Chapter S-1
Solutions for Chapter 1

S-1.1 Overlapping Charged Spheres

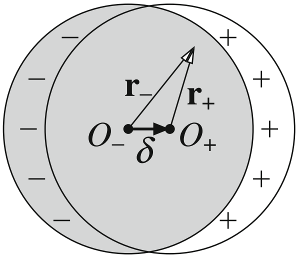

Fig. S-1.1

a) The electrostatic field at any point in space is the sum of the fields generated by each charged sphere (superposition principle). The field generated by a single uniformly charged sphere at its interior is $\mathbf{E}(\mathbf{r}) = 4\pi k_e \varrho_0 \mathbf{r}/3$, where ϱ_0 is the charge density and $\mathbf{r}$ is the position vector relative to the center of the sphere. Thus, the two spheres generate at their interiors the fields $\mathbf{E}_\pm = \pm 4\pi k_e \varrho_0 \mathbf{r}_\pm/3$, respectively, $\mathbf{r}_\pm$ being the position vectors relative to the two centers. We assume that the centers are located on the x axis at points $O_+ \equiv (+\delta/2, 0, 0)$ and $O_- \equiv (-\delta/2, 0, 0)$. We thus have $\mathbf{r}_\pm = \mathbf{r} \pm \boldsymbol{\delta}/2$, where $\mathbf{r}$ is the position vector relative to the origin $O \equiv (0, 0, 0)$. The total field in the overlap region is

$$\mathbf{E}_{\text{in}} = +\frac{4\pi k_e}{3}\varrho_0\left(\mathbf{r} - \frac{\boldsymbol{\delta}}{2}\right) - \frac{4\pi k_e}{3}\varrho_0\left(\mathbf{r} + \frac{\boldsymbol{\delta}}{2}\right) = -\frac{4\pi k_e}{3}\varrho_0\,\boldsymbol{\delta}\,. \qquad \text{(S-1.1)}$$

The internal field $\mathbf{E}_{in}$ is thus *uniform* and proportional to $-\boldsymbol{\delta}$.

b) The electrostatic field generated by a uniformly charged sphere, with volume charge density ϱ_0, *outside* its volume equals is the field of a point charge $Q = 4\pi R^3 \varrho_0/3$ located at its center. Thus, the electrostatic field in the outer region (outside both spheres) is the sum of the fields of two point charges $\pm Q$ located at O_+ and O_-, respectively. If $R \gg \delta$, this is equivalent to the field of an electric dipole of moment

$$\mathbf{p} = Q\boldsymbol{\delta} = \frac{4\pi R^3}{3}\varrho_0\,\boldsymbol{\delta} \qquad \text{(S-1.2)}$$

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located at the origin and lying on the x axis. The external field is thus

$$\mathbf{E}_{\mathrm{ext}}(\mathbf{r}) = k_{\mathrm{e}} \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3}, \tag{S-1.3}$$

where $\mathbf{r} = r\hat{\mathbf{r}}$ is the vector position relative to the origin.

In the two transition "shell" regions, of net charge densities $\pm\varrho_0$, the field is the sum of the inner field of one sphere and of the outer field of the other. We omit to write down the expression for brevity (Fig. S-1.1).

S-1.2 Charged Sphere with Internal Spherical Cavity

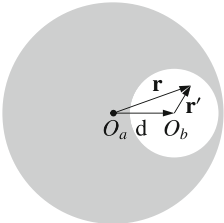

Fig. S-1.2

a) Once again we use the superposition principle. Our charged sphere with an internal spherical cavity can be thought of as the superposition of two uniformly charged spheres: a sphere of radius a centered in O_a, with charge density ϱ, and a smaller sphere of radius b centered in O_b, with charge density $-\varrho$. The electric field everywhere in space is the sum of the fields generated by the two spheres. The field generated by a uniformly charged sphere at its inside is $\mathbf{E} = (4\pi k_{\mathrm{e}}/3)\varrho\mathbf{r}$, where $\mathbf{r}$ is the distance from the center of the sphere. The total field inside the cavity at a point of vector position $\mathbf{r}$ relative to O_a, and vector position $\mathbf{r}'$ relative to O_b, is thus

$$\mathbf{E}_{\mathrm{cav}} = \frac{4\pi k_{\mathrm{e}}}{3}\varrho(\mathbf{r} - \mathbf{r}') = \frac{4\pi k_{\mathrm{e}}}{3}\varrho\mathbf{d}, \tag{S-1.4}$$

uniform and parallel to the straight line passing through O_a and O_b. If $\mathbf{d} = 0$ we obtain $\mathbf{E} = 0$, as expected from Gauss's law and symmetry considerations.

b) In an external field $\mathbf{E}_0$ the total force on the system is the sum of the forces that $\mathbf{E}_0$ would exert on the two point charges $Q_a = 4\pi a^3\varrho/3$ and $Q_b = -4\pi b^3\varrho/3$, located in O_a and O_b, respectively, so that

$$\mathbf{F} = \frac{4\pi}{3}\varrho(a^3 - b^3)\mathbf{E}_0 . \tag{S-1.5}$$

c) Since the vector sum of the forces is different from zero, the torque depends on our choice of the origin. The torque about the center of the sphere O_a is

$$\tau = \mathbf{d}\times\mathbf{F} = -\frac{4\pi}{3}\varrho b^3\mathbf{d}\times\mathbf{E}_0 . \tag{S-1.6}$$

Let us introduce a reference system with the x axis passing through O_a and O_b, and the origin in O_a. We denote by $\varrho_{\rm m} = \alpha\varrho$ the mass density, with α some constant value. Our coordinate origin is thus the location of the center of mass of the cavity-less sphere of radius a and mass $M_{\rm tot} = 4\pi a^3\varrho_{\rm m}/3$, while $x = d$ is the location of the center of mass of a sphere of radius b and mass $M_b = 4\pi b^3\varrho_{\rm m}/3$. Let us denote by $x_{\rm c}$ the center of mass of the sphere with cavity, of mass $M_{\rm c} = 4\pi(a^3 - b^3)\varrho_{\rm m}/3$. We have

$$0 = \frac{M_{\rm c}x_{\rm c} + M_b\,d}{M_{\rm tot}}, \quad \text{thus} \quad x_{\rm c} = -d\,\frac{M_b}{M_{\rm c}} = -d\,\frac{b^3}{a^3 - b^3}\,. \tag{S-1.7}$$

The torque about the center of mass $x_{\rm c}$ is thus

$$\tau_c = \frac{b^3 d}{a^3 - b^3}\,\frac{4\pi}{3}\,\varrho a^3\,\hat{\mathbf{x}}\times\mathbf{E}_0 - \left(d + \frac{b^3 d}{a^3 - b^3}\right)\frac{4\pi}{3}\,\varrho b^3\,\hat{\mathbf{x}}\times\mathbf{E}_0 = 0\,, \tag{S-1.8}$$

as was to be expected, since each charged volume element d^3r is subject to the force $\varrho\mathbf{E}_0\,\mathrm{d}^3r$, and acquires an acceleration

$$\mathbf{a} = \frac{\varrho\,\mathbf{E}_0\,\mathrm{d}^3r}{\varrho_{\rm m}\mathrm{d}^3r} = \frac{\mathbf{E}_0}{\alpha}\,, \tag{S-1.9}$$

equal for each charged volume element (Fig. S-1.2).

