\right]$$

$$-\mathrm{i}\,\frac{c^2}{2\pi q\omega}\left(q^2-k^2\right)E_0\,\mathrm{e}^{\mathrm{i}(ky-\omega t)}, \quad \text{(S-13.156)}$$

while the z component of **K** is zero because its presence would imply a nonzero y component of **B**.

(b) The time-averaged Poynting vector can be written as

$$\langle \mathbf{S}\rangle = \frac{c}{4\pi}\langle \mathbf{E}\times\mathbf{B}\rangle = \frac{c}{8\pi}\left[\hat{\mathbf{x}}\,\mathrm{Re}(\tilde{E}_y\tilde{B}_z^*)-\hat{\mathbf{y}}\,\mathrm{Re}(\tilde{E}_x\tilde{B}_z^*)\right], \tag{S-13.157}$$

where

$$\tilde{E}_y\tilde{B}_z^* = -\,\mathrm{sgn}(x)\frac{\mathrm{i}c}{q\omega}\left(q^2-k^2\right)|E_0|^2\,\mathrm{e}^{-2q|x|}, \tag{S-13.158}$$

$$\tilde{E}_x\tilde{B}_z^* = \frac{kc}{q^2\omega}\left(q^2-k^2\right)|E_0|^2\,\mathrm{e}^{-2q|x|}. \tag{S-13.159}$$

We thus obtain $\langle S_x\rangle = 0$ because $\tilde{E}_y\tilde{B}_z^*$, is purely imaginary, and the energy flow is in the $\hat{\mathbf{y}}$ direction only:

$$\langle \mathbf{S}\rangle = -\hat{\mathbf{y}}\,\frac{kc}{8\pi q^2\omega}\left(q^2-k^2\right)|E_0|^2\,\mathrm{e}^{-2q|x|}. \tag{S-13.160}$$

(c) Form Helmholtz's equation, we obtain

$$q^2-k^2+\frac{\omega^2}{c^2}=0. \tag{S-13.161}$$

(d) From (S-13.156) we can write, within our approximations,

$$\mathbf{J}=\frac{\mathbf{K}}{\ell}=-\hat{\mathbf{y}}\,\mathrm{i}\,\frac{c^2}{2\pi q\ell\omega}\left(q^2-k^2\right)E_0\,\mathrm{e}^{\mathrm{i}(ky-\omega t)} \tag{S-13.162}$$

and, combining with (13.11), we obtain

$$-\mathrm{i}\,\frac{c^2}{2\pi q\ell\omega}\left(q^2-k^2\right)E_0\,\mathrm{e}^{\mathrm{i}(ky-\omega t)} = 4\pi\mathrm{i}\,\frac{\omega_\mathrm{p}^2}{\omega}\,E_0\,\mathrm{e}^{\mathrm{i}(ky-\omega t)},$$

$$q^2-k^2=-8\pi^2\,\frac{\omega_\mathrm{p}^2}{c^2}\,q\ell. \tag{S-13.163}$$

where $\omega_{\rm p} = \sqrt{4\pi n_{\rm e} e^2/M_e}$ is the plasma frequency of the foil material. The product $2n_{\rm e}\ell$, appearing in the expression $\omega_{\rm p}^2\ell = 4\pi n_{\rm e}\ell e^2$, is the surface number density of the electrons in the foil, which is the relevant parameter in this problem.
(e) By comparing (S-13.161) and (S-13.163) we obtain

$$\omega^2 = 8\pi^2\omega_{\rm p}^2 q\ell = \Omega qc\,, \quad \text{where} \quad \Omega = \frac{8\pi^2\omega_{\rm p}^2\ell}{c}\,. \tag{S-13.164}$$

Solving (S-13.161) for q yields

$$cq^2 + \Omega q - k^2 c = 0 \quad \Rightarrow \quad q = \frac{\sqrt{\Omega^2 + (2kc)^2} - \Omega}{2c}\,, \tag{S-13.165}$$

where the root sign has been chosen so to have $q > 0$, as required by the boundary conditions, and in agreement with (13.10). Eventually, we obtain the dispersion relation:

$$\omega^2 = c^2k^2 - c^2q^2 = \frac{1}{2}\left[\Omega\sqrt{\Omega^2 + (2kc)^2} - \Omega^2 - (2kc)^2\right]. \tag{S-13.166}$$

S-13.13 The Fizeau Effect

(a) In the rest frame of the medium, S', we have $\omega'/k' = c/\mathrm{n}$. The Lorentz transformations from the laboratory frame S to S' lead to

$$\omega' = \gamma(\omega - uk) \simeq (\omega - uk)\,, \quad k' = \gamma\left(k - \frac{u\omega}{c^2}\right) \simeq \left(k - \frac{u\omega}{c^2}\right), \tag{S-13.167}$$

since $\gamma \simeq 1$ up to the first order in $\beta = u/c$. Dividing the two equations side by side we obtain

$$\frac{c}{\mathrm{n}} = \frac{\omega'}{k'} \simeq \frac{\omega - uk}{k - u\omega/c^2} = \frac{v_\varphi - u}{1 - v_\varphi u/c^2}\,, \tag{S-13.168}$$

where, in the last step, we have divided numerator and denominator by k, and substituted the phase velocity in the laboratory frame, $v_\varphi = \omega/k$. Multiplying the first and last term by $1 - v_\varphi u/c^2$ we obtain

$$\frac{c}{\mathrm{n}} - u\frac{v_\varphi}{c\mathrm{n}} = v_\varphi - u \quad \Rightarrow \quad v_\varphi\left(1 + \frac{u}{c\mathrm{n}}\right) = \frac{c}{\mathrm{n}} + u \quad \Rightarrow \quad v_\varphi = \frac{c(c + \mathrm{n}u)}{c\mathrm{n} + u}$$
$$\Rightarrow v_\varphi = c\,\frac{1 + \mathrm{n}\beta}{\mathrm{n} + \beta} \simeq c\left(\frac{1}{\mathrm{n}} + \frac{\mathrm{n}^2 - 1}{\mathrm{n}^2}\beta\right), \tag{S-13.169}$$

where, in the last step, we have approximated the fraction by its first-degree Taylor polynomial in β. The phase velocity in the laboratory frame S is thus

$$v_\varphi = \frac{c}{\mathrm{n}} + u\left(1 - \frac{1}{\mathrm{n}^2}\right). \tag{S-13.170}$$

The experiment was performed in 1851, with light propagating in flowing water parallel to the water velocity. Fizeau expected to measure a phase velocity equal to the phase velocity of light in water, c/n, plus the flow velocity of water, u, i.e., $v_\varphi = (c/\mathrm{n}) + u$, while the experimental result was in agreement with (S-13.170). This found a satisfactory explanation only 54 years later, in 1905, when Einstein published his theory of special relativity.

(b) Equation (S-13.170) takes into account the first-order correction to v_φ in $\beta = u/c$ for a non-dispersive medium. If the medium is dispersive according to a known law $\mathrm{n} = \mathrm{n}(\omega)$, we must also take into account that the frequency ω' observed in the rest-frame of the medium is different from the radiation frequency ω in the laboratory frame. We want to calculate the first-order correction to (S-13.170) in $\Delta\omega = \omega' - \omega$. We need to correct only the fist term of the right-hand side of (S-13.170), since the second term is already first-order, and a correction to it would be second-order. The first-order Doppler effect gives us

$$\Delta\omega = \omega' - \omega \simeq -\omega\,\frac{\mathrm{n}(\omega)\,u}{c}\,, \tag{S-13.171}$$

since the light velocity in the medium is $c/\mathrm{n}(\omega)$, an the medium is traveling away from the light source. Thus we have

$$\begin{aligned}
\frac{c}{\mathrm{n}(\omega')} &\simeq \frac{c}{\mathrm{n}(\omega)} + \Delta\omega\,\partial_\omega\left(\frac{c}{\mathrm{n}(\omega)}\right) \\
&= \frac{c}{\mathrm{n}(\omega)} + \left(-\omega\,\frac{\mathrm{n}(\omega)\,u}{c}\right)\left[-\frac{c}{\mathrm{n}^2(\omega)}\,\partial_\omega \mathrm{n}(\omega)\right] \\
&= \frac{c}{\mathrm{n}(\omega)} + \omega\,\frac{u}{\mathrm{n}(\omega)}\,\partial_\omega \mathrm{n}(\omega)\,,
\end{aligned} \tag{S-13.172}$$

and the first-order expression for the phase velocity in the case of a dispersive medium is

$$v_\varphi(\omega) = \frac{c}{\mathrm{n}(\omega)} + u\left[1 - \frac{1}{\mathrm{n}^2(\omega)} + \frac{\omega}{\mathrm{n}(\omega)}\,\partial_\omega \mathrm{n}(\omega)\right] + O(u^2)\,. \tag{S-13.173}$$

