

Chapter S-5

Solutions for Chapter 5

S-5.1 The Rowland Experiment

a) Neglecting the boundary effects, the electric field \mathbf{E}_0 in the regions between the disk and the plates is uniform, perpendicular to the disk surfaces, and its magnitude is $E_0 = V_0/h$ in both regions. In both regions, the field is directed outwards from the disk, according to the polarity of the source shown in Fig. 5.1. The charge densities of the lower and upper surfaces of the disk, σ , are equal in modulus and sign, because the field must be zero inside the disk. Thus we have $\sigma = E_0/(4\pi k_e) = V_0/(4\pi k_e h)$. In SI units we have $\sigma = \epsilon_0 V_0/h$, with $\epsilon_0 = 8.85 \times 10^{-12}$, $V_0 = 10^4$ V, $h = 5 \times 10^{-3}$ m, resulting in $\sigma = 1.77 \times 10^{-5}$ C/m². In Gaussian units we have $\sigma = V_0/(4\pi h)$, with $V_0 = 33.3$ statV and $h = 0.5$ cm, resulting in $\sigma = 5.3$ statC/cm².

b) We evaluate the magnetic field \mathbf{B}_c at the center of the disk by dividing its upper and lower surfaces into annuli of radius r (with $0 < r < a$) and width dr . On each surface, each annulus carries a charge $dq = \sigma dS = 2\pi\sigma r dr$. Due to the rotation of the disk, each annulus is equivalent to a coil with a current intensity $dI = \omega dq/(2\pi)$, that generates at its center a magnetic field $d\mathbf{B}_c = 2\pi k_m dI/r \hat{\omega}$, perpendicular to the disk plane. The total field at the center of the disk is thus given by the integral

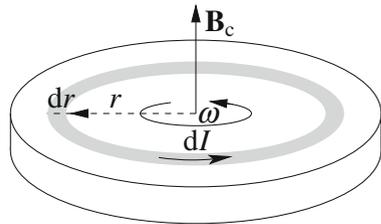


Fig. S-5.1

$$\begin{aligned}
 B_c &= 2 \int_0^a 2\pi k_m \frac{dI}{r} = 4\pi k_m \sigma \omega \int_0^a \frac{r dr}{r} \\
 &= 4\pi k_m \omega \sigma a = \begin{cases} \mu_0 \omega \sigma a \approx 1.4 \times 10^{-9} \text{ T} & \text{SI} \\ \frac{4\pi}{c} \omega \sigma a \approx 1.4 \times 10^{-5} \text{ G} & \text{Gaussian,} \end{cases} \quad (\text{S-5.1})
 \end{aligned}$$

where the factor of 2 in front of the first integral is due to contribution of both the upper and the lower surfaces of the disk to the magnetic field.

The magnetic field component B_r , parallel to the disk surface and close to it, can be evaluated by applying Ampère’s law to the closed rectangular path C shown Fig. S-5.2. The path is placed at a distance r from the rotation axis, with the sides parallel to the disk surfaces having length $\ell \ll r$, so that \mathbf{B}_r is approximately constant along the sides. The contribution of the vertical paths to the line integral cancel each other, thus

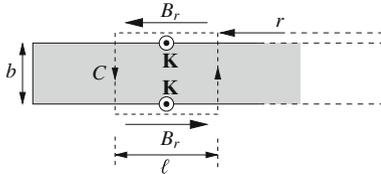


Fig. S-5.2

$$4\pi k_m I_c = \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} \simeq 2B_r \ell, \quad (\text{S-5.2})$$

where I_c is the current flowing through the rectangular loop C , and the antisymmetry of B_r with respect to the midplane has been used. The rotation of the disk leads to a surface current density $\mathbf{K} = \sigma \mathbf{v} = \sigma \omega r \hat{\boldsymbol{\phi}}$, resulting in a total current flowing through the rectangular loop $I_c = 2K\ell = 2\sigma\omega r\ell$. Thus, according to (S-5.2),

$$B_r(r) = \frac{2\pi k_m I_c}{\ell} = \frac{2\pi k_m}{\ell} 2\sigma\omega r\ell = 4\pi k_m \sigma\omega r. \quad (\text{S-5.3})$$

The maximum value of $B_r(r)$ occurs at $r = a$, where $B_r(a) = B_c$.

c) The deviation angle of the needle is given by $\tan \theta = B/B_\oplus$, hence

$$\theta \simeq \frac{B}{B_\oplus} = 2.8 \times 10^{-5} \text{ rad} = 1.6 \times 10^{-3} \text{ deg}. \quad (\text{S-5.4})$$

The expected angle is very small, and its measurement requires exceptional care.

