Chapter S-2
Solutions for Chapter 2

S-2.1 Metal Sphere in an External Field

a) The total electric field inside a conductor must be zero in static conditions. Thus, in the presence of an external field $\mathbf{E}_0$, the surface charge distribution of our sphere must generate a field $\mathbf{E}_{\text{in}} = -\mathbf{E}_0$ at its inside. As we found in Problem 1.1, a rigid displacement $-\boldsymbol{\delta}$ of the electron sphere (or "electron sea") with respect to the ion lattice gives origin to the internal uniform field (S-1.1)

$$\mathbf{E}_{\text{in}} = -k_{\text{e}} \frac{4\pi}{3} \varrho_0 \boldsymbol{\delta} \,, \tag{S-2.1}$$

where $\varrho_0 = e n_{\text{e}}$ is the charge density of the "electron sphere". The magnitude of the displacement δ is thus

$$\delta = \frac{3E_0}{4\pi k_{\text{e}} \varrho_0} \,. \tag{S-2.2}$$

For a rough numerical estimate for n_{e}, we can assume that each atom contributes a single conduction electron ($Z = 1$). If M is the atomic mass of our atoms, M grams of metal contain $N_{\text{A}} \simeq 6.0 \times 10^{23}$ atoms (Avogadro constant), and occupy a volume of M/ϱ_m cm^3, where ϱ_m is the mass density. Typical values for a metal are $M \sim 60$ and $\varrho_m \sim 8$ g/cm^3, leading to

$$n_{\text{e}} \sim \frac{N_A \varrho_m}{M} \sim 10^{22}\ \text{cm}^{-3} \,, \quad \text{and} \quad \varrho_0 = e n_{\text{e}} \sim 5 \times 10^{12}\ \text{statC/cm}^3 \,. \tag{S-2.3}$$

In SI units we have $n_{\text{e}} \sim 10^{29}$ m^{-3} and $\varrho_0 \sim 1.6 \times 10^{-10}$ C/m^3. Substituting into (S-2.2) and assuming $E_0 = 1000$ V/m (0.003 statV/cm), we finally obtain

$$\delta \sim 10^{-15}\ \text{cm} \,. \tag{S-2.4}$$

This value for δ is smaller by orders of magnitude than the spacing between the atoms in a crystalline lattice ($\sim 10^{-8}$ cm), therefore it makes sense to consider the

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charge as distributed on the surface. Formally, this is equivalent to take the limits $\delta \to 0$ and $\varrho_0 \to \infty$, keeping constant the product

$$\sigma_0 = \varrho_0 \delta = \frac{3E_0}{4\pi k_e} . \tag{S-2.5}$$

b) According to Problem 1.1, the field generated by the charge distribution of the metal sphere outside its volume equals the field of an electric dipole $\mathbf{p} = Q\boldsymbol{\delta}$, where $Q = (4\pi/3)R^3 en_e$, located at the center of the sphere. Replacing $\boldsymbol{\delta}$ by its value of (S-2.2) we have for the dipole moment

$$\mathbf{p} = \frac{R^3}{k_e}\mathbf{E}_0 . \tag{S-2.6}$$

The field outside the sphere ($r > R$) is the sum of $\mathbf{E}_0$ and the field generated by $\mathbf{p}$

$$\mathbf{E} = \mathbf{E}_0 + [3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0]\left(\frac{R}{r}\right)^3 . \tag{S-2.7}$$

c) The external field at the surface of the sphere is obtained by replacing r by R in (S-2.7)

$$\mathbf{E}_{\text{surf}} = \mathbf{E}_0 + 3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{E}_0 = 3(\mathbf{E}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} , \tag{S-2.8}$$

which is perpendicular to the surface, as expected. The surface charge density is

$$\sigma = \frac{1}{4\pi k_e}\mathbf{E}_{\text{surf}} \cdot \hat{\mathbf{r}} = k_e \frac{3}{4\pi} E_0 \cos\theta = \sigma_0 \cos\theta , \tag{S-2.9}$$

where $\sigma_0 = 3k_e E_0/(4\pi)$, and θ is the angle between $\hat{\mathbf{r}}$ and $\mathbf{E}_0$.

S-2.2 Electrostatic Energy with Image Charges

In all cases, the conducting (half-)planes divide the whole space into two regions: one free of charges (A), and one containing electrical charges (B),as shown in Fig. S-2.1. Since the charge distribution is finite, the electric potential φ equals zero at the boundaries of both regions, i.e., on the conducting surfaces and at infinity. We can thus use the uniqueness theorem for Poisson's equation. The potential φ (and therefore the electric field $\mathbf{E}$) is uniformly equal to zero in region A. The potential problem in region B is solved if we find an *image* charge distribution, located in region A, that replicates the boundary conditions of region B. The potential and the electric field (and thus the forces on the real charges) in region B are the same as if the image charges were real.

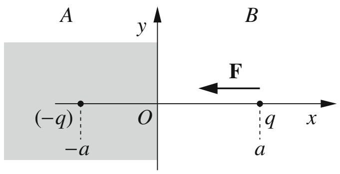

Fig. S-2.1

a) We introduce a reference frame with the x axis perpendicular to the conducting plane, and passing through the charge. The origin, and the y and z axes, lie on the plane. The charge is thus located at $(x = a, y = 0, z = 0)$, and the potential problem for $x > 0$ is solved by placing an image charge $q' = -q$ at $(x = -a, y = 0, z = 0)$. The force on the real charge is $F = -k_\mathrm{e} q^2/(4a^2)$. The electrostatic energy U_es of the system equals the work W done by the field when the real charge q moves from $x = a$ to $x = +\infty$. Simultaneously, the image charge will move from $x = -a$ to $x = -\infty$, but no additional work is needed for this, since what actually moves is the surface charge on the conducting plane, which is constantly at zero potential. Thus we have

$$U_\mathrm{es} = W = \int_a^\infty F\,\mathrm{d}x = -k_\mathrm{e}\,\frac{q^2}{4}\int_a^\infty \frac{\mathrm{d}x}{x^2} = -k_\mathrm{e}\,\frac{q^2}{4a}\,. \tag{S-2.10}$$

This is half the electrostatic energy U_real of a system comprising two *real* charges, q and $-q$, at a distance $2a$ from each other. The $1/2$ factor is due to the fact that, if two *real* charges move to infinity in opposite directions, the work done by the field is

$$W_\mathrm{real} = \int_{+a}^{+\infty} F\,\mathrm{d}x - \int_{-a}^{-\infty} (-F)\,\mathrm{d}x = 2\int_{+a}^{+\infty} F_x\,\mathrm{d}x = -k_\mathrm{e}\,\frac{q^2}{2a}\,, \tag{S-2.11}$$

since the force acting on $-q$ is the opposite of the force acting on q, and $U_\mathrm{real} = W_\mathrm{real}$.

The $1/2$ factor can also be explained by evaluating the electrostatic energies for our system, and for the system of the two real charges. In both cases, because of the cylindrical symmetry around the x axis, the electrostatic field is a function of the longitudinal coordinate x and of the radial distance $r = \sqrt{y^2 + z^2}$ only, i.e., $\mathbf{E} = \mathbf{E}(x, r)$. In the case of the two real charges we have

$$\begin{aligned} U_\mathrm{real} &= \frac{1}{8\pi k_\mathrm{e}}\int \mathrm{d}^3r\,\mathbf{E}^2 = \frac{1}{8\pi k_\mathrm{e}}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^\infty 2\pi r\,\mathrm{d}r\,\mathbf{E}^2(x,r) \\ &= 2\,\frac{1}{8\pi k_\mathrm{e}}\int_0^\infty \mathrm{d}x\int_0^\infty 2\pi r\,\mathrm{d}r\,\mathbf{E}^2(x,r)\,, \end{aligned} \tag{S-2.12}$$

since $\mathbf{E}(x, r) = -\mathbf{E}(-x, r)$, so that $\mathbf{E}^2(x, r) = \mathbf{E}^2(-x, r)$. In the case of the charge in front of a conducting plane we have

$$U_\mathrm{es} = \frac{1}{8\pi k_\mathrm{e}}\int_0^\infty \mathrm{d}x\int_0^\infty 2\pi r\,\mathrm{d}r\,\mathbf{E}^2(x,r)\,, \tag{S-2.13}$$

because $\mathbf{E} = 0$ for $x < 0$ (in region A), while the field is the same as in the "real" case for $x > 0$. Thus $U_\mathrm{es} = U_\mathrm{real}/2$. The electrostatic energy includes both the interaction energy between the charges, U_int, and the "self-energy", U_self, of each charge. For the "real" system we have

$$U_\mathrm{real} = U_\mathrm{self}(q) + U_\mathrm{self}(-q) + U_\mathrm{int}(q,-q) = 2U_\mathrm{self}(q) + U_\mathrm{int}(q,-q)\,, \tag{S-2.14}$$

since $U_\mathrm{self}(-q) = U_\mathrm{self}(q)$. For the charge in front of the conducting plane we have

$$U_{\text{es}} = U_{\text{self}}(q) + U_{\text{int}}(q, \text{plane}) \,, \tag{S-2.15}$$

since there is only one real charge. Actually, the self-energy U_{self} approaches infinity if we let the charge radius approach 0, but this issue is not really relevant here. In any case, the divergence may be treated by assuming an arbitrarily small, but non-zero radius for the charge. Since $U_{\text{es}} = U_{\text{real}}/2$, we also have $U_{\text{int}}(q, \text{plane}) = U_{\text{int}}(q, -q)/2$.

b) Again, we choose a reference frame with the x axis perpendicular to the conducting plane, so that q has coordinates $(a, d/2, 0)$ and $-q$ has coordinates $(a, -d/2, 0)$. The potential problem for the $x > 0$ half-space is solved by placing an image charge $-q$ at $(-a, d/2, 0)$, and an image charge q at $(-a, -d/2, 0)$. According to the arguments at the end of point **a)**, the electrostatic energy U_{es} of our system is one half of the energy U_{real} of a system of four charges, all of them real, at the same locations. We can evaluate U_{real} by inserting the four charges one by one, each interacting only with the previously inserted charges.

$$U_{\text{real}} = k_{\text{e}} \left(-2\,\frac{q^2}{d} - 2\,\frac{q^2}{2a} + 2\,\frac{q^2}{\sqrt{d^2 + 4a^2}} \right) . \tag{S-2.16}$$