(c) The refractive index of the free electron medium is $\mathrm{n}(\omega) = (1 - \omega_\mathrm{p}^2/\omega^2)^{1/2}$, where ω_p is the plasma frequency. Thus we have inside the square brackets of (S-13.173)

$$1 - \frac{1}{\mathrm{n}^2(\omega)} = -\frac{1}{1-\omega_\mathrm{p}^2/\omega^2} = \frac{\omega_\mathrm{p}^2}{\omega^2-\omega_\mathrm{p}^2}\,, \tag{S-13.174}$$

and

$$\frac{\omega}{\mathrm{n}(\omega)}\,\partial_\omega \mathrm{n}(\omega) = -\frac{\omega}{(1-\omega_\mathrm{p}^2/\omega^2)^{1/2}}\,\frac{1}{(1-\omega_\mathrm{p}^2/\omega^2)^{3/2}}\,\frac{\omega_\mathrm{p}^2}{\omega^3} = -\frac{\omega_\mathrm{p}^2}{\omega^2-\omega_\mathrm{p}^2}\,, \tag{S-13.175}$$

so that the two first-order corrections to $v_\varphi(\omega)$ cancel out, and the phase velocity is independent of the flow velocity of the medium up to the second order in β.

S-13.14 Lorentz Transformations for Longitudinal Waves

(a) The Lorentz transformations for the wave frequency and wavevector are, in the case of a boost along $\hat{\mathbf{x}}$,

$$\omega'_\mathrm{L} = \gamma(\omega_L - Vk_\mathrm{L})\,, \qquad k'_\mathrm{L} = \gamma\left(k_\mathrm{L} - \frac{V\omega_\mathrm{L}}{c^2}\right), \tag{S-13.176}$$

where $\gamma = 1/\sqrt{1-V^2/c^2}$. In the special case where the boost velocity equals the phase velocity, $V = v_\varphi = \omega_L/k_L$, we have $\omega'_L = 0$, and the fields are independent of time (*static*) in S'. Further, recalling that $k_\mathrm{L} = \omega_\mathrm{L}/v_\varphi$, we have

$$\begin{aligned} k'_\mathrm{L} &= \frac{1}{\sqrt{1-v_\varphi^2/c^2}}\left(\frac{\omega_\mathrm{L}}{v_\varphi} - \frac{v_\varphi\,\omega_\mathrm{L}}{c^2}\right) = \frac{1}{\sqrt{1-v_\varphi^2/c^2}}\,\frac{\omega_\mathrm{L}}{v_\varphi}\left(1-\frac{v_\varphi^2}{c^2}\right) \\ &= \frac{\omega_\mathrm{L}}{v_\varphi}\sqrt{1-v_\varphi^2/c^2} = \frac{k_\mathrm{L}}{\gamma}\,. \end{aligned} \tag{S-13.177}$$

If S' moves with velocity $\hat{\mathbf{x}}\,V = \hat{\mathbf{x}}\,v_\varphi$ relative to S, the fields in S' are obtained from (9.3) and are

$$\mathbf{E}' = \mathbf{E}'(x') = \hat{\mathbf{x}}\,E_0\,\mathrm{e}^{\mathrm{i}k'_\mathrm{L}x'}\,, \qquad \mathbf{B}' = 0\,, \tag{S-13.178}$$

i.e., $\mathbf{E}'$ is constant in time. The charge and current densities in S' can be obtained either by Lorentz transformations or directly from the equations

$$\varrho' = \frac{1}{4\pi}\boldsymbol{\nabla}'\cdot\mathbf{E}' = \frac{1}{4\pi}\partial_{x'}E'_x \quad \text{and} \quad 4\pi\mathbf{J}' + \partial'_t\mathbf{E}' = 0\,, \tag{S-13.179}$$

which lead to

$$\varrho' = \frac{\mathrm{i}k'_\mathrm{L}}{4\pi}E_0\,\mathrm{e}^{\mathrm{i}k'_\mathrm{L}x'}\,, \qquad \mathbf{J}' = 0\,. \tag{S-13.180}$$

(b) The Lorentz transformations for the case $V = c^2/v_\varphi = c^2 k_L/\omega_L$, and $v_\varphi > c$, lead to the following values for k'_L and ω'_L

$$k'_L = \gamma\left(k_L - \frac{V\omega_L}{c^2}\right) = 0,$$
$$\omega'_L = \gamma\left(\omega_L - \frac{k_L^2 c^2}{\omega}\right) = \gamma\omega_L\left(1 - \frac{V^2}{c^2}\right) = \frac{\omega_L}{\gamma}, \qquad \text{(S-13.181)}$$

which imply that the fields propagate in space with infinite phase velocity, oscillating with uniform phase at frequency ω'_L. The fields are

$$\mathbf{E}' = \mathbf{E}'(t') = \hat{\mathbf{x}} E_0 \, e^{-i\omega' t'}, \qquad \mathbf{B}' = 0, \qquad \text{(S-13.182)}$$

i.e., $\mathbf{E}'$ is uniform in space. We also obtain $\varrho' = 0$, and $\mathbf{J}' = \mathbf{J}/\gamma$.
(c) The Lorentz transformations of the wavevector and the frequency for a boost along the y axis are

$$k'_{Lx} = k_{Lx} = k_L,$$
$$k'_{Ly} = \gamma\left(k_{Ly} - \frac{V\omega_L}{c^2}\right) = -\gamma\frac{V\omega_L}{c^2},$$
$$\omega'_L = \gamma(\omega_L - Vk_{Ly}) = \gamma\omega_L. \qquad \text{(S-13.183)}$$

All fields and currents depend on space and time through a factor $e^{i(k'_{Lx}x' + k'_{Ly}y' - \omega'_L t')}$, thus, the propagation direction forms an angle

$$\theta' = \arctan(k'_{Ly}/k'_{Lx}) = -\arctan(\gamma V v_\varphi/c^2) \qquad \text{(S-13.184)}$$

with the x' axis. The wave has field amplitudes

$$E'_x = \gamma\left(E_x + \frac{V}{c} B_z\right) = \gamma E_0, \qquad \text{(S-13.185)}$$
$$B'_z = \gamma\left(B_z - \frac{V}{c^2} E_x\right) = -\gamma\frac{V}{c^2} E_0, \qquad \text{(S-13.186)}$$

all other field components being zero. Thus, in a frame moving transversally to the propagation direction, the wave is no longer purely longitudinal and electrostatic.

S-13.15 Lorentz Transformations for a Transmission Cable

(a) The continuity equation for a linear charge density is written $\partial_t\lambda = -\partial_z I$. Inserting the expressions for λ and I of (13.14) we obtain

$$-\mathrm{i}\omega\lambda_0 = -\mathrm{i}kI_0\,, \quad \Rightarrow \quad I_0 = \frac{\omega}{k}\lambda_0 = v_\varphi\lambda_0\,. \tag{S-13.187}$$

(b) The dispersion relation is

$$\omega = v_\varphi k = \frac{c}{\mathrm{n}}k = \frac{c}{\sqrt{\varepsilon}}k\,, \tag{S-13.188}$$

with

$$v_\varphi = \frac{c}{\sqrt{\varepsilon}} \qquad \text{and} \qquad k = \frac{\omega}{v_\varphi} = \frac{\omega\sqrt{\varepsilon}}{c}\,. \tag{S-13.189}$$

The electric field can be evaluated by applying Gauss's law to a cylindrical surface coaxial to the wire, of radius r and height h. Since the field is transverse, and we have cylindrical symmetry around the wire, the only nonzero component of $\mathbf{E}$ is E_r

$$\mathbf{E}(r,z,t) = \hat{\mathbf{r}}\,\frac{2\lambda}{\varepsilon r} = \hat{\mathbf{r}}\,E_r(r)\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\,, \quad \text{where} \quad E_r(r) = \frac{2\lambda_0}{\varepsilon r}\,. \tag{S-13.190}$$