S-5.2 Pinch Effect in a Cylindrical Wire

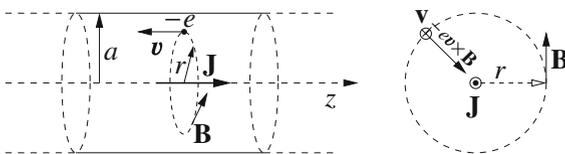


Fig. S-5.3

a) We use a cylindrical coordinate system (r, ϕ, z) with the z axis along the axis of the cylinder. The vectors \mathbf{J} and \mathbf{v} are along z . If we assume $J > 0$ we have $v < 0$ since $J = -n_e e v$.

The magnetic field \mathbf{B} is azimuthal for symmetry reasons. Its only component $B_\phi(r)$ can easily be evaluated by applying Ampère’s circuital law to a circular closed path coaxial with the cylinder axis, as shown in figure. We have

$$2\pi r B_\phi = 4\pi^2 r^2 k_m J = \begin{cases} \mu_0 \pi r^2 J, & \text{SI} \\ \frac{4\pi^2 r^2}{c} J, & \text{Gaussian,} \end{cases} \quad (\text{S-5.5})$$

so that

$$B_\phi = 2\pi k_m J r = -2\pi k_m n_e e v, \quad (\text{S-5.6})$$

and $B_\phi > 0$, since $v < 0$. Thus the field lines of \mathbf{B} are oriented counterclockwise with respect to the z axis.

The magnetic force $\mathbf{F}_m = -e b_m (\mathbf{v} \times \mathbf{B})$ is radial and directed towards the z axis

$$\mathbf{F}_m = -2\pi k_m b_m n_e e^2 v^2 = \begin{cases} -\frac{\mu_0 n_e e^2 v^2}{2} \mathbf{r}, & \text{SI} \\ -\frac{2\pi n_e e^2 v^2}{c^2} \mathbf{r}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.7})$$

Thus the magnetic force pulls the charge carriers toward the axis of the wire, independently of their sign. A beam of charged particles always gives origin to a magnetic field that tends to “pinch” the beam, i.e., to shrink it toward its axis. However, if the beam is propagating in vacuum, the Coulomb repulsion between the charged particles is dominant. In our case, or in the case of a plasma, the medium is globally neutral, and, initially, the positive and negative charge densities are uniform over the medium, so that the pinch effect can be observed, at least in principle.

b) The Lorentz force is $\mathbf{F}_L = -e(\mathbf{E} + b_m \mathbf{v} \times \mathbf{B})$. At equilibrium the r component of \mathbf{F}_L must be zero in the presence of conduction electrons (see Problem 1.9), so that the electrons flow only along the z axis. Thus the r component of the electric field, E_r , must be

$$E_r = -b_m v B = 2\pi k_m b_m n_e e v^2 r, \quad (\text{S-5.8})$$

while $E_z = J/\sigma$, where σ is the conductivity of the material. According to Gauss’s law, a charge density ϱ , uniform over the cylinder volume, generates a field $\mathbf{E} = 2\pi k_e \varrho \mathbf{r}$, and the required field E_r is generated by the charge density

$$\varrho = \frac{k_m b_m}{k_e} n_e e v^2 = n_e e \frac{v^2}{c^2}, \quad (\text{S-5.9})$$

independent of the system of units. On the other hand, the global charge density is $\varrho = e(Zn_i - n_e)$, so that

$$n_e = \frac{Zn_i}{1 - v^2/c^2}. \quad (\text{S-5.10})$$

Thus, the electron density is uniform over the wire volume, but it exceeds the value $n_{e0} = Zn_i$, corresponding to $\rho = 0$. This means that the number density of the electrons is increased by a factor $(1 - v^2/c^2)^{-1}$, and ρ is negative, *inside* the wire. The “missing” positive charge is uniformly distributed over the surface of the conductor.

c) For electrons in a usual Ohmic conductor we have $v \approx 1 \text{ cm/sec} = 10^{-2} \text{ m/sec}$, corresponding to $(v/c)^2 \approx 10^{-21}$, and the resulting “pinch” effect is so small that it cannot be observed. On the other hand, the effect may be strong in high density particle beams or plasma columns, where v is not negligible with respect to c .