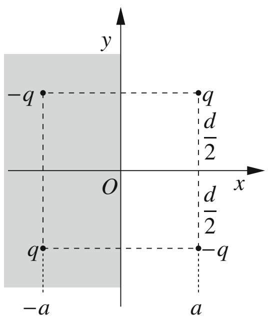

Fig. S-2.2

The same result is obtained by evaluating the work of the electric forces when the two real charges are moved to infinite distance from the plane, and infinite distance from each other. This can be done in two steps. First we move the charge at $(a, d/2, 0)$, then the charge at $(a, -d/2, 0)$. When we move the first charge, three forces are acting on it: $\mathbf{F}_1$, due to its own image, which is simultaneously moving to $-\infty$, and $\mathbf{F}_2$ and $\mathbf{F}_3$, due to the second real charge and to its image, at distances r_2 and r_3, respectively. The total work on the first charge is thus

$$\begin{aligned} W &= W_1 + W_2 + W_3 \\ &= \int_a^\infty \mathbf{F}_1 \cdot \mathrm{d}\mathbf{r} + \int_a^\infty \mathbf{F}_2 \cdot \mathrm{d}\mathbf{r} + \int_a^\infty \mathbf{F}_3 \cdot \mathrm{d}\mathbf{r} \,, \end{aligned} \tag{S-2.17}$$

where $\mathbf{r}$ is the position vector of the first charge, and the first integral is the same as the integral of (S-2.10) and equals $-k_{\text{e}}\, q^2/(4a)$. The second integral can be rewritten, in terms of the angle θ of Fig. S-2.3,

$$
\begin{aligned}
W_2 &= -k_e \int_a^\infty \frac{q^2}{r_2^2} \sin\theta \, dx \\
&= -k_e q^2 \int_0^{\pi/2} \frac{\cos^2\theta}{d^2} \sin\theta \frac{d}{\cos^2\theta} \, d\theta \\
&= -k_e \frac{q^2}{d} \int_0^{\pi/2} \sin\theta \, d\theta \\
&= -k_e \frac{q^2}{d} ,
\end{aligned} \tag{S-2.18}
$$

where we have used the facts that $r_2 = d/\cos\theta$ and $dx = (d/\cos^2\theta)\,d\theta$. The third integral of (S-2.17) can be treated analogously, in terms of the angle ψ of Fig. S-2.3,

$$
\begin{aligned}
W_3 &= k_e \int_a^\infty \frac{q^2}{r_3^2} \sin\psi \, dx \\
&= k_e \frac{q^2}{d} \int_{\psi_0}^{\pi/2} \sin\psi \, d\psi \\
&= k_e \frac{q^2}{\sqrt{4a^2+d^2}} ,
\end{aligned} \tag{S-2.19}
$$

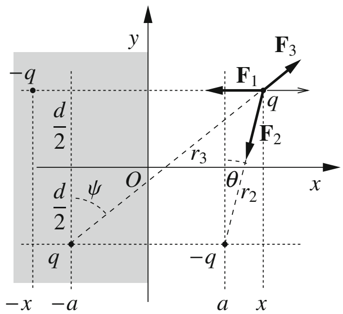

Fig. S-2.3

where ψ_0 is the value of ψ when q is at $x = a$, i.e., $\psi_0 = \arccos(d/\sqrt{4a^2+d^2})$. Thus, the work done by the electric field when the first charge is moved to infinity is

$$
W = k_e \left(-\frac{q^2}{4a} - \frac{q^2}{d} + \frac{q^2}{\sqrt{4a^2+d^2}} \right) . \tag{S-2.20}
$$

We must still move the second real charge to infinity, this is done in the presence of its own image charge only, and the work is $-k_e q^2/(4a)$. We finally have

$$
U_{es} = W - k_e \frac{q^2}{4a} = k_e \left(-\frac{q^2}{2a} - \frac{q^2}{d} + \frac{q^2}{\sqrt{4a^2+d^2}} \right) , \tag{S-2.21}
$$

i.e., one half of the value of U_{real} of (S-2.16), as expected.

c) We choose a reference frame with the half planes $(x = 0, y \geqslant 0)$ and $(y = 0, x \geqslant 0)$ coinciding with the two conducting half-planes. Thus, the real charge q is located at $(x = a, y = b, z = 0)$. If we add two image charges $q_1' = q_2' = -q$ at $(-a, b, 0)$ and $(a, -b, 0)$, respectively, and an image charge $q_3' = q$ at $(-a, -b, 0)$, the potential is zero on the $x = 0$ and $y = 0$ planes, and at infinity. This

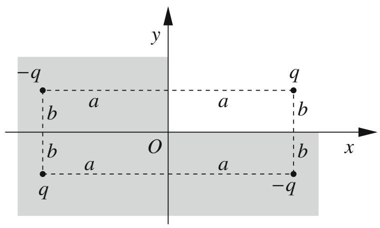

Fig. S-2.4

solves the potential problem in the dihedral angle where the real charge is located. Following the discussions of points **a)** and **b)**, the electrostatic energy of this system is one quarter of the energy of a system comprising four charges, all of them real, in the same locations, since the energy density is zero in three quarters of the whole space.

$$U_{\mathrm{es}} = \frac{1}{4} k_{\mathrm{e}} \left(\frac{-q^2}{a} - \frac{q^2}{b} + \frac{q^2}{\sqrt{b^2+a^2}} \right) . \tag{S-2.22}$$

Alternatively, we can calculate the work done by the electric field when the real charge is moved from $(a,b,0)$ to $(\infty,\infty,0)$.

S-2.3 Fields Generated by Surface Charge Densities

a) We use cylindrical coordinates (r,ϕ,z) with the origin O on the conducting plane, and the z axis perpendicular to the plane and passing through the real charge q. The real charge is located at $(0,\phi,z)$, and the image charge at $(0,\phi,-a)$, ϕ being irrelevant when $r=0$. The electric field on the conducting plane is perpendicular to the plane, and depends only on r. At a generic point $P \equiv (r,\phi,0)$ on the plane the magnitude of the field $\mathbf{E}^{\mathrm{real}}$ generated by the real charge is

$$E^{\mathrm{real}} = k_{\mathrm{e}} \frac{q}{b^2} = k_{\mathrm{e}} \frac{q}{a^2+r^2} . \tag{S-2.23}$$

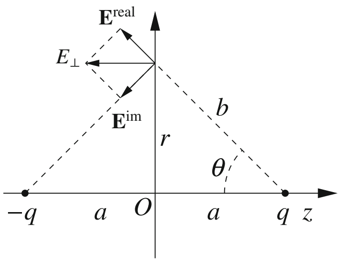

Fig. S-2.5

The field generated at P by the image charge, $\mathbf{E}^{\mathrm{im}}$, has the same magnitude, the same z component, but opposite r component of $\mathbf{E}^{\mathrm{real}}$, as in Fig. S-2.5. The total electric field in P is thus perpendicular to the plane and has magnitude

$$E(r) = 2E_z^{\mathrm{real}}(r) = -2k_{\mathrm{e}} \frac{q}{a^2+r^2} \frac{a}{\sqrt{(a^2+r^2)}} = -2k_{\mathrm{e}} \frac{q\,a}{(a^2+r^2)^{3/2}} . \tag{S-2.24}$$

The surface charge density is thus

$$\sigma(r) = \frac{1}{4\pi k_{\mathrm{e}}} E(r) = -\frac{1}{2\pi} \frac{q\,a}{(a^2+r^2)^{3/2}} , \tag{S-2.25}$$

and the annulus between r and $r+\mathrm{d}r$ on the conducting plain has a charge

$$\begin{aligned} \mathrm{d}q_{\mathrm{ind}} &= \sigma\, 2\pi r\, \mathrm{d}r = \frac{q a r\, \mathrm{d}r}{(a^2+r^2)^{3/2}} \\ &= -\frac{1}{2\pi} \frac{q}{a^2} cos^3\theta\, 2\pi a^2 \tan\theta \frac{\mathrm{d}\theta}{\cos^2\theta} = -q \sin\theta\, \mathrm{d}\theta , \end{aligned} \tag{S-2.26}$$

since $r = a\tan\theta$. The total induced charge on the conducting plane is

$$q_{\text{ind}} = \int_0^{\pi/2} \mathrm{d}q_{\text{ind}} = -q\int_0^{\pi/2} \sin\theta\,\mathrm{d}\theta = -q\,. \tag{S-2.27}$$

b) In the problem of a real charge q located on the z axis, at $z = a$, in front of a conducting plane, the only real charges are q and the surface charge distribution σ on the plane. What we observe is no field in the half-space $z < 0$, while in the half-space $z > 0$ we observe a field equivalent to the field of q, plus the field of an image charge $-q$ located on the z axis at $z = -a$. The field generated by the surface charge distribution alone is thus equivalent to the field of a charge $-q$ located at $z = +a$ in the half-space $z - 0$, and to the field of a charge $-q$ located at $z = -a$ in the half-space $z > 0$. In the half space $z < 0$, the field of the surface charge distribution and the field or the real charge cancel each other. The discontinuity of the field at $z = 0$ is due to the presence of a finite surface charge density on the conducting plane, which implies an infinite volume charge density.

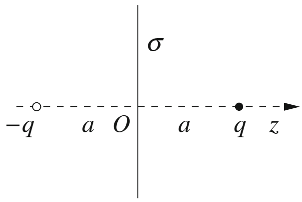

Fig. S-2.6

c) Let us introduce a spherical coordinate system (r,θ,ϕ) into Problem 2.4, with the origin O at the center of the conducting sphere and the z axis on the line through O and the real charge q. The electric potential outside the sphere, $r \geqslant a$, is obtained from (S-2.31) by replacing a by r, and q' and d' by their values of (S-2.37). We have

$$\varphi(r,\theta) = k_e \left[\frac{q}{\sqrt{r^2 + d^2 - 2dr\cos\theta}} - \frac{q\,\dfrac{a}{d}}{\sqrt{r^2 + \dfrac{a^4}{d^2} - 2r\,\dfrac{a^2}{d}\cos\theta}} \right], \tag{S-2.28}$$

independent of ϕ. The electric field at $r = a^+$, on the outer surface of the sphere, is

$$E_\perp(a^+,\theta) = -\,\partial_r V(r,\theta)\Big|_{r=a}\,, \tag{S-2.29}$$

and the surface charge density on the sphere is

$$\sigma(\theta) = \frac{1}{4\pi k_e} E_\perp(a^+,\theta)\,. \tag{S-2.30}$$

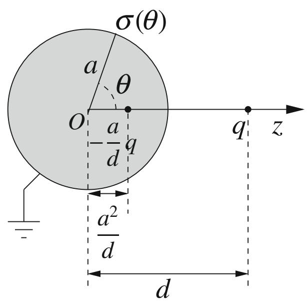

Fig. S-2.7

The actual evaluation does not pose particular difficulties, but is rather involved, and we neglect it here. But we can use the same arguments as in point **b)**. The only real charges of the problem are the real charge q, and the surface charge distribution of the sphere. There is no net field inside the sphere, and the field for $r > 0$ is equivalent to the field of q, plus the field of an image charge $-qa/d$ located at $z = a^2/d$. Thus, the surface charge distribution alone generates a field equivalent to a charge $-q$ located at $z = d$ inside the sphere, and a field equivalent to the field of the image charge $-qa/d$, located at $z = a^2/d$, outside the sphere.