The magnetic field can be evaluated by applying Stokes' theorem to a circle of radius r, coaxial to the wire. Because of symmetry, the only nonzero component of $\mathbf{B}$ is the azimuthal component B_ϕ

$$\mathbf{B}(r,z,t) = \hat{\boldsymbol{\phi}}\,\frac{2I}{rc} = \hat{\boldsymbol{\phi}}\,B_\phi(r)\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\,, \quad \text{where} \quad B_\phi(r) = \frac{2I_0}{rc} = \frac{2\omega\lambda_0}{krc}\,, \tag{S-13.191}$$

that can be rewritten as

$$B_\phi(r) = \frac{\varepsilon\omega}{kc}E_r(r) = \frac{\varepsilon v_\varphi}{c}E_r(r) = \frac{c}{v_\varphi}E_r(r)\,. \tag{S-13.192}$$

(c) The wave frequency ω' in the frame S', moving at the phase velocity $\hat{\mathbf{z}}\,v_\varphi$ relative to the laboratory frame S, is

$$\omega' = \gamma\left(\omega - v_\varphi k\right) = 0\,, \tag{S-13.193}$$

where we have used the second of (S-13.189). Thus the fields are static in S'. For our Lorentz boost we have

$$\boldsymbol{\beta} = \hat{\mathbf{z}}\,\frac{v_\varphi}{c} = \frac{\hat{\mathbf{z}}}{\sqrt{\varepsilon}}\,, \qquad \gamma = \frac{1}{\sqrt{1-1/\varepsilon}} = \sqrt{\frac{\varepsilon}{\varepsilon-1}}\,, \tag{S-13.194}$$

and the wave vector k' in S' can be written

$$k' = \gamma\left(k - \frac{v_\varphi\,\omega}{c^2}\right) = \sqrt{\frac{\varepsilon}{\varepsilon-1}}\left(\frac{\omega\sqrt{\varepsilon}}{c} - \frac{\omega}{c\sqrt{\varepsilon}}\right) = \frac{\omega}{c}\sqrt{\varepsilon-1}\,. \tag{S-13.195}$$

The (z', t')-dependence (actually, only z'-dependence) of our physical quantities in S' will thus be through a factor $e^{ik'z'}$. The amplitude of linear charge density in S' is

$$\lambda'_0 = \gamma\left(\lambda_0 - \frac{v_\varphi}{c^2} I_0\right) = \gamma\left(\lambda_0 - \frac{v_\varphi^2}{c^2}\lambda_0\right) = \gamma\left(\frac{1}{\gamma}\right)^2 \lambda_0 = \frac{\lambda_0}{\gamma}. \tag{S-13.196}$$

The amplitude of the current in S' is

$$I'_0 = \gamma\left(I_0 - v_\varphi \lambda_0\right) = 0. \tag{S-13.197}$$

The field amplitudes transform according to (9.3), thus we have

$$E'_r = \gamma\left(E_r - \beta B_\phi\right) = \gamma\left(E_r - \frac{v_\varphi}{c}\frac{c}{v_\varphi}E_r\right) = 0, \tag{S-13.198}$$

$$B'_\phi = \gamma\left(B_\phi - \frac{v_\varphi^2}{c^2}E_r\right) = \gamma\left(B_\phi - \frac{v_\varphi}{c}\frac{v_\varphi}{c}B_\phi\right) = \frac{B_\phi}{\gamma}. \tag{S-13.199}$$

It might seem surprising that, in S', we have $\lambda' \neq 0$ and $E'_r = 0$, while $I' = 0$ and $B'_\phi \neq 0$. The reason is that we must take into account also the polarization charge of the medium in contact with the wire, $\lambda_p(z,t)$, the presence of a polarization current, $\mathbf{J}_p(r,z,t)$, and their Lorentz transformations. In the laboratory frame S we must have $\lambda(z,t) + \lambda_p(z,t) = \lambda(z,t)/\varepsilon$, thus

$$\lambda_p(z,t) = -\frac{\varepsilon - 1}{\varepsilon}\lambda_0\, e^{ikz - i\omega t} = -\frac{\lambda_0}{\gamma^2}\, e^{ikz - i\omega t} = \lambda_0^{(p)}\, e^{ikz - i\omega t}, \tag{S-13.200}$$

where $\lambda_0^{(p)} = -\lambda_0/\gamma^2$. The electric field (S-13.190) generates a polarization of the medium

$$\mathbf{P}(z,r,t) = \hat{\mathbf{r}}\,\frac{\varepsilon - 1}{4\pi}E_r(r)\,e^{ikz - i\omega t} = \hat{\mathbf{r}}\,P_r(r)\,e^{ikz - i\omega t}, \tag{S-13.201}$$

where

$$P_r(r) = \frac{\varepsilon - 1}{4\pi}\frac{2\lambda_0}{\varepsilon r} = \frac{\varepsilon - 1}{\varepsilon}\frac{\lambda_0}{2\pi r} = \frac{1}{\gamma^2}\frac{\lambda_0}{2\pi r}. \tag{S-13.202}$$

A time-dependent polarization is associated to a polarization current density

$$\mathbf{J}_p = \partial_t \mathbf{P} = -\hat{\mathbf{r}}\,i\omega P_r(r)\,e^{ikz - i\omega t} = \hat{\mathbf{r}}\,J_r(r)\,e^{ikz - i\omega t} \tag{S-13.203}$$

where

$$J_r(r) = -i\omega P_r(r) = -i\,\frac{\omega}{\gamma^2}\frac{\lambda_0}{2\pi r}. \tag{S-13.204}$$

Thus, $\mathbf{J}_\mathrm{p}$ is radial in S. According to the first of (9.1), we have a polarization four-current

$$J_\mu^{(\mathrm{p})}(r,z,t) = \left[c\varrho_0^{(\mathrm{p})}(r), \hat{\mathbf{r}}\, J_r(r)\right] \mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\,, \tag{S-13.205}$$

where, for instance

$$\varrho_0^{(\mathrm{p})}(r) = \begin{cases} \dfrac{\lambda_0^{(\mathrm{p})}}{\pi r_0^2}\,, & \text{if} \quad r < r_0 \\ 0 & \text{if} \quad r > r_0 \end{cases}\,, \quad \text{so that} \quad \lambda_0^{(\mathrm{p})} = \int_0^\infty \varrho_0^{(\mathrm{p})}(r)\, 2\pi r\, \mathrm{d}r\,, \tag{S-13.206}$$

and we are interested in the limit $r_0 \to 0$. We can thus write

$$J_\mu^{(\mathrm{p})}(r,z,t) = G_\mu(r)\,\mathrm{e}^{\mathrm{i}kz-\mathrm{i}\omega t}\,, \quad \text{where} \quad G_\mu = \left[c\varrho_0^{(\mathrm{p})}, \hat{\mathbf{r}}\, J_r(r)\right]. \tag{S-13.207}$$

The four-vector G_μ transforms according to (9.2), thus we have in S'

$$G_0' = \gamma(G_0 - \boldsymbol{\beta}\cdot\mathbf{G}) = \gamma G_0\,, \tag{S-13.208}$$

since the spacelike component of G_μ, being radial, is perpendicular to $\boldsymbol{\beta}$. The amplitude of the linear polarization charge density in S' is

$$\lambda_0^{(\mathrm{p})\prime} = \gamma \int_0^\infty \frac{G_0}{c}\, 2\pi r\, \mathrm{d}r = \gamma\lambda_0^{(\mathrm{p})} = -\gamma\frac{\lambda_0}{\gamma^2} = -\frac{\lambda_0}{\gamma}\,, \tag{S-13.209}$$

which cancels (S-13.196), therefore we have $\mathbf{E}' = 0$. The radial component of $\mathbf{J}_\mathrm{p}$ does not contribute to the magnetic field, thus we are interested in

$$G_\parallel' = \gamma(G_\parallel - \beta G_0) = -\gamma\beta G_0 = -\gamma\beta\varrho_0^{(\mathrm{p})}(r)\, c\,, \tag{S-13.210}$$

which corresponds to a polarization current in S' of amplitude

$$I_0^{(\mathrm{p})\prime} = \int_\infty G_\parallel'\, 2\pi r'\, \mathrm{d}r' = -\gamma v_\varphi \lambda_0^{(\mathrm{p})} = \gamma v_\varphi \frac{\lambda_0}{\gamma^2} = \frac{I_0}{\gamma} \tag{S-13.211}$$

in agreement with (S-13.199).