S-1.3 Energy of a Charged Sphere

a) We can assemble the sphere by moving successive infinitesimal shells of charge from infinity to their final location. Let us assume that we have already assembled a sphere of charge density ϱ and radius $r < R$, and that we are adding a further shell of thickness $\mathrm{d}r$. The assembled sphere has charge $q(r) = \varrho(4\pi r^3/3)$, and its potential $\varphi(r, r')$ at any point at distance $r' \geq r$ from the center of the sphere is

$$\varphi(r, r') = k_{\rm e}\frac{q(r)}{r'} = k_{\rm e}\,\varrho\,\frac{4\pi r^3}{3}\,\frac{1}{r'}\,. \tag{S-1.10}$$

The work needed to move the new shell of charge $\mathrm{d}q = \varrho\,4\pi r^2\mathrm{d}r$ from infinity to r is

$$\mathrm{d}W = \varphi(r, r')\,\mathrm{d}q = k_{\rm e}\,\varrho\,\frac{4\pi r^3}{3}\,\frac{1}{r}\,\varrho\,4\pi r^2\mathrm{d}r = k_{\rm e}\,\frac{(4\pi\varrho)^2}{3}\,r^4\mathrm{d}r\,. \tag{S-1.11}$$

The total work needed to assemble the sphere is obtained by integrating $\mathrm{d}W$ from $r = 0$ (no sphere) up to the final radius R

$$U_0 = k_e \frac{(4\pi\varrho)^2}{3} \int_0^R r^4 \mathrm{d}r = k_e \frac{(4\pi\varrho)^2 R^5}{15} = \frac{3k_e}{5} \frac{Q^2}{R}, \tag{S-1.12}$$

where $Q = (\varrho 4\pi R^3)/3$ is the total charge of the sphere.
b) The electric field everywhere in space is, according to Gauss's law,

$$E(r) = k_e Q \times \begin{cases} \dfrac{r}{R^3}, & r \le R \\ \dfrac{1}{r^2}, & r \ge R, \end{cases} \tag{S-1.13}$$

and the integral of the corresponding energy density $u_E = E^2/(8\pi k_e)$ over the whole space is

$$\begin{aligned} U_0 &= \int_0^\infty \frac{E^2(r)}{8\pi k_e} 4\pi r^2 \,\mathrm{d}r = \frac{k_e^2 Q^2}{2k_e} \left[\int_0^R \left(\frac{r}{R^3}\right)^2 r^2 \mathrm{d}r + \int_R^\infty \left(\frac{1}{r^2}\right)^2 r^2 \mathrm{d}r \right] \\ &= k_e \frac{Q^2}{2} \left(\frac{1}{5R} + \frac{1}{R} \right) = \frac{3k_e}{5} \frac{Q^2}{R}. \end{aligned} \tag{S-1.14}$$

c) The electrostatic potential of the sphere everywhere in space is

$$\varphi(r) = k_e Q \times \begin{cases} -\dfrac{r^2}{2R^3} + \dfrac{3}{2R}, & r \le R \\ \dfrac{1}{r}, & r > R \end{cases} \tag{S-1.15}$$

where the constant $3k_e Q/(2R)$ appearing for $r \le R$ is needed for $\varphi(r)$ to be continuous at $r = R$. Since $\varrho = 0$ for $r > R$, we need only the integral of $\varrho\varphi/2$ inside the sphere

$$\begin{aligned} U_0 &= \frac{1}{2} \int_0^R \varrho k_e Q \left(-\frac{r^2}{2R^3} + \frac{3}{2R} \right) 4\pi r^2 \,\mathrm{d}r = k_e \frac{Q}{4} \frac{Q}{R^3/3} \left(-\frac{R^2}{5} + R^2 \right) \\ &= \frac{3k_e}{4} \frac{Q^2}{R^3} \frac{4R^2}{5} = \frac{3k_e}{5} \frac{Q^2}{R}. \end{aligned} \tag{S-1.16}$$

All methods, including **b)** and **c)**, lead to the correct result, as expected. However, a comparison between methods **b)** and **c)** shows that it is incorrect to interpret the "energy density" of the electric field as the "energy stored in a given region of space per unit volume". If we give this meaning to quantity $\mathbf{E}^2/(8\pi k_e)$, as in **b)**, we conclude that the energy is spread over the whole space. If, on the other hand, we assume the energy density to be $\frac{1}{2}\varrho\varphi$, as in **c)**, the energy is "stored" only inside the volume of the sphere, i.e., "where the charge is". Thus, the concept of energy density is ambiguous, while the total electrostatic energy of the system is a well defined quantity, at least in the absence of point charges.

S-1.4 Plasma Oscillations

a) Assuming $\delta > 0$, the collective rigid displacement of the conduction electrons due to the external field gives origin to the charge density

$$\varrho(x) = \begin{cases} 0, & x < 0, \\ +en, & 0 < x < \delta, \\ 0, & \delta < x < h, \\ -en, & h < x < h+\delta, \\ 0, & x > h+\delta. \end{cases} \qquad \text{(S-1.17)}$$

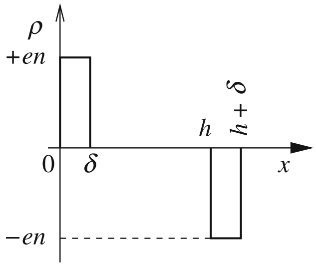

Fig. S-1.3

The electrostatic field $\mathbf{E}(x)$ generated by this charge distribution is obtained by integrating the equation $\boldsymbol{\nabla}\cdot\mathbf{E} = \partial_x E_x = 4\pi k_e \varrho$ with the boundary condition $\mathbf{E}(-\infty) = 0$:

$$E_x(x) = 4\pi e n k_e \begin{cases} 0, & x < 0, \\ x, & 0 < x < \delta, \\ \delta, & \delta < x < h, \\ h+\delta-x, & h < x < h+\delta, \\ 0, & x > h+\delta. \end{cases} \qquad \text{(S-1.18)}$$

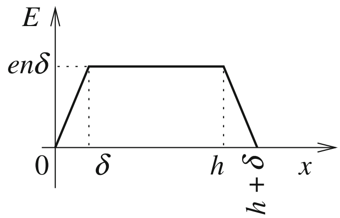

Fig. S-1.4

If we assume a negative displacement $-\delta$ (with $\delta > 0$) the charge density and the electric field are

$$\varrho(x) = \begin{cases} 0, & x < -\delta, \\ -en, & -\delta < x < 0, \\ 0, & 0 < x < h-\delta, \\ +en, & h-\delta < x < h, \\ 0, & x > h. \end{cases} \qquad E_x(x) = 4\pi e n k_e \begin{cases} 0, & x < -\delta, \\ -x-\delta, & -\delta < x < 0, \\ -\delta, & 0 < x < h-\delta, \\ x-h-\delta, & h-\delta < x < h, \\ 0, & x > h. \end{cases} \qquad \text{(S-1.19)}$$