S-2.4 A Point Charge in Front of a Conducting Sphere

a) We have a conducting grounded sphere of radius a, and an electric charge q located at a distance $d > a$ from its center O. Again, the whole space is divided into two regions: the inside (A) and the outside (B) of the sphere. The electrostatic potential is uniformly equal to zero in region A because the sphere is grounded. We try to solve the potential problem in region B by locating an image charge q' inside the sphere, on the line through O and q, at a distance d' from the center O. The problem is solved if we can find values for q' and d' such that the electric potential φ is zero everywhere on the surface of the sphere. This would replicate the boundary conditions for region B, with $\varphi = 0$ both on the surface of the sphere and at infinity, and only the real charge q in between. Let us evaluate the potential $\varphi(P)$ at a generic point P of the sphere surface, such that the line segment OP forms an angle θ with the line segment Oq, as shown in Fig. S-2.8. We must have

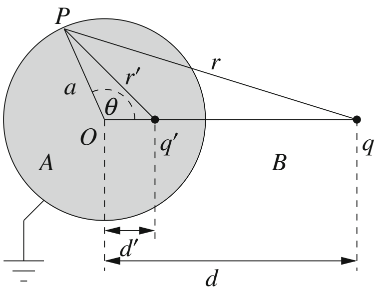

Fig. S-2.8

$$0 = \varphi(P) = k_e\left(\frac{q}{r} + \frac{q'}{r'}\right) = k_e\left(\frac{q}{\sqrt{a^2 + d^2 - 2ad\cos\theta}} + \frac{q'}{\sqrt{a^2 + d'^2 - 2ad'\cos\theta}}\right), \tag{S-2.31}$$

where r is the distance from P to q, r' the distance from P to q', and we have used the cosine rule. We see that the sign of q' must be the opposite of the sign of q. If we take the square of (S-2.31) we have

$$q^2(a^2 + d'^2 - 2ad'\cos\theta) = q'^2(a^2 + d^2 - 2ad\cos\theta)\,, \tag{S-2.32}$$

which must hold for any θ. We must thus have separately

$$q^2(a^2 + d'^2) = q'^2(a^2 + d^2)\,, \quad \text{and} \tag{S-2.33}$$

$$2q^2ad'\cos\theta = 2q'^2ad\cos\theta\,. \tag{S-2.34}$$

Equation (S-2.34) leads to

$$q'^2 = q^2 \frac{d'}{d}, \quad q' = -q\sqrt{\frac{d'}{d}}, \tag{S-2.35}$$

which can be inserted into (S-2.33), leading to

$$dd'2 - (a^2 + d'^2)d' + a^2 d = 0\,, \tag{S-2.36}$$

which has the two solutions $d' = d$ and $d' = a^2/d$. The first solution is not acceptable because it is larger than the radius of the sphere a (it actually corresponds to the trivial solution of superposing a charge $-q$ to the charge q). Thus we are left with $d' = a^2/d$, which can be substituted into (S-2.35), leading to our final solution

$$q' = q\frac{a}{d}, \quad d' = \frac{a^2}{d}\,. \tag{S-2.37}$$

If the sphere is isolated and has a net charge Q, the problem in region B is solved by placing an image charge q' at d', as above, and a further point charge $q'' = Q - q'$ in O, so that the potential is uniform over the sphere surface, and the total charge of the sphere is Q. The case $Q = 0$ corresponds to an uncharged, isolated sphere.

b) The total force $\mathbf{f}$ on q equals the sum of the forces exerted on q by the image charge q' located in d', $q'' = -q'$ and Q, both located in O. Thus $\mathbf{f} = \mathbf{f}' + \mathbf{f}'' + \mathbf{f}'''$, with

$$f' = k_e \frac{qq'}{(d-d')^2} = -k_e \frac{q^2 ad}{(d^2-a^2)^2}\,, \quad f'' = k_e \frac{q^2}{d^3}\,, \quad f''' = k_e \frac{qQ}{d^2}\,. \tag{S-2.38}$$

with $f'' = f''' = 0$ if the sphere is grounded.

c) The electrostatic energy U of the system equals the work of the electric field if the real charge q is moved to infinity. When q is at a distance x from O we evaluate the force on it by simply replacing d by x in (S-2.38). The work is thus the sum of the three terms

$$\begin{aligned} W_1 &= \int_d^\infty f'\,\mathrm{d}x = k_e\left[\frac{q^2 a}{2(x^2-a^2)}\right]_d^\infty = -k_e\frac{q^2 a}{2(d^2-a^2)}\,,\\ W_2 &= \int_d^\infty f''\,\mathrm{d}x = -k_e\left[\frac{q^2 a}{2x^2}\right]_d^\infty = k_e\frac{q^2 a}{2d^2}\,,\\ W_3 &= \int_d^\infty f'''\,\mathrm{d}x = k_e\frac{qQ}{d}\,. \end{aligned} \tag{S-2.39}$$

Thus we have $U = W_1$ for the grounded sphere, $U = W_1 + W_2$ for the isolated chargeless sphere, and $U = W_1 + W_2 + W_3$ for the isolated charged sphere.

It is interesting to compare this result for the energy of the isolated chargeless sphere with the electrostatic energy U_{real} of a system comprising three real charges q, q', and $-q'$, located in d, d' and O, respectively:

$$U_{\text{real}} = k_e \sum_{i<j} \frac{q_i q_j}{r_{ij}} = k_e \left(\frac{qq'}{d-d'} - \frac{qq'}{d} - \frac{q'^2}{d'} \right)$$
$$= k_e \left(-\frac{q^2 a}{d^2 - a^2} + \frac{q^2 a}{d^2} - \frac{q^2}{d} \right) . \tag{S-2.40}$$

We see that U is obtained from U_{real} by halving the interaction energies of the real charge with the two image charges, and neglecting the interaction energy between the two image charges.

S-2.5 Dipoles and Spheres

a) We consider the case of the grounded sphere first, so that its potential is zero. We can treat the dipole as a system of two point charges $\pm q$, separated by a distance $2h$ as in Fig. S-2.9. Eventually, we shall let q approach ∞, and h approach zero, with the product $p = 2hq$ remaining constant. Following Problem 2.4, the two charges induce two images

$$\pm q' = \mp q \frac{a}{\sqrt{d^2 + h^2}}, \tag{S-2.41}$$

respectively, each at a distance

$$d' = \frac{a^2}{\sqrt{d^2 + h^2}} \tag{S-2.42}$$

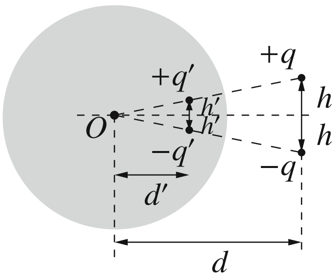

Fig. S-2.9

from the center of the sphere O, each lying on the straight line passing through O and the corresponding real charge. Since we are interested in the limit $h \to 0$ (thus, $h \ll d$), we can use the approximations

$$\pm q' \simeq \mp q \frac{a}{d}, \quad \text{and} \quad d' = \frac{a^2}{d} . \tag{S-2.43}$$

The two image charges are separated by a distance

$$2h' = 2h \frac{d'}{d} = h \left(\frac{a}{d} \right)^2 , \tag{S-2.44}$$

so that the moment of the image dipole is

$$\mathbf{p}' = 2q'\mathbf{h}' = -2q\mathbf{h} \left(\frac{a}{d} \right)^3 = -\mathbf{p} \left(\frac{a}{d} \right)^3 . \tag{S-2.45}$$

The image dipole is antiparallel to the real dipole, i.e., the two dipoles lie on parallel straight lines, but point in opposite directions. The sum of the image charges, which equals the total induced charge on the sphere surface, is zero. Therefore this solution is valid also for an isolated uncharged sphere.

b) Also in this case, we consider the grounded sphere first. Again, the dipole can be treated as a system of two charges $\pm q$, separated by a distance $h = p/q$. This time the charge $+q$ is at distance d from the center of the sphere O, while $-q$ is at distance $d+h$.

Thus, the images q' of $+q$, and q'' of $-q$, have different absolute values, and are located at different distances from O, d' and d'', respectively. We have

$$q' = -q\frac{a}{d} = -\frac{p}{h}\frac{a}{d}\,, \quad d' = \frac{a^2}{d}\,, \qquad \text{(S-2.46)}$$

$$q'' = +q\frac{a}{d+h} = +\frac{p}{h}\frac{a}{d+h}\,, \quad d'' = \frac{a^2}{d+h}\,.$$

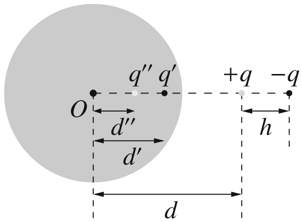

Fig. S-2.10

The absolute values of q' and q'' remain different from each other also at the limits $h \to 0, q \to \infty$, so that a net image charge q''' is superposed to the image dipole

$$q''' = \lim_{h\to 0}(q' + q'') = \lim_{h\to 0} -p\frac{a}{h}\frac{h}{d(d+h)} = -p\,\frac{a}{d^2}\,. \qquad \text{(S-2.47)}$$

The moment of the image electric dipole can be calculated as the limit of the absolute value of q' times $(d' - d'')$

$$p' = \lim_{h\to 0}|q'|(d' - d'') = \lim_{h\to 0}\frac{p}{h}\frac{a}{d}\frac{a^2 h}{d(d+h)} = p\left(\frac{a}{d}\right)^3\,, \qquad \text{(S-2.48)}$$

the same result is obtained by evaluating the limit of $q''(d' - d'')$. Thus the real dipole $\mathbf{p}$ and the image dipole $\mathbf{p}'$ lie on the same straight line and point in the same direction. The image dipole is located at a distance a^2/d from O.

Since a net charge q''' is needed to have zero potential on the surface of the sphere, this solution is valid only in the case of a grounded sphere. The solution for an isolated uncharged sphere requires an image charge $-q''' = +pa/d^2$ at the center of the sphere, so that the total image charge is zero and the surface of the sphere is equipotential.

c) We start from the case of the grounded sphere, and use a Cartesian reference frame with the origin located at the center of the sphere, O, the x axis passing through the dipole $\mathbf{p}$, and the y axis lying in the plane of the dipole. We denote by θ the angle between the electric dipole $\mathbf{p}$ and the x axis, as in Fig. S-2.11. We can decompose the dipole into the vector sum of its x and y components

$$\mathbf{p}_x = p\cos\theta\,\hat{\mathbf{x}}\,, \quad \text{and} \quad \mathbf{p}_y = p\sin\theta\,\hat{\mathbf{y}}\,. \qquad \text{(S-2.49)}$$

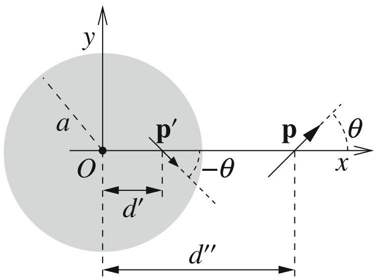

Fig. S-2.11

Both components generate images located on the x axis at a distance $d' = a^2/d$ from O. According to **a)** and **b)**, $\mathbf{p}_y$ and $\mathbf{p}_y$ generate the images

$$\mathbf{p}'_y = -\left(\frac{a}{d}\right)^3 p\sin\theta\,\hat{\mathbf{y}}$$
$$\mathbf{p}'_x = \left(\frac{a}{d}\right)^3 p\cos\theta\,\hat{\mathbf{x}}, \qquad \text{(S-2.50)}$$

resulting in an image dipole $\mathbf{p}'$, of modulus $p' = p(a/d)^3$, forming an angle $-\theta$ with the x axis, and superposed to a net charge $q''' = +p\cos\theta(a/d^2)$, since now it is the "tail" of $\mathbf{p}_x$ which points toward O. In the case of an isolated uncharged sphere, we must add a point charge $-q'''$ in O, so that the net charge of the sphere is zero.