S-13.16 A Waveguide with a Moving End

(a) The electric field of the TE_{10} must be parallel to the two conducting planes, thus it must vanish on them, and be of the form $\mathbf{E}(x,y,t) = \hat{\mathbf{z}}\, E_0 \cos(\pi y/a) f(x,t)$. The dispersion relation is

$$\omega^2 = \omega_{\rm co}^2 + k^2 c^2\,, \quad \text{where} \quad \omega_{\rm co} = \frac{\pi c}{a} \tag{S-13.212}$$

is the cutoff frequency of the waveguide. In our terminated waveguide, the global electric field is the superposition of the fields of the wave incident on the terminating wall at $x = 0$, and of the reflected wave. Incident and reflected wave have equal amplitudes, thus

$$\mathbf{E}(x, y, t) = \hat{\mathbf{z}}\, E_0 \cos\left(\frac{\pi y}{a}\right) \sin(kx)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{S-13.213}$$

where the phase has been chosen so that $\mathbf{E}(0, y, t) = 0$. The magnetic field can be obtained from the relation $\partial_t \mathbf{B} = -c\boldsymbol{\nabla}\times\mathbf{E}$, and has the components

$$B_x = -\frac{\mathrm{i}c}{\omega}\,\partial_y E_z = \frac{\mathrm{i}\pi c}{\omega a}\, E_0 \sin\left(\frac{\pi y}{a}\right) \sin(kx)\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{S-13.214}$$

$$B_y = \frac{\mathrm{i}c}{\omega}\,\partial_x E_z = -\frac{kc}{\omega}\, E_0 \cos\left(\frac{\pi y}{a}\right) \cos(kx)\, \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-13.215}$$

Notice that $B_x(0, y, t) = 0$, as required.
(b) In the frame S' where the waveguide termination is at rest ($\boldsymbol{v}' = 0$), the incident wave has frequency and wavevector

$$\omega_{\rm i}' = \gamma(\omega - \beta k c)\,, \qquad k_{\rm i}' = \gamma(k - \beta\omega/c)\,, \tag{S-13.216}$$

where $\beta = v/c$. Since we assumed $\beta < kc/\omega$, we have $k_{\rm i}' > 0$ (notice that $\omega_{\rm i}' > 0$ anyway because $k < \omega/c$). In S' the reflected wave has frequency and wavevector

$$\omega_{\rm r}' = \omega_{\rm i}'\,, \qquad k_{\rm r}' = -k_{\rm i}'\,. \tag{S-13.217}$$

By transforming back into the laboratory frame S we obtain

$$\omega_{\rm r} = \gamma(\omega_{\rm r}' + \beta k_{\rm r}' c) = \gamma^2\left[(1+\beta^2)\,\omega - 2\beta k c\right]\,, \tag{S-13.218}$$

$$k_{\rm r} = \gamma\left(-k + \beta\frac{\omega}{c}\right) = \gamma^2\left[-(1+\beta^2)\,k + 2\beta\frac{\omega}{c}\right]\,, \tag{S-13.219}$$

As a check, at the limit $a \to \infty$ we have $\omega_{\rm co} \to 0$ and $k \to \omega/c$, and we obtain (S-9.54) of Problem 9.6 for the frequency reflected by a moving mirror. With some algebraic manipulations we obtain

$$
\begin{aligned}
\omega_{\mathrm{r}}^2 - c^2 k_{\mathrm{r}}^2 &= \gamma^4 \left\{ \left[(1+\beta^2)\,\omega - 2\beta kc\right]^2 - c^2\left[-(1+\beta^2)\,k + 2\beta\,\frac{\omega}{c}\right]^2 \right\} \\
&= \gamma^4 \left[(1+\beta^2)\,\omega - 2\beta kc + (1+\beta^2)\,kc - 2\beta\omega\right] \times \\
&\quad \times \left[(1+\beta^2)\,\omega - 2\beta kc - (1+\beta^2)\,kc + 2\beta\omega\right] \\
&= \gamma^2 \left[(1-\beta)^2\omega + (1-\beta)^2 kc\right]\left[(1+\beta)^2\omega + (1+\beta)^2 kc\right] \\
&= (\omega + kc)(\omega - kc) = \omega^2 - k^2c^2 \,. \qquad \text{(S-13.220)}
\end{aligned}
$$

(c) If $v > kc^2/\omega$, in S' we have $k_{\mathrm{i}}' < 0$, the incident wave propagates parallel to $-\hat{\mathbf{x}}'$, and cannot reach the waveguide termination. In these conditions there is no reflected wave. The condition is equivalent to $v > v_{\mathrm{g}}$, the group velocity in the waveguide.

S-13.17 A "Relativistically" Strong Electromagnetic Wave

(a) The equations of motion for p_x, p_y, and p_z in the presence of the electromagnetic fields of the wave are

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -eE_x + \frac{e}{c}\,v_z B_y \,, \qquad \text{(S-13.221)}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = -eE_y - \frac{e}{c}\,v_z B_x \,, \qquad \text{(S-13.222)}$$

$$\frac{\mathrm{d}p_z}{\mathrm{d}t} = -\frac{e}{c}\,v_x B_y + \frac{e}{c} v_y B_z \,. \qquad \text{(S-13.223)}$$

In general the magnetic contribution is not negligible, since v_z is not necessarily much smaller than c. However, if we assume $v_z = 0$, the magnetic force vanishes. In these conditions the solutions of (S-13.221-S-13.222) are

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -eE_0 \cos\omega t \,, \qquad \frac{\mathrm{d}p_y}{\mathrm{d}t} = -eE_0 \sin\omega t \,, \qquad \text{(S-13.224)}$$

$$p_x = -\frac{eE_0}{\omega} \sin\omega t \,, \qquad p_y = +\frac{eE_0}{\omega} \cos\omega t \,. \qquad \text{(S-13.225)}$$

Inserting these solutions into (S-13.223) we have

$$
\begin{aligned}
\frac{dp_z}{dt} &= \frac{e}{m_{\mathrm{e}}\gamma}(-p_x B_y + p_y B_x) = \\
&= \frac{e}{m_{\mathrm{e}}\gamma}\,\frac{eE_0^2}{\omega c}(-\sin\omega t \cos\omega t + \cos\omega t \sin\omega t) = 0 \,, \qquad \text{(S-13.226)}
\end{aligned}
$$

so that a p_z is constant in time. Either assuming $v_z = 0$ as initial condition or by a proper change of reference frame, $v_z = 0$ is a self-consistent assumption.

(b) Since $\mathbf{p}^2 = p_x^2 + p_y^2 = (eE_0/\omega)^2$ does not depend on time, the Lorentz factor $\gamma = \sqrt{1 + p^2/(m_e c)^2} = \sqrt{1 + [eE_0/(m_e \omega c)]^2}$ is a constant. This implies

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m_e \frac{\mathrm{d}(\gamma \boldsymbol{v})}{\mathrm{d}t} = \gamma m_e \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} . \quad \text{(S-13.227)}$$

The equations of motion have the same form as in the non-relativistic case if we make the replacement $m_e \to \gamma m_e$. The relativistic behavior can be obtained by attributing an "effective mass" γm_e, dependent on the wave intensity, to the electron.

(c) Accordingly, the refractive index for the relativistic case can be simply obtained by replacing $m_e \to m_e \gamma$ into the non-relativistic expression, so that $\omega_p = \sqrt{(4\pi n_e e^2/m_e)} \to \omega_p/\sqrt{\gamma}$. We thus obtain

$$\mathrm{n}^2(\omega) = 1 - \frac{4\pi n_e e^2}{m_e \gamma \omega^2} = 1 - \frac{\omega_p^2}{\gamma \omega^2} . \quad \text{(S-13.228)}$$

(d) The dispersion relation corresponding to $\mathrm{n}^2(\omega)$ in (S-13.228) is

$$\omega^2 = k^2 c^2 + \frac{\omega_p^2}{\gamma} . \quad \text{(S-13.229)}$$

The cutoff frequency is $\omega_{co} = \omega_p/\sqrt{\gamma}$ and depends on the wave amplitude. Since $\gamma > 1$, a plasma can be opaque to a low-intensity wave for which $\omega_p > \omega$, but transparent to a high-intensity wave of the same frequency if $\gamma > \omega_p/\omega$.

It should be stressed, however, that the concept of a refractive index dependent on the wave intensity deserves some care. What we have discussed above is just a special case of "relativistically induced transparency", applying to a plane, monochromatic, infinite wave. In the case of a real light beam, of finite duration and extension, different parts of the beam can have different amplitudes, and thus can have different phase velocities, resulting in a complicated nonlinear dispersion.[1] However, (S-13.229) can be of help to a *qualitative* discussion of some important nonlinear effects observed for a relativistically strong wave. An important example is the propagation of a strong beam of finite width, for which the effective refractive index is higher at the boundaries (where the intensity is lower and γ is smaller) than on the beam axis. This can compensate diffraction, analogously to what occurs in an optical fiber (see Problem 12.6), and can cause *self-focusing*.