In order to get further insight into the size of the effect, let us consider an Ohmic cylindrical conductor (wire) of radius a . We assume that the electron density is increased in a central cylindrical region of radius $a - d$, where $n_e^{\text{pinch}} = Zn_i/(1 - v^2/c^2)$, and the volume charge density is

$$\rho^{\text{pinch}} = e(Zn_i - n_e^{\text{pinch}}) = -eZn_i \frac{v^2}{c^2} \frac{1}{1 - v^2/c^2} < 0, \quad (\text{S-5.11})$$

while the cylindrical shell between $r = a - d$ and $r = a$ is depleted of conduction electrons, so that its charge density ρ^{surf} is $\rho^{\text{surf}} = eZn_i$. The thickness d of the depleted cylindrical shell can be estimated by the constraint of charge conservation.

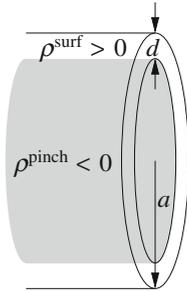


Fig. S-5.4

A slice of wire of length ℓ must be globally neutral, thus, assuming $d \ll a$, we must have

$$\begin{aligned} \pi(a-d)^2 \ell \rho^{\text{pinch}} &= -2\pi a d \ell \rho^{\text{surf}} \\ \pi(a-d^2) e Zn_i \frac{v^2}{c^2} \frac{1}{1 - v^2/c^2} &= 2\pi a d e Zn_i \\ (a^2 - 2ad + d^2) \frac{v^2}{c^2} \frac{1}{1 - v^2/c^2} &= 2ad, \quad (\text{S-5.12}) \end{aligned}$$

and, since $v \ll c$ and $d \ll a$, we can approximate

$$a^2 \frac{v^2}{c^2} \approx 2ad, \quad \text{so that} \quad d \approx \frac{a v^2}{2 c^2}. \quad (\text{S-5.13})$$

Remembering that v^2/c^2 is of the order of 10^{-21} , we see that a value of d of the order of the crystal lattice spacing ($\approx 10^{-10} \text{ m}$) would require a wire of radius $a \approx 10^{11} \text{ m}$, a remarkably large radius!

S-5.3 A Magnetic Dipole in Front of a Magnetic Half-Space

a) The analogy between the magnetostatic equations in the absence of free currents ($\nabla \times \mathbf{H} = 0$, $\nabla \cdot \mathbf{B} = 0$) with those for the electrostatics of dielectric in the absence of free charges ($\nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{D} = 0$) indicates that the solution of this problem will be

similar to that of Problem (3.2). Thus, analogously to Problem (3.2), we treat the vacuum half-space and the medium-filled half-space separately, with separate educated guesses for the magnetic field in each half-space.

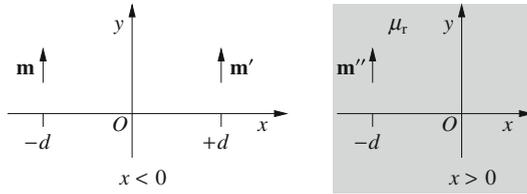


Fig. S-5.5

This in order to exploit the uniqueness theorem for the Poisson equation (5.5). Our guess for half-space 1 ($x < 0$) is that the field is same as if the magnetic medium were removed from half-space 2 (thus, vacuum in the whole space), and replaced by an image magnetic dipole \mathbf{m}' located symmetrically to \mathbf{m} with respect to the $x = 0$ plane, at $x = d$. Our guess for half-space 2 ($x > 0$) is that the field is the same as if the magnetic medium filled the whole space, and the magnetic dipole \mathbf{m} were replaced by a different magnetic dipole \mathbf{m}'' , placed at the same location. Thus we look for values of \mathbf{m}' and \mathbf{m}'' originating a magnetic field \mathbf{B}_1 in half-space 1, and a magnetic field \mathbf{B}_2 in half-space 2, satisfying the interface conditions at $x = 0$