S-2.6 Coulomb's Experiment

a) The zeroth-order solution is obtained by neglecting the induction effects, considering the charges as uniformly distributed over the surfaces of the two spheres. Thus, at zeroth order, the force between the two spheres equals the force between two point charges, each equal to Q, located at their centers. In order to evaluate higher-order solutions, it is convenient to introduce the dimensionless parameter $\alpha = (a/r) < 1$, where a is the radius of the two spheres, and r the distance between their centers. The solution of order n is obtained by locating inside each sphere a point charge q of the same order of magnitude as Q at its center, plus increasingly smaller point charges $q', q'', \ldots, q^{(n)}$ at appropriate positions, with orders of magnitude $|q'| \sim \alpha Q$, $|q''| \sim \alpha^2 Q$, $\ldots$, $|q^{(n)}| \sim \alpha^n Q$. The charges must obey the normalization condition $q + q' + q'' + \cdots + q^{(n)} = Q$.

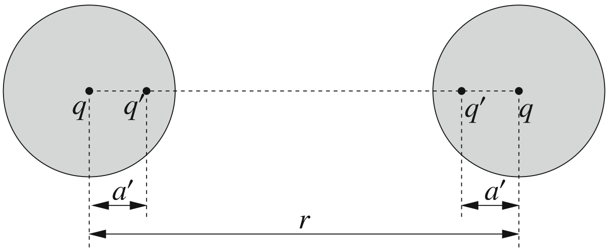

Fig. S-2.12

At the first order, the point charge q at the center of each sphere induces an image charge $q' = -\alpha q$ inside the other sphere, located at a distance $a' = a^2/r = r\alpha^2$ from its center (see Problem 2.4), as shown in Fig. S-2.12. Thus, by solving the simultaneous equations $q' = -\alpha q$, and $q + q' = Q$, we obtain for the values of the two charges

$$q = \frac{1}{1-\alpha}\,Q\,, \quad q' = -\frac{\alpha}{1-\alpha}\,Q\,. \tag{S-2.51}$$

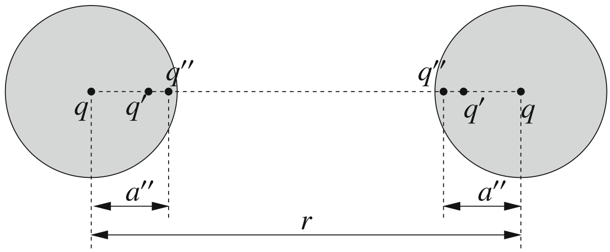

Fig. S-2.13

At the second order, the first-order charge q' inside each sphere induces an image charge q'' inside the other sphere, located a distance a'' from its center, as shown in Fig. S-2.13. Since the distance of q' from the center of the other sphere is $r - a' = r(1-\alpha^2)$, we have

$$q'' = -q'\frac{a}{r-a'} = -q'\frac{\alpha}{1-\alpha^2}\,, \qquad a'' = \frac{a^2}{r-a'} = r\frac{\alpha^2}{1-\alpha^2}\,. \tag{S-2.52}$$

Combining the above equation for q'' with equations $q' = -\alpha q$ and $q + q' + q'' = Q$, we finally obtain

$$q = Q\frac{1-\alpha^2}{1-\alpha+\alpha^3}\,, \quad q' = -Q\frac{\alpha(1-\alpha^2)}{1-\alpha+\alpha^3}\,, \quad q'' = Q\frac{\alpha^2}{1-\alpha+\alpha^3}\,. \tag{S-2.53}$$

Higher order approximations are obtained by iterating the procedure. Thus we obtain a sequence of image charges $q, q', q'', q''', \dots$ inside each sphere. At each iteration, the new image charge is of the order of α times the charge added at the previous iteration. Therefore, the smaller the value of $\alpha = a/r$, the sooner one may truncate the sequence obtaining a good approximation.

b) We obtain the first order approximation of the force between the two spheres by considering only the charges of (S-2.51) for each sphere. To this approximation, the force between the spheres is the sum of four terms. The first term is the force between the two zeroth-order charges q, at a distance r from each other. The second and third terms are the forces between the zeroth order charge q of one sphere and the first-order charge q' of the other. The distance between these charges is $r - a' = r(1-\alpha^2)$. The fourth term is the force between the two first-order charges q', at a distance $r - 2a' = r(1-2\alpha^2)$ from each other. Summing up all these contributions we obtain

$$F = k_e \frac{Q^2}{r^2} \frac{1}{(1-\alpha)^2} \left[1 - \frac{2\alpha}{(1-\alpha^2)^2} + \left(\frac{\alpha}{1-2\alpha^2} \right)^2 \right]. \tag{S-2.54}$$

From the Taylor expansion, valid for $x < 1$,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + O(x^4), \tag{S-2.55}$$

we obtain, to the fourth order,

$$\frac{1}{(1-\alpha^2)^2} = 1 + 2\alpha^2 + 3\alpha^4 + O(\alpha^6), \tag{S-2.56}$$

and

$$\frac{1}{(1-2\alpha^2)^2} = 1 + 4\alpha^2 + 12\alpha^4 + O(x^6), \tag{S-2.57}$$

so that

$$\begin{aligned} F &= k_e \frac{Q^2}{r^2} \left(1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots\right)\left(1 - 2\alpha + \alpha^2 - 4\alpha^3 + \dots\right) \\ &= k_e \frac{Q^2}{r^2} \left[1 - 4\alpha^3 + O(\alpha^4)\right], \end{aligned} \tag{S-2.58}$$

since all the terms of order α and α^2 vanish. The first non vanishing correction to the "Coulomb" force is thus at the third order in a/r,

$$F = k_e \frac{Q^2}{r^2} \left(1 - 4 \frac{a^3}{r^3} \right). \tag{S-2.59}$$

This result can be interpreted in terms of multipole expansions of the charge distributions of the spheres. The first two multipole moments of the charge distribution of each sphere are a monopole equal to the total charge Q, and an electric dipole $\mathbf{p} = -q'a'\hat{\mathbf{r}} = -(\alpha Q)(\alpha^2 r)\hat{\mathbf{r}} = -\alpha^3 Qr\hat{\mathbf{r}}$, with $\hat{\mathbf{r}}$ pointing toward the center of the opposite sphere. The contribution of the monopole moments to the total force is $F_{mm} = k_e Q^2/r^2$. Now we need the force exerted by the monopole terms of each sphere on the dipole term of the other. The monopole of, say, the left sphere generates a field $\mathbf{E}^{(0)} = k_e Q/r^2$ at the center of the right sphere. We can consider the dipole moment of the right sphere as the limit for $h \to 0$ of two charges, $-q'$ located at $r-h$ from the center of the left sphere, and q' located at r, with $q'h = |p|$. The force between the left monopole and the right dipole is thus

$$\begin{aligned} F_{md} &= \lim_{h\to 0} k_e Qq' \left[-\frac{1}{(r-h)^2} + \frac{1}{r^2} \right] \simeq \lim_{h\to 0} k_e Qq' \left(-\frac{1}{r^2} - \frac{2h}{r^3} + \frac{1}{r^2} \right) \\ &= -\lim_{h\to 0} k_e Qq' \frac{2h}{r^3} = -2k_e \frac{Qp}{r^3} = -2k_e \alpha^3 \frac{Q^2}{r^3}, \end{aligned} \tag{S-2.60}$$

where we have used the first-order Taylor expansion of $(r-h)^{-2}$. Adding the force between the right monopole and the left dipole, the total force is thus

$$F = F_{\mathrm{mm}} + 2F_{\mathrm{md}} = k_{\mathrm{e}} \frac{Q^2}{r^2}\left(1 - 4\frac{a^3}{r^3}\right), \tag{S-2.61}$$

in agreement with (S-2.59). The same result can be obtained by applying the formula for the force between a point charge and an electric dipole, $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$. See also Problem 1.10 on this subject.

From (S-2.59) we find that a ratio $a/r \simeq 0.13$ is enough to reduce the systematic deviation from the pure inverse-square law below 1%.

S-2.7 A Solution Looking for a Problem

a) The total electric potential in a point of position vector $\mathbf{r}$ is the sum of the dipole potential and of the potential of the external uniform electric field,

$$\varphi(\mathbf{r}) = k_{\mathrm{e}} \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} - Ez = k_{\mathrm{e}} \frac{p\cos\theta}{r^2} - Er\cos\theta, \tag{S-2.62}$$

where θ is the angle between $\mathbf{r}$ and the z axis. Note that it is not possible to take the reference point for the electrostatic potential at infinity, since the potential of our uniform electric field diverges for $z \to \pm\infty$. Thus we have chosen $\varphi = 0$ on the xy plane, which is an equipotential surface both for the dipole and for the uniform electric field. Now we look for a possible further equipotential surface on which $\varphi = 0$. On this surface we must have

$$\varphi = k_{\mathrm{e}} \frac{p\cos\theta}{r^2} - Er\cos\theta = 0, \tag{S-2.63}$$

and, in addition to the solution $\theta = \pi/2$, corresponding to the xy plane, we have the θ-independent solution

$$r = k_{\mathrm{e}}^{1/3}\left(\frac{p}{E}\right)^{1/3} \equiv R, \tag{S-2.64}$$

corresponding to a sphere of radius R. Note that the two equipotential surfaces intersect each other on the circumference $x^2 + y^2 = R^2$ on the $z = 0$ plane. This is possible because the electric field of the dipole on the intersection circumference is

$$\mathbf{E}_{\mathrm{dip}} = k_{\mathrm{e}} \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} = -k_{\mathrm{e}} \frac{\mathbf{p}}{R^3} = -\mathbf{E}, \tag{S-2.65}$$

so that the total field on the circumference is zero, i.e., the only field that can be perpendicular to both equipotential surfaces.

b) We must find a solution for the potential φ that satisfies the condition $\varphi = 0$ at the surface of the conducting sphere, i.e. $\varphi(|\mathbf{r}| = a) = 0$, and such that at large distance from the conductor the field is $\mathbf{E}_0$.