[1] In some cases, nonlinearity effects can compensate dispersion for particular wavepacket shapes, these special solutions can propagate without changing their envelope shape, and are known as *solitons*.

S-13.18 Electric Current in a Solenoid

(a) This problem originated from the question: "can the electric field in a solenoid have circular field lines, as seems to be required for driving the current in each turn of the coil?" The answer is obviously *no* in static conditions, since $\boldsymbol{\nabla}\times\mathbf{E}$ must be zero. But a *uniform* field is sufficient to drive the current, since the coil of a real solenoid does not consist of single circular circular loops perpendicular to the axis (each loop would require its own current source, in this case!) The winding of a real solenoid is actually a helix, of small, but nonzero pitch. The current is driven by the component of $\mathbf{E}$ *parallel* to the wire, equal to (assuming $\mathbf{E} = E\hat{\mathbf{z}}$)

$$E_{\|} = E\sin\theta \simeq E\frac{a}{\pi b}\,. \tag{S-13.230}$$

The perpendicular component $E_{\perp}$ is compensated by the electrostatic fields generated by the surface charge distribution of the wire, analogously to Problem 3.11. Thus, the current density and intensity in the wire are

$$J = \sigma E_{\|} \simeq \sigma E\frac{a}{\pi b}\,, \qquad I = J\pi a^2 \simeq \frac{\sigma a^3}{b}E\,. \tag{S-13.231}$$

Neglecting boundary effects, the current generates a uniform field $\mathbf{B}^{(\mathrm{int})} = \hat{\mathbf{z}}B_z$, with

$$B_z = \frac{4\pi nI}{c} = \frac{4\pi I}{2ac} \simeq \frac{2\pi\, a^2\sigma}{bc}E\,, \tag{S-13.232}$$

inside the solenoid, since $n = 1/(2a)$ is the number of turns per unit length. The field outside the solenoid, $\mathbf{B}^{(\mathrm{ext})}$, is generated by the total current I flowing parallel to $\hat{\mathbf{z}}$. Thus in the external central region $b < r \ll h$, $|z| \ll h$, the field is azimuthal, $\mathbf{B}^{(\mathrm{ext})} = \hat{\boldsymbol{\phi}}B_\phi$, with

$$B_\phi \simeq \frac{2I}{cr} = \frac{2\pi J a^2}{cr} \simeq \frac{2\sigma a^3}{bcr}E\,, \qquad b < r \ll h\,, \quad |z| \ll h\,, \tag{S-13.233}$$

where the z origin is located at the center of the solenoid.

(b) In the external central region $b < r \ll h$, $|z| \ll h$, the fields E_z and B_ϕ are associated to a a Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} = -\hat{\mathbf{r}}\frac{c}{4\pi}E_z B_\phi = -\hat{\mathbf{r}}\frac{\sigma a^3}{2\pi br}E^2\,, \tag{S-13.234}$$

with an entering flux through the lateral surface of a coaxial cylinder of length ℓ

$$\Phi_{\mathrm{in}} = 2\pi r\ell\,\frac{\sigma a^3}{2\pi br}E^2 = \frac{\sigma a^3\ell}{b}E^2\,. \tag{S-13.235}$$

The power dissipated by Joule heating in a solenoid portion of length ℓ is obtained by multiplying the power dissipated in single turn

$$W_{\text{turn}} = I^2 R = \left(\frac{\sigma a^3}{b} E\right)^2 \frac{2\pi b}{\pi a^2 \sigma} = \frac{2\sigma a^4}{b} E^2 \,, \tag{S-13.236}$$

by the number of turns $\ell/(2a)$

$$W(\ell) = \frac{2\sigma a^4}{b} E^2 \frac{\ell}{2a} = \frac{\sigma a^3 \ell}{b} E^2 \,, \tag{S-13.237}$$

in agreement with Poynting's theorem.

S-13.19 An Optomechanical Cavity

(a) In the following we omit the vector notation for the electric fields, since the results are independent of the polarization. The general expression for the electric field of a monochromatic plane wave propagating along x is, in complex notation,

$$E(x,t) = E(x)\mathrm{e}^{-\mathrm{i}\omega t} = \left(E_1 \mathrm{e}^{+\mathrm{i}kx} + E_2 \mathrm{e}^{-\mathrm{i}kx}\right)\mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{S-13.238}$$

where $k = \omega/c$. The boundary conditions at the two perfectly conducting walls are $E(\pm d/2) = 0$, thus we must have

$$E_1 \mathrm{e}^{+\mathrm{i}kd/2} + E_2 \mathrm{e}^{-\mathrm{i}kd/2} = 0 \,, \qquad E_1 \mathrm{e}^{-\mathrm{i}kd/2} + E_2 \mathrm{e}^{+\mathrm{i}kd/2} = 0 \,. \tag{S-13.239}$$

This system of two equations has nontrivial solutions for E_1 and E_2 only if the determinant is zero,

$$\mathrm{e}^{\mathrm{i}kd} - \mathrm{e}^{-\mathrm{i}kd} = 2\mathrm{i}\sin(kd) = 0, \tag{S-13.240}$$

from which we obtain

$$kd = n\pi \quad (n = 1, 2, 3, \ldots) \qquad\qquad \omega = kc = n\frac{\pi c}{d} \,, \tag{S-13.241}$$

$$E_2 = -E_1 \mathrm{e}^{\mathrm{i}n\pi} = (-1)^{n+1} E_1 \,. \tag{S-13.242}$$

Thus, the electric field of the n-th mode is

$$E_n(x) = \frac{E_0}{2}\left[\mathrm{e}^{\mathrm{i}n\pi x/d} + (-1)^{n+1}\mathrm{e}^{-\mathrm{i}n\pi x/d}\right] . \tag{S-13.243}$$

The magnetic field can be obtained from $\partial_t \mathbf{B} = -\boldsymbol{\nabla}\times\mathbf{E}$:

$$B_n(x) = \frac{E_0}{2}\left[e^{in\pi x/d} - (-1)^{n+1}e^{-in\pi x/d}\right]. \qquad \text{(S-13.244)}$$

(b) The field is thus the superposition of two plane monochromatic waves of equal frequency ω and amplitude $E_0/2$, propagating in opposite directions. The radiation pressure on each reflecting wall is thus the pressure exerted by a normally incident wave of intensity $I = c|E_0/2|^2/8\pi$, evaluated in Problem 8.5,

$$P = \frac{2I}{c} = \frac{|E_0|^2}{16\pi}. \qquad \text{(S-13.245)}$$

(c) The energy per unit surface inside the cavity is independent of time and can be evaluated as

$$U = \int_{-d/2}^{+d/2} \frac{1}{8\pi}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) dx. \qquad \text{(S-13.246)}$$

We have

$$|\mathbf{E}|^2 = \frac{|E_0|^2}{2}\left[1 + (-1)^{n+1}\cos\left(\frac{2n\pi x}{d}\right)\right], \qquad \text{(S-13.247)}$$

$$|\mathbf{B}|^2 = \frac{|E_0|^2}{2}\left[1 - (-1)^{n+1}\cos\left(\frac{2n\pi x}{d}\right)\right]. \qquad \text{(S-13.248)}$$

Integrating over x, the oscillating terms of both expressions average to zero, and we finally have

$$U = \frac{|E_0|^2 d}{16\pi}\pi = Pd. \qquad \text{(S-13.249)}$$

(d) At mechanical equilibrium, the force due to the radiation pressure on the walls must balance the recoil force of the springs. Assuming that each wall is displaced by δ from its equilibrium position in the absence of fields, we have

$$PS = K\delta = M\Omega^2\delta, \qquad \text{(S-13.250)}$$

where $\Omega = \sqrt{K/M}$ is the free oscillation frequency of the walls. Thus

$$\frac{M\Omega^2\delta}{S} = P = \frac{|E_0|^2}{16\pi}, \qquad \text{(S-13.251)}$$

from which we obtain $\delta = \alpha\left|E_0^2\right|$ where

$$\alpha = \frac{S}{16\pi M\Omega^2} = \frac{S}{16\pi K}. \qquad \text{(S-13.252)}$$

The length of the cavity is now $d+2\delta$, and the resonance condition is

$$d+2\delta = n\frac{\lambda_n}{2} = n\frac{\pi c}{\omega_n}, \quad (n=1,2,\dots), \tag{S-13.253}$$

where the mode frequencies are

$$\omega_n = \frac{2\pi c}{\lambda_n} = \frac{n\pi c}{d+2\alpha|E_0|^2}. \tag{S-13.254}$$

This is a simple classical example of a resonant cavity where the frequency and amplitude of the wave depend on each other (and on the cavity length), the link being due to radiation pressure effects; this is called an *optomechanical* cavity [2].