$$B_{1\perp}(x = 0^-) = B_{2\perp}(x = 0^+), \quad B_{1\parallel}(x = 0^-) = \frac{1}{\mu_r} B_{2\parallel}(x = 0^+), \quad (\text{S-5.14})$$

The subscripts \parallel and \perp stand for parallel and perpendicular to the $x = 0$ plane, respectively. Thus, at a generic point $P \equiv (0, y, z)$ of the $x = 0$ plane, we must have

$$\begin{aligned} B_x(0^-, y, z) &= B_x(0^+, y, z) \\ B_y(0^-, y, z) &= \frac{1}{\mu_r} B_y(0^+, y, z) \\ B_z(0^-, y, z) &= \frac{1}{\mu_r} B_z(0^+, y, z). \end{aligned} \quad (\text{S-5.15})$$

The field generated by a magnetic dipole \mathbf{m} in a medium of relative magnetic permittivity μ_r is

$$\mathbf{B}(\mathbf{r}) = \frac{k_m}{b_m} \mu_r \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3} \quad (\text{S-5.16})$$

where \mathbf{r} is the distance vector directed from \mathbf{m} to the point where we evaluate the field, and $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector along \mathbf{r} . Note that, differently from Problem (3.2), here we do not have cylindrical symmetry around the x axis, because the real magnetic dipole \mathbf{m} is not lying on x . It is convenient to introduce the angles $\theta = \arcsin(d/r)$ and $\phi = \arctan(z/y)$, and write the Cartesian components of \mathbf{B}

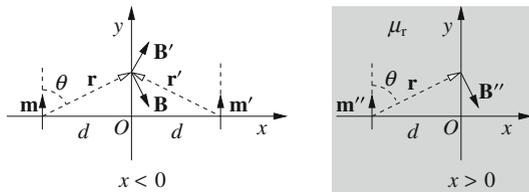


Fig. S-5.6

separately

$$\begin{aligned}
 B_x &= \frac{k_m}{b_m} \mu_r \frac{3m \cos \theta \sin \theta}{r^3} \\
 B_y &= \frac{k_m}{b_m} \mu_r \frac{3m \cos^2 \theta \cos \phi - m}{r^3} \\
 B_z &= \frac{k_m}{b_m} \mu_r \frac{3m \cos^2 \theta \sin \phi}{r^3}, \tag{S-5.17}
 \end{aligned}$$

where $r = \sqrt{d^2 + y^2 + z^2}$. If we replace (S-5.17) into (S-5.15) and divide by (k_m/b_m) the boundary conditions become

$$\begin{aligned}
 \frac{3m \cos \theta \sin \theta}{r^3} - \frac{3m' \cos \theta \sin \theta}{r^3} &= \mu_r \frac{3m'' \cos \theta \sin \theta}{r^3} \\
 \frac{3m \cos^2 \theta \cos \phi - m}{r^3} + \frac{3m' \cos^2 \theta \cos \phi - m'}{r^3} &= \frac{3m'' \cos^2 \theta \cos \phi - m''}{r^3} \\
 \frac{3m \cos^2 \theta \sin \phi}{r^3} + \frac{3m' \cos^2 \theta \sin \phi}{r^3} &= \frac{3m'' \cos^2 \theta \sin \phi}{r^3}, \tag{S-5.18}
 \end{aligned}$$

which can be further simplified into

$$\begin{aligned}
 \mathbf{m} - \mathbf{m}' &= \mu_r \mathbf{m}'' \\
 \mathbf{m} + \mathbf{m}' &= \mathbf{m}'' \tag{S-5.19}
 \end{aligned}$$

leading to the solution

$$\mathbf{m}' = -\frac{\mu_r - 1}{\mu_r + 1} \mathbf{m}, \quad \mathbf{m}'' = \frac{2}{\mu_r + 1} \mathbf{m}. \tag{S-5.20}$$