According to point **a)**, the field outside the sphere must equal $\mathbf{E}_0$ plus the field of an electric dipole $\mathbf{p}_\mathrm{i}$, parallel to $\mathbf{E}_0$ and located at the center of the sphere. The moment of the dipole is obtained by substituting $R = a$ into (S-2.64),

$$\mathbf{p}_\mathrm{i} = k_\mathrm{e}^{-1} a^3\,\mathbf{E}_0 = \frac{3}{4\pi k_\mathrm{e}} V_a \mathbf{E}_0 \,, \tag{S-2.66}$$

where V_a is the volume of the sphere. The potential for $r \geqslant a$ is thus

$$\varphi = k_\mathrm{e} \frac{\mathbf{p}_i \cdot \mathbf{r}}{r^3} - E_0 z \,, \tag{S-2.67}$$

while $\varphi = 0$ for $\leqslant a$. The total charge induced on the sphere is zero, so that the solution is the same for a grounded and for an isolated, uncharged sphere. The solution is identical to the one obtained in Problem 2.1 via a different (heuristic) approach.

c) For the dipole at the center of a spherical conducting cavity, the boundary condition is $\varphi = 0$ at $r = b$. The polarization charges on the inner surface must generate a uniform field $\mathbf{E}_\mathrm{i}$ parallel to $\mathbf{p}_0$ and, according to (S-2.64), of intensity

$$E_\mathrm{i} = k_\mathrm{e} \frac{p_0}{b^3} = k_\mathrm{e} \frac{4\pi p_0}{3V_b} = k_\mathrm{e} \frac{p_0}{b^3} \,. \tag{S-2.68}$$

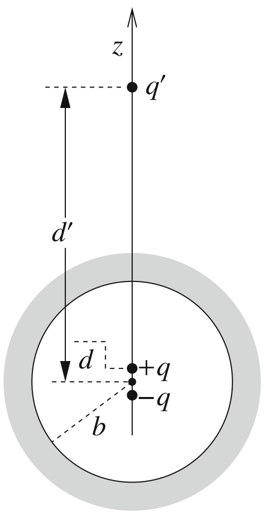

Fig. S-2.14

As in the preceding case, the total induced charge is zero and thus it does not matter whether the shell is grounded, or isolated and uncharged.

d) We can think of the dipole as a system of two point charges $\pm q$, respectively located at $z = \pm d$, with $p = 2qd$, as in Fig. S-2.14. According to the method of the image charges, the charge $+q$ modifies the charge distribution of the inner surface of the shell, so that it generates a field inside the sphere, equivalent to the field of an image charge $q' = -qb/d$, located at $z = d' = b^2/d$. Also the presence of the charge $-q$ affects the surface charge distribution, so that the total field inside the shell is the sum of the fields of the two real charges, plus the field of two image charges $\mp qb/d$ located at $z = \pm b^2/d$, respectively. Letting $d \to 0$ and $q \to \infty$, keeping the product $2qd = p$ constant, the field of the real charges approaches the field of a dipole $\mathbf{p} = p\hat{\mathbf{z}}$ located at the center of the shell, while the field of the image charges approaches a uniform field. Let us evaluate the field of the image charges at the center of the shell:

$$E_\mathrm{c} = 2k_\mathrm{e} \frac{qb}{d} \left(\frac{d}{b^2}\right)^2 = k_\mathrm{e} \frac{2qd}{b^3} = k_\mathrm{e} \frac{p}{b^3} \,, \tag{S-2.69}$$

in agreement with the result of point **c**). The method of the image charges can also be used to obtain the result of point **b**).

S-2.8 Electrically Connected Spheres

a) To the zeroth order in a/d and b/d, we assume the surface charges to be uniformly distributed. The electrostatic potential generated by each sphere outside its volume is thus equal to the potential of a point charge located at the center of the sphere. Let us denote by Q_a and Q_b the charges on each sphere, with $Q_a + Q_b = Q$. The charge on the wire is negligible because we have assumed that its capacitance is negligible. The electrostatic potentials of the spheres with respect to infinity are

$$V_a \simeq k_e \frac{Q_a}{a}, \qquad V_b \simeq k_e \frac{Q_b}{b}, \tag{S-2.70}$$

respectively. Since the spheres are electrically connected, $V_a = V_b \equiv V$. Solving for the charges we obtain

$$Q_a \simeq Q \frac{a}{a+b}, \qquad Q_b \simeq Q \frac{b}{a+b}, \tag{S-2.71}$$

so that $Q_a > Q_b$.
b) From the results of point **a**) it follows

$$V \simeq k_e \frac{Q}{a+b}, \qquad C \simeq \frac{a+b}{k_e}. \tag{S-2.72}$$

c) The electric fields at the sphere surfaces are

$$E_a \simeq k_e \frac{Q_a}{a^2} = k_e \frac{Q}{a(a+b)}, \qquad E_b \simeq k_e \frac{Q_b}{b^2} = k_e \frac{Q}{b(a+b)}, \tag{S-2.73}$$

with $E_b > E_a$. At the limit $b \to 0$ we have $E_a \to k_e Q/a^2$, while $E_b \to \infty$.

d) We proceed as in Problem 2.6. To zeroth order, we consider the field of each sphere outside its volume as due to a point charge at the sphere center. We denote by q_a and q_b the values of these point charges. To the first orders in a/d and b/d, we consider that each zeroth-order charge induces an image charge inside the other sphere, with values

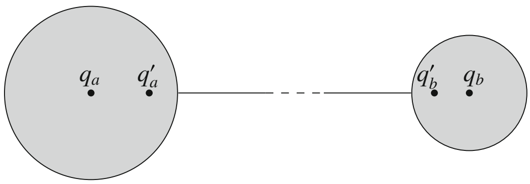

Fig. S-2.15

$$q'_a = -q_b \frac{a}{d}\,, \qquad q'_b = -q_a \frac{b}{d}\,, \tag{S-2.74}$$

at distances a^2/d and b^2/d from the centers, respectively. At each successive order, we add the images of the images added at the previous order. This leads to image charges of higher and higher orders in a/d and b/d.

Up to the first order, we thus have four point charges with the condition $q_a + q_b + q'_a + q'_b = Q$. A further condition is that the potentials at the sphere surfaces are

$$V_a \simeq k_\mathrm{e} \frac{q_a}{a}\,, \qquad V_b \simeq k_\mathrm{e} \frac{q_b}{b}\,, \tag{S-2.75}$$

since, at the surface of each sphere, the potentials due to the external zeroth-order charge and to the internal first-order charge cancel each other. Finally, we must have $V_a = V_b$, because the spheres are connected by the wire, so that

$$q_a \simeq \frac{Q}{1 + b/a - 2b/d}\,, \qquad q_b \simeq \frac{Q}{1 + a/b - 2a/d}\,. \tag{S-2.76}$$

S-2.9 A Charge Inside a Conducting Shell

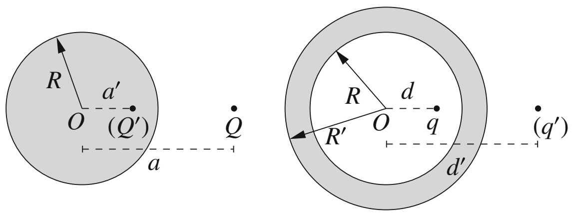

Fig. S-2.16

a) Let us first recall Problem 2.4 , now with a point charge Q at a distance a from the center O of a conducting, grounded sphere, of radius $R < a$. We introduce a spherical coordinate system, with the origin in O. We shall need only the radial coordinate r.

We have seen that the boundary conditions for $r \geqslant R$ are replicated by locating an image charge $Q' = Q(a/R)$ inside the sphere, at a distance $a' = R^2/a$ from O, on the line joining O and Q. In the present case we are dealing with the reverse problem, and we can obtain the solution in the region $r \leqslant R$ by reversing the roles of the real and image charges. The real charge q is now inside the cavity of a spherical conducting, grounded shell of internal radius R, at a distance $d < R$ from the center O. The boundary conditions inside the cavity are replicated by locating an *external* image charge $q' = q(R/d)$ at a distance $d' = R^2/d$ from O, on the straight line through O and q, as in Fig. S-2.16. Thus, the electric potential inside the cavity equals the sum of the potentials of q and q'. The potential φ in the region $R \leqslant r \leqslant R'$ is constant because here we are inside a conductor in static conditions, and equal to zero because the shell is grounded. We have $\varphi \equiv 0$ also for $r \geqslant R'$, because $\varphi = 0$ both on the spherical surface at $r = R$, and at infinity, and there are no charges in between.

b) The force between q and the shell equals the Coulomb force between q and its image charge q', and is attractive

$$F = k_\mathrm{e}\,\frac{qq'}{(d'-d)^2} = -k_\mathrm{e}\,\frac{q^2Rd}{(R^2-d^2)^2}\,. \tag{S-2.77}$$

c) Let us consider a spherical surface of radius R'', centered in O, with $R < R'' < R'$. The flux of the electric field through this closed surface is zero, because the field is zero everywhere inside a conductor. The total charge inside the sphere must thus be zero according to Gauss's law. This implies that the charge induced on the inner surface of the shell is $-q$, as may be verified directly by calculating the surface charge and integrating over the whole surface.

d) The electric potential must still be constant for $R \leqslant r \leqslant R'$, but it is no longer constrained to be zero. The electric potential in the region $r \leqslant R$ is still equivalent to the potential generated by the charges q and q' of point **a)**, plus a constant quantity φ_0 to be determined. The electric field in the region $R \leqslant r \leqslant R'$ is still zero, so that the potential is constant and equal to φ_0. Since the total charge on the shell must be zero, we must distribute a charge q over its external surface, of radius R', to compensate the charge $-q$ distributed over the internal surface, of radius R. Since the real charge q, and the charge $-q$ distributed over the surface of radius R generate a constant potential for $r \geqslant R$, the charge q must be distributed uniformly over the external surface in order to keep the total potential constant in the region $R \leqslant r \leqslant R'$.

The potential in the region $r \geqslant R'$ is equivalent to the potential generated by a point charge q located in O. Thus we have $\varphi(r) = k_\mathrm{e} q/r$ for $r \geqslant R'$, if we choose $\varphi(\infty) = 0$. Thus $\varphi_0 = \varphi(R') = k_\mathrm{e} q/R'$, and $\varphi(r) = \varphi_0$ for $R \leqslant r \leqslant R'$. For $r \leqslant R$ we have

$$\varphi(r) = k_\mathrm{e}\left(\frac{q}{r_q} + \frac{q'}{r_{q'}}\right) + \varphi_0, \tag{S-2.78}$$

where r_q is the distance of the point from the real charge q, and $r_{q'}$ is the distance of the point from the image charge q'. The field inside the cavity is the same for a grounded or for an isolated shell.