S-13.20 Radiation Pressure on an Absorbing Medium

We assume the incident wave to be linearly polarized parallel to $\hat{\mathbf{y}}$ for definiteness (the generalization to a different polarization is straightforward). The electric field of the wave is thus $\mathbf{E}(x,t) = \hat{\mathbf{y}}\,E_y(x,t)$, with

$$E_y(x,t) = \begin{cases} \mathrm{Re}\left(E_\mathrm{i}\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t} + E_\mathrm{r}\,\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}\omega t}\right), & (x<0), \\ \mathrm{Re}\left(E_\mathrm{t}\mathrm{e}^{\mathrm{i}k\mathrm{n}x-\mathrm{i}\omega t}\right), & (x>0), \end{cases} \tag{S-13.255}$$

where $E_\mathrm{i} = \sqrt{8\pi I_\mathrm{i}/c}$, and

$$E_\mathrm{r} = \frac{1-\mathrm{n}}{1+\mathrm{n}}E_\mathrm{i}, \qquad E_\mathrm{t} = \frac{2}{1+\mathrm{n}}E_\mathrm{i} \tag{S-13.256}$$

(Fresnel formulas at normal incidence). The magnetic field of the wave can be obtained from $\partial_t\mathbf{B} = -\boldsymbol{\nabla}\times\mathbf{E}$, we have $\mathbf{B}(x,t) = \hat{\mathbf{z}}\,B_z(x,t)$, with

$$B_z(x,t) = \begin{cases} \mathrm{Re}\left(E_\mathrm{i}\,\mathrm{e}^{\mathrm{i}kx-\mathrm{i}\omega t} - E_\mathrm{r}\,\mathrm{e}^{-\mathrm{i}kx-\mathrm{i}\omega t}\right), & (x<0), \\ \mathrm{Re}\left(\mathrm{n}E_\mathrm{t}\,\mathrm{e}^{\mathrm{i}k\mathrm{n}x-\mathrm{i}\omega t}\right), & (x>0). \end{cases} \tag{S-13.257}$$

The field for $x>0$ is exponentially decaying, since

$$E_\mathrm{t}\,\mathrm{e}^{\mathrm{i}k(\mathrm{n}_1+\mathrm{i}\mathrm{n}_2)x-\mathrm{i}\omega t} = E_\mathrm{t}\,\mathrm{e}^{\mathrm{i}k\mathrm{n}_1-\mathrm{i}\omega t}\,\mathrm{e}^{-k\mathrm{n}_2 x}, \tag{S-13.258}$$

the decay length being $(k\mathrm{n}_2)^{-1} = \lambda/(2\pi\mathrm{n}_2) \gg \lambda/\mathrm{n}_1$, where $\lambda = 2\pi c/\omega$ is the wavelength in vacuum.

The cycle-averaged value of the Poynting vector at the $x=0$ plane gives the flux of electromagnetic energy entering the medium. Since the field decays with increasing x, there is no net flux of energy for $x\to\infty$, and all the energy entering the medium is eventually absorbed. Using (S-13.255) and (S-13.257) we find

$$\begin{aligned}\langle S_x(0^+)\rangle &= \left\langle \frac{c}{4\pi}E_y(0^+,t)B_z(0^+,t)\right\rangle = \frac{1}{2}\frac{c}{4\pi}\,\mathrm{Re}(E_\mathrm{t}\mathrm{n}^* E_\mathrm{t}^*) = \frac{c}{8\pi}|E_\mathrm{t}|^2\,\mathrm{Re}(\mathrm{n}^*)\\ &= \frac{c}{8\pi}\mathrm{n}_1|E_\mathrm{t}|^2 = \frac{c}{8\pi}\mathrm{n}_1\frac{4}{|1+\mathrm{n}|^2}|E_\mathrm{i}|^2 = \frac{4\mathrm{n}_1}{|1+\mathrm{n}|^2}I_\mathrm{i} \equiv AI_\mathrm{i}\,. \end{aligned} \qquad \text{(S-13.259)}$$

The reflection coefficient $R = |E_\mathrm{r}/E_\mathrm{i}|^2 = |1-\mathrm{n}|^2/|1+\mathrm{n}|^2$. Thus

$$1-R = 1-\left|\frac{1-\mathrm{n}}{1+\mathrm{n}}\right|^2 = \frac{2(\mathrm{n}+\mathrm{n}^*)}{|1+\mathrm{n}|^2} = \frac{4\,\mathrm{Re}(\mathrm{n})}{|1+\mathrm{n}|^2} = \frac{4\mathrm{n}_1}{|1+\mathrm{n}|^2} = A\,. \qquad \text{(S-13.260)}$$

(b) The pressure on the medium is the flow of electromagnetic momentum through the $x=0$ surface. Such flow is given, in the present conditions, by $P_\mathrm{rad} = -\langle T_{xx}(x=0)\rangle$ where T_{ij} is Maxwell stress tensor (see Problem 8.5). Since

$$T_{xx}(0,t) = T_{11}(0,t) = -\frac{1}{8\pi}\left(\mathbf{E}^2(0,t)+\mathbf{B}^2(0,t)\right) \qquad \text{(S-13.261)}$$

we obtain

$$\begin{aligned}\langle T_{11}(0,t)\rangle &= -\frac{1}{16\pi}|E_\mathrm{t}|^2\left(1+|\mathrm{n}|^2\right) = -\frac{1}{4\pi}|E_\mathrm{i}|^2\frac{1+|\mathrm{n}|^2}{|1+\mathrm{n}|^2} = \frac{1}{4\pi}|E_\mathrm{i}|^2\frac{|1+\mathrm{n}|^2-2\,\mathrm{Re}(\mathrm{n})}{|1+\mathrm{n}|^2}\\ &= -\frac{1}{8\pi}|E_\mathrm{i}|^2\left(2-\frac{4\mathrm{n}_1}{|1+\mathrm{n}|^2}\right) = -\frac{I_\mathrm{i}}{c}(1+R) \equiv -P_\mathrm{rad}. \end{aligned} \qquad \text{(S-13.262)}$$

The same result can also be obtained by calculating the total average force per unit surface exerted on the medium by the electromagnetic field

$$P_\mathrm{EM} = \int_0^{+\infty}\langle(\mathbf{J}\times\mathbf{B})_x\rangle\,\mathrm{d}x\,, \qquad \text{(S-13.263)}$$

since the electric term gives no contribution. The current density $\mathbf{J}$ inside the medium can be obtained from the equation $\mathbf{J} = (c\boldsymbol{\nabla}\times\mathbf{B}-\partial_t\mathbf{E})/4\pi$, obtaining

$$\begin{aligned} J_y &= \mathrm{Re}\left(-\frac{\mathrm{i}kc\mathrm{n}}{4\pi}\frac{\mathrm{n}}{c}E_\mathrm{t}\,\mathrm{e}^{-\mathrm{i}k\mathrm{n}x-\mathrm{i}\omega t} + \frac{\mathrm{i}\omega}{4\pi}E_\mathrm{t}\,\mathrm{e}^{-\mathrm{i}k\mathrm{n}x-\mathrm{i}\omega t}\right)\\ &= \mathrm{Re}\left(\frac{\mathrm{i}\omega}{4\pi}\left(1-\mathrm{n}^2\right)E_\mathrm{t}\,\mathrm{e}^{-\mathrm{i}k\mathrm{n}x-\mathrm{i}\omega t}\right). \end{aligned} \qquad \text{(S-13.264)}$$

A further way to obtain this result is recalling the relation between conductivity and dielectric permittivity for complex fields, i.e.,

$$\sigma(\omega) = -\frac{\mathrm{i}\omega}{4\pi}\left[\varepsilon_r(\omega)-1\right] = -\frac{\mathrm{i}\omega}{4\pi}\left[\mathrm{n}^2(\omega)-1\right]. \qquad \text{(S-13.265)}$$

