Chapter S-4
Solutions for Chapter 4

S-4.1 The Tolman-Stewart Experiment

a) The equation of motion for the "free" (conduction) electrons in a metal is, according to the Drude model,

$$m\frac{\mathrm{d}\langle\mathbf{v}\rangle}{\mathrm{d}t} = \mathbf{F} - m\eta\,\langle\mathbf{v}\rangle\,, \tag{S-4.1}$$

where $\langle\mathbf{v}\rangle$ is the "average" electron velocity, $\mathbf{F}$ is the external force on the electrons, and $m\eta\langle\mathbf{v}\rangle$ is a phenomenological friction force. In a steady state ($\mathrm{d}\langle\mathbf{v}\rangle/\mathrm{d}t = 0$) in the presence of an external electric field $\mathbf{E}$, so that $\mathbf{F} = -e\mathbf{E}$, the electrons have a constant average velocity

$$\langle\mathbf{v}\rangle = -\frac{e}{m\eta}\mathbf{E}\,. \tag{S-4.2}$$

The current density is $\mathbf{J} = -e\,n_\mathrm{e}\langle\mathbf{v}\rangle$, where n_e is the volume density of free electrons. From this we obtain the microscopic form of Ohm's law

$$\mathbf{J} = \frac{n_\mathrm{e}e^2}{m\eta}\mathbf{E} \equiv \sigma\mathbf{E}\,. \tag{S-4.3}$$

The value of the damping frequency η for copper is

$$\eta = \frac{n_\mathrm{e}e^2}{m\sigma} = \frac{8.5\times10^{28}(1.6\times10^{-19})^2}{9.1\times10^{-31}\times10^7} \simeq 2.4\times10^{14}\ \mathrm{s}^{-1}\,, \tag{S-4.4}$$

($m = m_e = 9.1\times10^{-31}$ kg).

At $t = 0$ the electron tangential velocity is $\mathbf{v}_0 = a\omega$. For $t > 0$, due to the absence of external forces the solution of Eq. (S-4.1) is

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$$v = v_0\, e^{-\eta t}\,. \tag{S-4.5}$$

The total current is thus given by

$$I = I_0\, e^{-\eta t}\,, \qquad I_0 = -(e n_{\rm e} v_0)\, S\,, \tag{S-4.6}$$

and the decay time is $\tau = 1/\eta \simeq 4\times 10^{-15}$ s.

b) The total charge flown in the ring is

$$Q = \int_0^\infty I(t)\,{\rm d}t = \frac{I_0}{\eta} = -\frac{m}{e}\sigma S\, v_0\,. \tag{S-4.7}$$

Thus, measuring σ, S, v_0 and Q the value of e/m can be obtained. In the original experiment, Tolman and Stewart were able to measure Q using a ballistic galvanometer in a circuit coupled with a rotating coil.

S-4.2 Charge Relaxation in a Conducting Sphere

a) For symmetry reasons the electric field is radial, and it is convenient to use a spherical coordinate system (r,θ,ϕ) with the origin located at the center of the sphere. Coordinates θ and ϕ are irrelevant for this problem. Let us denote by $q(r,t)$ the electric charge contained inside the sphere $r < a$, at time $t \geqslant 0$. If we apply Gauss's law to the surface of our sphere we obtain

$$E(r,t) = k_{\rm e}\,\frac{q(r,t)}{r^2}. \tag{S-4.8}$$

According to the continuity equation, the flux of the current density $\mathbf{J} = \sigma\mathbf{E}$ through our spherical surface equals the time derivative of $q(r,t)$:

$$\oint \mathbf{J}\cdot{\rm d}\mathbf{S} = 4\pi r^2 J(r,t) = 4\pi r^2 \sigma E(r,t) = -\partial_t q(r,t). \tag{S-4.9}$$

By substituting (S-4.8) into (S-4.9) we obtain

$$\partial_t q(r,t) = -4\pi k_{\rm e}\,\sigma\, q(r,t), \tag{S-4.10}$$

with solution

$$q(r,t) = q(r,0)\,{\rm e}^{-t/\tau}, \quad \text{where} \quad \tau = \frac{1}{4\pi k_{\rm e}\,\sigma}. \tag{S-4.11}$$

Since at $t = 0$ the charge density $\varrho(r,t)$ is uniform all over the volume of the sphere of radius a, we have

$$\varrho(r,0) = \varrho_0 = Q\,\frac{3}{4\pi a^3}\,, \quad r < a\,, \quad \text{so that} \quad q(r,0) = Q\,\frac{r^3}{a^3}\,. \tag{S-4.12}$$

Thus, according to (S-4.11), the density $\varrho(r,t)$ remains uniform over the sphere volume (independent of r) at any time $t > 0$

$$\varrho(r,t) = \varrho(t) = \varrho_0\,\mathrm{e}^{-t/\tau}\,. \tag{S-4.13}$$

The *surface* charge density $q_\mathrm{s}(t)$ (we have already used the Greek letter σ for the conductivity) can also be evaluated from the continuity equation, since

$$\partial_t q_\mathrm{s}(t) = +J(a,t) = \sigma E(a,t) = k_\mathrm{e}\,\sigma\frac{Q}{a^2}\,\mathrm{e}^{-t/\tau} = \frac{Q}{4\pi a^2\tau}\,\mathrm{e}^{-t/\tau}\,, \tag{S-4.14}$$

so that, asymptotically,

$$q_\mathrm{s}(\infty) = \int_0^\infty \partial_t q_\mathrm{s}\,\mathrm{d}t = \frac{Q}{4\pi a^2\tau}\int_0^\infty \mathrm{e}^{-t/\tau}\,\mathrm{d}t = \frac{Q}{4\pi a^2}. \tag{S-4.15}$$