S-2.10 A Charged Wire in Front of a Cylindrical Conductor

a) We have $r = \sqrt{(x+a)^2+y^2}$ and $r' = \sqrt{(x-a)^2+y^2}$, x and y being the coordinates of Q. Thus, squaring the equation $r/r' = K$ we get

$$\frac{(x+a)^2+y^2}{(x-a)^2+y^2} = K^2$$
$$x^2+2ax+a^2+y^2 = K^2x^2-2K^2ax+K^2a^2+K^2y^2$$
$$-(x^2+y^2)(K^2-1)+2ax(K^2+1) = a^2(K^2-1)$$
$$x^2+y^2-2\,\frac{K^2+1}{K^2-1}\,ax = a^2\,. \tag{S-2.79}$$

On the other hand, the equation of a circumference centered at $(x_0, 0)$ and radius R is

$$(x-x_0)^2+y^2 = R^2$$
$$x^2+y^2-2x_0x = R^2-x_0^2\,. \tag{S-2.80}$$

Comparing (S-2.80) to (S-2.79) we see that the curves defined by the equation $r/r' = K$ are circumferences centered at

$$x_0(K) = \frac{K^2+1}{K^2-1}\,a\,, \quad y_0 = 0\,, \tag{S-2.81}$$

of radius

$$R(K) = \frac{2K}{|K^2-1|}\,a\,. \tag{S-2.82}$$

Note that

$$x_0\left(\frac{1}{K}\right) = -x_0(K)\,, \quad \text{and} \quad R\left(\frac{1}{K}\right) = R(K)\,. \tag{S-2.83}$$

Thus, we may restrict ourselves to $K > 1$, so that $x_0(K) > a > 0$, and omit the absolute-value sign in the expression for $R(K)$. The circumferences corresponding to $0 < K < 1$ are obtained by reflection across the y axis of the circumferences corresponding to $1/K$.

b) According to Gauss's law, the electrostatic field and potential generated by an infinite straight wire with linear charge density λ are

$$E(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{and} \quad \varphi(r) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{r_0}\right), \tag{S-2.84}$$

where r is the distance from the wire and r_0 an arbitrary constant, corresponding to the distance at which we pose $\varphi = 0$. The potential generated by two parallel wires of charge densities λ and $-\lambda$, respectively, is

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r}{r_0}\right)+\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r'}{r'_0}\right) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r'}{r}\right)+\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r'_0}\right), \tag{S-2.85}$$

where r'_0 is a second arbitrary constant, analogous to r_0. The term

$$\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_0}{r'_0}\right) \tag{S-2.86}$$

is actually a single arbitrary constant, which we can set equal to zero. With this choice the electrostatic potential is zero on the $x = 0$ plane of a Cartesian reference frame where the two wires lie on the straight lines $(x = -a, y = 0)$ and $(x = +a, y = 0)$. The equation for the equipotential surfaces in this reference frame is

$$\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r'}{r}\right) = \varphi, \tag{S-2.87}$$

which leads to

$$\frac{r}{r'} = \mathrm{e}^{-2\pi\varepsilon_0\varphi/\lambda}. \tag{S-2.88}$$

Thus we can substitute $K = \mathrm{e}^{-2\pi\varepsilon_0\varphi/\lambda}$ into (S-2.81) and (S-2.82). We see that the equipotential surfaces are infinite cylindrical surfaces whose axes have the equations

$$x_0(\varphi) = \frac{\mathrm{e}^{-4\pi\varepsilon_0\varphi/\lambda} + 1}{\mathrm{e}^{-4\pi\varepsilon_0\varphi/\lambda} - 1}, \quad y_0 = 0, \tag{S-2.89}$$

and their radii are

$$R(\varphi) = \frac{2\,\mathrm{e}^{-2\pi\varepsilon_0\varphi/\lambda}}{|\mathrm{e}^{-4\pi\varepsilon_0\varphi/\lambda} - 1|}\, a. \tag{S-2.90}$$

By multiplying the numerators and denominators of the above expressions by $\mathrm{e}^{2\pi\varepsilon_0\varphi/\lambda}$ we finally obtain

$$x_0(\varphi) = \frac{\mathrm{e}^{-2\pi\varepsilon_0\varphi/\lambda} + \mathrm{e}^{2\pi\varepsilon_0\varphi/\lambda}}{\mathrm{e}^{-2\pi\varepsilon_0\varphi\lambda} - \mathrm{e}^{2\pi\varepsilon_0\varphi/\lambda}}\, a = -a \coth\left(\frac{2\pi\varepsilon_0\varphi}{\lambda}\right) \tag{S-2.91}$$

and

$$R(\varphi) = \frac{2}{|\mathrm{e}^{-2\pi\varepsilon_0\varphi/\lambda} - \mathrm{e}^{2\pi\varepsilon_0\varphi/\lambda}|}\, a = \left|\frac{a}{\sinh(2\pi\varepsilon_0\varphi/\lambda)}\right|. \tag{S-2.92}$$

If the negative wire is located on the $(x = -a, y = 0)$ straight line, the $\varphi > 0$ equipotential cylinders are located in the $x < 0$ half space ($r < r'$ in Fig. 2.8), and the $\varphi < 0$ equipotentials in the $x > 0$ half space.

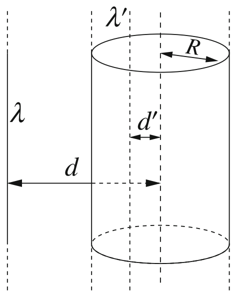

Fig. S-2.17

c) We can solve the problem by locating an image wire with charge density $\lambda' = -\lambda$ inside the cylinder. In Fig. (2.8), let the real wire intersect the xy plane at $P \equiv (-a, 0)$, and the image wire at $P' \equiv (a, 0)$. The surface of the conducting cylinder intersects the xy plane on one of the circumferences $r/r' = K$. This is always possible as far as $d > R$. With these locations of the real and image wires the potential of the cylinder surface is constant and equal to a certain value φ_0. Given R and d, we can find the values of a and d' by first defining the dimensionless constant $\varphi' = 2\pi\varepsilon_0\varphi_0/\lambda$, and then solving the simultaneous equations

$$2a + d' = d\,, \quad a + d' = x_0 = a\coth\varphi'\,, \quad \frac{a}{\sinh\varphi'} = R\,. \tag{S-2.93}$$

From the first equation we obtain $a = (d - d')/2$, which we substitute into the other two equations

$$\frac{d+d'}{2} = \frac{d-d'}{2}\coth\varphi'\,, \quad \frac{d-d'}{2} = R\sinh\varphi'\,, \tag{S-2.94}$$

and the latter equation leads to

$$\sinh\varphi' = \frac{d-d'}{2R}\,, \tag{S-2.95}$$

independent of λ. From the relations

$$\cosh^2 x - \sinh^2 x = 1\,, \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x}\,,$$

we obtain

$$\coth\varphi' = \frac{\sqrt{4R^2 + (d-d')^2}}{d-d'}\,, \tag{S-2.96}$$

which, substituted into the first of (S-2.93) leads to

$$\frac{d+d'}{2} = \frac{d-d'}{2}\,\frac{\sqrt{4R^2 + (d-d')^2}}{d-d'}\,. \tag{S-2.97}$$

Disregarding the trivial solution $d' = d$ (corresponding to two superposed wires of linear charge density λ and $-\lambda$, generating zero field in the whole space), we have

$$d' = \frac{R^2}{d}\,, \quad a = \frac{d^2+R^2}{2d}\,, \quad \varphi' = \operatorname{arccosh}\left(\frac{d^2+3R^2}{d^2+R^2}\right). \tag{S-2.98}$$

Alternatively, may proceed analogously to the well-known problem of the potential of a point charge in front of a grounded, conducting sphere.

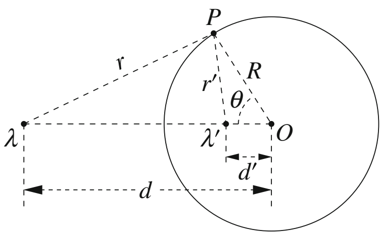

Fig. S-2.18

Figure S-2.18 shows the intersection with the xy plane of the conducting cylinder of radius R, the real charged wire at distance d from the cylinder axis, and the image wire at distance d' from the axis. We have translational symmetry perpendicularly to the figure. The potential φ generated by the real wire of linear charge density λ, and by the image wire of linear of linear charge density λ' must be constant over the cylinder surface. The potential at a generic point P of the surface is

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r - \frac{\lambda'}{2\pi\varepsilon_0}\ln r' = \text{const} \tag{S-2.99}$$

where r is the distance of P from the real wire and r' the distance of P from the image wire. Multiplying by $-2\pi\varepsilon_0$ we obtain

$$\lambda \ln r + \lambda \ln r' = \text{const}, \tag{S-2.100}$$

which can be rewritten by expressing r and r' in terms of d, d', R and the angle θ between r and the radius joining P to the intersection of the cylinder axis with the xy plane, O, and applying the law of cosines,

$$\lambda \ln\left(\sqrt{d^2 + R^2 - 2Rd\cos\theta}\right) + \lambda' \ln\left(\sqrt{d'^2 + R^2 - 2Rd'\cos\theta}\right) = \text{const}. \tag{S-2.101}$$

Differentiating with respect to θ we obtain

$$\frac{\lambda Rd\sin\theta}{d^2 + R^2 - 2Rd\cos\theta} = -\frac{\lambda' Rd'\sin\theta}{d'^2 + R^2 - 2Rd'\cos\theta}, \tag{S-2.102}$$

implying that λ and λ' must have opposite signs. Dividing both sides by $R\sin\theta$ we obtain, after some algebra,

$$\begin{aligned} \lambda d\left(d'^2 + R^2 - 2Rd'\cos\theta\right) &= -\lambda' d'\left(d^2 + R^2 - 2Rd\cos\theta\right) \\ \lambda\left(dd'^2 + dR^2 - 2Rdd'\cos\theta\right) &= -\lambda'\left(d'd^2 + d'R^2 - 2Rdd'\cos\theta\right), \end{aligned} \tag{S-2.103}$$

which requires $\lambda' = -\lambda$ in order to make the equation independent of θ, and, disregarding the trivial solution $d' = d$, we finally obtain

$$d' = \frac{R^2}{d}. \tag{S-2.104}$$

S-2.11 Hemispherical Conducting Surfaces

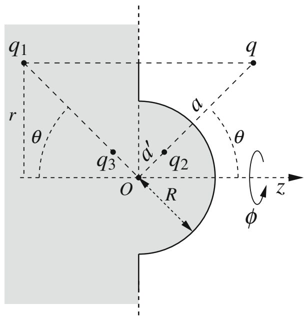

Fig. S-2.19

a) We choose a cylindrical coordinate system (r,ϕ,z) with the symmetry axis of the problem as z axis, so that the point charge is located in $(a\sin\theta,\phi,a\cos\theta)$, with ϕ a given fixed angle, as in Fig. S-2.19. The conductor surface, comprising the hemispherical boss and the plane part, is equipotential with $\varphi = 0$. If the conductor surface were simply plane, with no boss, the problem would be solved by locating an image charge $q_1 = -q$ in $(a\sin\theta,\phi,-a\cos\theta)$, as in Fig. S-2.19. On the other hand, if the conductor were a grounded spherical surface of radius R, the problem would be solved by locating an image charge $q_2 = -q(R/a)$ in $(a'\sin\theta,\phi,a'\cos\theta)$, with $a' = R^2/a$. The real charge q, with the two image charges q_1 and q_2, gives origin to a potential