We thus have

$$\begin{aligned}\langle J_y B_z \rangle &= \frac{1}{2}\frac{\omega}{4\pi}\,\mathrm{Re}\left\{\left[\mathrm{i}\left(1-\mathrm{n}^2\right)E_\mathrm{t}\,\mathrm{e}^{(-\mathrm{i}k\mathrm{n}_1-\mathrm{n}_2)x}\right]\left[\mathrm{n}^* E_\mathrm{t}^*\mathrm{e}^{(+\mathrm{i}k\mathrm{n}_1-\mathrm{n}_2)x}\right]\right\}\\ &= \frac{1}{8\pi}|E_\mathrm{t}|^2\,\mathrm{Re}\left[\mathrm{i}\left(1-\mathrm{n}^2\right)\mathrm{n}^*\right]\mathrm{e}^{-2\mathrm{n}_2 x}\,. \end{aligned} \quad \text{(S-13.266)}$$

Now

$$\begin{aligned}\mathrm{Re}\left[\mathrm{i}\left(1-\mathrm{n}^2\right)\mathrm{n}^*\right] &= \mathrm{Re}\left[\left(1-\mathrm{n}_1^2+\mathrm{n}_2^2-2\mathrm{i}\mathrm{n}_1\mathrm{n}_2\right)(\mathrm{i}\mathrm{n}_1+\mathrm{n}_2)\right] = \mathrm{n}_2\left(1+\mathrm{n}_1^2+\mathrm{n}_2^2\right)\\ &= \mathrm{n}_2\left(1+|\mathrm{n}|^2\right), \end{aligned} \quad \text{(S-13.267)}$$

thus, by substituting in (S-13.263) and comparing to (S-13.262) we obtain

$$\begin{aligned} P_\mathrm{EM} &= \int_0^\infty \langle J_y B_z\rangle\,\mathrm{d}x = \frac{1}{2}\frac{1}{4\pi}|E_\mathrm{t}|^2\mathrm{n}_2\left(1+|\mathrm{n}|^2\right)\int_0^\infty \mathrm{e}^{-2\mathrm{n}_2 x}\mathrm{d}x\\ &= \frac{1}{2\pi}|E_\mathrm{i}|^2\frac{1+|\mathrm{n}|^2}{1-|\mathrm{n}|^2}\mathrm{n}_2\frac{1}{2\mathrm{n}_2} = P_\mathrm{rad}\,. \end{aligned} \quad \text{(S-13.268)}$$

S-13.21 Scattering from a Perfectly Conducting Sphere

(a) Since the radius of the sphere, a, is much smaller than the radiation wavelength, λ, we can consider the electric field of the incident wave as uniform over the whole volume of the sphere. As shown in Problems 1.1 and 2.1, the "electron sea" is displaced by an amount $\boldsymbol{\delta}$ with respect to the ion lattice in order to keep the total electric field equal to zero inside the sphere. According to (S-2.2) we have

$$\boldsymbol{\delta} = -\hat{\mathbf{y}}\,\frac{3}{4\pi\varrho_0}\,E_0\cos(\omega t)\,, \quad \text{(S-13.269)}$$

where ϱ_0 is the volume charge density of the ion lattice. This corresponds to a volume polarization $\mathbf{P}$

$$\mathbf{P} = -\rho\boldsymbol{\delta} = \hat{\mathbf{y}}\,\frac{3}{4\pi}\,E_0\cos(\omega t)\,, \quad \text{(S-13.270)}$$

and to a total dipole moment of the conducting sphere

$$\mathbf{p} = \frac{4\pi}{3}a^3\,\mathbf{P} = \mathbf{E}_0 a^3 = \hat{\mathbf{y}}\,a^3 E_0\cos(\omega t)\,. \quad \text{(S-13.271)}$$

The scattered, time-averaged power is thus

$$W_\mathrm{scatt}^{(\mathrm{el})} = \frac{1}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{\omega^4 a^6}{3c^3}\,E_0^2\,. \quad \text{(S-13.272)}$$

The intensity of the incident wave is $I = (c/8\pi)E_0^2$, so we obtain for the scattering cross section

$$\sigma_{\rm scatt}^{\rm (el)} = \frac{W_{\rm scatt}^{\rm (el)}}{I} = \frac{8\pi}{3}\frac{\omega^4 a^6}{c^4} = 128\pi^4(\pi a^2)\left(\frac{a}{\lambda}\right)^4 . \tag{S-13.273}$$

(b) Due to the condition $a \ll \lambda$, also the magnetic field of the wave can be considered as uniform inside the sphere

$$\mathbf{B}(t) = \hat{\mathbf{z}}\, B_0 \cos(\omega t) = \hat{\mathbf{z}}\, E_0 \cos(\omega t) . \tag{S-13.274}$$

Analogously to what seen above for the electric polarization, the sphere must acquire also a uniform magnetization $\mathbf{M}$ in order to cancel the magnetic field of the wave at its interior. As shown by (S-5.72) of Problem 5.10, we must have

$$\mathbf{M}(t) = -\hat{\mathbf{z}}\,\frac{3}{8\pi}\, B_0 \cos(\omega t) = -\hat{\mathbf{z}}\,\frac{3}{8\pi}\, E_0 \cos(\omega t) , \tag{S-13.275}$$

corresponding to a magnetic dipole moment of the sphere

$$\mathbf{m} = \frac{4\pi a^3}{3}\,\mathbf{M} = -\hat{\mathbf{z}}\,\frac{a^3}{2}\, E_0 \cos(\omega t) , \tag{S-13.276}$$

Thus the power scattered by the magnetic dipole is one fourth of the electric dipole contribution:

$$W_{\rm scatt}^{\rm (magn)} = \frac{1}{3c^3}|\ddot{\mathbf{m}}|^2 = \frac{\omega^4 a^6}{12c^3}E_0^2 . \tag{S-13.277}$$

The total cross section is thus 5/4 times the value due to the electric dipole only:

$$\sigma_{\rm scatt}^{\rm (el,magn)} = 160\pi^4(\pi a^2)\left(\frac{a}{\lambda}\right)^4 . \tag{S-13.278}$$

A discussion on how the magnetic dipole term contributes to the angular distribution of the scattered radiation can be found in Reference [3].

S-13.22 Radiation and Scattering from a Linear Molecule

(a) At the initial time $t = 0$, we assume the center of mass of the molecule to be at rest at the origin of our Cartesian coordinate system. The center of mass will remain at rest, since the net force acting on the molecule is zero. However, the field $\mathbf{E}_0$ exerts a torque $\tau_0 = \mathbf{p}_0 \times \mathbf{E}_0$, and the molecule rotates around the z axis. The equation of motion is $\mathcal{I}\ddot{\theta} = \tau_0$, or

$$\mathcal{I}\ddot{\theta} = -p_0 E_0 \sin\theta\,, \tag{S-13.279}$$

where $\theta = \theta(t)$ is the angle between $\mathbf{p}_0$ and the x axis. The potential energy of the molecule is

$$V(\theta) = -\mathbf{p}_0 \cdot \mathbf{E}_0 + C = -p_p E_0 \cos\theta + C\,, \tag{S-13.280}$$

where C is an arbitrary constant. The molecule has two equilibrium positions, at $\theta = 0$ (stable), and $\theta = \pi$ (unstable), respectively. For small oscillations around the stable equilibrium position we can approximate $\sin\theta \simeq \theta$, and (S-13.279) turns into the equation for the harmonic oscillator

$$\ddot{\theta} \simeq -\frac{p_0 E_0}{\mathcal{I}}\theta \equiv -\omega_0^2\theta\,, \quad \text{where} \quad \omega_0^2 = \frac{E_0 p_0}{\mathcal{I}}\,. \tag{S-13.281}$$

