Chapter S-3
Solutions for Chapter 3

S-3.1 An Artificial Dielectric

a) According to (S-2.6) of Problem 2.1, a metal sphere in a uniform external field $\mathbf{E}$ acquires a dipole moment

$$\mathbf{p} = \frac{a^3}{k_e}\mathbf{E} = \frac{3}{4\pi k_e} V\,\mathbf{E}\,, \tag{S-3.1}$$

where $V = {}^4\!/_3\,\pi a^3$ is the volume of the sphere. The polarization of our suspension is

$$\mathbf{P} = n\,\mathbf{p} = \frac{3n}{4\pi k_e} V\,\mathbf{E}\,. \tag{S-3.2}$$

In SI units we have $\mathbf{P} = \varepsilon_0\chi\mathbf{E}$, and $\chi = 3f$, while in Gaussian units we have $\mathbf{P} = \chi\mathbf{E}$, and $\chi = 3f/(4\pi)$. In both cases $f = nV$ is the fraction of the volume occupied by the spheres. Since the minimum distance between the centers of two spheres is $2a$, we have

$$f \le \frac{4\pi a^3}{3}\frac{1}{8a^3} = \frac{\pi}{6}\,, \tag{S-3.3}$$

leading to $\chi \le \pi/2$ in SI units, and $\chi \le 1/8$ in Gaussian units.
b) The average distance ℓ between two sphere centers is of the order of $n^{-1/3}$. The electric field of a dipole at a distance ℓ is of the order of

$$E_{\mathrm{dip}} \simeq k_e\frac{p}{\ell^3} \simeq k_e\frac{a^3}{k_e}E\,n = a^3 E\,n. \tag{S-3.4}$$

Thus, the condition $E_{\mathrm{dip}} \ll E$ requires $n \ll 1/a^3$.

A. Macchi et al., *Problems in Classical Electromagnetism*,
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S-3.2 Charge in Front of a Dielectric Half-Space

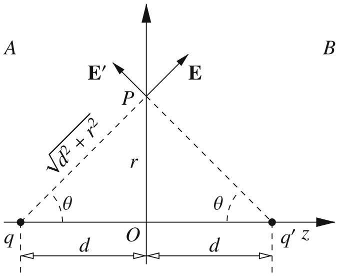

Fig. S-3.1

We denote by A the $z<0$, vacuum, half-space, containing the real charge q, and by B the $z>0$, dielectric, half-space, containing no free charge. We shall treat the two half spaces separately, making educated guesses, in order to apply the uniqueness theorem for the Poisson equation. We use cylindrical coordinates (r,ϕ,z), with the real charge located at $(0,\phi,-d)$. All our formulas will be independent of the azimuthal coordinate ϕ, which is not determined, and not relevant, when $r=0$.

a) We treat the field in the half-space A assuming vacuum in the whole space, including the half-space B. As ansatz, we locate an image charge q', of value to be determined, at $(0,0,+d)$, in the half space that we are not considering, as in Fig. S-3.1. Now we evaluate the electric field $\mathbf{E}^{(-)}$ in a generic point $P\equiv(r,\phi,0^-)$ of the plane $z=0^-$. The distance between P and q is $\sqrt{d^2+r^2}$ and forms an angle $\theta=\arccos(d/\sqrt{d^2+r^2})$ with the z axis. Also the distance between P and q' will be $\sqrt{d^2+r^2}$. The field at P, $\mathbf{E}^{(-)}$, is the vector sum of the fields $\mathbf{E}$ due to the real charge q, and $\mathbf{E}'$ do to the image charge q'. The components of $\mathbf{E}^{(-)}$, perpendicular and parallel to the $z=0$ plane are, respectively

$$\begin{aligned} E_\perp^{(-)} &= k_\mathrm{e}\,\frac{q}{d^2+r^2}\cos\theta - k_\mathrm{e}\,\frac{q'}{d^2+r^2}\cos\theta = k_\mathrm{e}\,\frac{d}{(d^2+r^2)^{3/2}}\,(q-q') \\ E_\parallel^{(-)} &= k_\mathrm{e}\,\frac{q}{d^2+r^2}\sin\theta + k_\mathrm{e}\,\frac{q'}{d^2+r^2}\sin\theta = k_\mathrm{e}\,\frac{r}{(d^2+r^2)^{3/2}}\,(q+q') \\ &. \end{aligned} \tag{S-3.5}$$

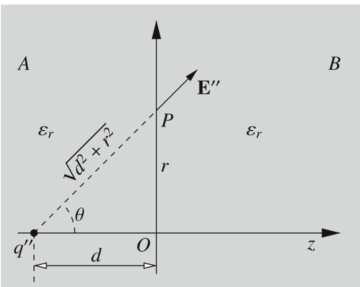

Fig. S-3.2

We treat the half-space B assuming that the whole space, including the half-space A, is filled by a dielectric medium of relative permittivity ε_r. We are not allowed to introduce charges or alter anything in B, but, as an educated guess, we replace the real charge q, located in the half-space A that we are not treating, by a charge q'', of value to be determined (Fig. S-3.2). We evaluate the field $\mathbf{E}^{(+)}$ at the same point P as before, but on the $z=0^+$ plane. The components of $\mathbf{E}^{(+)}$ perpendicular and parallel to the $z=0$ plane are

$$E_{\perp}^{(+)} = \frac{k_e}{\varepsilon_r}\frac{q''}{d^2+r^2}\cos\theta = \frac{k_e}{\varepsilon_r}\frac{d}{(d^2+r^2)^{3/2}}q''$$
$$E_{\parallel}^{(+)} = \frac{k_e}{\varepsilon_r}\frac{q''}{d^2+r^2}\sin\theta = \frac{k_e}{\varepsilon_r}\frac{r}{(d^2+r^2)^{3/2}}q'' \,. \quad \text{(S-3.6)}$$

If our educated guesses are correct, the dielectric boundary conditions must hold at $z = 0$. This implies $E_{\perp}^{(-)} = \varepsilon_r E_{\perp}^{(+)}$ and $E_{\parallel}^{(-)} = E_{\parallel}^{(+)}$, corresponding to the equations

$$q - q' = q'' \,, \quad \text{and} \quad q + q' = \frac{q''}{\varepsilon_r} \,, \quad \text{(S-3.7)}$$

with solutions

$$q' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} q \,, \quad \text{and} \quad q'' = \frac{2\varepsilon_r}{\varepsilon_r + 1} q \,. \quad \text{(S-3.8)}$$

We can easily check that, at the limit $\varepsilon_r \to 1$ (vacuum in the whole space), we have $q' \to 0$ and $q'' \to q$, i.e., in the whole space we have the field of charge q only. At the limit $\varepsilon_r \to \infty$ (dielectric $\to$ conductor limit) we have $q' \to -q$ and $q'' \to 2q$, i.e., the field of the real charge q and its image $-q$ in the half-space A, and zero field in the half space B, as at point **a)** of Problem 2.2. The finite value of q'' is irrelevant for the field in the half-space B, because of the infinite value of ε_r.

Notice that we can also write equations (S-3.6) without ε_r in the denominators, thus including the dielectric bound charge into q''. This leads to the equations

$$q - q' = \varepsilon_r q'' \,, \quad \text{and} \quad q + q' = q'' \quad \text{(S-3.9)}$$

with solutions

$$q' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} q \,, \quad \text{and} \quad q'' = \frac{2}{\varepsilon_r + 1} q \,, \quad \text{(S-3.10)}$$

which give the same expressions for the electric field as for the choice (S-3.6).
b) The polarization charge density on the $z = 0$ plane, $\sigma_b(r)$, is

$$\sigma_b(r) = -\frac{1}{4\pi k_e}\left(E_{\perp}^{(-)} - E_{\perp}^{(+)}\right) = -\frac{1}{4\pi}\frac{d}{(d^2+r^2)^{3/2}}\left(q - q' - \frac{q''}{\varepsilon_r}\right)$$
$$= -\frac{1}{2\pi}\frac{d}{(d^2+r^2)^{3/2}}\frac{\varepsilon_r - 1}{\varepsilon_r + 1} q = \frac{1}{2\pi}\frac{d}{(d^2+r^2)^{3/2}} q' \,. \quad \text{(S-3.11)}$$

The total polarization charge on the $z = 0$ plane is

$$q_p = \int_0^\infty \sigma_b(r)\, 2\pi r\, \mathrm{d}r = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1} q \int_0^{\pi/2} \cos\theta\, \mathrm{d}\theta = q' \,, \quad \text{(S-3.12)}$$

where we have substituted $\cos\theta = d/\sqrt{d^2+r^2}$, $r = d/\cos\theta$ and $\mathrm{d}r = d\,\mathrm{d}\theta/\cos^2\theta$.

c) The polarization charge of the $z = 0$ plane generates an electric field equal to the field of a charge $q' = -q(\varepsilon_r - 1)/(\varepsilon_r + 1)$ located at $(0,0,+d)$ in the half space $z < 0$, and equal to the field of a charge q' located at $(0,0,-d)$ in the half space $z > 0$.