The equation for the time evolution of the electric field inside the sphere ($r < a$) is

$$E(r,t) = k_\mathrm{e}\frac{q(r,t)}{r^2} = k_\mathrm{e}\,Q\,\frac{r}{a^3}\,\mathrm{e}^{-t/\tau}\,, \quad r < a\,, \tag{S-4.16}$$

while the electric field is independent of time outside the sphere

$$E(r,t) = E(r) = k_\mathrm{e}\,\frac{Q}{r^2}\,, \quad r > a\,. \tag{S-4.17}$$

The time constant $\tau = 1/(4\pi k_\mathrm{e}\sigma)$ is extremely short in a good conductor. For copper we have (in SI units) $\sigma \simeq 6\times 10^7\ \Omega^{-1}\mathrm{m}^{-1}$ at room temperature, thus

$$\tau = \frac{\varepsilon_0}{\sigma} \simeq \frac{8.854\times 10^{-12}}{6\times 10^7}\ \mathrm{s} \sim 1.4\times 10^{-19}\ \mathrm{s} = 0.14\ \mathrm{as} \tag{S-4.18}$$

(1 as = 1 *attosecond* = 10^{-18} s: *atten* means *eighteen* in Danish). This extremely short value should be not surprising, since there is no need for the electrons to travel distances even of the order of the atomic spacing within the relaxation time; a very small collective displacement of the electrons is sufficient to reach a condition of mechanical equilibrium (see also Problem 2.1).

b) We can easily evaluate the variation of electrostatic energy ΔU_es during the charge relaxation by noticing that the electric field $E(r,t)$ is constant outside the conducting sphere ($r > a$). The electric field inside the sphere decays from the initial profile $E(r,0) = (Q/4\pi\varepsilon_\theta)(r/a^3)$ to $E(r,\infty) = 0$. Thus, using the "energy density" $u_{es} = E^2/(8\pi k_e)$ we can write

$$\begin{aligned}\Delta U_{es} &= -\frac{1}{8\pi k_e}\int_{\text{sphere}} E^2(r,0)\,\mathrm{d}^3x = -\frac{1}{8\pi k_e}\left(\frac{k_e^2 Q^2}{a^3}\right)\int_0^a r^2\,4\pi r^2\,\mathrm{d}r \\ &= -\frac{k_e}{10}\frac{Q^2}{a}. \end{aligned} \tag{S-4.19}$$

c) The time derivative of the electrostatic energy can be written as

$$\begin{aligned}\partial_t U_{es} &= \frac{1}{8\pi k_e}\partial_t\int_0^\infty E^2(r,t)\,4\pi r^2\,\mathrm{d}r = \frac{1}{8\pi k_e}\int_0^a \partial_t E^2(r,t)\,4\pi r^2\mathrm{d}r \\ &= \frac{1}{8\pi k_e}\int_0^a\left(-\frac{2}{\tau}\right)E^2(r,t)\,4\pi r^2\mathrm{d}r = -\frac{1}{4\pi k_e\tau}\int_0^a k_e^2 Q^2\frac{r^2}{a^6}\,\mathrm{e}^{-2t/\tau}4\pi r^2\mathrm{d}r \\ &= -\frac{k_e}{5\tau}\frac{Q^2}{a}\,\mathrm{e}^{-2t/\tau} = -\frac{4\pi k_e^2\sigma Q^2}{5a}\,\mathrm{e}^{-2t/\tau}, \end{aligned} \tag{S-4.20}$$

where we used (S-4.16) and (S-4.17). The power loss due to Joule heating is

$$\begin{aligned}P_d &= \int_0^\infty \mathbf{J}\cdot\mathbf{E}\,4\pi r^2\mathrm{d}r = \int_0^a \sigma E^2(r,t)\,4\pi r^2\mathrm{d}r \\ &= \frac{4\pi k_e^2\sigma Q^2}{5a}\,\mathrm{e}^{-2t/\tau}, \end{aligned} \tag{S-4.21}$$

since $\mathbf{J} = \sigma\mathbf{E}$ for $r < a$, and $\mathbf{J} = 0$ for $r > a$. Thus $P_d = -\partial_t U_{es}$, and all the electrostatic energy lost by the sphere during the relaxation process is turned into Joule heat.

S-4.3 A Coaxial Resistor

a) We use a cylindrical coordinate system (r,ϕ,z), with the z axis coinciding with the common axis of the cylindrical plates. The material between the plates can be considered as a series of infinitesimal cylindrical-shell resistors, each of internal and external radii r and $r+\mathrm{d}r$ and of height h. The resistance of the cylindrical shell between r and $r+\mathrm{d}r$ is

$$\mathrm{d}R = \rho\frac{\mathrm{d}r}{S(r)} = \rho\frac{\mathrm{d}r}{2\pi r h}, \tag{S-4.22}$$

since $\mathrm{d}r$ is the "length" of our resistor, and $S(r) = 2\pi r$ its "cross-sectional area". The resistance of the material is thus

$$R = \frac{\rho}{2\pi h}\int_a^b\frac{\mathrm{d}r}{r} = \rho\frac{\ln(b/a)}{2\pi h}. \tag{S-4.23}$$

b) The capacity of a cylindrical capacitor of radii a and b, and length h, is

$$C_0 = \frac{h}{2k_e \ln(b/a)}, \tag{S-4.24}$$

assuming that the space between the plates is filled with vacuum. Thus, in our case we have

$$R = \frac{\rho}{4\pi k_e C_0}. \tag{S-4.25}$$

Equation (S-4.25) is actually of much more general validity, and is a very good approximation for evaluating the resistance between two electrodes of high conductivity and calculable capacity immersed in a medium of known resistivity. As an example, consider two highly-conducting square plates immersed in an ohmic medium, and connected to a voltage source by insulated cables, as in Fig. S-4.1. The current that flows, for instance, from the left plate, can be written