As expected, the expressions for m' and m'' as functions of m are identical to (S-3.8) for the image charges q' and q'' as functions of the real charge q (although Problem 3.2 involves point charges, the generalization to electric dipoles is immediate).

b) The force exerted by the magnetic half-space on \mathbf{m} equals the force that would be exerted on \mathbf{m} by a real magnetic dipole \mathbf{m}' located at $x = +d$. The force between two magnetic dipoles at a distance \mathbf{r} from each other is

$$\mathbf{f} = -\frac{k_m}{b_m} \nabla \left[\frac{\mathbf{m} \cdot \mathbf{m}' - 3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{m}' \cdot \hat{\mathbf{r}})}{r^3} \right], \tag{S-5.21}$$

with, in our case, $\hat{\mathbf{r}} = \hat{\mathbf{x}}$, $r = 2d$, and $\mathbf{m} \cdot \hat{\mathbf{r}} = \mathbf{m}' \cdot \hat{\mathbf{r}} = 0$, so that the force on \mathbf{m} is

$$\mathbf{f} = -\frac{k_m}{b_m} \frac{3m^2}{r^4} \left(\frac{\mu_r - 1}{\mu_r + 1} \right) \hat{\mathbf{x}} = \frac{k_m}{b_m} \frac{3m^2}{16d^4} \left(\frac{\mu_r - 1}{\mu_r + 1} \right) \hat{\mathbf{x}}. \tag{S-5.22}$$

The force is repulsive (antiparallel to $\hat{\mathbf{x}}$) for $\mu_r < 1$ (diamagnetic material), and attractive (parallel to $\hat{\mathbf{x}}$) for $\mu_r > 1$ (paramagnetic material). At the limit $\mu_r \rightarrow 0$ we have a perfect diamagnetic material (superconductor), and $\mathbf{m}' \rightarrow \mathbf{m}$, the two dipoles are *parallel* and the force is repulsive, as expected. In this case $\mathbf{m}'' \neq 0$, so that $\mathbf{H} \neq 0$

in the half-space 2, where, however, $\mu_r = 0$ so that $\mathbf{B} = \mu_0\mu_r\mathbf{H} = 0$. The situation is *opposite* to that of a perfect conductor in electrostatics, where an electric dipole would induce an *opposite* image dipole, and the force would be *attractive*.

At the limit of $\mu_r \rightarrow \infty$ (perfect ferromagnetic material), we have $\mathbf{m}' \rightarrow -\mathbf{m}$, corresponding to an *attractive* force, while $\mathbf{m}'' \rightarrow 0$ and $\mathbf{H} \rightarrow 0$ inside the material. This situation is analogous to the case of a conductor in electrostatics. Notice that \mathbf{B} is finite inside the material (since $\mu_r\mathbf{m}'' \rightarrow 2\mathbf{m}$) and given by

$$\mathbf{B} = \frac{k_m}{b_m} \mu_r \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}'') - \mathbf{m}''}{4\pi r^3} \rightarrow 2 \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{4\pi r^3}, \quad (x > 0), \quad (\text{S-5.23})$$

so that the paramagnetic material doubles the value of the magnetic field in vacuum in the limit $\mu_r \rightarrow \infty$.

S-5.4 Magnetic Levitation

a) The radial component B_r of the magnetic field close to the z axis can be evaluated by applying Gauss's law $\nabla \cdot \mathbf{B} = 0$ to a small closed cylinder of radius r , coaxial with the z axis, and with the bases at z and $z + \Delta z$, as shown in Fig. S-5.7. The flux of \mathbf{B} through the total surface of the cylinder must be zero, thus we have

$$0 = \oint_{\text{cylinder}} \mathbf{B} \cdot d\mathbf{S} \quad (\text{S-5.24})$$