The plots are obtained from Figs. S-1.3 and S-1.4, respectively, by flipping around the x axis and translating by δ towards the negative x values.

b) The electrostatic energy of the system, in the case of a positive displacement, can be evaluated by integrating the "energy density" $u = E_x^2/(8\pi k_e)$ over the whole space:

$$\begin{aligned} U_{es} &= \int \frac{E_x^2}{8\pi k_e}\, d^3 r = \frac{L^2}{8\pi k_e}\int_0^{h+\delta} E_x^2\, dx \\ &= \frac{L^2}{8\pi k_e}(4\pi e n)^2 \left[\int_0^{\delta} x^2\, dx + \int_{\delta}^{h} \delta^2\, dx + \int_h^{h+\delta} (h+\delta-x)^2\, dx\right] \\ &= 2\pi k_e (enL)^2 \left[\frac{\delta^3}{3} + \delta^2(h-\delta) + \frac{\delta^3}{3}\right] = 2\pi k_e (enL)^2 \left(h\delta^2 - \frac{\delta^3}{3}\right), \end{aligned} \qquad \text{(S-1.20)}$$

where, due to the symmetry of the problem, we used $\mathrm{d}^3 r = L^2\,\mathrm{d}x$. Exactly the same result is obtained for a negative displacement by $-\delta$. The δ appearing in the last line of (S-1.20) is actually to be interpreted as the absolute value $|\delta|$.

c) At the limit $\delta \ll h$ we can neglect the third-order term in δ of (S-1.20), and approximate $U_{\mathrm{es}} \simeq 2\pi k_{\mathrm{e}}(enL)^2 h\delta^2$, which is the potential energy of a harmonic oscillator. The force on the "electron slab" is thus

$$F = -\frac{\partial U_{\mathrm{es}}}{\partial \delta} = -4\pi k_{\mathrm{e}}(enL)^2 h\delta\,, \tag{S-1.21}$$

where δ can be positive or negative. The equation of motion for the electrons is

$$M\ddot{\delta} = F \equiv -M\omega^2\delta\,, \tag{S-1.22}$$

where $M = m_{\mathrm{e}} n L^2 h$ is the total mass of the conduction-electron slab. We thus have

$$\omega^2 = \frac{4\pi k_{\mathrm{e}} n e^2}{m_{\mathrm{e}}} \equiv \omega_{\mathrm{p}}^2\,, \tag{S-1.23}$$

where ω_{p} is called the *plasma frequency*, and is an intrinsic property of the given conductor, dependent only on the density of free electrons.

S-1.5 Mie Oscillations

a) In Problem 1.1 we showed that the electric field is uniform and equal to $-4\pi k_{\mathrm{e}}\varrho_0\,\delta/3$, with $\varrho_0 = en_{\mathrm{e}}$, in the region where the conduction-electrons sphere overlap,

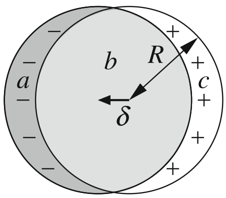

Fig. S-1.5

$$\mathbf{E}_{\mathrm{int}} = -\frac{4\pi k_{\mathrm{e}}}{3}\, en_{\mathrm{e}}\boldsymbol{\delta}\,. \tag{S-1.24}$$

We assume that the displacement δ is sufficiently small for the volumes a (only conduction electrons) and c (only ion lattice) of Fig. S-1.5 to be negligible compared to the overlap volume b, an order of magnitude for δ is found in (S-2.4) of Solution S-2.1. Assuming further that conduction electrons behave like a "rigid" body, oscillating in phase with the same displacement $\boldsymbol{\delta} = \boldsymbol{\delta}(t)$ from their rest positions, the equation of motion for the single electron is

$$m_{\mathrm{e}}\frac{\mathrm{d}^2\boldsymbol{\delta}}{\mathrm{d}t^2} = -e\mathbf{E}_{\mathrm{int}} = -e\frac{4\pi k_{\mathrm{e}}}{3}en_{\mathrm{e}}\boldsymbol{\delta} = -m_{\mathrm{e}}\frac{\omega_{\mathrm{p}}^2}{3}\boldsymbol{\delta}\,, \tag{S-1.25}$$

where ω_p is the plasma frequency (S-1.23). Thus, the displacement of each conduction electron from its rest position is

$$\boldsymbol{\delta}(t) = \boldsymbol{\delta}(0)\cos\left(\frac{\omega_\mathrm{p}}{\sqrt{3}}\,t\right), \tag{S-1.26}$$

where $\boldsymbol{\delta}(0)$ is a constant and $\omega_\mathrm{p}/\sqrt{3}$ is called the Mie frequency. This type of motion is known as Mie oscillation (or surface plasmon of the sphere).
b) The electrostatic energy of the system is given by the integral

$$U_\mathrm{es} = \int \frac{E^2}{8\pi k_\mathrm{e}}\,\mathrm{d}^3r\,. \tag{S-1.27}$$

For δ approaching 0, the electric field is given by (S-1.24) inside the sphere ($r < R$), and by (S-1.3), i.e., an electric dipole field, outside the sphere ($r > R$). Thus we can split the integral of (S-1.27) into the sum of two terms, corresponding to the integration domains $r < R$ and $r > R$, respectively

$$U_\mathrm{es} = U_\mathrm{es}^\mathrm{in} + U_\mathrm{es}^\mathrm{out} = \frac{1}{8\pi k_\mathrm{e}}\left(\int_{r<R} E^2\,\mathrm{d}^3r + \int_{r<R} E^2\,\mathrm{d}^3r\right). \tag{S-1.28}$$

For $r < R$ the field is uniform and we immediately find

$$U_\mathrm{es}^\mathrm{in} = \frac{1}{8\pi k_\mathrm{e}}\left(k_\mathrm{e}\frac{4\pi}{3}en_\mathrm{e}\delta\right)^2\frac{4\pi}{3}R^3 = k_\mathrm{e}\,\frac{8\pi^2}{27}\,(en_\mathrm{e}\delta)^2R^3\,. \tag{S-1.29}$$

For evaluating the contribution of the outer region, we substitute

$$\mathrm{d}^3r = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,,\quad \text{and}\quad E^2 = \left(\frac{k_\mathrm{e}p}{r^3}\right)^2(3\cos^2\theta+1)\,, \tag{S-1.30}$$

where $\mathbf{p} = Q\boldsymbol{\delta} = \boldsymbol{\delta}\,en_\mathrm{e}\,4\pi R^3/3$ (see Problem 1.1), into the second integral at the right-hand side of (S-1.28)

$$U_\mathrm{es}^\mathrm{out} = 2\pi\,\frac{1}{8\pi k_\mathrm{e}}\,k_\mathrm{e}^2p^2\int_R^{+\infty} r^2\,\mathrm{d}r\int_0^\pi \sin\theta\,\mathrm{d}\theta\,\frac{3\cos^2\theta+1}{r^6}\,, \tag{S-1.31}$$

and obtain

$$U_\mathrm{es}^\mathrm{out} = k_\mathrm{e}\frac{p^2}{3R^3} = k_\mathrm{e}\,\frac{4\pi}{9}\,(en_\mathrm{e})^2\,\frac{4\pi}{3}\,R^3 = k_\mathrm{e}\,\frac{16\pi^2}{27}\,(en_\mathrm{e}\delta)^2R^3\,, \tag{S-1.32}$$

thus $U_{\mathrm{es}}^{\mathrm{out}} = 2U_{\mathrm{es}}^{\mathrm{in}}$. For the total energy we have finally

$$U_{\mathrm{es}} = U_{\mathrm{es}}^{\mathrm{in}} + U_{\mathrm{es}}^{\mathrm{out}} = k_{\mathrm{e}}\frac{8\pi^2}{9}(en_{\mathrm{e}}\delta)^2 R^3 \,. \tag{S-1.33}$$