S-3.3 An Electrically Polarized Sphere

a) Since the polarization $\mathbf{P}$ of the sphere is uniform, we have no volume bound-charge density, according to $\varrho_b = \boldsymbol{\nabla}\cdot\boldsymbol{P}$. If we choose a spherical coordinate system (r,θ,ϕ) with the azimuthal axis parallel to $\mathbf{P}$, as shown in Fig. S-3.3, we see that the surface bound-charge density of the sphere is $\sigma_b = P\cos\theta$, according to $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. Thus, in principle, we can evaluate the electric field everywhere in space as the field generated by the bound-charge distribution on the sphere surface.

However, it is easier to consider the polarized sphere as the superposition of two uniformly charged spheres, both of radius a, one of volume charge density ϱ, and one of volume charge density $-\varrho$. The centers of the two spheres are separated by a small distance $\boldsymbol{\delta}$, as in Fig. 1.1 of Problem 1.1. Thus, two initially superposed infinitesimal volume elements d^3r of the two spheres, of charge $\pm\varrho\,\mathrm{d}^3r$, respectively, give origin to an infinitesimal electrical dipole moment $\mathrm{d}\mathbf{p} = \boldsymbol{\delta}\varrho\,\mathrm{d}^3r$ after the displacement.

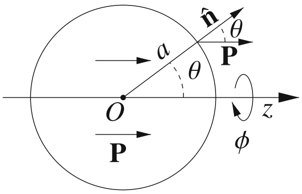

Fig. S-3.3

This corresponds to a polarization $\mathrm{d}\mathbf{p}/\mathrm{d}^3r = \varrho\boldsymbol{\delta}$, we must have $\mathbf{P} = \varrho\boldsymbol{\delta}$, and are interested in the limit $|\delta| \to 0$, $\rho \to \infty$, with $\varrho\boldsymbol{\delta} = \mathbf{P} =$ constant. Now we can follow the solution of Problem 1.1. According to (S-1.1), the electric field inside the sphere is uniform and equals

$$\mathbf{E}_{\mathrm{in}} = -\frac{4\pi k_e}{3}\varrho\boldsymbol{\delta} = -\frac{4\pi k_e}{3}\mathbf{P}\,. \qquad \text{(S-3.13)}$$

The problem of the field outside the sphere is solved at point **b)** of Problem 1.1, we have

$$\mathbf{E}_{\mathrm{ext}}(\mathbf{r}) = k_e\,\frac{3\hat{\mathbf{r}}\,(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3}\,, \qquad \text{(S-3.14)}$$

where $\mathbf{p} = \mathbf{P}(4\pi a^3/3)$ is the total dipole moment of the sphere.

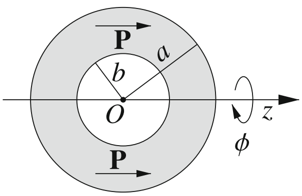

Fig. S-3.4

b) The problem can be solved by the superposition principle. The hole of radius b can be regarded as a sphere of uniform electrical polarization $-\mathbf{P}$ superposed to the sphere of radius a and polarization $\mathbf{P}$. The sphere of radius b generates a field

$$\mathbf{E}_{\mathrm{in}}^{(b)} = \frac{4\pi k_e}{3}\mathbf{P} \qquad \text{(S-3.15)}$$

at its interior. Thus, the total field inside the spherical hole is $\mathbf{E}_{\rm in}^{(a+b)} = 0$. The field inside the spherical shell $b < r < a$ is the sum of the uniform field (S-3.13) and the field generated by an electric dipole of moment $\mathbf{p}^{(b)}$ located at the center O, with

$$\mathbf{p}^{(b)} = -\frac{4\pi b^3}{3}\mathbf{P}\,. \tag{S-3.16}$$

Finally, the external field ($r > a$) equals the field generated by a single dipole $\mathbf{p}^{(a+b)}$ located in O with

$$\mathbf{p}^{(a+b)} = \frac{4\pi\left(a^3 - b^3\right)}{3}\mathbf{P}\,. \tag{S-3.17}$$

S-3.4 Dielectric Sphere in an External Field

a) As an educated guess. Let us assume that the external field induces a uniform electric polarization $\mathbf{P}$ in the sphere. We have seen in Problem 3.3 that a sphere of uniform electric polarization $\mathbf{P}$ generates a uniform electric field $\mathbf{E}^{\rm pol} = -(4\pi k_{\rm e}/3)\mathbf{P}$ at its interior. The difference is that in the present case $\mathbf{P}$ is not permanent but it is induced by the local electric field, and

$$\mathbf{P} = \frac{\varepsilon_{\rm r} - 1}{4\pi k_{\rm e}}\mathbf{E}^{\rm diel}\,, \tag{S-3.18}$$

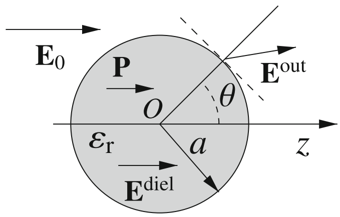

Fig. S-3.5

where $\mathbf{E}^{\rm diel}$ is the field inside the dielectric sphere, which is the sum of the external and the induced fields:

$$\mathbf{E}^{\rm diel} = \mathbf{E}_0 + \mathbf{E}^{\rm pol}\,. \tag{S-3.19}$$

We thus have

$$\mathbf{E}^{\rm diel} = \mathbf{E}_0 - \frac{4\pi k_{\rm e}}{3}\mathbf{P} = \mathbf{E}_0 - \frac{\varepsilon_{\rm r} - 1}{3}\mathbf{E}^{\rm diel}\,, \tag{S-3.20}$$

which can be solved for $\mathbf{E}^{\rm diel}$:

$$\mathbf{E}^{\rm diel} = \frac{3}{\varepsilon_{\rm r} + 2}\mathbf{E}_0\,. \tag{S-3.21}$$

Since $\varepsilon_{\rm r} > 1$, the field inside the dielectric sphere is smaller than $\mathbf{E}_0$.

The electric field outside the sphere $\mathbf{E}^{\rm out}$ will be given by the sum of $\mathbf{E}_0$ and the field of a dipole

$$\mathbf{p} = \frac{4\pi a^3}{3}\mathbf{P} = \frac{a^3}{3k_\mathrm{e}}\frac{\varepsilon_\mathrm{r}-1}{\varepsilon_\mathrm{r}+2}\mathbf{E}_0 \tag{S-3.22}$$

located at the center of the of the sphere. Thus

$$\mathbf{E}^\mathrm{out} = \mathbf{E}_0 + k_\mathrm{e}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3} = \mathbf{E}_0 + \frac{a^3}{3k_\mathrm{e}}\frac{\varepsilon_\mathrm{r}-1}{\varepsilon_\mathrm{r}+2}[3(\mathbf{E}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{E}_0]\,. \tag{S-3.23}$$

It is instructive, and useful for the following, to check that the above solution satisfies the boundary conditions at the surface of the sphere. Let us then restart the problem by assuming that the field $\mathbf{E}^\mathrm{diel}$ inside the sphere ($r < a$) is uniform and parallel to the external field $\mathbf{E}_0$, and that the field $\mathbf{E}^\mathrm{out}$ outside the sphere ($r > a$) is the sum of the external field and that of a dipole $\mathbf{p}$ located at the center of the sphere and also parallel to $\mathbf{E}_0$. Thus we can write $\mathbf{E}^\mathrm{diel} = \alpha\mathbf{E}_0$ and $\mathbf{p} = \eta\mathbf{E}_0$, with the constants α and η to be determined by the boundary conditions at $r = a$. Choosing a spherical coordinate system with the origin O at the center of the sphere, and the polar axis z parallel to $\mathbf{E}_0$, we have

$$\begin{aligned}
E_r^\mathrm{diel} &= \alpha E_0\cos\theta & E_r^\mathrm{out} &= E_0\cos\theta + k_\mathrm{e}\eta E_0\frac{2\cos\theta}{r^3}\\
E_\theta^\mathrm{diel} &= \alpha E_0\sin\theta & E_\theta^\mathrm{out} &= E_0\sin\theta - k_\mathrm{e}\eta E_0\frac{\sin\theta}{r^3}\\
E_\phi^\mathrm{diel} &= 0 & E_\phi^\mathrm{out} &= 0\,.
\end{aligned} \tag{S-3.24}$$

The boundary conditions at the surface of the sphere are $\varepsilon_\mathrm{r}E_\perp^\mathrm{diel} = E_\perp^\mathrm{out}$ and $E_\parallel^\mathrm{diel} = E_\parallel^\mathrm{out}$ which yields, in spherical coordinates,

$$\varepsilon_\mathrm{r}E_r^\mathrm{diel}(r=a^-) = E_r^\mathrm{out}(r=a^+) \qquad E_\theta^\mathrm{diel}(r=a^-) = E_\theta^\mathrm{out}(r=a^+)\,. \tag{S-3.25}$$