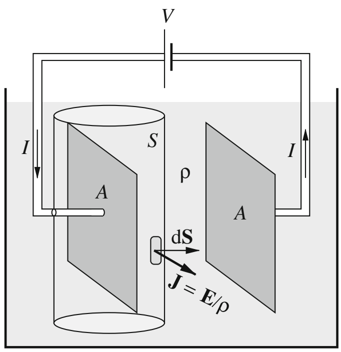

Fig. S-4.1

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \frac{1}{\rho}\int \mathbf{E}\cdot d\mathbf{S}, \tag{S-4.26}$$

where the flux is calculated through a surface enclosing the electrode, except for the area through which the current enters it, like the cylindrical closed surface of Fig. S-4.1. In most cases the contribution of the excluded area to the flux of **E** is negligible in an electrostatic problem, while, according to Gauss's law, we have

$$\oint \mathbf{E}\cdot d\mathbf{S} = 4\pi k_e Q \tag{S-4.27}$$

where Q is the charge on the electrode that would produce the field **E**. Within the approximation of considering the last integral of (S-4.26) as equal to the integral through the whole closed surface, we have

$$I = \frac{4\pi k_e}{\rho} Q. \tag{S-4.28}$$

On the other hand, if we consider the two electrodes as the plates of a capacitor of capacitance C_0 we have

$$Q = C_0 V, \tag{S-4.29}$$

where V is the potential difference between them. We thus have

$$I = \frac{4\pi k_e}{\rho} C_0 V = \frac{V}{R}, \quad \text{(S-4.30)}$$

from which (S-4.25) follows.

S-4.4 Electrical Resistance between two Submerged Spheres (1)

a) We start by evaluating the capacitance C_0 of the two spheres in vacuum, with the same geometry of the problem. Let the sphere of radius a carry a charge Q, and the sphere of radius b a charge $-Q$. With our assumptions $a \ll x$ and $b \ll x$ the electric potentials φ_a and φ_b of the two spheres are given approximately by

$$\varphi_a \simeq k_e Q \left(\frac{1}{a} - \frac{1}{x}\right) \quad \text{and} \quad \varphi_b \simeq k_e Q \left(-\frac{1}{b} + \frac{1}{x}\right), \quad \text{(S-4.31)}$$

where we have assumed the potential φ to be zero at infinity, and have neglected the induction effects between the two spheres, discussed in Problem 2.6. The capacitance of the two spheres can thus be approximated as

$$C_0 = \frac{Q}{\varphi_a - \varphi_b} \simeq \frac{1}{k_e}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right)^{-1}, \quad \text{(S-4.32)}$$

and, according to (S-4.25), the resistance between them is

$$R = \frac{\rho}{4\pi k_e C_0} \simeq \frac{\rho}{4\pi}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right), \quad \text{(S-4.33)}$$

which can be further approximated to

$$R \simeq \frac{\rho}{4\pi}\left(\frac{1}{a} + \frac{1}{b}\right), \quad \text{(S-4.34)}$$

independent of the distance between x between the centers of the spheres.

b) In this case the resistance between the spheres is twice the value found at point **a)**, since at point **a)** we can introduce a horizontal plane passing through the centers of the spheres, which divides the fluid into two equivalent halves, each of resistance $2R$, so that, in parallel, they are equivalent to a resistance R. In the present case the upper half is replaced by vacuum, so that only the resistance $2R$ of the lower half remains. This problem is of interest in connection with electrical circuits that use the ground as a return path. In this case ρ is the resistivity of the earth (of course, the assumption that ρ is uniform is a very rough approximation). In practical applications, the resistivity of the earth in the neighborhood of the electrodes can be decreased by moistening the ground around them.

S-4.5 Electrical Resistance between two Submerged Spheres (2)

a) According to (S-4.32) of Problem 4.4, and remembering that now the medium has a relative dielectric permittivity ε_r, the charge of each sphere is

$$Q \simeq \varepsilon_\mathrm{r} C_0 V = \frac{\varepsilon_\mathrm{r}}{k_\mathrm{e}}\left(\frac{2}{a}-\frac{2}{\ell}\right)^{-1} V = \frac{\varepsilon_\mathrm{r}}{k_\mathrm{e}}\,\frac{\ell}{\ell-a}\,\frac{a}{2}\,V$$
$$\simeq \frac{\varepsilon_\mathrm{r}\,a}{2k_\mathrm{e}}\left(1+\frac{a}{\ell}\right)V \simeq \frac{\varepsilon_\mathrm{r}\,a}{2k_\mathrm{e}}\,V\,, \qquad \text{(S-4.35)}$$

where the last two terms are the first and the zeroth order approximations in a/ℓ.

b) According to (S-4.33) and (S-4.34) we have

$$R \simeq \frac{\rho}{4\pi}\left(\frac{2}{a}-\frac{2}{\ell}\right) \simeq \frac{\rho}{2\pi a}\,, \qquad \text{(S-4.36)}$$

again to the zeroth order in a/ℓ. The current I is thus

$$I = \frac{V}{R} = \frac{2\pi\,a}{\rho}\,V\,. \qquad \text{(S-4.37)}$$

This result can be checked by introducing a cylindrical coordinate system (r,ϕ,z) with the z axis through the centers of the two spheres and the origin O so that the sphere centers are at $(0,\phi,-\ell/2)$ and $(0,\phi,+\ell/2)$, respectively, and evaluating the flux of the current density $\mathbf{J}$ through the plane $z=0$