$$= 2\pi r \Delta z B_r(r) + \pi r^2 [B_z(z + \Delta z) - B_z(z)],$$

leading to

$$B_r = -\frac{r[B_z(z + \Delta z) - B_z(z)]}{2\Delta z}$$

$$\simeq -\frac{B_0}{2L} r \quad (\text{S-5.25})$$

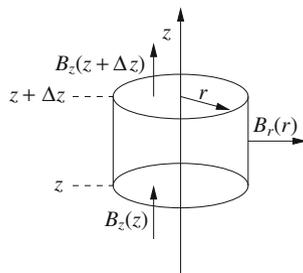


Fig. S-5.7

b) According to Table 5.1, the force exerted by an external magnetic field \mathbf{B} on a magnetic dipole \mathbf{m} is $\mathbf{f} = (\mathbf{m} \cdot \nabla)\mathbf{B}$. If we assume that the dipole is moving in a region free of electric current densities, so that $\nabla \times \mathbf{B} = 0$, the work done on the dipole when it performs an infinitesimal displacement $d\mathbf{r} \equiv (dx, dy, dz)$ is¹

¹We have

$$dW = [(\mathbf{m} \cdot \nabla)\mathbf{B}] \cdot d\mathbf{r} = \sum_{i,j} m_i \partial_i B_j dx_j = \sum_{i,j} m_i \partial_j B_i dx_j = \sum_i m_i dB_i = \mathbf{m} \cdot d\mathbf{B}, \quad (\text{S-5.26})$$

where, as usual, $x_{1,2,3} = x, y, z$, and $\partial_{1,2,3} = \partial_x, \partial_y, \partial_z$. We have used the property $\partial_i B_j = \partial_j B_i$, trivial for $i = j$, while the condition $\nabla \times \mathbf{B} = 0$ implies $\partial_i B_j - \partial_j B_i = 0$ also for $i \neq j$.

$$dW = \mathbf{f} \cdot d\mathbf{r} = [(\mathbf{m} \cdot \nabla)\mathbf{B}] \cdot d\mathbf{r} = \mathbf{m} \cdot d\mathbf{B}. \quad (\text{S-5.27})$$

For a permanent magnetic dipole this leads to the well known expression for the potential energy of a dipole located at \mathbf{r}

$$U(\mathbf{r}) = -\mathbf{m} \cdot \mathbf{B}(\mathbf{r}). \quad (\text{S-5.28})$$

Here, however, the magnetic dipole is not permanent. Rather, we have an induced dipole $\mathbf{m} = \alpha\mathbf{B}$. Thus we have

$$\begin{aligned} U(\mathbf{r}_2) - U(\mathbf{r}_1) &= - \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{m} \cdot d\mathbf{B} = -\alpha \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{B} \cdot d\mathbf{B} = -\frac{\alpha}{2} \int_{\mathbf{r}_1}^{\mathbf{r}_2} dB^2 \quad (\text{S-5.29}) \\ &= \frac{\alpha}{2} [B^2(\mathbf{r}_1) - B^2(\mathbf{r}_2)] = \frac{1}{2} [\mathbf{m}(\mathbf{r}_1) \cdot \mathbf{B}(\mathbf{r}_1) - \mathbf{m}(\mathbf{r}_2) \cdot \mathbf{B}(\mathbf{r}_2)]. \end{aligned}$$

and the potential energy for the induced dipole at \mathbf{r} is written

$$U(\mathbf{r}) = -\frac{1}{2} \mathbf{m}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}). \quad (\text{S-5.30})$$

For the present problem, this leads to

$$U(\mathbf{r}) = -\frac{1}{2} \mathbf{m}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) = -\frac{1}{2} \alpha B^2(\mathbf{r}) = -\frac{1}{2} \frac{\alpha B_0^2}{L^2} \left(z^2 + \frac{r^2}{4} \right). \quad (\text{S-5.31})$$

c) The potential energy U has a minimum in the origin ($r = 0, z = 0$) if $\alpha < 0$ (diamagnetic particle). The force is

$$\mathbf{f} = -\nabla U = -\frac{1}{2} |\alpha| \frac{2B_0^2}{L^2} \left(z \hat{\mathbf{z}} + \frac{r}{4} \mathbf{r} \right). \quad (\text{S-5.32})$$