Using (S-3.24) and deleting the common factors we obtain

$$\varepsilon_\mathrm{r}\alpha = 1 + \frac{2k_\mathrm{e}}{a^3}\eta\,, \qquad \alpha = 1 - \frac{k_\mathrm{e}}{a^3}\eta\,, \tag{S-3.26}$$

whose solutions for α and η are

$$\alpha = 3/(\varepsilon_\mathrm{r}+2)\,, \qquad \eta = \frac{a^3}{k_\mathrm{e}}\frac{\varepsilon_\mathrm{r}-1}{\varepsilon_\mathrm{r}-2}\,, \tag{S-3.27}$$

and we eventually recover (S-3.21) and (S-3.22).

b) We make an educated guess analogous to the one of the previous point, i.e., we assume that the field inside the cavity, $\mathbf{E}^\mathrm{cav}$, is uniform and parallel to $\mathbf{E}_\mathrm{d}$, and that the field in the dielectric medium, $\mathbf{E}^\mathrm{diel}$, is the sum of $\mathbf{E}_\mathrm{d}$ and the field of an electric dipole $\mathbf{p}_c$, located at the center of the cavity and parallel to $\mathbf{E}_\mathrm{d}$. Thus we can write

$$\mathbf{E}^{\text{cav}} = \alpha\,\mathbf{E}_{\text{d}}\,, \quad \mathbf{p}_c = \eta\mathbf{E}_{\text{d}}\,; \tag{S-3.28}$$

where, again, α and η are constants to be determined by the boundary conditions. Using again spherical coordinates with the origin at the center of the spherical cavity and the z axis parallel to $\mathbf{E}_{\text{d}}$, the expressions analogous to (S-3.24) are

$$\begin{aligned} E_r^{\text{cav}} &= \alpha E_d \cos\theta & E_r^{\text{med}} &= E_d \cos\theta + k_{\text{e}}\,\eta E_d\,\frac{2\cos\theta}{r^3} \\ E_\theta^{\text{cav}} &= \alpha E_d \sin\theta & E_\theta^{\text{med}} &= E_d \sin\theta - k_{\text{e}}\eta E_d\,\frac{\sin\theta}{r^3} \\ E_\phi^{\text{cav}} &= 0 & E_\phi^{\text{med}} &= 0\,, \end{aligned} \tag{S-3.29}$$

with the boundary conditions

$$E_r^{\text{cav}}(r=a^-) = \varepsilon_{\text{r}} E_r^{\text{med}}(r=a^+) \qquad E_\theta^{\text{cav}}(r=a^-) = E_\theta^{\text{med}}(r=a^+)\,. \tag{S-3.30}$$

The values for α and η may thus be easily obtained by solving a linear system of two equations as in point **a**). However, we can immediately obtain the solution by noticing that (S-3.29) and (S-3.30) are identical to (S-3.24) and (S-3.25) but for the replacements $E_d \leftrightarrow E_0$, $E^{\text{cav}} \leftrightarrow E^{\text{diel}}$, $E^{\text{med}} \leftrightarrow E^{\text{out}}$, and $\varepsilon_{\text{r}} \leftrightarrow 1/\varepsilon_{\text{r}}$. Thus, the solutions for $\mathbf{E}^{\text{cav}}$ and $\mathbf{p}_c$ are obtained from those for $\mathbf{E}^{\text{diel}}$ and $\mathbf{p}$, (S-3.21) and (S-3.22), by substituting E_d for E_0 and $1\varepsilon_{\text{r}}$ for ε_{r}:

$$\mathbf{E}^{\text{cav}} = \frac{3}{1/\varepsilon_{\text{r}}+2}\mathbf{E}_{\text{d}} = \frac{3\varepsilon_{\text{r}}}{1+2\varepsilon_{\text{r}}}\mathbf{E}_{\text{d}}\,, \tag{S-3.31}$$

$$\mathbf{p}_c = \frac{a^3}{3k_{\text{e}}}\,\frac{1/\varepsilon_{\text{r}}-1}{1/\varepsilon_{\text{r}}+2}\mathbf{E}_d = \frac{a^3}{3k_{\text{e}}}\,\frac{1-\varepsilon_{\text{r}}}{1+2\varepsilon_{\text{r}}}\mathbf{E}_d\,. \tag{S-3.32}$$

Thus $E^{\text{cav}} > E_{\text{d}}$, i.e. the field inside the cavity is stronger than that outside it, and $\mathbf{p}_c$ is antiparallel to $\mathbf{E}_{\text{d}}$.

S-3.5 Refraction of the Electric Field at a Dielectric Boundary

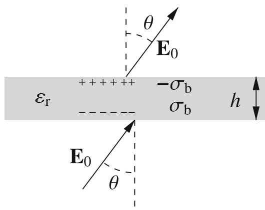

Fig. S-3.6

a) First, we note that the electric field $\mathbf{E}_0$ outside the dielectric slab equals the field that we would have in vacuum in the absence of the slab. Neglecting the boundary effects, the bound surface charge densities of slab are analogous to the surface charge densities of a parallel-plate capacitor. These generate a uniform electric field inside the capacitor, but no field outside. Thus, the electric field inside the slab is the sum of $\mathbf{E}_0$ and the field generated by the surface polarization charge densities. If we denote by $\mathbf{E}'$ the internal electric field, the boundary conditions at the dielectric surfaces are

$$E_{0\perp} = \varepsilon_r E'_\perp , \quad E_{0\|} = \varepsilon_r E'_\| , \tag{S-3.33}$$

where the subscripts $\perp$ and $\|$ denote the field components perpendicular and parallel to the interface surface, respectively. In terms of the angles θ and θ' of Fig. S-3.6

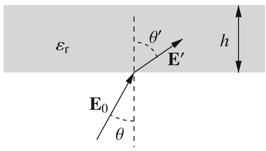

Fig. S-3.7

we have

$$\begin{aligned} E_0 \cos\theta &= \varepsilon_r E' \cos\theta' \\ E_0 \sin\theta &= E' \sin\theta' . \end{aligned} \tag{S-3.34}$$

If we divide the second of (S-3.34) by the first we obtain

$$\frac{1}{\varepsilon_r} \tan\theta' = \tan\theta , \tag{S-3.35}$$

and, since $\varepsilon_r > 1$, we have $\theta' > \theta$.

b) From Gauss's law we obtain

$$\sigma_b = \frac{1}{4\pi k_e} (E_{o,\perp} - E'_\perp) = \frac{1}{4\pi k_e} E_0 \cos\theta \left(1 - \frac{1}{\varepsilon_r}\right) . \tag{S-3.36}$$

c) The electrostatic energy density inside the slab is

$$\begin{aligned} u_{es} &= \frac{\varepsilon_r}{8\pi k_e} \mathbf{E}'^2 = \frac{\varepsilon_r}{8\pi k_e}, (E'^2_\perp + E'^2_\|) = \frac{\varepsilon_r}{8\pi k_e} E_0^2 \left(\frac{\cos^2\theta}{\varepsilon_r^2} + \sin^2\theta\right) \\ &= \frac{1}{8\pi k_e \varepsilon_r} E_0^2 \left[(\varepsilon_r^2 - 1)\sin^2\theta + 1\right] , \end{aligned} \tag{S-3.37}$$

so that u_{es} increases with increasing θ, and we expect a torque τ tending to rotate the slab toward the angle of minimum energy, i.e., $\theta = 0$. Neglecting boundary effects, the total electrostatic energy of the slab is $U_{es} = V u_{es}$, where V is the volume of the slab, and the torque exerted by the electric field is

$$\tau = -\frac{\partial U_{es}}{\partial\theta} = -\frac{1}{8\pi k_e \varepsilon_r} E_0^2 V \left(\varepsilon_r^2 - 1\right) \sin 2\theta < 0 . \tag{S-3.38}$$

S-3.6 Contact Force between a Conducting Slab and a Dielectric Half-Space

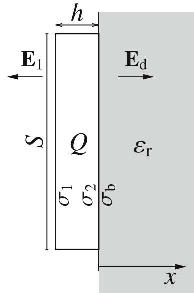

Fig. S-3.8

a) Neglecting boundary effects at the edges of the slab, the electric field is parallel to the x axis in all the regions of interest because of symmetry reasons. Thus, we can omit the vector notation, and we shall use positive numbers for vectors whose unit vector is $\hat{\mathbf{x}}$, negative numbers otherwise.