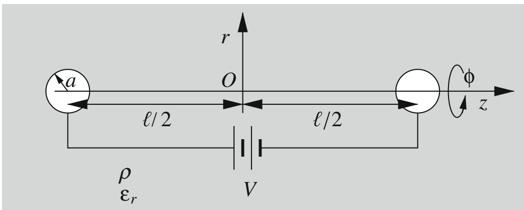

Fig. S-4.2

$$I = \int \mathbf{J}\cdot \mathrm{d}\mathbf{S}$$
$$= \frac{1}{\rho}\int_0^\infty E_z(r)\,2\pi r\,\mathrm{d}r\,,$$

where

$$E_z(r) = \frac{2k_\mathrm{e}Q}{\varepsilon_\mathrm{r}}\,\frac{\ell/2}{[(\ell/2)^2+r^2]^{3/2}}\,, \qquad \text{(S-4.38)}$$

so that

$$I = \frac{k_\mathrm{e}Q}{\rho\varepsilon_\mathrm{r}}\int_0^\infty \frac{\ell r}{[(\ell/2)^2+r^2]^{3/2}}\,\mathrm{d}r = \frac{4\pi k_\mathrm{e}}{\rho\varepsilon_\mathrm{r}}\,Q = \frac{2\pi a}{\rho}\,V. \qquad \text{(S-4.39)}$$

c) Having a system equivalent to a capacitor in parallel with a resistor, we expect an exponential decay of the charge Q, with a time constant $\tau = RC$. The time constant is independent of the geometry of the problem because the capacitance C of the system is $\varepsilon_r C_0$, where C_0 is the capacitance when the medium is replaced by vacuum, while, according to (S-4.25), the resistance is $R = \rho/(4\pi k_e C_0)$, so that

$$\tau = RC = \frac{\varepsilon_r \rho}{4\pi k_e}. \tag{S-4.40}$$

This relations holds for any "leaky capacitor", if the discharge occurs only through leakage. In the present case we obtain from the continuity equation

$$\frac{dQ}{dt} = -I = -\frac{4\pi k_e}{\varepsilon_r \rho} Q, \qquad Q(t) = Q(0)\,e^{-t/\tau}, \qquad \tau = \frac{\varepsilon_r \rho}{4\pi k_e}. \tag{S-4.41}$$

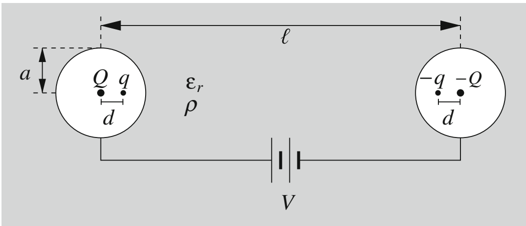

Fig. S-4.3

d) To the first order in a/ℓ the electrostatic induction effects can be described by regarding the electric field outside the two spheres as due to two charges $\pm Q$ located at the centers of the spheres, and two charges $\pm q = \pm(a/\ell)Q$ located at distances $d = a^2/\ell$ from the centers, on the line connecting the two centers, each toward the other sphere, as in the figure.

Thus, to the first order, the potential of each sphere is $\simeq \pm k_e Q/(\varepsilon_r a)$, since the contribution of the charge $\mp Q$ on the other sphere is canceled by the image charge $\pm q$ present in the sphere. We have $Q \simeq \varepsilon_r a V/(2k_e)$, while the absolute value of the total charge on each sphere is $Q + q = Q(1 + a/\ell)$. The capacitance of the system is thus

$$C = \frac{Q}{V} = \frac{\varepsilon_r a}{2k_e}\left(1 + \frac{a}{\ell}\right). \tag{S-4.42}$$

The same result is obtained from (S-4.32) of Problem 4.4

$$C = \varepsilon_r C_0 = \frac{\varepsilon_r}{k_e}\left(\frac{2}{a} - \frac{2}{\ell}\right)^{-1} = \frac{\varepsilon_r a}{2k_e}\,\frac{\ell}{\ell - a} \simeq \frac{\varepsilon_r a}{2k_e}\left(1 + \frac{a}{\ell}\right), \tag{S-4.43}$$

where the image charges have been disregarded, but the effect of the charge on each sphere on the potential of the other has been taken into account.

According to (S-4.33) the resistance now becomes

$$R = \frac{\rho}{4\pi k_e C_0} = \frac{\rho}{2\pi a}\frac{\ell - a}{a}\,, \qquad \text{(S-4.44)}$$

so that the time constant $\tau = RC = \varepsilon_r \rho/(4\pi k_e)$ is unchanged.

S-4.6 Effects of non-uniform resistivity

a) We use a cylindrical coordinate system (r, ϕ, z), with the z axis along the common axis of the two cylinders, and the origin O on the surface separating the two cylinders as in Fig. S-4.4. We denote the volume charge density by q_v, since the Greek letter ρ is already used to denote the resistivities. In a steady state we must have $\partial_t q_v = 0$ everywhere, otherwise the volume charge density would increase, or decrease, indefinitely. Thus, according to the continuity equation, we have also

$$\nabla\cdot\mathbf{J} = -\partial_t q_v = 0\,. \qquad \text{(S-4.45)}$$

On the other hand, from $\nabla\cdot\mathbf{E} = 4\pi k_e\, q_v$ and $\mathbf{J} = \mathbf{E}/\rho$ we obtain

$$0 = \nabla\cdot\mathbf{J} = \frac{1}{\rho}\nabla\cdot\mathbf{E} = \frac{4\pi k_e}{\rho}\, q_v\,, \qquad \text{(S-4.46)}$$

showing that also the volume charge density $q_\mathbf{v}$ must be zero everywhere inside a conductor in stationary conditions. This does not exclude the presence of surface charge densities on the surfaces delimiting a conductor.