The derivative of U_{es} with respect to δ gives the force associated to the displacement of the electron sphere:

$$F = -\frac{\partial U_{\mathrm{es}}}{\partial \delta} = -k_{\mathrm{e}}\frac{16\pi^2}{9}R^3\,(en_{\mathrm{e}})^2\delta \,. \tag{S-1.34}$$

The equation of motion for the rigid sphere of electrons is $M\,\mathrm{d}^2\boldsymbol{\delta}/(\mathrm{d}t^2) = \mathbf{F}$, where $M = m_{\mathrm{e}}n_{\mathrm{e}}4\pi R^3/3$ is the total mass of electrons. Thus

$$\frac{\mathrm{d}^2\boldsymbol{\delta}}{\mathrm{d}t^2} = -k_{\mathrm{e}}\frac{4\pi n_{\mathrm{e}}e^2}{3m_{\mathrm{e}}}\boldsymbol{\delta} \equiv -\frac{\omega_{\mathrm{p}}^2}{3}\boldsymbol{\delta} \,, \tag{S-1.35}$$

and we are back to the oscillations at the Mie frequency of (S-1.26).

S-1.6 Coulomb Explosions

a) The electric field has radial symmetry, $\mathbf{E} = E(r)\,\hat{\mathbf{r}}$. According to Gauss's law we have $4\pi r^2 E(r) = 4\pi k_{\mathrm{e}} Q_{\mathrm{int}}(r)$, where $Q_{\mathrm{int}}(r)$ is the charge inside the sphere of radius r. At $t = 0$ we have $Q_{\mathrm{int}} = Q(r/R)^3$ for $r < R$, and $Q_{\mathrm{int}} = Q$ for $r > R$, thus the electric field inside and outside the cloud is, respectively,

$$E(r) = k_{\mathrm{e}}Q \times \begin{cases} \dfrac{r}{R^3}\,, & r \le R\,, \\ \dfrac{1}{r^2}\,, & r \ge R\,. \end{cases} \tag{S-1.36}$$

Due to the spherical symmetry of the problem, we have $E = -\partial\varphi/\partial r$, where φ is the electric potential, and the potential at $t = 0$ can be obtained by a simple integration:

$$\varphi(r) = k_{\mathrm{e}}Q \times \begin{cases} -\dfrac{r^2}{2R^3} + \dfrac{3}{2R}\,, & r \le R\,, \\ \dfrac{1}{r}\,, & r \ge R\,. \end{cases} \tag{S-1.37}$$

As in Equation (S-1.15) of Problem 1.3, the integration constants have been chosen so that $\varphi(\infty) = 0$ and $\varphi(r)$ is continuous at $r = R$. The potential energy of a *test* charge q_{t} located at distance r from the center is thus $q_{\mathrm{t}}\,\varphi(r)$.

b) Under the action of the electric field, the test charge would move and convert all its potential energy into kinetic energy *if the field remained stationary during the charge motion*, i.e., *if all the source charges of the field remained fixed*. At $t = 0$ the electric field inside the spherical cloud increases with r. Thus, the "outer" particles, located at larger r, have a higher acceleration than the "inner" particles, located at smaller r. After an infinitesimal time interval the "outer" particles will acquire a higher velocity (we are assuming that all particles are at rest at $t = 0$), and will not be overtaken by the "inner" ones. Moreover, also the acceleration has radial symmetry, and thus any spherical shell preserves its shape in time. These arguments can be iterated for any following time, proving the validity of our assumptions that the particles do not overtake one another, and that the spherical symmetry is preserved.

Let us denote by $r_s(r_0,t)$ the position of a particle initially located at r_0. Since the particles do not overtake one another, the charge inside a sphere of radius $r_s(r_0,t)$ is constant. The electric field intensity at $r_s(r_0,t)$ can be evaluated by applying Gauss's law: from

$$4\pi r_s^2(r_0,t)\,E\,[r_s(r_0,t)] = 4\pi k_e Q\left(\frac{r_0}{R}\right)^3 \tag{S-1.38}$$

we obtain

$$E\,[r_s(r_0,t)] = k_e\frac{Q}{r_s^2(r_0,t)}\left(\frac{r_0}{R}\right)^3 , \tag{S-1.39}$$

from which Equation (1.16) can be derived. The forces on the particles, and thus their accelerations, increase with increasing r_0, in agreement with our "non-overtaking" result. Note that the electric field, and thus the force, at $t = 0$ is proportional to r_0, not to r_0^3, because we have $r_s^2(r_0,0) = r_0^2$ at the denominator.

c) Each infinitesimal spherical shell expands from its initial radius r_0 to its final radius $r_s(r_0,\infty) = \infty$ under the action of the force (1.16). The final kinetic energy of a particle belonging to the shell, $K_{fin}(r_0)$, equals the work done by the force on the particle

$$K_{fin}(r_0) = \int_{r_{i0}}^{+\infty} k_e\frac{qQ}{r_i^2}\left(\frac{r_0}{R}\right)^3 \mathrm{d}r = k_e\frac{qQ}{r_0}\left(\frac{r_0}{R}\right)^3 = k_e\,q\,Q\frac{r_0^2}{R^3} . \tag{S-1.40}$$

Quantity $K_{fin}(r_0)$ is a monotonically increasing function of r_0, thus its maximum value K_{max} is observed for $r_0 = R$

$$K_{max} = K_{fin}(R) = k_e\frac{qQ}{R}. \tag{S-1.41}$$

This means that the particles initially located at $r_0 = R$, i.e., at the cloud surface, acquire the maximum final kinetic energy.

d) The energy distribution, or energy spectrum, function $f(K)$ is defined so that the number $\mathrm{d}N$ of particles with kinetic energy in the interval $(K, K+\mathrm{d}K)$ equals $f(K)\,\mathrm{d}K$, therefore $f(K) = \mathrm{d}N/\mathrm{d}K$. A particle belonging to the shell $r_0 < r_s < r_0+\mathrm{d}r_0$ at $t = 0$ has a kinetic energy in the interval $(K_{fin}(r_0), K_{fin}(r_0)+\mathrm{d}K_{fin})$ at $t = \infty$, where

$$\mathrm{d}K_{\mathrm{fin}} = \frac{2k_{\mathrm{e}}\,qQ\,r_0}{R^3}\,\mathrm{d}r_0\,. \tag{S-1.42}$$