According to Gauss's law, a uniformly charged plane with surface charge density σ_a generates uniform fields at both its sides, of intensities $E_a = \pm\sigma_a/2\varepsilon_0$, respectively. In our problem we have three charged parallel plane surfaces: we denote by σ_1 the surface charge density on the left surface of the slab, by σ_2 charge the density on its right surface, and by σ_b the *bound* surface charge density of the dielectric material on its surface, as shown in Fig. S-3.8. Since the total *free* charge on the slab is Q, we have

$$\sigma_1 + \sigma_2 = \frac{Q}{S} = \sigma_{\mathrm{tot}} \,. \tag{S-3.39}$$

At any point in space the total electric field is the sum of the fields generated by the three surface charges. Now, the electric field must be zero inside the conducting slab. Thus the sum of all surface charge densities (including *both* free and bound charges) at the left of the slab must equal the sum of all surface charge densities at the right, so that their respective fields cancel out inside the slab. This conclusion holds both when the slab is in contact with the dielectric, and when there is a vacuum gap between them. Thus, we have

$$\sigma_1 = \sigma_2 + \sigma_\mathrm{b} \,. \tag{S-3.40}$$

The electric field E_d inside the dielectric medium is $E_\mathrm{d} = 4\pi k_\mathrm{e}\,(\sigma_2 + \sigma_\mathrm{b})$. This implies for the dielectric polarization of the medium P

$$P = \frac{\varepsilon_\mathrm{r} - 1}{4\pi k_\mathrm{e}}\, E_\mathrm{d} = (\varepsilon_\mathrm{r} - 1)\,\sigma_1 \,. \tag{S-3.41}$$

Since we also have $\sigma_\mathrm{b} = -\mathbf{P}\cdot\hat{\mathbf{x}} = -P$, we obtain the additional relation

$$\sigma_\mathrm{b} = -(\varepsilon_\mathrm{r} - 1)\,(\sigma_2 + \sigma_\mathrm{b})\,, \tag{S-3.42}$$

that leads to

$$\sigma_\mathrm{b} = -\frac{\varepsilon_\mathrm{r} - 1}{\varepsilon_\mathrm{r}}\,\sigma_2 \,. \tag{S-3.43}$$

From (S-3.39), (S-3.40) and (S-3.43) we finally obtain

$$\sigma_1 = \frac{1}{\varepsilon_r + 1}\sigma_{tot}\,, \qquad \sigma_2 = \frac{\varepsilon_r}{\varepsilon_r + 1}\sigma_{tot}\,, \qquad \sigma_b = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\sigma_{tot}\,. \tag{S-3.44}$$

The magnitudes of the electric field at the left of the slab E_1, and of the electric field inside the dielectric medium E_d, can be evaluated from Gauss's law, recalling that the field is zero inside the slab. We have

$$E_1 = -4\pi k_e \sigma_1 = -\frac{4\pi k_e}{\varepsilon_r + 1}\sigma_{tot} \quad \text{and} \quad E_d = 4\pi k_e (\sigma_2 + \sigma_b) = -E_1\,. \tag{S-3.45}$$

In the case of a vacuum gap between the conducting slab and the dielectric medium, as shown in Fig. S-3.9, the field E_2 in the gap is

$$E_2 = 4\pi k_e \sigma_2 = 4\pi k_e \frac{\varepsilon_r}{\varepsilon_r + 1}\sigma_{tot} = -\varepsilon_r E_1\,. \tag{S-3.46}$$

The values of E_1 and E_d are not affected by the presence of the vacuum gap.

As an alternative approach we can assume, following Problem 3.2, that the free charge layers σ_1 and σ_2 induce image charge layers σ_1' and σ_2' in the dielectric,

$$\sigma_1' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\sigma_1\,, \qquad \sigma_2' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\sigma_2\,, \tag{S-3.47}$$

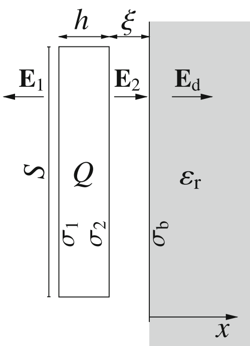

Fig. S-3.9

with the image planes located in position symmetrical with respect to the dielectric surface. Due to Gauss's law the bound surface charge density is the sum of the image charge densities,

$$\sigma_p = \sigma_1' + \sigma_2' = -\frac{\varepsilon_r - 1}{\varepsilon_r + 1}\frac{Q}{S}\,. \tag{S-3.48}$$

The free charge densities can be now found by requiring the field to vanish inside the slab: omitting a common multiplying factor we have

$$0 = \sigma_1 - \sigma_2 - \sigma_1' - \sigma_2' = 2\,\frac{\varepsilon_r\sigma_1 - \sigma_2}{\varepsilon_r + 1}\,, \tag{S-3.49}$$

from which we obtain $\varepsilon_r\sigma_1 = \sigma_2$, and we eventually recover the free and bound surface charge densities of (S-3.44).

b) In order to evaluate the electrostatic force acting on the conducting slab, we first assume the presence of a small vacuum gap of width ξ between the slab and the dielectric medium, as shown in Fig. S-3.9.

We can evaluate the total electrostatic force F acting on the conducting slab in three equivalent ways:

(i) We can evaluate the variation of the total electrostatic energy U_{es} when the slab is displaced by an infinitesimal amount dx toward the right, thus decreasing the gap. In this case U_{es} increases by $E_1^2 S\,\mathrm{d}x/(8\pi k_{\mathrm{e}})$ at the left of the slab, because the width of region "filled" by the field E_1 increases by dx, and correspondingly decreases by $E_2^2 S\,\mathrm{d}x/(8\pi k_{\mathrm{e}})$ at its right. Thus, $\mathrm{d}U_{\mathrm{es}} = (E_1^2 - E_2^2) S\,\mathrm{d}x/(8\pi k_{\mathrm{e}})$, from which we obtain the force per unit surface

$$F = -\frac{\mathrm{d}U_{\mathrm{es}}}{\mathrm{d}x} = \frac{S}{8\pi k_{\mathrm{e}}}(E_1^2 - E_2^2) = \frac{S}{8\pi k_{\mathrm{e}}}(\varepsilon_{\mathrm{r}}^2 - 1)\left(\frac{4\pi k_{\mathrm{e}}}{\varepsilon_{\mathrm{r}}+1}\right)^2\left(\frac{Q}{S}\right)^2$$
$$= 2\pi k_e \frac{\varepsilon_{\mathrm{r}} - 1}{\varepsilon_{\mathrm{r}} + 1}\frac{Q^2}{S}\,. \qquad \text{(S-3.50)}$$

We have $F > 0$, meaning that the slab is attracted by the dielectric medium.

(ii) We can multiply the charge of the slab Q by the *local* field, i.e., by the field generated by all charges excluding the charges of the slab. In our case the local field is the field E_p generated by the bound surface charge density σ_{b}. We have

$$E_p = -2\pi k_{\mathrm{e}}\sigma_{\mathrm{b}} = 2\pi k_{\mathrm{e}}\frac{\varepsilon_{\mathrm{r}} - 1}{\varepsilon_{\mathrm{r}} + 1}\sigma_{\mathrm{tot}}\,, \quad \text{and} \quad F = 2\pi k_{\mathrm{e}}\frac{\varepsilon_{\mathrm{r}} - 1}{\varepsilon_{\mathrm{r}} + 1}\frac{Q^2}{S}\,. \qquad \text{(S-3.51)}$$

(iii) We can evaluate the force on the slab by summing the forces F_1 on its left and F_2 on its right surface. These are obtained by multiplying the respective charges $Q_1 = S\sigma_1$ and $Q_2 = S\sigma_2$ by the average fields at the surfaces

$$F = F_1 + F_2 = Q_1\frac{E_1}{2} + Q_2\frac{E_2}{2} = -\frac{Q}{\varepsilon_{\mathrm{r}}+1}\frac{2\pi k_{\mathrm{e}}}{\varepsilon_{\mathrm{r}}+1}\sigma_{\mathrm{tot}} + \frac{\varepsilon_{\mathrm{r}} Q}{\varepsilon_{\mathrm{r}}+1}\frac{\varepsilon_{\mathrm{r}} 2\pi k_{\mathrm{e}}}{\varepsilon_{\mathrm{r}}+1}\sigma_{\mathrm{tot}}$$
$$= 2\pi k_{\mathrm{e}}\frac{\varepsilon_{\mathrm{r}}^2 - 1}{(\varepsilon_{\mathrm{r}}+1)^2}\frac{Q^2}{S} = 2\pi k_{\mathrm{e}}\frac{\varepsilon_{\mathrm{r}} - 1}{\varepsilon_{\mathrm{r}} + 1}\frac{Q^2}{S}\,. \qquad \text{(S-3.52)}$$

The force F is independent of ξ, thus the above result should be valid also at the limit $\xi \to 0$, i.e., when there is contact between the metal slab and the dielectric. One may argue, however, that in these conditions the field at $x = 0^+$, i.e., at the right of the slab, is given by $E_d = -E_1$, so that following the approach (iii) one would write

$$F = F_1 + F_2 \overset{?}{=} Q_1\frac{E_1}{2} + Q_2\frac{E_d}{2} \neq Q_1\frac{E_1}{2} + Q_2\frac{E_2}{2}\,. \qquad \text{(S-3.53)}$$

This discrepancy comes out because actually the average field on the *free* charges located on the right surface of the slab is *not* $E_d/2$, which is the average field across the two merging layers of free and bound charges; however, the force on the slab must be calculated by taking the average field on free charges only.