$\varphi(\mathbf{r}) = \varphi_q(\mathbf{r}) + \varphi_{q_1}(r) + \varphi_{q_2}(r)$ which is different from zero both on the plane surface, where it equals $\varphi_{q_2}(r)$, since $\varphi_q(\mathbf{r}) + \varphi_{q_1}(r) = 0$ on the plane, and on the hemispherical surface, where it equals $\varphi_{q_1}(r)$. The problem is solved by adding a third image charge $q_3 = q(R/a)$ at $(a' \sin\theta, \phi, -a' \cos\theta)$, so that the pairs $\{q, q_1\}$ and $\{q_2, q_3\}$ generate a potential $\varphi = 0$ on the plane surface, and the pairs $\{q, q_2\}$ and $\{q_1, q_3\}$ generate a potential $\varphi = 0$ on the spherical (and hemispherical!) surface. According to Gauss's law, the total charge induced on the conductor equals the sum of the image charges

$$q_{\text{ind}} = q_1 + q_2 + q_3 = -q + \left(-\frac{R}{a}q\right) + \left(\frac{R}{a}q\right) = -q\,. \tag{S-2.105}$$

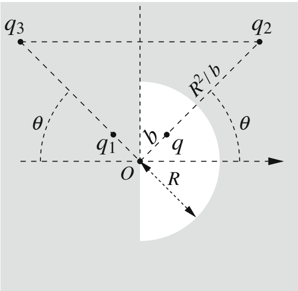

Fig. S-2.20

Note that, since the electric field generated by the real charge plus the three image charges is always perpendicular to the conductor surface, it must be zero on the circumference $(R, \phi, 0)$, here with ϕ any, where the hemisphere joins the plane.

b) Now the real charge q is located at $(b \sin\theta, \phi, b \cos\theta)$ inside the hemispherical cavity of radius $R > b$ in the conductor, as in Fig. S-2.20. The solution is analogous to the solution of point **a)**: we locate three image charges in the conductor, outside of the cavity, namely, $q_1 = -q$ in $(b \sin\theta, \phi, -b \cos\theta)$, $q_2 = -(R/b)q$ in $(b' \sin\theta, \phi, b' \cos\theta)$, with $b' = R^2/b > R$, and $q_3 = -q_2 = (R/b)q$ in $(b' \sin\theta, \phi, -b' \cos\theta)$.

S-2.12 The Force between the Plates of a Capacitor

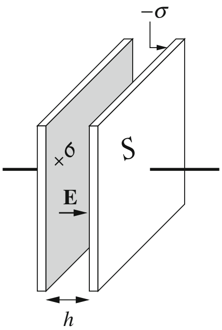

Fig. S-2.21

We present this simple problem in order to point out, and prevent, two typical recurrent errors. The first error regards the electrostatic pressure at the surface of a conductor, the second the derivation of the force from the energy of a system.

a) Let us consider the electrostatic pressure first. If Q is the charge of the capacitor, and S the surface of its plates, the surface charge density (which is located on the *inner* surfaces only!) is $\pm\sigma = \pm Q/S$. Within our approximations, the electric field is uniform between the two charged surfaces, $E = 4\pi k_e \sigma$, and zero everywhere else. This leads to an electrostatic pressure

$$P = \frac{1}{2}\sigma E = 2\pi k_e \sigma^2\,. \tag{S-2.106}$$

Here, the typical mistake is to forget the $1/2$ factor and to write $P \overset{\mathrm{w}}{=} \sigma E$ (the "w" on the "=" sign stands for wrong!). In fact, only one half of the electric field is due to the charge on the other plate. The force F is attractive because the two plates have opposite charges, and we can write

$$F = -PS = -2\pi k_{\mathrm{e}} \frac{Q^2}{S^2} S = -2\pi k_{\mathrm{e}} \frac{Q^2}{S} \,. \qquad \text{(S-2.107)}$$

Thus the force depends on Q only, and is independent of the distance h between the plates. (S-2.107) is valid both for an isolated capacitor, and for a capacitor connected to a voltage source maintaining a fixed potential difference V. But, in the latter case, the charge is no longer constant, and it is convenient to replace Q by the product CV, remembering that the capacity of a parallel-plate capacitor is $C = S/(4\pi k_{\mathrm{e}} h)$. Thus

$$F = -2\pi k_{\mathrm{e}} \frac{(CV)^2}{S} = -\frac{V^2 S}{8\pi k_{\mathrm{e}} h^2} \,. \qquad \text{(S-2.108)}$$

b) In the case of an isolated capacitor, the force between the plates can also be evaluated as minus the derivative of the electrostatic energy U_{es} of the capacitor with respect to the distance between the plates, h. It is convenient to write U_{es} as a function of the charge Q, which is constant for an isolated capacitor,

$$U_{\mathrm{es}} = \frac{Q^2}{2C} = 2\pi k_{\mathrm{e}} \frac{Q^2 h}{2S} \,, \qquad \text{(S-2.109)}$$

so that the force between the plates is

$$F = -\partial_h U_{\mathrm{es}} = -2\pi k_{\mathrm{e}} \frac{Q^2}{S} \,, \qquad \text{(S-2.110)}$$

in agreement with (S-2.107).

If the capacitor is connected to a voltage source, the potential difference V between the plates is the constant quantity. Thus, it is more convenient to write U_{es} as a function of V

$$U_{\mathrm{es}} = \frac{1}{2} C V^2 = \frac{1}{8\pi k_{\mathrm{e}}} \frac{V^2 S}{h} \,. \qquad \text{(S-2.111)}$$

At this point, it is tempting, but wrong, to evaluate the force between the plates as minus the derivative of U_{es} with respect to h. We would get

$$F \overset{\mathrm{w}}{=} -\partial_h U_{\mathrm{es}} = +\frac{1}{8\pi k_{\mathrm{e}}} \frac{V^2 S}{h^2} \,, \qquad \text{(S-2.112)}$$

and, if the "+" sign were correct, now the force would be repulsive, although equal in magnitude to (S-2.108)! Of course, this cannot be true, since the plates have opposite charges and attract each other. The error is that the force equals minus the gradient of the potential energy of an *isolated system*, which now includes also the voltage source. And the voltage source has to do some work to keep the potential difference of the capacitor constant while the capacity is changing. Let us consider an infinitesimal variation of the plate separation, dh which leads to an infinitesimal variation of the capacity, dC. The voltage source must move a charge $\mathrm{d}Q = V\mathrm{d}C$ across the potential difference V, in order to keep V constant. The source thus does a work

$$\mathrm{d}W = V\mathrm{d}Q = V^2\mathrm{d}C\,, \tag{S-2.113}$$

and its internal energy (whatever its nature: mechanical, chemical, ...) must change by the amount

$$\mathrm{d}U_{\text{source}} = -\mathrm{d}W = -V^2\mathrm{d}C\,. \tag{S-2.114}$$

Since at the same time the electrostatic energy of the capacitor changes by $\frac{1}{2}\,V^2\mathrm{d}C$, the variation of the *total* energy of the isolated system, $\mathrm{d}U_{\text{tot}}$, is

$$\mathrm{d}U_{\text{tot}} = \mathrm{d}U_{\text{source}} + \mathrm{d}U_{\text{es}} = -V^2\mathrm{d}C + \frac{V^2}{2}\mathrm{d}C = -\frac{V^2}{2}\mathrm{d}C = -\mathrm{d}U_{\text{es}}\,. \tag{S-2.115}$$

Thus, the force is

$$F = -\partial_h U_{\text{tot}} = +\partial_h U_{\text{es}} = -\frac{V^2 S}{8\pi k_{\text{e}} h^2}\,, \tag{S-2.116}$$

in agreement with (S-2.108).

S-2.13 Electrostatic Pressure on a Conducting Sphere

a) The surface charge is $\sigma = Q/S$, where $S = 4\pi a^2$ is the surface of the sphere. The electric field at the surface is $E = 4\pi k_{\text{e}}\sigma$, so that the pressure is

$$P = \frac{1}{2}\sigma E = 2\pi k_{\text{e}}\sigma^2 = k_{\text{e}}\frac{Q^2}{8\pi a^4}\,. \tag{S-2.117}$$

b) According to Gauss's law, the electric field of the sphere is

$$E(r) = \begin{cases} 0\,, & r < a\,, \\ k_{\text{e}}\dfrac{Q}{r^2}\,, & r > a\,, \end{cases} \tag{S-2.118}$$

and thus the electrostatic energy is

$$U_{\text{es}} = \int \frac{1}{8\pi k_{\text{e}}} E^2(\mathbf{r})\,\mathrm{d}^3\mathbf{r} = \int_a^\infty \frac{k_{\text{e}}}{8\pi}\left(\frac{Q}{r^2}\right)^2 4\pi r^2\,\mathrm{d}r = k_{\text{e}}\frac{Q^2}{2a}\,. \tag{S-2.119}$$

The derivative of U_{es} with respect to a, which has the dimensions of a force, can be interpreted as the integral of the electrostatic pressure over the surface of the sphere. Since the pressure is uniform for symmetry reasons, we can write

$$P = \frac{1}{4\pi a^2}\left(-\frac{\mathrm{d}U_{es}}{\mathrm{d}a}\right) = \frac{1}{4\pi a^2}\frac{k_e Q^2}{2a^2} = k_e\frac{Q^2}{8\pi a^4}\,, \tag{S-2.120}$$

in agreement with (S-2.117).

c) This problem is equivalent to locating a charge Q on the sphere, such that the potential difference between the sphere and infinity is V. The problem can also be seen as a spherical capacitor with internal radius a and external radius b, potential difference V, at the limit of b approaching infinity. The capacity is

$$C = \lim_{b\to\infty}\frac{1}{k_e}\frac{ab}{b-a} = \frac{a}{k_e}\,, \tag{S-2.121}$$

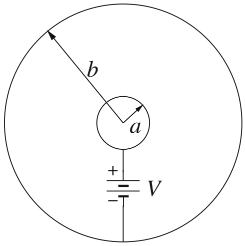

Fig. S-2.22

while the electric potential inside the capacitor is

$$\varphi(r) = \begin{cases} V & (r<a) \\ V\dfrac{a}{r} & (r>a) \end{cases} \tag{S-2.122}$$

so that the charge on the sphere of radius a is $Q = aV/k_e$. By substituting Q in (S-2.117) we obtain

$$P = \frac{V^2}{8\pi k_e a^2}\,. \tag{S-2.123}$$

Alternatively, we can write the electrostatic energy (S-2.119) as a function of V,

$$U_{es} = \frac{1}{2}CV^2 = \frac{aV^2}{2k_e}\,, \tag{S-2.124}$$

and remember from Problem 2.12 that, if the radius a is increased by da at constant voltage, the electrostatic energy of our "capacitor" changes by $\mathrm{d}U_{es}$, and, simultaneously, the voltage source does a work $\mathrm{d}W = 2\mathrm{d}U_{es}$, so that the variation of the "total" energy is

$$\mathrm{d}U_{tot} = \mathrm{d}U_{es} - \mathrm{d}W = -\mathrm{d}U_{es}\,, \tag{S-2.125}$$

and the pressure is

$$P = \frac{1}{4\pi a^2}\left(-\frac{\mathrm{d}U_{tot}}{\mathrm{d}a}\right) = \frac{1}{4\pi a^2}\left(\frac{\mathrm{d}U_{es}}{\mathrm{d}a}\right) = \frac{V^2}{8\pi k_e a^2}\,, \tag{S-2.126}$$

in agreement with (S-2.123).