Thus, we have a harmonic force both for radial and axial displacements, with corresponding oscillation frequencies

$$\omega_z = \sqrt{\frac{|\alpha| B_0^2}{ML^2}}, \quad \omega_r = \frac{\omega_z}{2}. \quad (\text{S-5.33})$$

S-5.5 Uniformly Magnetized Cylinder

a) The volume magnetization current (bound current) density \mathbf{J}_m is zero all over the cylinder volume because the cylinder magnetization \mathbf{M} is uniform, and $\mathbf{J}_m = \nabla \times \mathbf{M}/b_m$. For the surface magnetization current density \mathbf{K}_m we have $\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{n}}/b_m$, where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the cylinder lateral surface, and b_m is the system dependent constant defined in (5.1). Since \mathbf{M} and $\hat{\mathbf{n}}$ are perpendicular to each other, we have $K_m = |\mathbf{K}_m| = |\mathbf{M}|/b_m$.

b) The magnetized cylinder is equivalent to a solenoid with $nI = K_m$, where n is the number of coils per unit length, and I is the electric current circulating in each coil. Thus, at the $h \gg R$ limit, the magnetic field is uniformly zero outside the cylinder, and it is uniform and equal to

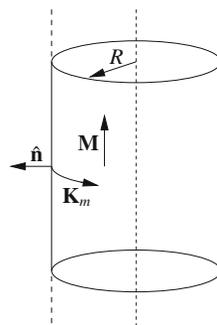


Fig. S-5.8

$$B = \frac{4\pi k_m}{b_m} nI = \frac{4\pi k_m}{b_m} K_m = \frac{4\pi k_m}{b_m} M = \begin{cases} \mu_0 M, & \text{SI} \\ 4\pi M, & \text{Gaussian,} \end{cases} \quad (\text{S-5.34})$$

inside. The auxiliary field \mathbf{H} is zero both inside and outside the cylinder because

$$\mathbf{H}_{\text{in}} = \begin{cases} \frac{\mathbf{B}_{\text{in}}}{\mu_0} - \mathbf{M} = 0, & \text{SI} \\ \mathbf{B}_{\text{in}} - 4\pi \mathbf{M} = 0, & \text{Gaussian.} \end{cases} \quad \mathbf{H}_{\text{out}} = \begin{cases} \frac{\mathbf{B}_{\text{out}}}{\mu_0} = 0, & \text{SI} \\ \mathbf{B}_{\text{out}} = 0, & \text{Gaussian.} \end{cases} \quad (\text{S-5.35})$$

c) At the “flat cylinder” limit, $R \gg h$, the cylinder is equivalent to a single coil of radius R carrying a current $I = hK_m = hM/b_m$. Thus we have for the field at its center

$$B_0 = 2\pi k_m \frac{I}{R} = \begin{cases} \frac{\mu_0 I}{2R} = \frac{\mu_0 M h}{2R}, & \text{SI} \\ \frac{2\pi I}{cR} = \frac{2\pi M h}{cR}, & \text{Gaussian,} \end{cases} \quad (\text{S-5.36})$$

and \mathbf{B}_0 approaches zero as $h/r \rightarrow 0$.

d) The equivalent magnetic charge density is defined as $\varrho_m = -\nabla \cdot \mathbf{M}$, thus $\varrho_m \equiv 0$ inside the cylinder volume, while the two bases of the cylinder carry surface magnetic charge densities $\sigma_m = \mathbf{M} \cdot \hat{\mathbf{n}} = \pm M$. Therefore our flat magnetized cylinder is the “magnetostatic” equivalent of an electrostatic parallel-plate capacitor. The equivalent magnetic charge “generates” the auxiliary magnetic field \mathbf{H} , which is uniform, and equal to $H = -\sigma_m = -M$, inside the volume of the flat cylinder, and zero outside. Thus $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ is zero everywhere (more realistically, it is zero far from the boundaries).

The field of a magnetized cylinder and its electrostatic analog are further discussed in Problem 13.1.