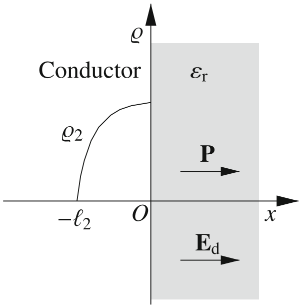

Fig. S-3.10

To illustrate this issue, let us assume for a moment the free charges at the slab surfaces to be distributed in a layer of small but finite width, so that we can localize exactly where free charges are without merging them with bound charges. In particular, let the free surface charge layer have thickness ℓ_2 and volume charge density $\varrho_2(x)$ such that

$$\int_{-\ell_2}^{0} \varrho_2 \,\mathrm{d}x = \sigma_2 \,, \tag{S-3.54}$$

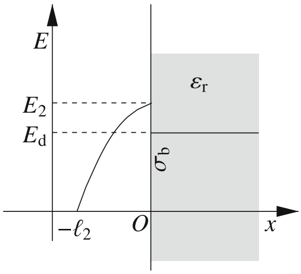

Fig. S-3.11

as shown in Fig. S-3.10. The electric field is still directed along the x axis for symmetry reasons. Gauss's law in one dimension gives $\partial_x E = 4\pi k_e \varrho$. Since $E(-\ell_2) = 0$ (as deep into the conductor the field should vanish) we have for the electric field in the $-\ell_2 \leqslant x \leqslant 0$ region

$$E(x) = 4\pi k_e \int_{-\ell_2}^{x} \varrho_2(x')\,\mathrm{d}x' \,. \tag{S-3.55}$$

The total force *on the free charges only* can thus be evaluated as

$$\begin{aligned} F_2 &= S \int_{-\ell_2}^{0} E(x)\varrho_2(x)\,\mathrm{d}x \\ &= \frac{S}{4\pi k_e} \int_{-\ell_2}^{0} E(x)\,\partial_x E(x)\,\mathrm{d}x = \frac{S}{8\pi k_e} \int_{-\ell_2}^{0} \partial_x E^2(x)\,\mathrm{d}x \\ &= \frac{S}{8\pi k_e} E_2^2 = \frac{S\,(4\pi k_e \sigma_2)^2}{8\pi k_e} = 2\pi k_e\, S\,\sigma_2^2 , \end{aligned} \tag{S-3.56}$$

the electric field at $x = 0^-$ is E_2, as shown in Fig. S-3.11, and the resulting electrostatic pressure is $p_2 = F_2/S = 2\pi k_e\,\sigma_2^2$, independent of the particular distribution $\varrho_2(x)$, and in agreement with the previous result (S-3.52). However, the electric field at $x = 0^+$ is E_d because of the presence of the surface bound charge. [1]

c) If the dielectric medium is actually a slab limited at $x = w$, as shown in Fig. S-3.12, a further bound surface charge density $-\sigma_b$, opposite to the density σ_b at $x = 0$, appears at its $x = w$ surface. This charge distribution is identical to that of a plane capacitor, so that the bound charges generate *no* field outside the dielectric

[1] We might assume that also the polarization charge fills a layer of small, but finite width ℓ_d at the surface of the dielectric. However, this would only imply that the field becomes E_d at $x \geqslant \ell_d$, and would not affect our conclusions on the forces on the conductor.

slab. As trivial consequences the surface charge densities on the conducting slab are $\sigma_1 = -\sigma_2 = Q/2S$, the field inside the dielectric is $E_1/\varepsilon_r = Q/(2\varepsilon_0\varepsilon_r S)$, and there is no force between the slab and the dielectric. Moreover, this result is independent of w, and therefore should be valid also in the limit $w \to \infty$.

The apparent contradiction with the results of points **a)** and **b)** is that, in two attempts to approximate real conditions by objects of infinite size, we are assuming different boundary conditions at infinity. To discuss this issue let us look again at Fig. 3.5, showing the slab of charge Q located in front of a dielectric hemisphere of radius R. At the limit $R \to \infty$, the field in the dielectric half-space approaches the field that we would have if the dielectric medium filled the whole space, and the surface S had surface charge density $\sigma'' = 2\varepsilon_r\sigma/(\varepsilon_r + 1)$, see Problem 3.2. Thus, the field, the polarization, and the polarization surface charge density all approach zero at the hemispherical surface. Part b) of Fig. 3.5 is an enlargement of the area enclosed in the dashed rectangle of part a) of the same figure, and the vanishing charge density on the hemisphere surface does not contribute to the field in this area, according to the result of Problem 1.11. This motivates the boundary condition assumed in points **a)** and **b)**. In contrast, in point **c)** the bound surface charge density does not vanish at infinity and generates a uniform field, which in vacuum cancels out the field generated by the dielectric surface at $x = 0$.

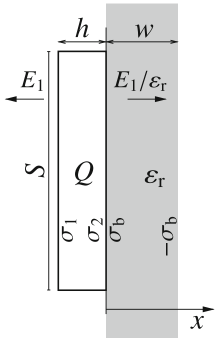

Fig. S-3.12

S-3.7 A Conducting Sphere between two Dielectrics

a) We use a spherical coordinate system (r, θ, ϕ) with the origin O at the center of the sphere, and the zenith direction perpendicular to the plane separating the two dielectric media, as shown in Fig. S-3.13. The electric field inside the conducting sphere is zero. The electric field outside the sphere, $\mathbf{E}(r, \theta, \phi)$, is independent of ϕ because of the symmetry of our problem. Since the sphere is conducting, the electric field $\mathbf{E}(R^+, \theta, \phi)$ must be perpendicular to its surface, and its only nonzero component is E_r. If we write Maxwell's equation $\nabla \times \mathbf{E} = 0$ in spherical coordinates over the spherical surface $r = R^+$ (see Table A.1 of the Appendix), we see that the r and θ components of the curl are automatically zero because $E_\phi = 0$, $E_\theta = 0$, and all derivatives with respect to ϕ are zero. The condition that also the ϕ component of the curl must be zero is

$$\partial_\theta E_r = \partial_r (r E_\theta) = E_\theta + r\partial_r E_\theta \,. \tag{S-3.57}$$

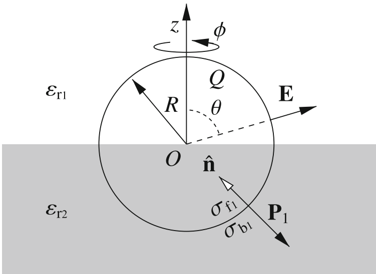

Fig. S-3.13

The right-hand side of (S-3.57) is zero because $E_\theta(R^-) = E_\theta(R^+) = 0$, implying that also $\partial_r E_\theta(R) = 0$. Thus, $\partial_\theta E_r(R^+) = 0$, and $E_r(R^+)$ does not depend on θ (and, consequently, on the dielectric medium). If we denote by $\sigma_{\rm tot}$ the sum of the free surface charge density $\sigma_{\rm f}$ and the bound charge density $\sigma_{\rm b}$, the relation

$$\sigma_{\rm tot} = \frac{E(R^+,\theta,\phi)}{4\pi k_{\rm e}} = \frac{E(R^+)}{4\pi k_{\rm e}} \tag{S-3.58}$$

shows that $\sigma_{\rm tot}$ is constant over the whole surface of the sphere. Thus the electric field in the whole space outside the sphere equals the field of a point charge $Q_{\rm tot}$ located in O, with $Q_{\rm tot} = Q + Q_{\rm b}$, where $Q_{\rm b}$ is the total polarization (bound) charge:

$$\mathbf{E}(r,\theta,\phi) = \mathbf{E}(r) = 4\pi\, k_{\rm e}\, \sigma_{\rm tot}\, \frac{R^2}{r^2}\,\hat{\mathbf{r}}, \quad r > R, \tag{S-3.59}$$

since the field depends on r only. The polarization charge densities on the surfaces of the two dielectrics in contact with the sphere are, respectively,

$$\begin{aligned}\sigma_{\rm b1} &= \hat{\mathbf{n}}\cdot\mathbf{P}_1 = -\frac{\varepsilon_{\rm r1}-1}{4\pi k_{\rm e}}\,E(R) = -(\varepsilon_{\rm r1}-1)\,\sigma_{\rm tot}\\ \sigma_{\rm b2} &= \hat{\mathbf{n}}\cdot\mathbf{P}_2 = -\frac{\varepsilon_{\rm r2}-1}{4\pi k_{\rm e}}\,E(R) = -(\varepsilon_{\rm r2}-1)\,\sigma_{\rm tot}\end{aligned} \tag{S-3.60}$$

where the unit vector $\hat{\mathbf{n}}$ points toward the center of the sphere. The free surface charge densities in the regions in contact with the two dielectrics, $\sigma_{\rm f1}$ and $\sigma_{\rm f2}$, are, respectively,

$$\begin{aligned}\sigma_{\rm f1} &= \sigma_{\rm tot} - \sigma_{\rm b1} = \varepsilon_{\rm r1}\sigma_{\rm tot}\\ \sigma_{\rm f2} &= \sigma_{\rm tot} - \sigma_{\rm b2} = \varepsilon_{\rm r2}\sigma_{\rm tot}\,.\end{aligned} \tag{S-3.61}$$