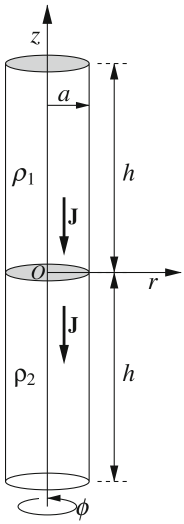

Fig. S-4.4

If we assume that $h \gg a$, it follows from $\nabla\cdot\mathbf{J} = 0$ and $\nabla\times\mathbf{E} = 0$ that $\mathbf{J}$ is uniform inside the cylinders, pointing downwards along the z direction. Since $\mathbf{E}$ and $\mathbf{J}$ are proportional to each other inside each cylinder, it follows that also $\mathbf{E}$ is uniform inside each cylinder. The current density $\mathbf{J}$ must be continuous through the surface separating the two cylinders, otherwise charge would accumulate indefinitely on the surface. Thus, $\mathbf{J}$ is uniform throughout the whole conductor, and the current is $I = J\pi a^2$.

The resistances $R_{1,2}$ of the two cylinders are, respectively,

$$R_{1,2} = \rho_{1,2}\,\frac{h}{\pi a^2}\,, \tag{S-4.47}$$

leading to a total resistance R of the system

$$R = R_1 + R_2 = (\rho_1 + \rho_2)\,\frac{h}{\pi a^2}\,. \tag{S-4.48}$$

The current, and the current density, flowing in the system are

$$I = \frac{V}{R} = \frac{\pi a^2 V}{h(\rho_1 + \rho_2)}\,, \qquad J = \frac{V}{h(\rho_1 + \rho_2)}\,. \tag{S-4.49}$$

Since we have the same current density in two conductors of different resistivities, and $\mathbf{E} = \rho\mathbf{J}$, the electric fields in the two conductors must be different, namely

$$E_1 = \rho_1 J = \frac{\rho_1 V}{h(\rho_1 + \rho_2)}\,, \qquad E_2 = \rho_2 J = \frac{\rho_2 V}{h(\rho_1 + \rho_2)}\,. \tag{S-4.50}$$

b) The surface charge density on the surface separating the two cylinders can be evaluated from Gauss's law

$$\sigma = \frac{1}{4\pi k_\mathrm{e}}\,(E_2 - E_1) = \frac{1}{4\pi k_\mathrm{e}}\,\frac{(\rho_2 - \rho_1)\,V}{h(\rho_1 + \rho_2)}\,. \tag{S-4.51}$$

Assuming that the electric field is zero above the upper base and below the lower base of the conductor, the surface charge densities at the two bases are also obtained from Gauss's law as

$$\sigma_1 = \frac{E_1}{4\pi k_\mathrm{e}} = \frac{1}{4\pi k_\mathrm{e}}\,\frac{\rho_1 V}{h(\rho_1 + \rho_2)}\,, \qquad \sigma_2 = -\frac{E_2}{4\pi k_\mathrm{e}} = -\frac{1}{4\pi k_\mathrm{e}}\,\frac{\rho_2 V}{h(\rho_1 + \rho_2)}\,. \tag{S-4.52}$$

S-4.7 Charge Decay in a Lossy Spherical Capacitor

a) We use a spherical coordinate system (r,θ,ϕ), with the origin O at the center of the capacitor. We have $\mathbf{E} = 0$ for $r < a$ and $r > b$. For symmetry reasons, the electric field $\mathbf{E}$ is radial and depends on r and t only in the spherical shell $a < r < b$. The flux of $\varepsilon_\mathrm{r}\mathbf{E}$ through a spherical surface centered in O and of radius r is independent of r and equals

$$\varepsilon_\mathrm{r} \oint \mathbf{E}(r,t)\cdot \mathrm{d}\mathbf{S} = 4\pi k_\mathrm{e}\, Q(t)\,, \tag{S-4.53}$$

where $Q(t)$ is the *free* charge contained in the surface, i.e, the free surface charge of the conducting sphere of radius a. Thus we have

$$\mathbf{E}(r,t) = \frac{k_\mathrm{e}}{\varepsilon_\mathrm{r}}\frac{Q(t)}{r^2}\, r_\mathrm{c}\,. \tag{S-4.54}$$

In addition to the free charge, our system contains surface polarization charges at $r = a$ and $r = b$, of values $\mp Q\,(\varepsilon_\mathrm{r}-1)/\varepsilon_\mathrm{r}$, respectively. No volume polarization charge is present, because

$$\boldsymbol{\nabla}\cdot\mathbf{P} = \frac{\varepsilon_\mathrm{r}-1}{4\pi k_\mathrm{e}}\,\boldsymbol{\nabla}\cdot\mathbf{E}(r,t) = 0\,. \tag{S-4.55}$$

The electric field $\mathbf{E}(r,t)$, in the presence of an electrical conductivity σ, gives origin to a current density $\mathbf{J}$

$$\mathbf{J} = \sigma\mathbf{E} = \sigma\frac{k_\mathrm{e}}{\varepsilon_\mathrm{r}}\frac{Q(t)}{r^2}\,\hat{\mathbf{r}}\,, \tag{S-4.56}$$

so that we have a total charge flux rate (electric current) through the surface

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \oint \mathbf{J}\cdot\mathrm{d}\mathbf{S} = \frac{4\pi\sigma k_\mathrm{e}}{\varepsilon_\mathrm{r}}\,Q(t)\,. \tag{S-4.57}$$

The charge crossing the surface is subtracted from the free charge on the internal conducting sphere, so that

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\frac{4\pi\sigma k_\mathrm{e}}{\varepsilon_\mathrm{r}}\,Q(t)\,, \tag{S-4.58}$$

leading to

$$Q(t) = Q_0\,\mathrm{e}^{-t/\tau}\,, \quad \text{with} \quad \tau = \frac{\varepsilon_\mathrm{r}}{4\pi\sigma k_\mathrm{e}}\,, \tag{S-4.59}$$

and the decay constant is independent of the sizes of the capacitor, in agreement with (S-4.40).