S-2.14 Conducting Prolate Ellipsoid

a) Let us consider a line segment of length $2c$, of uniform linear electric charge density λ, so that the total charge of the segment is $Q = 2c\lambda$. We start using a system of cylindrical coordinates (r,ϕ,z), such that the end points of the segment have coordinates $(0,\phi,\pm c)$, the value of ϕ being irrelevant when $r = 0$. The electric potential $\varphi(P)$ of a generic point P, of coordinates (r,ϕ,z), is

$$\varphi(P) = k_e \int_{-c}^{+c} \frac{\lambda\,dz'}{s} = k_e\lambda \int_{-c}^{+c} \frac{dz'}{\sqrt{(z-z')^2 + r^2}}, \tag{S-2.127}$$

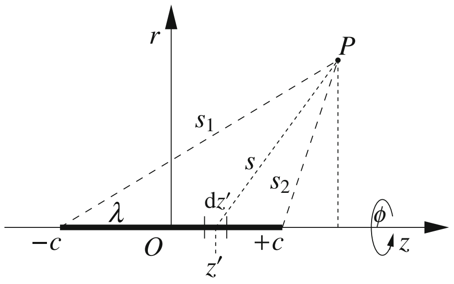

Fig. S-2.23

where s is the distance from P to the point of the charged segment of coordinate z', as shown in Fig. S-2.23. The indefinite integral is

$$\int \frac{dz'}{\sqrt{(z-z')^2 + r^2}} = -\ln\left[2\sqrt{(z-z')^2 + r^2} + 2z - 2z'\right] + C, \tag{S-2.128}$$

as can be checked by evaluating the derivative, leading to

$$\varphi(P) = k_e\lambda \ln\left[\frac{\sqrt{(z+c)^2 + r^2} + z + c}{\sqrt{(z-c)^2 + r^2} + z - c}\right] = k_e \frac{Q}{2c} \ln\left(\frac{s_1 + z + c}{s_2 + z - c}\right), \tag{S-2.129}$$

where $s_1 = \sqrt{(z+c)^2 + r^2}$ and $s_2 = \sqrt{(z-c)^2 + r^2}$ are the distances of P from the end points of the charged line segment, as shown in Fig. S-2.23. We now introduce the elliptic coordinates u and v

$$u = \frac{s_1 + s_2}{2c}, \quad v = \frac{s_1 - s_2}{2c}, \tag{S-2.130}$$

so that

$$s_1 = c(u+v), \quad s_2 = c(u-v),$$

and

$$uv = \frac{s_1^2 - s_2^2}{4} = \frac{z}{c}. \tag{S-2.131}$$

Because of (S-2.130), we have $u \geq 1$, and $-1 \leq v \leq 1$. The surfaces $u =$ const are confocal ellipsoids of revolution, and the surfaces $v =$ const are confocal hyperboloids of revolution, as shown in Fig. S-2.24. The surface $u = 1$ is the degenerate case of an ellipsoid with major radius $a = c$ and minor radius $b = 0$, coinciding with segment $(-c, c)$. The surface $v = 0$ is the degenerate case of the plane $z = 0$, while $v = \pm 1$ correspond to the degenerate cases of hyperboloids collapsed to the half-lines $(c, +\infty)$ and $(-c, -\infty)$. In terms of u and v, equation (S-2.129) becomes

$$\begin{aligned}\varphi(P) &= k_e \frac{Q}{2c} \ln\left[\frac{c(u+v)+cuv+c}{c(u-v)+cuv-c}\right] = k_e \frac{Q}{2c} \ln\left[\frac{(u+1)(v+1)}{(u-1)(v+1)}\right] \\ &= k_e \frac{Q}{2c} \ln\left(\frac{u+1}{u-1}\right),\end{aligned} \tag{S-2.132}$$

Thus, the electric potential depends only on the elliptical coordinate u, and is constant on the ellipsoidal surfaces u = const. The surfaces v = const are perpendicular to the equipotential surfaces u = const, so that the intersections of the surfaces v = const with the planes ϕ = const (confocal hyperbolae) are the field lines of the electric field. If we let u approach infinity, i.e., for $s_1 + s_2 \gg c$, we have $s_1 \simeq s_2$ and

$$\frac{u+1}{u-1} \simeq 1 + \frac{2}{u}, \qquad \ln\left(1 + \frac{2}{u}\right) \simeq \frac{2}{u}, \tag{S-2.133}$$

and

$$\lim_{u\to\infty} \varphi(P) = k_e \frac{Q}{2c}\frac{2}{u} = k_e \frac{Q}{cu} \simeq k_e \frac{Q}{s_1}, \tag{S-2.134}$$

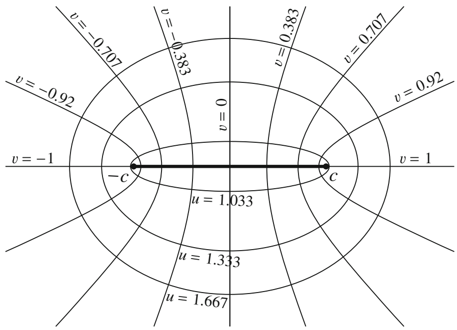

Fig. S-2.24

since $s_1 \simeq s_2$. This is what expected for a point charge. In other words, the ellipsoidal equipotential surfaces approach spheres as $u \to \infty$.

b) For a prolate ellipsoid of revolution of major and minor radii a and b, respectively, the distance between the center O and a focal point, c, is

$$c = \sqrt{a^2 - b^2}. \tag{S-2.135}$$

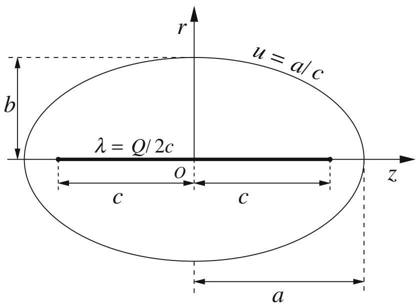

Fig. S-2.25

At each point of the surface of the ellipsoid we have $s_1 + s_2 = 2a$, so that the equation of the surface in elliptic coordinates is $u = a/c$. A uniformly charged line segment with end points at $(0,\phi,-c)$ and $(0,\phi,c)$, and linear charge density $\lambda = Q/(2c)$, generates a constant electric potential $\varphi(a,b)$ on the surface of the ellipsoid

$$\varphi(a,b) = k_\mathrm{e}\,\frac{Q}{2c}\,\ln\left(\frac{u+1}{u-1}\right) = k_\mathrm{e}\,\frac{Q}{2\sqrt{a^2-b^2}}\,\ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right). \tag{S-2.136}$$

On the other hand, the potential generated by the charged segment at infinity is zero, and there are no charges between the surface of the ellipsoid and infinity. The flux of the electric field through any closed surface containing the ellipsoid is Q. Thus, the potential, and the electric field, generated by the charged segment outside the surface of the ellipsoid equal the potential, and the electric field, generated by the conducting ellipsoid carrying a charge Q, and this solves the problem. The capacity of the ellipsoid is thus

$$C = \frac{Q}{\varphi(a,b)} = \frac{2\sqrt{a^2-b^2}}{k_\mathrm{e}}\left[\ln\left(\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}}\right)\right]^{-1}. \tag{S-2.137}$$

The denominator of the argument of the logarithm can be rationalized, leading to

$$\frac{a+\sqrt{a^2-b^2}}{a-\sqrt{a^2-b^2}} = \frac{\left(a+\sqrt{a^2-b^2}\right)^2}{a-a^2+b^2} = \left(\frac{a+\sqrt{a^2-b^2}}{b},\right)^2 \tag{S-2.138}$$

and the capacity of the prolate ellipsoid can be rewritten

$$C = \frac{\sqrt{a^2-b^2}}{k_\mathrm{e}}\left[\ln\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right]^{-1}. \tag{S-2.139}$$

The plates of a confocal ellipsoidal capacitor are the surfaces of two prolate ellipsoids of revolution, sharing the same focal points located at $\pm c$ on the z axis, and of major radii a_1 and a_2, respectively, with $a_1 < a_2$. According to (S-2.135) and (S-2.136) the potential on the two plates are

$$\varphi_{1,2} = k_\mathrm{e}\,\frac{Q}{2c}\,\ln\left(\frac{a_{1,2}+c}{a_{1,2}-c}\right) \tag{S-2.140}$$

so that the capacity is

$$C = \frac{Q}{\varphi_1-\varphi_2} = \frac{2c}{k_\mathrm{e}\ln\left(\dfrac{a_1+c}{a_1-c}\,\dfrac{a_2-c}{a_2+c}\right)} = \frac{2c}{k_\mathrm{e}\ln\left(\dfrac{a_1a_2-c^2+c\,(a_2-a_1)}{a_1a_2-c^2-c\,(a_2-a_1)}\right)}. \tag{S-2.141}$$

c) A straight wire of length h and diameter $2b$, with $h \gg b$, can be approximated by an ellipsoid prolate in the extreme, with major radius $a = h/2$ and minor radius b, with, of course, $b \ll a$. From

$$\sqrt{a^2 - b^2} \simeq a - \frac{b^2}{a} \quad \text{valid for} \quad b \ll a\,, \tag{S-2.142}$$

and (S-2.139) we have

$$C_{\text{wire}} \simeq \frac{a}{2k_{\text{e}}\ln(2a/b)} = \frac{h}{k_{\text{e}}\ln(h/b)}\,. \tag{S-2.143}$$