Thus, if the molecule starts at rest at a small initial angle $\theta(0) = \theta_0$, we have $\theta(t) \simeq \theta_0 \cos\omega_0 t$. The potential energy of the molecule can be approximated as

$$V(\theta) \simeq -p_0 E_0 \left(1 - \frac{\theta^2}{2}\right) + C = \frac{1}{2}\,p_0 E_0\,\theta^2 = \frac{1}{2}\,\mathcal{I}\omega_0^2\theta^2\,, \tag{S-13.282}$$

where we have chosen $C = p_0 E_0$, in order to have $V(0) = 0$. The kinetic energy of the molecule is

$$K(\dot{\theta}) = \frac{1}{2}\,\mathcal{I}\dot{\theta}^2\,. \tag{S-13.283}$$

(b) In our coordinate system the instantaneous dipole moment has components

$$p_x = p_0 \cos\theta \simeq p_0\,, \qquad p_y = p_0 \sin\theta \simeq p_0\theta_0 \cos(\omega_0 t)\,, \tag{S-13.284}$$

so that, for small oscillations, the radiation emitted by the molecule is equivalent to the radiation of an electric dipole parallel to $\hat{\mathbf{y}}$, and of frequency ω_0. The radiation is linearly polarized, and the angular distribution of the emitted power is $\sim \cos^2\alpha$, where α is the observation angle relative to $\mathbf{E}_0$. Thus, the radiated power per unit solid angle is maximum in the xz plane and vanishes in the $\hat{\mathbf{y}}$ direction. The time-averaged total emitted power is

$$P_{\text{rad}} = \frac{1}{3c^3}\,|\ddot{\mathbf{p}}|^2 = \frac{1}{3c^3}\,\omega_0^4 p_0^2 \theta_0^2\,. \tag{S-13.285}$$

We assume that the decay time is much longer than the oscillation period, so that we can write

$$\theta(t) \simeq \theta_{\text{s}}(t) \cos\omega_0 t\,, \tag{S-13.286}$$

with $\theta_{\text{s}}(0) = \theta_0$, and $\theta_{\text{s}}(t)$ decaying in time so slowly that it is practically constant over a single oscillation. In these conditions the total energy of the molecule during a single oscillation period can be written

$$U(t) = K(\dot{\theta}) + V(\theta) \simeq \frac{1}{2}\mathcal{I}\omega_0^2\theta_s^2(t)\,. \tag{S-13.287}$$

The rate of energy loss due to the emitted radiation is

$$\frac{dU}{dt} = \omega_0^2\mathcal{I}\theta_s\frac{d\theta_s}{dt} = -P_{rad}(\theta_s)\,, \tag{S-13.288}$$

from which we obtain

$$\frac{d\theta_s}{dt} = -\frac{1}{3c^3}\frac{\omega_0^2 p_0^2}{\mathcal{I}}\theta_s\,. \tag{S-13.289}$$

Thus the oscillation amplitude decays exponentially in time

$$\theta_s(t) = \theta_0\,e^{-t/\tau}\,, \quad \text{with} \quad \tau = \frac{3\mathcal{I}c^3}{\omega_0^2 p_0^2}\,. \tag{S-13.290}$$

(c) Since $kd \ll 1$, the electric field of the wave can be considered as uniform over the molecule, and we can write $\mathbf{E}_1(0,t) \simeq \mathbf{E}_1 e^{-i\omega t}$ in complex notation. The torque exerted by the wave is $\tau_1 = \mathbf{p}_0 \times \mathbf{E}_1$. The complete equation of motion for the molecule is thus

$$\mathcal{I}\ddot{\theta} = -p_0E_0\sin\theta - p_0E_1\cos\theta\,e^{-i\omega t}\,, \tag{S-13.291}$$

which, at the limit of small oscillations ($\sin\theta \simeq \theta$, $\cos\theta \simeq 1$) becomes

$$\ddot{\theta} = -\omega_0^2\theta - \omega_1^2 e^{-i\omega t}\,, \quad \text{with} \quad \omega_1^2 = \frac{E_1 p_0}{\mathcal{I}} = \omega_0^2\frac{E_1}{E_0}\,. \tag{S-13.292}$$

The general solution of (S-13.292) is the sum of the homogeneous solution considered at point **(a)**, which describes free oscillations, and of a particular solution of the complete equation. A particular solution can be found in the form $\theta(t) = \theta_f e^{-i\omega t}$, which, substituted into (S-13.292), gives

$$\theta_f = \frac{\omega_1^2}{\omega^2 - \omega_0^2}\,. \tag{S-13.293}$$

For simplicity, we neglected the possible presence of friction in (S-13.281). However, in principle a friction term such as $-\dot{\theta}/\tau$ should appear because of the energy loss by radiation. In the presence of the plane wave the friction term is relevant only close to the $\omega = \omega_0$ resonance, because $\tau^{-1} \ll \omega_0$.

(d) After a transition time of the order of τ possible initial oscillations at ω_0 are damped, and the the molecule reaches a steady state where it oscillates at frequency ω. Assuming, as in **(b)**, small-amplitude oscillations, we have an oscillating dipole component $p_y \simeq p_0\,\theta_f\,e^{-i\omega t}$. The scattered power is

$$P_{scatt} = \frac{1}{3c^3}|\ddot{p}_y|^2 = \frac{p_0^2}{3c^3}\frac{\omega^4\omega_1^4}{(\omega^2-\omega_0^2)^2} = \frac{p_0^4 E_1^2}{3\mathcal{I}^2c^3}\frac{\omega^4}{(\omega^2-\omega_0^2)^2}\,. \qquad \text{(S-13.294)}$$

The intensity of the wave is $I = (c/4\pi)E_1^2$, thus the scattering cross section is

$$\sigma_{scatt} = \frac{P_{scatt}}{I} = \frac{4\pi p_0^4}{3\mathcal{I}^2c^4}\frac{\omega^4}{(\omega^2-\omega_0^2)^2}\,. \qquad \text{(S-13.295)}$$

An order-of-magnitude estimate for a simple molecule such as H_2 can be performed by noticing that $p_0 \sim ed$ and $\mathcal{I} \sim md$, with $m \sim m_p$ the mass of the nuclei, so that $(p_0^4/\mathcal{I}^2c^4) \sim (e^2/m_pc^2)^2$.

S-13.23 Radiation Drag Force

(a) The electric field of the wave in complex notation is

$$\mathbf{E} = \hat{\mathbf{y}}\,\mathrm{Re}\left(E_0\,e^{ikx-i\omega t}\right). \qquad \text{(S-13.296)}$$

Neglecting the magnetic field, the particle oscillates in the $\hat{\mathbf{y}}$ direction without changing its x and z coordinates. Thus, assuming the particle to be initially located at the origin of our Cartesian system, and looking for a solution of the form $\boldsymbol{v} = \mathrm{Re}(\boldsymbol{v}_0\,e^{-i\omega t})$, we obtain by substitution into (13.22):

$$\boldsymbol{v}_0 = \hat{\mathbf{y}}\,\frac{iq}{m(\omega+i\nu)}\,E_0\,. \qquad \text{(S-13.297)}$$

(b) The power developed by the electromagnetic force is $q\mathbf{E}\cdot\boldsymbol{v}$. Thus

$$P_{abs} = \langle q\mathbf{E}\cdot\boldsymbol{v}\rangle = \frac{q}{2}\mathrm{Re}\left(E_0 v_0^*\right) = \frac{q^2}{2m}\frac{\nu}{\omega^2+\nu^2}|E_0|^2. \qquad \text{(S-13.298)}$$

(c) The electric dipole moment of the particle is $\mathbf{p} = q\mathbf{r}$. Using Larmor's formula for the radiated power we obtain

$$P_{rad} = \frac{2}{3c^3}\left\langle\ddot{\mathbf{p}}^2\right\rangle = \frac{2q^2}{3c^3}\left\langle\ddot{\mathbf{r}}^2\right\rangle = \frac{q^4}{3m^2c^3}\frac{\omega^2}{\omega^2+\nu^2}|E_0|^2. \qquad \text{(S-13.299)}$$

Assuming $P_{rad} = P_{abs}$, we obtain

$$\nu = \frac{2q^2\omega^2}{3mc^3}. \qquad \text{(S-13.300)}$$

(d) We must evaluate

$$F_x = \left\langle \frac{q}{c} v_y B_z \right\rangle, \qquad \text{(S-13.301)}$$

where for v_y we use the result of **(a)**, while the amplitude of the magnetic field is $B_0 = E_0$. Thus we have

$$F_x = \frac{q}{2c} \mathrm{Re}\left(v_0 E_0^*\right) = \frac{P_{\mathrm{abs}}}{c}. \qquad \text{(S-13.302)}$$

Thus, the ratio between the energy and the momentum absorbed by the particle from the electromagnetic field equals c.
(e) The radiation from a cluster smaller than one wavelength is coherent and thus scales as N^2, so does the total force. The cluster mass scales as N, thus the acceleration scales as $N^2/N = N$. In other terms, a cluster of many particles may be accelerated much more efficiently than a single particle: the higher the number of particles (within the limits of our approximations), the stronger the acceleration. This is the basis of a concept of "coherent" acceleration using electromagnetic waves, formulated by V. I. Veksler. [4]

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