b) The power dissipated over the volume of the capacitor is

$$\begin{aligned} P_d &= \int \mathbf{J}\cdot\mathbf{E}\,\mathrm{d}^3x = \sigma\int \mathbf{E}^2\,\mathrm{d}^3x = \sigma\int_a^b \left[\frac{k_\mathrm{e}}{\varepsilon_\mathrm{r}}\frac{Q(t)}{r^2}\right]^2 4\pi r^2\,\mathrm{d}r \\ &= \frac{4\pi\sigma k_\mathrm{e}^2}{\varepsilon_\mathrm{r}^2}\,Q_0^2\,\mathrm{e}^{-2t/\tau}\int_a^b \frac{\mathrm{d}r}{r^2} = \frac{4\pi\sigma k_\mathrm{e}^2\,(b-a)}{\varepsilon_\mathrm{r}^2\,ab}\,Q_0^2\,\mathrm{e}^{-2t/\tau}\,. \end{aligned} \tag{S-4.60}$$

The electrostatic energy of the capacitor is

$$U_{\rm es} = \frac{1}{2}\frac{Q^2(t)}{C} = \frac{k_{\rm e}\,(b-a)}{2\varepsilon_{\rm r}\,ab}\,Q_0^2\,{\rm e}^{-2t/\tau}\,, \tag{S-4.61}$$

so that

$$\frac{{\rm d}U_{\rm es}}{{\rm d}t} = -\frac{k_{\rm e}(b-a)}{\tau\,\varepsilon_{\rm r}\,ab}\,Q_0^2\,{\rm e}^{-2t/\tau} = -\frac{4\pi\sigma k_{\rm e}^2(b-a)}{\varepsilon_{\rm r}^2\,ab}\,Q_0^2\,{\rm e}^{-2t/\tau} = -P_d\,. \tag{S-4.62}$$

Thus, the electrostatic energy of the capacitor is dissipated into Joule heating.

S-4.8 Dielectric-Barrier Discharge

a) We denote by E_1 and E_2 the electric fields in the gas and in the dielectric layers, respectively. Since the voltage drop between the plates is V, we must have

$$E_1 d_1 + E_2 d_2 = V\,. \tag{S-4.63}$$

In the absence of free surface charges the normal component of $\varepsilon_{\rm r}\mathbf{E}$ is continuous through the surface separating the two layers, so that

$$E_1 = \varepsilon_{\rm r} E_2\,. \tag{S-4.64}$$

Combining (S-4.63) and (S-4.64) we obtain

$$E_1 = \frac{\varepsilon_{\rm r} V}{\varepsilon_{\rm r} d_1 + d_2}\,, \qquad E_2 = \frac{V}{\varepsilon_{\rm r} d_1 + d_2}\,. \tag{S-4.65}$$

b) In steady-state conditions the current density in the gas, $\mathbf{J}$, must be zero, otherwise the free charge on the surface separating the gas and the dielectric material would increase steadily. Since the current density is $J = E_1/\rho$, we must have $E_1 = 0$. On the other hand (S-4.63) still holds, so that $E_2 = V/d_2$. The free charge density on the surface separating the layers in steady conditions is

$$\sigma_s = \frac{1}{4\pi k_{\rm e}}\,(\varepsilon_{\rm r} E_2 - E_1) = \frac{\varepsilon_{\rm r}}{4\pi k_{\rm e}}\,E_2 = \frac{\varepsilon_{\rm r}}{4\pi k_{\rm e}}\,\frac{V}{d_2}\,. \tag{S-4.66}$$

c) The continuity equation for σ and J is

$$\partial_t \sigma = J = \frac{E_1}{\rho}\,. \tag{S-4.67}$$

From (S-4.66), now with $E_1 \neq 0$ (discharge conditions), we have

$$E_1 = \varepsilon_r E_2 - 4\pi k_e \sigma\,, \tag{S-4.68}$$

which, combined with (S-4.63), leads to

$$E_1 = \varepsilon_r \left(\frac{V}{d_2} - \frac{d_1}{d_2} E_1 \right) - 4\pi k_e \sigma\,, \quad \text{i.e.,} \quad E_1 = \frac{\varepsilon_r V}{\varepsilon_r d_1 + d_2} - \frac{4\pi k_e d_2}{\varepsilon_r d_1 + d_2}\sigma\,. \tag{S-4.69}$$

Equation (S-4.69), substituted into (S-4.67), gives

$$\partial_t \sigma = -\frac{4\pi k_e d_2}{\rho(\varepsilon_r d_1 + d_2)}\sigma + \frac{\varepsilon_r V}{\rho(\varepsilon_r d_1 + d_2)}\,. \tag{S-4.70}$$

with solution

$$\sigma = \frac{\varepsilon_r V}{4\pi k_e d_2}\left(1 - e^{-t/\tau}\right) \equiv \sigma_s \left(1 - e^{-t/\tau}\right), \quad \text{where} \quad \tau = \frac{\rho(\varepsilon_r d_1 + d_2)}{4\pi k_e d_2}\,. \tag{S-4.71}$$

This problem shows the concept of the “*dielectric-barrier discharge*” (DBD). This scheme, where the dielectric layer acts as a current limiter, is used in various electrical discharge devices, for example in plasma TV displays, where the discharge acts as an ultraviolet micro-source to activate the phosphors in each pixel of the screen.