On the other hand, at $t = 0$ the number of particles in the shell $(r_0, r_0 + \mathrm{d}r_0)$ is

$$\mathrm{d}N = N\frac{3}{4\pi R^3}\,4\pi r_0^2\,\mathrm{d}r_0 = N\frac{3r_0^2}{R^3}\,\mathrm{d}r_0\,, \tag{S-1.43}$$

and the number of particles in a given shell is constant during the motion, since the particles do not overtake one another. Thus, inserting (S-1.40) and (S-1.41), we obtain

$$f(K_{\mathrm{fin}}) = \frac{\mathrm{d}N}{\mathrm{d}K_{\mathrm{fin}}} = N\frac{3r_0^2}{R^3}\,\frac{R^3}{2k_{\mathrm{e}}qQr_0} = N\frac{3r_0}{2k_{\mathrm{e}}qQ} = \frac{3N}{2K_{\mathrm{max}}^{3/2}}\,\sqrt{K_{\mathrm{fin}}}\,, \tag{S-1.44}$$

valid for $K_{\mathrm{fin}} \le K_{\mathrm{max}}$.

The final total kinetic energy is

$$\begin{aligned} K_{\mathrm{tot}} &= \int_0^{K_{\mathrm{max}}} K\,f(K)\,\mathrm{d}K = \frac{3N}{2K_{\mathrm{max}}^{3/2}}\int_0^{K_{\mathrm{max}}} K^{3/2}\mathrm{d}K = \frac{3N}{5}K_{\mathrm{max}} \\ &= \frac{3k_{\mathrm{e}}}{5}\frac{NqQ}{R} = \frac{3k_{\mathrm{e}}}{5}\frac{Q^2}{R}\,, \end{aligned} \tag{S-1.45}$$

which equals the total electrostatic energy stored in the charged sphere at $t = 0$ (Problem 1.3). Here we have substituted $Nq = Q$. Thus, all the electrostatic energy stored in the initial configuration is eventually converted into kinetic energy.

It is a relatively common error to assume that the final kinetic energy of a particle initially in the shell $r_0 < r_{\mathrm{s}} < r_0 + \mathrm{d}r$ is equal to the potential energy of the same particle at $t = 0$, i.e., that $K_{\mathrm{fin}} = q\varphi(r_0)$, where φ is given by (S-1.37). This is obviously wrong, because a particle initially at $r_0 = 0$ has the highest possible initial potential energy, $\varphi(0) = 3k_{\mathrm{e}}Q/(2R)$, while it undergoes the lowest possible gain in kinetic energy (zero)! Moreover, this behavior would not preserve the total energy of the system, because the initial potential energy of the sphere is (see Problem 1.3)

$$U(0) = \frac{1}{2}\sum_i q\varphi[r_i(0)]\,, \quad \textit{not} \quad U(0) = \sum_i q\varphi[r_i(0)]\,.$$

The point is that while the field is electrostatic ($\boldsymbol{\nabla}\times\mathbf{E} = 0$) at any time, it is *time dependent*. Thus, φ can be defined for any value of t, but it *cannot* be used to evaluate the final kinetic energy, because φ changes as the particles move.

The gain in kinetic energy equals the initial potential energy $q\varphi(R, t = 0)$ for the particles initially at $r_i = R$, i.e., for the most external ones. Only these particles are accelerated by a field that can be treated as static, being simply equal to the field of a point charge Q located at $r = 0$ at any time.

e) If we introduce the new variable $x(t) = r_{\mathrm{s}}(r_0, t)/r_0$, (1.16) becomes

$$m\frac{d^2x}{dt^2} = k_e\frac{qQ}{R^3x^2}, \tag{S-1.46}$$

which is independent of r_0. The solution of (S-1.46), $x = x(t)$, with the initial condition $x(0) = 1$, is thus valid for all the particles of the cloud. Thus, if two shells, labeled 1 and 2, have initial radii r_{10} and r_{20}, with $r_{20} > r_{10}$, their subsequent radii will be $r_1(t) = r_s(r_{10},t) = r_{10}\,x(t)$ and $r_2(t) = r_s(r_{20},t) = r_{20}x(t)$. It will always be $r_2(t) > r_1(t)$, and the internal shell cannot overtake the external one. The number of particles contained between the layers 1 and 2 is constant and equal to

$$\delta N_{12} = \frac{N}{R^3}\left(r_{20}^3 - r_{10}^3\right). \tag{S-1.47}$$

Thus the particle density between the two layers at time t is

$$n(t) = \frac{3}{4\pi\left[r_2^3(t) - r_1^3(t)\right]}\,\delta N_{12} = \frac{3N}{4\pi R^3}\,\frac{r_{20}^3 - r_{10}^3}{(r_{20}^3 - r_{10}^3)\,x^3(t)} = \frac{3N}{4\pi R^3}\,x^{-3}(t). \tag{S-1.48}$$

This result does not depend on the particular choice of the two layers, and the particle density is uniform at any time t, and decreases with increasing time as $x^{-3}(t)$.

S-1.7 Plane and Cylindrical Coulomb Explosions

a) The electric field is parallel to the x axis and independent of the y and z coordinates for symmetry reasons, thus we have $\mathbf{E}(x,y,z) = E(x)\,\hat{\mathbf{x}}$. Again for symmetry reasons, the electric field is antisymmetric with respect to the $x = 0$ plane, so that $E(-x) = -E(x)$. Thus it is sufficient to consider the field for $x \geq 0$. The charge density at $t = 0$ is $\varrho(x) = qn_0\Theta(a/2 - |x|)$, where $\Theta(x)$ is the Heaviside step function, defined as $\Theta(x) = 1$ for $x > 0$, and $\Theta(x) = 0$ for $x < 0$. The electric field at $t = 0$ can be evaluated by integrating the equation $\nabla\cdot\mathbf{E} = \partial_x E_x = 4\pi k_e\varrho(x)$, with the boundary condition $\mathbf{E}(0) = 0$, obtaining

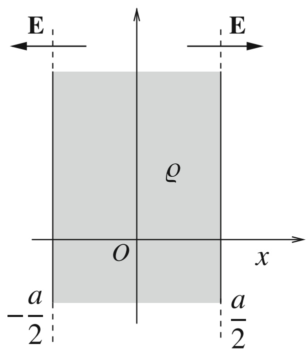

Fig. S-1.6

$$E(x) = 4\pi k_e \begin{cases} qn_0x, & x < \dfrac{a}{2}, \\ qn_0\dfrac{a}{2}, & x > \dfrac{a}{2}. \end{cases} \tag{S-1.49}$$

Since the particles are at rest at t=0, and the electric field increases with increasing x, the particles cannot overtake one another. The motion of a particle initially at x_0 is described by an equation $x_s = x_s(x_0, t)$. Let us consider a parallelepiped of base S lying on the $x = 0$ plane and height $x_s(x_0, t)$. The charge inside our parallelepiped is constant in time since no particle can cross the moving base. We can apply Gauss's law to evaluate the electric field on the particle located at $x_s(x_0, t)$

$$E[x_s(x_0,t),t]\,S = 4\pi k_e Q_{in}(t) = 4\pi k_e Q_{in}(0) = 4\pi k_e q n_0 x_0 S\,. \tag{S-1.50}$$