Since $2\pi R^2(\sigma_{\rm f1}+\sigma_{\rm f2}) = Q$, we finally obtain

$$\begin{aligned}\sigma_{\rm tot} &= \frac{Q}{2\pi R^2(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})} & \mathbf{E}(r) &= 2k_{\rm e}\,\frac{Q}{(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})\,r^2}\,\hat{\mathbf{r}}\\ \sigma_{\rm f1} &= \frac{\varepsilon_{\rm r1}\,Q}{2\pi R^2(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})} & \sigma_{\rm f2} &= \frac{\varepsilon_{\rm r2}\,Q}{2\pi R^2(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})}\\ \sigma_{\rm b1} &= -\frac{(\varepsilon_{\rm r1}-1)\,Q}{2\pi R^2(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})} & \sigma_{\rm b2} &= -\frac{(\varepsilon_{\rm r2}-1)\,Q}{2\pi R^2(\varepsilon_{\rm r1}+\varepsilon_{\rm r2})}\,.\end{aligned} \tag{S-3.62}$$

b) The electrostatic pressures on the two hemispherical surfaces equal the electrostatic energy densities in the corresponding dielectric media, and are, respectively,

$$\mathcal{P}_{\mathrm{f}i} = \frac{2\pi k_\mathrm{e}}{\varepsilon_{\mathrm{r}i}}\,\sigma_{\mathrm{f}i}^2 = \frac{2\pi k_\mathrm{e}}{\varepsilon_{\mathrm{r}i}}\left[\frac{\varepsilon_{\mathrm{r}i}\,Q}{2\pi R^2(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})}\right]^2 = k_\mathrm{e}\,\frac{\varepsilon_{\mathrm{r}i}\,Q^2}{2\pi R^4(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2} \qquad \text{(S-3.63)}$$

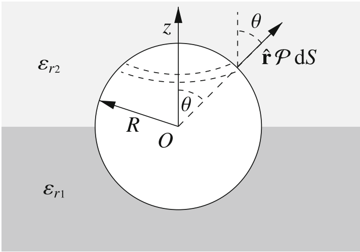

Fig. S-3.14

with $i = 1, 2$. Thus $\mathcal{P}_1 > \mathcal{P}_2$ because $\varepsilon_{\mathrm{r}1} > \varepsilon_{\mathrm{r}2}$, and the pressure pushes the sphere towards the medium of higher permittivity. The force on the sphere surface element $\mathrm{d}S = R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ is $\mathrm{d}\mathbf{F} = \hat{\mathbf{r}}\mathcal{P}_{\mathrm{f}i}\,\mathrm{d}S$, with $i = 1$ if $\theta > \pi/2$, and $i = 2$ if $\theta < \pi/2$. The total force acting on the upper hemisphere ($\theta < \pi/2$) is thus

$$\begin{aligned}\mathbf{F}_2 &= \hat{\mathbf{z}}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\mathrm{d}\theta\,R^2\sin\theta\cos\theta\,k_\mathrm{e}\,\frac{\varepsilon_{\mathrm{r}2}\,Q^2}{2\pi(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2R^4}\\ &= \hat{\mathbf{z}}\pi R^2 k_\mathrm{e}\,\frac{\varepsilon_{\mathrm{r}2}Q^2}{2\pi(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2R^4} = \hat{\mathbf{z}}\pi R^2\,\mathcal{P}_2 = \hat{\mathbf{z}}k_\mathrm{e}\,\frac{\varepsilon_{\mathrm{r}2}Q^2}{2\,(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2R^2}\,,\end{aligned} \qquad \text{(S-3.64)}$$

directed upwards, since the force components perpendicular to the z axis cancel out. Note that $\mathbf{F}_2$ simply equals $\mathcal{P}_{\mathrm{f}2}$ times the section of the sphere πR^2. The total force acting on the lower hemisphere ($\theta > \pi/2$) is, analogously,

$$\mathbf{F}_1 = -\hat{\mathbf{z}}k_\mathrm{e}\,\frac{\varepsilon_{\mathrm{r}1}\,Q^2}{2\,(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2R^2}\,. \qquad \text{(S-3.65)}$$

The total electrostatic force acting on the conducting sphere is thus

$$\mathbf{F}_{\mathrm{tot}} = \mathbf{F}_1 + \mathbf{F}_2 = -\hat{\mathbf{z}}k_\mathrm{e}\,\frac{(\varepsilon_{\mathrm{r}1}-\varepsilon_{\mathrm{r}2})\,Q^2}{2\,(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2\,R^2}\,. \qquad \text{(S-3.66)}$$

If the sphere is at equilibrium when half of its volume is submerged, $\mathbf{F}_{\mathrm{tot}}$ plus the sphere weight must balance Archimedes' buoyant force

$$k_\mathrm{e}\,\frac{(\varepsilon_{\mathrm{r}1}-\varepsilon_{\mathrm{r}2})\,Q^2}{2\,(\varepsilon_{\mathrm{r}1}+\varepsilon_{\mathrm{r}2})^2\,R^2} = g\,\frac{2\pi R^3}{3}\,(\varrho_1+\varrho_2-2\varrho)\,, \qquad \text{(S-3.67)}$$

where g is the gravitational acceleration. Thus, at equilibrium, the electric charge on the sphere must be

$$Q = \sqrt{\frac{4\pi R^5\,(\varepsilon_{r1}+\varepsilon_{r2})^2\,(\varrho_1+\varrho_2-2\varrho)\,g}{3\,k_e\,(\varepsilon_{r1}-\varepsilon_{r2})}}\,. \tag{S-3.68}$$

S-3.8 Measuring the Dielectric Constant of a Liquid

The partially filled capacitor is equivalent to two capacitors connected in parallel, one with vacuum between the plates, and the other filled by the dielectric liquid. The two capacitors have the same internal and external radii, a and b, but different lengths, $\ell - h$ and h, respectively. The total capacitance is

$$C = \frac{\ell - h}{2k_e \ln(b/a)} + \frac{\varepsilon_r h}{2k_e \ln(b/a)} = \frac{\ell + (\varepsilon_r - 1)\,h}{2k_e \ln(b/a)}\,, \tag{S-3.69}$$

and the electrostatic energy of the capacitor is

$$U_{es} = \frac{1}{2}\,CV^2 = \frac{\ell + (\varepsilon_r - 1)\,h}{4k_e \ln(b/a)}\,V^2\,. \tag{S-3.70}$$

If the liquid raises by an amount dh the capacity increases by

$$\mathrm{d}C = \frac{(\varepsilon_r - 1)\,\mathrm{d}h}{2k_e \ln(b/a)}\,, \tag{S-3.71}$$

and, if the potential difference V across the capacitor plates is kept constant, the electrostatic energy of the capacitor increases by an amount

$$\mathrm{d}U_{es} = \frac{1}{2}V^2\mathrm{d}C = \frac{(\varepsilon_r - 1)\,\mathrm{d}h}{4k_e \ln(b/a)}\,V^2\,. \tag{S-3.72}$$

Simultaneously the voltage source does a work

$$\mathrm{d}W = V\mathrm{d}Q = V^2\mathrm{d}C = \frac{(\varepsilon_r - 1)\,\mathrm{d}h}{2k_e \ln(b/a)}\,V^2\,, \tag{S-3.73}$$

because the charge of the capacitor must increase by $\mathrm{d}Q = V\mathrm{d}C$ in order to keep the potential difference across the plates constant, and this implies moving a charge dQ from one plate to the other. The energy of the voltage source changes by

$$\mathrm{d}U_{source} = -\frac{(\varepsilon_r - 1)\,\mathrm{d}h}{2k_e \ln(b/a)}\,V^2 = -2\,\mathrm{d}U_{es}\,. \tag{S-3.74}$$

We must still evaluate the increase in gravitational potential energy of the liquid. When the liquid raises by $\mathrm{d}h$ an infinitesimal annular cylinder of mass $\mathrm{d}m = \varrho\pi(b^2 - a^2)\,\mathrm{d}h$ is added at its top, and the gravitational energy increases by

$$\mathrm{d}U_\mathrm{g} = g\varrho\pi(b^2 - a^2)\,h\,\mathrm{d}h\,. \tag{S-3.75}$$

The *total* energy variation is thus

$$\begin{aligned}\mathrm{d}U_\mathrm{tot} &= \mathrm{d}U_\mathrm{es} + \mathrm{d}U_\mathrm{source} + \mathrm{d}U_\mathrm{g} = -\mathrm{d}U_\mathrm{es} + \mathrm{d}U_\mathrm{g} \\ &= -\frac{(\varepsilon_\mathrm{r} - 1)\,\mathrm{d}h}{4k_\mathrm{e}\ln(b/a)}\,V^2 + g\varrho\pi(b^2 - a^2)\,h\,\mathrm{d}h\,,\end{aligned} \tag{S-3.76}$$

and the total force is

$$F = -\frac{\partial U_\mathrm{tot}}{\partial h} = \frac{\varepsilon_\mathrm{r} - 1}{4k_\mathrm{e}\ln(b/a)}\,V^2 - g\varrho\pi(b^2 - a^2)\,h\,. \tag{S-3.77}$$