S-4.9 Charge Distribution in a Long Cylindrical Conductor

a) As we saw in point **a)** of Problem 4.6, the volume charge density $q_\mathbf{v}$ is zero everywhere inside our conducting cylinder, while **E** and **J** are uniform. The presence of an electric field requires the presence of a charge distribution generating it, and, since there cannot be volume charge densities inside a conductor in steady conditions, the charges generating the fields must be distributed on the conductor surfaces. Consider the thin cylindrical conductor shown in Fig. S-4.5, of radius a and length $2h$, with $h \ll a$, connected to a voltage source V_0. In this case, neglecting boundary effects, the surface charge densities σ_B and $-\sigma_B$ on the two bases are sufficient to generate the uniform electric field **E** inside the conductor. This leads also to a uniform current density $\mathbf{J} = \mathbf{E}/\rho$. Neglecting the boundary effects we have

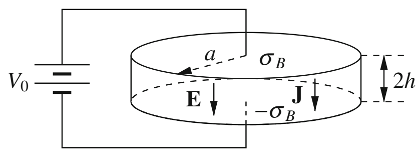

Fig. S-4.5

$$E = \frac{V_0}{2h}\,, \quad \sigma_B = \frac{E}{4\pi k_e} = \frac{V_0}{8\pi k_e h}\,. \tag{S-4.72}$$

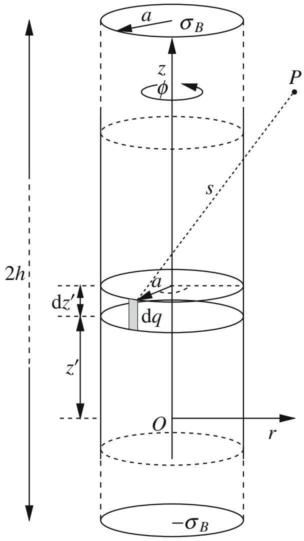

Fig. S-4.6

But here we are dealing with the opposite case, when the potential difference V_0 is applied to the bases of a very elongated cylinder, with $h \gg a$. Without loss of generality, we assume the potential to be $+V_0/2$ at the upper base, and $-V_0/2$ at the lower base. With this geometry, the surface charge densities $\pm\sigma_B$ on the two bases alone cannot generate a uniform electric field inside the whole conductor. We need another charge density σ_L, not necessarily uniform, distributed on the lateral surface of the cylinder. In order to treat the problem, we introduce a cylindrical coordinate system (r,ϕ,z), with the z axis coinciding with the axis of the cylinder, the origin O being located so that the upper and lower bases are at $z = \pm h$, respectively (this is not apparent from Fig. S-4.6 for practical reasons).

Because of symmetry reasons, σ_L cannot depend on ϕ. And it cannot be constant along the lateral surface, otherwise, neglecting the boundary effects, it would generate no field inside the conductor. Thus, σ_L must be a function of z, and z only. As an educated guess, we assume that σ_L is proportional to z, so that we have

$$\sigma_L(z) = \gamma z\,, \tag{S-4.73}$$

with γ a constant. This choice leads to $\sigma_L(0) = 0$ at $z = 0$, and $|\sigma_L|$ increasing, with opposite signs toward the upper and lower bases. Let us evaluate the electric potential in a point $P \equiv (r,0,z)$, with $r \ll h$, not necessarily inside the conductor. The choice of $\phi = 0$ does not affect the generality of the approach because of the rotational symmetry around the z axis. The contribution of the charge element $\mathrm{d}q = \gamma z' a\,\mathrm{d}\phi\,\mathrm{d}z'$, located on the lateral surface of the conductor at (a,ϕ,z'), to the potential $\Phi(r,0,z)$ is

$$\mathrm{d}\Phi = k_\mathrm{e}\frac{\mathrm{d}q}{s}\,,$$

where s is the distance between the points (a,ϕ,z') and $(r,0,z)$. The distance s can be evaluated by the cosine formula,

$$\begin{aligned} s &= \sqrt{(z'-z)^2 + a^2 + r^2 - 2ar\cos\phi} \\ &= \sqrt{(z'-z)^2 + a^2\left(1 + \frac{r^2}{a^2} - 2\frac{r}{a}\cos\phi\right)} \\ &= \sqrt{(z'-z^2) + a^2 f(r,\phi)}\,, \end{aligned} \tag{S-4.74}$$

where we have defined

$$f(r,\phi) = \left(1 + \frac{r^2}{a^2} - 2\frac{r}{a}\cos\phi\right). \tag{S-4.75}$$

We thus have

$$\mathrm{d}\Phi = k_\mathrm{e} a\gamma z' \frac{\mathrm{d}\phi\,\mathrm{d}z'}{\sqrt{(z'-z)^2 + a^2 f(r,\phi)}}, \tag{S-4.76}$$

and the electric potential in P is

$$\Phi(P) = k_\mathrm{e} a\gamma \int_0^{2\pi} \mathrm{d}\phi \int_{-h}^{h} \frac{z'\,\mathrm{d}z'}{\sqrt{(z'-z)^2 + a^2 f(r,\phi)}}. \tag{S-4.77}$$

In order to evaluate the integral we introduce a new variable $\zeta = z' - z$, so that

$$\Phi(P) = k_\mathrm{e}\, a\gamma \int_0^{2\pi} \mathrm{d}\phi \int_{-h-z}^{h-z} \frac{(z+\zeta)\,\mathrm{d}\zeta}{\sqrt{\zeta^2 + a^2 f(r,\phi)}}. \tag{S-4.78}$$

The indefinite integrals needed in the formula are

$$\int \frac{\mathrm{d}\zeta}{\sqrt{\zeta^2+b}} = \ln\left(2\zeta + 2\sqrt{\zeta^2+b}\right), \quad \text{and} \quad \int \frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\zeta^2+b}} = \sqrt{\zeta^2+b}. \tag{S-4.79}$$