Thus, the field accelerating each particle is constant in time, and equals

$$E(x_0) = 4\pi k_e q n_0 x_0\,, \tag{S-1.51}$$

where x_0 is the particle position at $t = 0$. The equation of motion is thus

$$m\frac{d^2 x_s(x_0,t)}{dt^2} = qE(x_0) = 4\pi k_e q^2 n_0 x_0\,, \tag{S-1.52}$$

with the initial conditions $x_s(x_0, 0) = x_0$ and $\dot{x}_s(x_0, 0) = 0$. The solution is

$$x_s(x_0,t) = x_0 + 2\pi k_e \frac{q^2 n_0 x_0}{m} t^2 = x_0\left(1 + \frac{\omega_p^2 t^2}{2}\right), \tag{S-1.53}$$

where $\omega_p = \sqrt{4\pi k_e q^2 n_0/m}$ is the "plasma frequency" of the infinite charged layer at $t = 0$ (see Problem 1.4). Thus, the acceleration of an infinitesimal plane layer of thickness dx is proportional to its initial x coordinate, and more external layers are faster than more internal ones. The velocity, and the kinetic energy, of each layer grow indefinitely with time, which is not surprising since the system has an infinite initial potential energy (Fig. S-1.6).

If we introduce the dimensionless variable $\xi = x_s/x_0$, its equation of motion

$$m\frac{d^2\xi}{dt^2} = qE(x_0) = 4\pi k_e q^2 n_0, \qquad \xi(0) = 1\,, \qquad \frac{d\xi(0)}{dt} = 0\,, \tag{S-1.54}$$

is independent of x_0. Thus the position of any particle can be written in the form

$$x_s(x_0,t) = x_0\,\xi(t)\,, \tag{S-1.55}$$

which shows that the particle density, and the charge density, remain uniform during the explosion.

b) The case of the Coulomb explosion of a system of charged particles initially confined, at rest, inside an infinite cylinder of radius a, is similar. We use cylindrical coordinates (r, ϕ, z), and assume that the initial particle density is uniform and equal to n_0 for $r < a$, and zero for $r > a$. All particles have mass m and charge q. According to Gauss's law, at $t = 0$ the field at position (r_0, ϕ, z), with $r_0 < a$, is

$$E(r_0) = 2\pi k_e n_0 q r_0 . \tag{S-1.56}$$

Again, the particles cannot overtake one another because the electric field increases with increasing r_0. A particle initially at r_0 will move along the r coordinate according to a law $r_s = r_s(r_0,t)$, with $r_s(r_0,0) = r_0$. The field acting on the particle at time t is

$$E(r_0,t) = \frac{2\pi k_e n_0 q r_0^2}{r_s(r_0,t)}, \tag{S-1.57}$$

and its equation of motion is

$$m\frac{d^2 r_s(r_0,t)}{dt^2} = q\frac{2\pi k_e n_0 q r_0^2}{r_s(r_0,t)}. \tag{S-1.58}$$

It is not possible to solve (S-1.58) for $r_s(r_0,t)$ in a simple way, however, we can multiply both sides by $dr_s(r_0,t)/dt$, obtaining

$$m\frac{dr_s(r_0,t)}{dt}\frac{d^2 r_s(r_0,t)}{dt^2} = 2\pi k_e n_0 q^2 r_0^2 \frac{1}{r_s(r_0,t)}\frac{dr_s(r_0,t)}{dt}. \tag{S-1.59}$$

Equation (S-1.59) can be rewritten

$$\frac{m}{2}\frac{d}{dt}\left[\frac{dr_s(r_0,t)}{dt}\right]^2 = 2\pi k_e n_0 q^2 r_0^2 \frac{d}{dt}\ln[r_s(r_0,t)], \tag{S-1.60}$$

which can be integrated with respect to time, leading to

$$\begin{aligned}\frac{m}{2}\left[\frac{dr_s(r_0,t)}{dt}\right]^2 &= 2\pi k_e n_0 q^2 r_0^2 \ln[r_s(r_0,t)] + C \\ &= 2\pi k_e n_0 q^2 r_0^2 \ln\left(\frac{r_s(r_0,t)}{r_0}\right),\end{aligned} \tag{S-1.61}$$

where the integration constant C has been determined by the condition that the kinetic energy of the particle must be zero at $t = 0$, when $r_s(r_0,t) = r_0$. The first side of (S-1.61) is the kinetic energy $K(r_s)$ at time t, when the particle is located at $r_s(r_0,t)$, which we can simply denote by r_s. Thus we have the following, seemingly time-independent equation for the kinetic energy of a particle initially located at r_0

$$K(r_s) = 2\pi k_e n_0 q^2 r_0^2 \ln\left(\frac{r_s}{r_0}\right). \tag{S-1.62}$$

At the limit $r_s \to \infty, t \to \infty$, the integral diverges logarithmically. Again, this is due to the infinite potential energy initially stored in the system.

S-1.8 Collision of two Charged Spheres

a) The electrostatic energy of a uniformly charged sphere of radius R and total charge Q is, according to the result of Problem 1.3,

$$U_0 = \frac{3}{5}k_e\frac{Q^2}{R}, \tag{S-1.63}$$

so that the initial energy of our system of two spheres is

$$U_{tot} = 2U_0 = \frac{6}{5}k_e\frac{Q^2}{R}. \tag{S-1.64}$$

b) Let us denote by x the distance between the centers of the two spheres. When x is such that the interaction energy $U_{int}(x)$ is no longer negligible with respect to U_0, but still larger than $2R$, the total potential energy $U_{pot}(x)$ of the system is

$$U_{pot}(x) = 2U_0 + U_{int}(x). \tag{S-1.65}$$

As long as $x \geqslant 2R$ the force between the spheres is identical to the force between two point charges $\pm Q$ located at the centers of the spheres, and

$$U_{int}(x) = -k_e\frac{Q^2}{x}. \tag{S-1.66}$$

Both the total momentum and the total energy of the system are conserved. Thus, the velocities of the two sphere are always equal and opposite. As long as $x \geqslant 2R$ the total energy of the system $U_{tot} = 2U_0$ equals the sum of the potential and kinetic energies of the system

$$U_{tot} = 2\,\frac{1}{2}Mv^2(x) + 2U_0 + U_{int}(x), \tag{S-1.67}$$

where M is the mass of each sphere, and $\pm v(x)$ are the velocities of the two spheres. Thus

$$\frac{6}{5}k_e\frac{Q^2}{R} = Mv^2(x) + \frac{6k_e}{5}\frac{Q^2}{R} - k_e\frac{Q^2}{x}, \quad \text{and} \quad v(x) = \sqrt{k_e\frac{Q^2}{Mx}}. \tag{S-1.68}$$