At equilibrium we have $F = 0$, which corresponds to

$$h = \frac{(\varepsilon_\mathrm{r} - 1)\,V^2}{4\pi k_\mathrm{e}\,g\varrho(a^2 - b^2)\ln(b/a)} \quad \text{and} \quad \varepsilon_\mathrm{r} = 1 + \frac{4\pi k_\mathrm{e}\,g\varrho(a^2 - b^2)\ln(b/a)}{V^2}\,h\,. \tag{S-3.78}$$

For the electric susceptibility χ, we have in SI units $\chi = \varepsilon_\mathrm{r} - 1$, and

$$\chi = \frac{g\varrho(a^2 - b^2)\ln(b/a)}{\varepsilon_0 V^2}\,h\,, \tag{S-3.79}$$

while in Gaussian units we have $\chi = (\varepsilon_\mathrm{r} - 1)/4\pi$, and

$$\chi = \frac{g\varrho(a^2 - b^2)\ln(b/a)}{V^2}\,h\,. \tag{S-3.80}$$

S-3.9 A Conducting Cylinder in a Dielectric Liquid

a) We choose a cylindrical coordinate system (r, ϕ, z) with the longitudinal axis z superposed to the axis of the conducting cylinder, and the origin O at the height of the boundary surface between the dielectric liquid and the vacuum above it. The azimuthal angle ϕ is irrelevant for the present problem. The electric field $\mathbf{E}(r, \phi, z)$ is perpendicular to the surface of the cylinder, thus we have $\mathbf{E}(r, \phi, z) \equiv [E_r(r, z), 0, 0]$. The field is continuous at the dielectric-vacuum boundary surface, since it is parallel to it. We thus have $E_r(r, z) = E_r(r)$, independently of z. Let us denote by σ_1 and σ_2 the free-charge surface densities on the cylinder lateral surface for $z > 0$ and $z < 0$, respectively. Quantities σ_1 and σ_2 are related to the electric field at the cylinder surface, $E_r(a)$, to ε_r and Q by

$$\sigma_1 = \frac{E_r(a)}{4\pi k_e}, \qquad \sigma_2 = \frac{\varepsilon_r E_r(a)}{4\pi k_e},$$
$$Q = 2\pi a\,[\sigma_1 h + \sigma_2(L-h)]\ . \tag{S-3.81}$$

We thus have

$$E_r(a) = Q\,\frac{2k_e}{a[h+\varepsilon_r(L-h)]} = Q\,\frac{2k_e}{a[\varepsilon_r L-(\varepsilon_r-1)h]}\ . \tag{S-3.82}$$

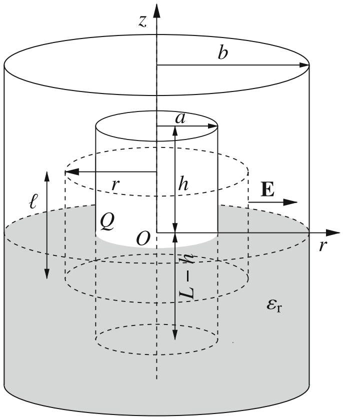

Fig. S-3.15

b) The electric field $E_r(r)$ in the region $a < r < b$ can be evaluated by applying Gauss's law to a closed cylindrical surface of radius r and height $\ell \ll L$, coaxial to the conducting cylinder. Neglecting the boundary effects, the flux of the electric field through the bases of the Gaussian surface is zero, and we have

$$2\pi r\,\ell E_r(r) = 4\pi k_e Q_{int}\,,$$
$$E_r(r) = \frac{2k_e Q_{int}}{r\ell}\,, \tag{S-3.83}$$

where Q_{tot} is the *total* charge inside the Gaussian surface, including both free and polarization charges. If we let r approach a keeping ℓ constant, Q_{int} remains constant and we have

$$\lim_{r\to a} E_r = \frac{2k_e Q_{int}}{a\ell} = E_r(a), \quad \text{so that} \quad E_r(r) = E_r(a)\,\frac{a}{r} \tag{S-3.84}$$

and, inserting (S-3.82),

$$E_r(r) = \frac{2k_e Q}{r[\varepsilon_r L-(\varepsilon_r-1)h]}\ . \tag{S-3.85}$$

c) The electrostatic energy of the system is

$$\begin{aligned}U_{\rm es} &\simeq \frac{1}{8\pi k_{\rm e}}\left[\varepsilon_{\rm r}\int_0^{L-h}{\rm d}z\int_a^b E_r^2(r)\,2\pi r\,{\rm d}r+\int_{L-h}^{L}{\rm d}z\int_a^b E_r^2(r)\,2\pi r\,{\rm d}r\right]\\ &= \frac{1}{8\pi k_{\rm e}}\left[\frac{2k_{\rm e}Q}{[\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h]}\right]^2\left\{\left[\varepsilon_{\rm r}(L-h)+h\right]\int_a^b\frac{2\pi}{r}\,{\rm d}r\right\}\\ &= k_{\rm e}\left[\frac{Q}{[\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h]}\right]^2\left[\varepsilon_{\rm r}(L-h)+h\right]\ln\left(\frac{b}{a}\right)\\ &= k_{\rm e}\,\frac{Q^2\,\ln(b/a)}{\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h}\,,\end{aligned} \tag{S-3.86}$$

i.e., the electrostatic energy of two cylindrical capacitors connected in parallel, with total charge Q. Both capacitors have internal radius a and external radius b, one has length $L-h$ and is filled with the dielectric material, the other has length h and vacuum between the plates. The electrostatic force, directed along z, is

$$F_{\rm es} = -\frac{{\rm d}U_{\rm es}}{{\rm d}h} = -k_{\rm e}\,\frac{(\varepsilon_{\rm r}-1)\ln(b/a)\,Q^2}{[\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h]^2} < 0\,. \tag{S-3.87}$$

The electrostatic forces tends to decrease h, i.e., to sink the cylinder into the liquid.
d) The sum of the gravitational and buoyant (due to Archimedes' principle) forces on the cylinder is

$$F_{\rm g} = -Mg+\varrho g(L-h)\,\pi a^2\,, \tag{S-3.88}$$

and the cylinder is in equilibrium when $F_{\rm es}+F_{\rm g}=0$, i.e., when

$$\varrho g(L-h)\,\pi a^2 - Mg = k_{\rm e}\,\frac{(\varepsilon_{\rm r}-1)\ln(b/a)\,Q^2}{[\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h]^2}\,. \tag{S-3.89}$$

Given L, h and $\varepsilon_{\rm r}$, we have equilibrium for

$$Q = [\varepsilon_{\rm r}L-(\varepsilon_{\rm r}-1)\,h]\sqrt{\frac{\varrho g(L-h)\,\pi a^2 - Mg}{k_{\rm e}(\varepsilon_{\rm r}-1)\ln(b/a)}}\,. \tag{S-3.90}$$

S-3.10 A Dielectric Slab in Contact with a Charged Conductor

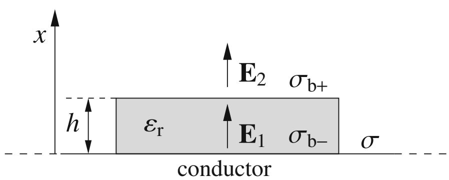

Fig. S-3.16

a) Within our approximations, the electric fields are perpendicular to the conducting surface. We choose a Cartesian reference frame with the origin on the conductor surface and the x axis perpendicular the surface, as in Fig. S-3.16, so that the only nonzero component of the electric fields is their x component. We denote by E_1 the electric field field inside the dielectric slab, and by E_2 the electric field in vacuum, while the field will is zero inside the conductor.

The fields E_1 and E_2 can be evaluated by applying equation $\nabla \cdot (\varepsilon_r \mathbf{E}) = 4\pi k_e \varrho_f$ to two Gaussian "pillboxes", crossing the $x = 0$ and the $x = h$ surfaces, respectively, as in Fig. S-3.17. We see that $\varepsilon_r E$ is discontinuous at $x = 0$ surface, and continuous at $x = h$:

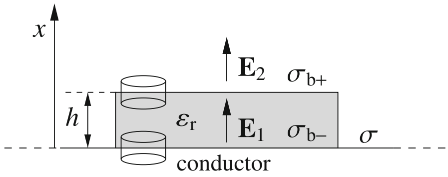

Fig. S-3.17

$$\begin{aligned} \varepsilon_r E_1 &= 4\pi k_e \sigma\,, \\ E_2 &= \varepsilon_r E_1\,, \end{aligned} \qquad \text{(S-3.91)}$$

which lead to

$$E_1 = \frac{4\pi k_e}{\varepsilon_r}\sigma\,, \quad E_2 = 4\pi k_e \sigma\,. \qquad \text{(S-3.92)}$$

b) We denote by σ_{b-} and σ_{b+} the surface polarization charge densities at $x = 0$ and $x = h$, respectively. These quantities can be calculated by applying Gauss's law $\nabla \cdot \mathbf{E} = 4\pi k_e(\varrho_f + \varrho_b)$ to the two "pillboxes" of Fig. S-3.17, and obtaining

$$E_1 = 4\pi k_e(\sigma + \sigma_{b-})\,, \quad E_2 - E_1 = 4\pi k_e \sigma_{b+}\,, \qquad \text{(S-3.93)}$$

introducing (S-3.91) into (S-3.93) we finally have

$$\sigma_{b+} = -\sigma_{b-} = \left(1 - \frac{1}{\varepsilon_r}\right)\sigma = \frac{\varepsilon_r - 1}{\varepsilon_r}\sigma\,. \qquad \text{(S-3.94)}$$

c) In the vacuum region between the conductor and the dielectric slab the field is $E = 4\pi k_e \sigma = E_2$, independent of s. The electric field inside the dielectric slab, and above the slab, are E_1 and E_2, respectively, as in the case of $s = 0$, thus independently of s.