We can split $\Phi(P)$ into the sum of two terms $\Phi(P) = \Phi_1(P) + \Phi_2(P)$, where

$$\begin{aligned}\Phi_1(P) &= k_\mathrm{e}\, a\gamma z \int_0^{2\pi} \mathrm{d}\phi \int_{-h-z}^{h-z} \frac{\mathrm{d}\zeta}{\sqrt{\zeta^2 + a^2 f(r,\phi)}} \\ &= k_\mathrm{e}\, a\gamma z \int_0^{2\pi} \mathrm{d}\phi \, \ln\left[\frac{h - z + \sqrt{(h-z)^2 + a^2 f(r,\phi)}}{-h - z + \sqrt{(h+z)^2 + a^2 f(r,\phi)}}\right]\end{aligned} \tag{S-4.80}$$

and

$$\begin{aligned}\Phi_2(P) &= k_\mathrm{e}\, a\gamma \int_0^{2\pi} \mathrm{d}\phi \int_{-h-z}^{h-z} \frac{\zeta\,\mathrm{d}\zeta}{\sqrt{\zeta^2 + a^2 f(r,\phi)}} \\ &= k_\mathrm{e}\, a\gamma \int_0^{2\pi} \mathrm{d}\phi \left[\sqrt{(h-z)^2 + a^2 f(r,\phi)} - \sqrt{(h+z)^2 + a^2 f(r,\phi)}\right].\end{aligned} \tag{S-4.81}$$

The square roots appearing in the integrals can be approximated as

$$\sqrt{(h \pm z)^2 + a^2 f(r,\phi)} \simeq h \pm z + \frac{a^2}{2\,(h \pm z)}\, f(r,\phi) \tag{S-4.82}$$

up to the second order in a/h and r/h. The second order is needed only in the denominator of the argument of the logarithm appearing in (S-4.80), where the first order cancels out with $-h-z$. Thus, $\Phi_1(P)$ can be approximated as

$$\begin{aligned}\Phi_1(P) &\simeq k_\mathrm{e}\, a\gamma z \int_0^{2\pi} \mathrm{d}\phi\, \ln\left\{\frac{2(h-z)}{a^2 f(r,\phi)/[2(h+z)]}\right\} \\ &= 2\pi k_\mathrm{e}\, a\gamma z \int_0^{2\pi} \mathrm{d}\phi\, \ln\left[\frac{4(h^2-z^2)}{a^2 f(r,\phi)}\right] \simeq 2\pi k_\mathrm{e}\, acz \int_0^{2\pi} \mathrm{d}\phi\, \ln\left[\frac{4h^2}{a^2 f(r,\phi)}\right],\end{aligned} \tag{S-4.83}$$

while the approximation for $\Phi_2(P)$ is

$$\Phi_2(P) \simeq -k_\mathrm{e}\, a\gamma \int_0^{2\pi} 2z\, \mathrm{d}\phi = -4\pi k_\mathrm{e}\, acz\,. \tag{S-4.84}$$

The two contributions sum up to

$$\begin{aligned}\Phi(P) = \Phi_1(P) + \Phi_2(P) &\simeq 2\pi k_\mathrm{e}\, a\gamma z \left\{\int_0^{2\pi} \mathrm{d}\phi\, \ln\left[\frac{4h^2}{a^2 f(r,\phi)}\right] - 2\right\} \\ &= 2\pi k_\mathrm{e}\, a\gamma z \left\{2\pi \ln\left(\frac{h^2}{a^2}\right) + \int_0^{2\pi} \mathrm{d}\phi\, \ln\left[\frac{4}{f(r,\phi)}\right] - 2\right\} \\ &= 2\pi k_\mathrm{e}\, a\gamma z \left\{4\pi \ln\left(\frac{h}{a}\right) + \int_0^{2\pi} \mathrm{d}\phi\, \ln\left[\frac{4}{f(r,\phi)}\right] - 2\right\}.\end{aligned} \tag{S-4.85}$$

If h is sufficiently large, the first terms in braces is dominant, and we have

$$\Phi(r,z) \simeq 8\pi^2 k_\mathrm{e}\, a\gamma \ln\left(\frac{h}{a}\right) z, \tag{S-4.86}$$

thus independent of r, within our approximations, as expected. Since we have assumed $\Phi(r,h) = V_0/2$, we must have

$$\frac{V_0}{2} = 8\pi^2 k_\mathrm{e}\, a\gamma \ln\left(\frac{h}{a}\right) h, \tag{S-4.87}$$

which leads to

$$\gamma = \frac{V_0}{16\pi^2 k_e ah \ln(h/a)} \quad \text{and} \quad \sigma_L(z) = \frac{V_0}{16\pi^2 k_e ah \ln(h/a)}\, z\,. \tag{S-4.88}$$

S-4.10 An Infinite Resistor Ladder

Let us denote by R_L the resistance measured between the terminals A and B. If a further unit of three resistors is added to the left of the ladder, as in Fig. S-4.7, the "new" resistance measured between terminals A' and B' must equal the "old' resistance R_L. The "old" resistor ladder at the right of terminals A and B can be replaced by the equivalent resistance R_L, leading to the configuration of Fig. S-4.8. We see that the resistance between terminals A' and B' is the solution of

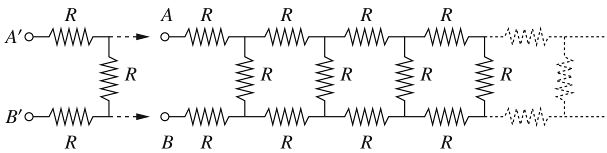

Fig. S-4.7

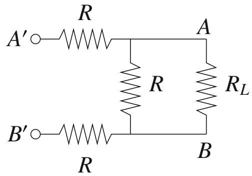

Fig. S-4.8

$$\begin{aligned} R_L &= 2R + \frac{RR_L}{R+R_L} \\ RR_L + R_L^2 &= 2R^2 + 2RR_L + RR_L \\ R_L^2 - 2RR_L - 2R^2 &= 0\,, \end{aligned} \tag{S-4.89}$$

and, disregarding the negative solution, we have

$$R_L = R\left(1+\sqrt{3}\right). \tag{S-4.90}$$