When $x = 2R$, the velocity is

$$v(2R) = \sqrt{k_e\frac{Q^2}{2MR}}. \tag{S-1.69}$$

c) When the two spheres overlap completely, the charge density and the electrostatic field are zero over the whole space, so that also the electrostatic energy is zero. This means that all the initial energy has been converted into kinetic energy, i.e.,

$$2\,\frac{1}{2}\,Mv^2(0) = 2U_0\,, \tag{S-1.70}$$

from which we obtain

$$v(0) = \sqrt{\frac{6}{5}\,k_e\,\frac{Q^2}{MR}}\,. \tag{S-1.71}$$

S-1.9 Oscillations in a Positively Charged Conducting Sphere

a) At equilibrium, the remaining $(1-f)N$ conduction electrons must be subject to zero electric field. For symmetry reasons, this is possible only if they occupy a spherical volume of radius $b < a$ concentric with the conducting sphere, where the where e is the elementary electric charge, and n_e and n_i are the conduction-electron density and the ion density, respectively. Thus, we must have $n_e = n_i$, with total charge density ϱ is zero. We thus have

$$\varrho(r) = e(n_i - n_e) = 0 \quad \text{for} \quad r < b\,, \tag{S-1.72}$$

where

$$n_i = \frac{3N}{4\pi a^3} \quad \text{and} \quad n_e = \frac{3\,(1-f)N}{4\pi b^3}\,, \tag{S-1.73}$$

and we get

$$b = a\,\sqrt[3]{1-f}\,, \tag{S-1.74}$$

and

$$\varrho(r) = \frac{3Ne}{4\pi a^3} \quad \text{for} \quad b < r < a\,. \tag{S-1.75}$$

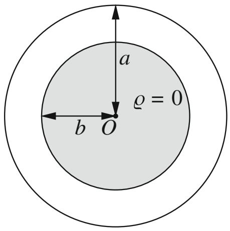

Fig. S-1.7

Note that the electric field is nonzero in the spherical shell $b < r < a$. However, this region is not conducting, since the conduction electrons are confined in the inner region $r < b$ (Fig. S-1.7).

b) Now the conduction electron sphere is rigidly displaced by an amount $\boldsymbol{\delta}$ relative to the metal sphere centered in O, so that its center is in O', as in Fig. S-1.8. The electric field

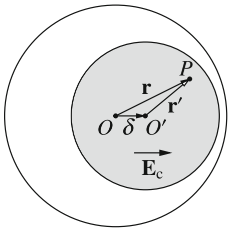

Fig. S-1.8

in any point of space can be evaluated by superposition, adding the field generated by the ion lattice, of charge density $\varrho_i = en_i$ and the field generated by the conduction electrons of charge density $\varrho_e = -en_e$. The electric field $\mathbf{E}_c$ in a point P inside the conduction-electron sphere, of vector position $\mathbf{r}$ relative to O, and $\mathbf{r}'$ relative to O', is

$$\mathbf{E}_c = \frac{4\pi k_e}{3}(\varrho_i \mathbf{r} - \varrho_e \mathbf{r}') = \frac{4\pi k_e}{3}\frac{3Ne}{4\pi a^3}(\mathbf{r}-\mathbf{r}') = \frac{k_e Ne}{a^3}\boldsymbol{\delta}, \qquad \text{(S-1.76)}$$

spatially uniform uniform and parallel to $\boldsymbol{\delta}$.

c) Each conduction electron is subject to the force

$$\mathbf{F}_c = -e\mathbf{E}_c = -\frac{k_e Ne^2}{a^3}\boldsymbol{\delta}, \qquad \text{(S-1.77)}$$

proportional to the displacement from its equilibrium position. The equation of motion for each electron is thus

$$m_e \frac{d^2\boldsymbol{\delta}}{dt^2} = \mathbf{F}_c = -\frac{k_e Ne^2}{a^3}\boldsymbol{\delta} = -m_e\,\omega_M^2\,\boldsymbol{\delta}, \qquad \text{(S-1.78)}$$

where m_e is the electron mass, and ω_M the oscillation frequency for the resulting harmonic motion. The oscillation frequency is thus

$$\omega_M = \sqrt{\frac{k_e Ne^2}{m_e a^3}}, \qquad \text{(S-1.79)}$$

i.e., the Mie frequency of (S-1.26).

S-1.10 Interaction between a Point Charge and an Electric Dipole

The potential energy of an electric dipole $\mathbf{p}$ in the presence of an external electric field $\mathbf{E}$ is

$$U = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta, \qquad \text{(S-1.80)}$$

and, in our case, we have

$$\mathbf{E} = k_e \frac{q}{r^2}\hat{\mathbf{r}}, \quad \text{and} \quad U = -k_e \frac{qp\cos\theta}{r^2}. \qquad \text{(S-1.81)}$$

a) The force acting on the dipole is thus

$$\mathbf{F} = -\boldsymbol{\nabla} U = -2k_e \frac{qp\cos\theta}{r^3}\hat{\mathbf{r}}. \qquad \text{(S-1.82)}$$

Assuming that the point charge q is positive, the force is attractive if $\cos\theta > 0$, repulsive if $\cos\theta < 0$.

b) The torque acting on the dipole is

$$\tau = -\partial_\theta U\,\hat{\mathbf{z}} = -k_e \frac{qp\sin\theta}{r^2}\,\hat{\mathbf{z}}, \tag{S-1.83}$$

where $\hat{\mathbf{z}}$ is the unit vector perpendicular to the plane determined by the vectors $\mathbf{r}$ and $\mathbf{p}$, pointing out of paper in Fig. 1.5. If $q > 0$, the torque tends to align $\mathbf{p}$ to $\mathbf{r}$.

Alternatively, we can think of the dipole $\mathbf{p}$ as the limit approached by a system of two opposite charges q' and $-q'$, at a distance $2h$ from each other (as in Fig. S-1.9), as $h \to 0$ and $q' \to \infty$, the product $2hq' = p$ being constant. We assume that the point charges q and q' are positive. The distances r_1 of q' from q, and r_2 of $-q'$ from q, can be written, as functions of r, h, and θ,

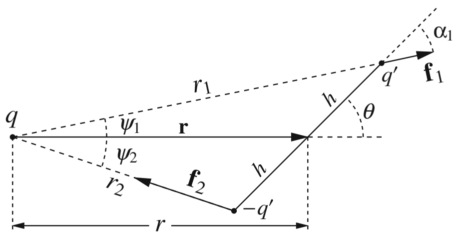

Fig. S-1.9

$$r_1 = \sqrt{r^2 + h^2 + 2rh\cos\theta}, \quad r_2 = \sqrt{r^2 + h^2 - 2rh\cos\theta}, \tag{S-1.84}$$

where we have used the law of cosines. The force $\mathbf{f}_1$ acting on the positive charge q' of the dipole is

$$\mathbf{f}_1 = k_e \frac{qq'}{r_1^2}\,\hat{\mathbf{r}}_1 = k_e \frac{qq'}{r^2 + h^2 + 2rh\cos\theta}\,\hat{\mathbf{r}}_1, \tag{S-1.85}$$

and the force $\mathbf{f}_2$ acting on $-q'$ is

$$\mathbf{f}_2 = -k_e \frac{qq'}{r_1^2}\,\hat{\mathbf{r}}_2 = -k_e \frac{qq'}{r^2 + h^2 - 2rh\cos\theta}\,\hat{\mathbf{r}}_2, \tag{S-1.86}$$

the minus sign meaning that the force is attractive for $q > 0$. Both angles ψ_1 and ψ_2 approach zero as $h \to 0$, and therefore

$$\lim_{h\to 0} \hat{\mathbf{r}}_1 = \hat{\mathbf{r}}, \quad \text{and} \quad \lim_{h\to 0} \hat{\mathbf{r}}_1 = \hat{\mathbf{r}}. \tag{S-1.87}$$