The net electrostatic force on the slab is zero, independently of s, since the forces on the upper and lower surfaces of the slab are exactly opposite. Further, if we evaluate the electrostatic energy of the system as the volume integral of $\varepsilon_r E^2/(8\pi k_e)$, we see that also this quantity is independent of s, within our approximations.

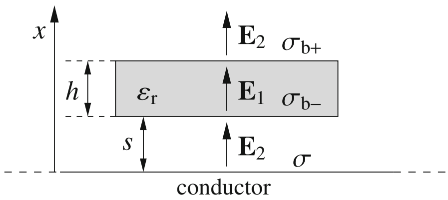

Fig. S-3.18

S-3.11 A Transversally Polarized Cylinder

We choose a cylindrical coordinate system (r,ϕ,z), with the cylinder axis as z axis, and the reference plane (the plane from which the angle ϕ is measured) parallel to $\mathbf{P}$. We have translational symmetry along z, so that, mathematically, the problem is two-dimensional. The surface charge polarization density of the cylinder is $\sigma(\phi) = \mathbf{P}\cdot\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outgoing unit vector perpendicular to the cylinder surface, thus

$$\sigma(\phi) = P\cos\phi\,. \qquad \text{(S-3.95)}$$

Similarly to Problem 1.1, our transversally polarized cylinder can be considered as the limit for $h \to 0$ and $\varrho \to \infty$ of two partially overlapping cylinders, of volume charge density $\pm\varrho$, respectively. The two cylinder axes are the straight lines $x = \pm h/2$, both out of paper in Fig. S-3.19. The product ϱh is constant, and equals the polarization P of the original cylinder. The electrostatic potential $\varphi^{\text{ext}}_{\pm}(A)$, generated by each charged cylinder at an external point $A \equiv (r,\phi,z)$, equals the potential of an infinite line charge of linear charge density $\lambda_\pm = \pm\pi a^2\varrho$, located on the cylinder axis,

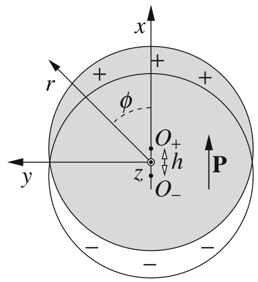

Fig. S-3.19

$$\varphi^{\text{ext}}_{\pm}(A) = \mp 2k_e\pi a^2\varrho \ln\left(\frac{r_\pm}{R_\pm}\right), \qquad \text{(S-3.96)}$$

where

$$r_\pm \simeq r \mp \frac{h}{2}\cos\phi \qquad \text{(S-3.97)}$$

are the distances of A from the axes of the two cylinders, see Fig. S-3.20. Quantities $R_\pm$ are two arbitrary constants, such that $\varphi^{\text{ext}}_{\pm}(r_\pm,\phi,z) = 0$ on the cylindrical surfaces $r_\pm = R_\pm$. It is convenient to choose $R_+ = R_-$, so that $\ln(R_+/R_-) = 0$ will cancel out in the following computations, leaving the potential equal to zero at $r = \infty$. The electrostatic potential generated by both cylinders is thus

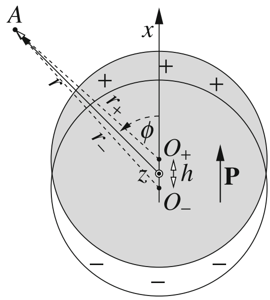

Fig. S-3.20

$$\begin{aligned}\varphi^{\rm ext}(A) &= \varphi_+^{\rm ext}(A)+\varphi_-^{\rm ext}(A) = 2k_{\rm e}\pi a^2\varrho\ln\left(\frac{r_-}{r_+}\right)+2k_{\rm e}\pi a^2\varrho\ln\left(\frac{R_+}{R_-}\right)\\ &\simeq 2k_{\rm e}\pi a^2\varrho\ln\left[\frac{r+(h/2)\cos\phi}{r-(h/2)\cos\phi}\right] = 2k_{\rm e}\pi a^2\varrho\ln\left[\frac{1+(h/2r)\cos\phi}{1-(h/2r)\cos\phi}\right]\\ &= 2k_{\rm e}\pi a^2\varrho\left[\ln\left(1+\frac{h}{2r}\cos\phi\right)-\ln\left(1-\frac{h}{2r}\cos\phi\right)\right]\\ &\simeq 2k_{\rm e}\pi a^2\varrho\frac{h}{r}\cos\phi = 2k_{\rm e}\pi a^2\,\frac{P\cos\phi}{r} = 2k_{\rm e}\pi a^2\,\frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r}\,, \end{aligned} \tag{S-3.98}$$

where $\hat{\mathbf{r}}$ is the unit vector of the cylindrical coordinate r. Thus, the potential of our *two-dimensional electric dipole* decreases as r^{-1}, while the potential of the ordinary electric dipole decreases as r^{-2}. In Cartesian coordinates we have

$$\varphi^{\rm ext}(x,y,z) = 2k_{\rm e}\pi a^2\,\frac{Px}{x^2+y^2}\,, \tag{S-3.99}$$

where the x and y axes are the ones shown in Fig. S-3.19.

The external electric field is obtained by evaluating $\mathbf{E}^{\rm ext} = -\nabla\varphi^{\rm ext}$. The cylindrical components are, from Table A.1 of the Appendix,

$$\begin{aligned} E_r^{\rm ext} &= -\partial_r\varphi^{\rm ext} = 2k_{\rm e}\pi a^2\,\frac{P\cos\phi}{r^2}\,,\\ E_\phi^{\rm ext} &= -\frac{1}{r}\partial_\phi\varphi^{\rm ext} = 2k_{\rm e}\pi a^2\,\frac{P\sin\phi}{r^2}\,,\\ E_z^{\rm ext} &= -\partial_z\varphi^{\rm ext} = 0\,, \end{aligned} \tag{S-3.100}$$

the field decreases proportionally to r^{-2}, while the field of the usual electric dipole decreases as r^{-1}. The Cartesian components of the field are

$$\begin{aligned} E_x^{\rm ext} &= -\partial_x\varphi^{\rm ext} = 2k_{\rm e}\pi a^2 P\,\frac{x^2-y^2}{(x^2+y^2)^2}\,,\\ E_y^{\rm ext} &= -\partial_y\varphi^{\rm ext} = 2k_{\rm e}\pi a^2 P\,\frac{2xy}{(x^2+y^2)^2}\,,\\ E_z^{\rm ext} &= -\partial_z\varphi^{\rm ext} = 0\,. \end{aligned} \tag{S-3.101}$$

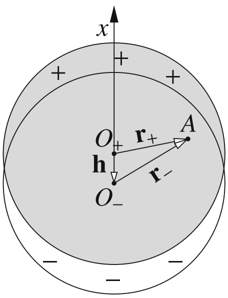

Fig. S-3.21

The electric field generated by each cylinder at its interior is, according to Gauss's law, $\mathbf{E}_\pm^{\rm int} = \pm 2\pi k_{\rm e}\varrho\,\mathbf{r}_\pm$, where $\mathbf{r}_\pm$ is the distance from the respective axis, see Fig. S-3.21. The two contributions sum up to a uniform internal field

$$\mathbf{E}^{\rm int}(A) = 2\pi k_{\rm e}\varrho\,(\mathbf{r}_+-\mathbf{r}_-) = -2\pi k_{\rm e}\varrho\,h\,\hat{\mathbf{x}} = -2\pi k_{\rm e}\,\mathbf{P}\,. \tag{S-3.102}$$

The electrostatic potential inside the cylinder, in Cartesian and cylindrical coordinates, is thus

$$\varphi^{\text{int}} = 2\pi k_{\text{e}}\, x + C = 2\pi k_{\text{e}}\, r\cos\phi + C\ , \qquad \text{(S-3.103)}$$

where C is an arbitrary constant. Since the potential most be continuous, we must have, in cylindrical coordinates,

$$\varphi^{\text{int}}(a,\phi,z) = \varphi^{\text{ext}}(a,\phi,z)\ , \qquad \text{(S-3.104)}$$

which is verified if we choose $C = 0$.