Thus, the total force $\mathbf{F}$ acting on the dipole can be written

$$\mathbf{F} = \lim_{\substack{h\to 0\\ q'\to\infty\\ 2hq'=p}} (\mathbf{f}_1 + \mathbf{f}_2) = \lim_{\substack{h\to 0\\ q'\to\infty\\ 2hq'=p}} \hat{\mathbf{r}}\, k_\mathrm{e}\, qq' \left(\frac{1}{r^2 + h^2 + 2rh\cos\theta} - \frac{1}{r^2 + h^2 - 2rh\cos\theta} \right)$$

$$= \lim_{\substack{h\to 0\\ q'\to\infty\\ 2hq'=p}} \hat{\mathbf{r}}\, k_\mathrm{e}\, qq' \frac{-4rh\cos\theta}{(r^2 + h^2 + 2rh\cos\theta)(r^2 + h^2 - 2rh\cos\theta)}$$

$$= -2k_\mathrm{e} \frac{qp\cos\theta}{r^3} \hat{\mathbf{r}}, \tag{S-1.88}$$

in agreement with (S-1.82).

As $h \to 0$ and $\psi_1 \to 0$, the angle α_1 approaches θ, and the limit of the torque of $\mathbf{f}_1$ on the dipole is

$$\tau_1 = -\hat{\mathbf{z}} \lim_{\substack{h\to 0\\ q'\to\infty\\ 2hq'=p}} h f_1 \sin\alpha_1 = -\hat{\mathbf{z}} \lim_{\substack{h\to 0\\ q'\to\infty\\ 2hq'=p}} k_\mathrm{e} \frac{qq' h \sin\alpha_1}{r^2 + h^2 + 2rh\cos\theta} = -k_\mathrm{e} \frac{qp\sin\theta}{2r^2} \hat{\mathbf{z}}, \tag{S-1.89}$$

analogously, the limit of the torque of $\mathbf{f}_2$ is

$$\tau_2 = -k_\mathrm{e} \frac{qp\sin\theta}{2r^2} \hat{\mathbf{z}}, \tag{S-1.90}$$

and the total torque on the dipole is

$$\tau = \tau_1 + \tau_2 = -k_\mathrm{e} \frac{qp\sin\theta}{r^2} \hat{\mathbf{z}}, \tag{S-1.91}$$

in agreement with (S-1.83).

S-1.11 Electric Field of a Charged Hemispherical surface

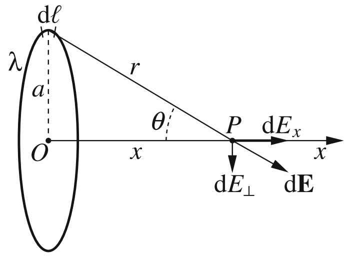

Fig. S-1.10

We start from the electric field generated by a ring of radius a and linear charge density λ in a generic point P on its axis, at a distance x from the center O of the ring. With reference to Fig. S-1.10, the infinitesimal ring arc $\mathrm{d}\ell$, of charge $\lambda\,\mathrm{d}\ell$, generates a field $\mathrm{d}\mathbf{E}$ in P. The magnitude of $\mathrm{d}\mathbf{E}$ is

$$\mathrm{d}E = k_\mathrm{e} \frac{\lambda\,\mathrm{d}\ell}{a^2 + x^2}. \tag{S-1.92}$$

The field d**E** has a component $\mathrm{d}E_x$ parallel to the ring axis, and a component $\mathrm{d}E_\perp$ perpendicular to the axis. We need only the parallel component

$$\mathrm{d}E_x = \cos\theta\,\mathrm{d}E = k_\mathrm{e}\,\frac{\lambda\,\mathrm{d}\ell}{a^2+x^2}\,\frac{x}{\sqrt{a^2+x^2}} = k_\mathrm{e}\,\frac{\lambda x\,\mathrm{d}\ell}{\left(a^2+x^2\right)^{3/2}}\,, \tag{S-1.93}$$

because the perpendicular component cancels out because of symmetry when we integrate over the whole ring. When we integrate, θ and r do not depend on the position of $\mathrm{d}\ell$, and the total field in P is

$$E_x = k_\mathrm{e}\frac{\lambda x}{\left(a^2+x^2\right)^{3/2}}\int_0^{2\pi a}\mathrm{d}\ell = k_\mathrm{e}\frac{2\pi a\lambda x}{\left(a^2+x^2\right)^{3/2}} = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\,\dfrac{2\pi a\lambda x}{\left(a^2+x^2\right)^{3/2}} & \text{SI} \\ \dfrac{2\pi a\lambda x}{\left(a^2+x^2\right)^{3/2}} & \text{Gaussian,} \end{cases} \tag{S-1.94}$$

which can be rewritten

$$\mathbf{E} = \hat{\mathbf{x}}\,k_\mathrm{e}\,\frac{Q\,x}{\left(a^2+x^2\right)^{3/2}}\,, \tag{S-1.95}$$

where $Q = 2\pi a\lambda$ is the total charge of the ring.

The charged hemispherical surface can be divided into infinitesimal strips between "parallels" of colatitude θ and $\theta+\mathrm{d}\theta$ with respect to the symmetry axis of the hemisphere, as in Fig. S-1.11. Each infinitesimal strip is equivalent to a charged ring of radius $R\sin\theta$ and total charge $\mathrm{d}Q = \sigma 2\pi R^2\sin\theta\,\mathrm{d}\theta$. The curvature center of the hemisphere is located on the axis of the rings, at a distance $x = R\cos\theta$ from the center of each ring. Thus, the contribution of each strip to the field at the center is

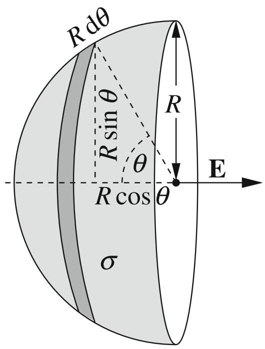

Fig. S-1.11

$$\mathrm{d}E = k_\mathrm{e}\,\frac{\sigma 2\pi R^2\sin\theta\,\mathrm{d}\theta\,R\cos\theta}{\left(R^2\sin^2\theta + R^2\cos^2\theta\right)^{3/2}} = k_\mathrm{e}\,\frac{\sigma 2\pi R^3\cos\theta\sin\theta\,\mathrm{d}\theta}{R^3} = k_\mathrm{e}\,\sigma 2\pi\cos\theta\sin\theta\,\mathrm{d}\theta\,, \tag{S-1.96}$$

and the total field is

$$E = k_e \sigma 2\pi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = k_e \pi\sigma \,, \tag{S-1.97}$$

independent of the radius R.