S-5.6 Charged Particle in Crossed Electric and Magnetic Fields

a) We choose a Cartesian laboratory frame of reference xyz with the y axis parallel to the electric field \mathbf{E} , the z axis parallel to the magnetic field \mathbf{B} , and the origin O located so that the particle is initially at rest in O . The Lorentz force on the particle

$$\mathbf{f} = q[\mathbf{E} + b_m \mathbf{v} \times \mathbf{B}]$$

has no z component, and the motion of the particle occurs in the xy plane. The equations of motion are thus

$$\begin{aligned} m\ddot{x} &= b_m q B \dot{y}, \\ m\ddot{y} &= -b_m q B \dot{x} + qE. \end{aligned} \quad (\text{S-5.37})$$

It is convenient to introduce two new variables x' , and y' , such that

$$x = x' + v_0 t, \quad y = y', \quad (\text{S-5.38})$$

where v_0 is a constant velocity, which we shall determine in order to simplify the equations of motion. The initial conditions for the primed variables are

$$\begin{aligned} x'(0) &= 0, & \dot{x}'(0) &= -v_0, \\ y'(0) &= 0, & \dot{y}'(0) &= -0. \end{aligned} \quad (\text{S-5.39})$$

Differentiating (S-5.38) with respect to time we obtain

$$\begin{aligned} \dot{x} &= v_0 + \dot{x}', & \ddot{x} &= \ddot{x}', \\ \dot{y} &= \dot{y}', & \ddot{y} &= \ddot{y}', \end{aligned} \quad (\text{S-5.40})$$

which we substitute into (S-5.37), thus obtaining the following equations for the time evolution of the primed variables

$$\begin{aligned} m\ddot{x}' &= b_m q B \dot{y}', \\ m\ddot{y}' &= -b_m q B v_0 - b_m q B \dot{x}' + qE. \end{aligned} \quad (\text{S-5.41})$$

Now we choose the constant velocity v_0 to be

$$v_0 = \frac{E}{b_m B} = \begin{cases} \frac{E}{B}, & \text{SI,} \\ \frac{E}{B} c, & \text{Gaussian,} \end{cases} \quad (\text{S-5.42})$$

independently of the charge and mass of the particle, so that the terms qE and $-b_m qB v_0$ cancel each other in the second of (S-5.41). The equations reduce to

$$\begin{aligned}\ddot{x}' &= b_m \frac{qB}{m} \dot{y}' , \\ \ddot{y}' &= -b_m \frac{qB}{m} \dot{x}' ,\end{aligned}\tag{S-5.43}$$

which are the equations of a uniform circular motion with angular velocity $\omega = -b_m qB/m$. The rotation is clockwise if $q > 0$, counterclockwise if $q < 0$. Since, according to (S-5.39), $\dot{x}'(0) = -v_0$ and $\dot{y}'(0) = 0$, the radius of the circular path is

$$r = \frac{mv_0}{b_m qB} = \frac{mE}{b_m^2 qB^2} .\tag{S-5.44}$$

The time evolution of the primed variables is thus

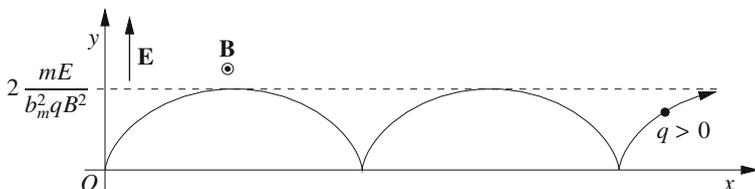


Fig. S-5.9

$$x' = x'_0 + r \cos(\omega t + \phi) = r \sin(\omega t) = -\frac{mE}{b_m^2 q B^2} \sin\left(\frac{b_m q B}{m} t\right),\tag{S-5.45}$$

$$y' = y'_0 + r \sin(\omega t + \phi) = r - r \cos(\omega t) = \frac{mE}{b_m^2 q B^2} \left[1 - \cos\left(\frac{b_m q B}{m} t\right)\right],$$

where we have chosen the constants $\phi = -\pi/2$, $x'_0 = 0$, and $y'_0 = -r$, in order to reproduce the initial conditions. The time evolution of the unprimed variables is

$$\begin{aligned}x &= \frac{E}{b_m B} t - \frac{mE}{b_m^2 q B^2} \sin\left(\frac{b_m q B}{m} t\right), \\ y &= \frac{mE}{b_m^2 q B^2} \left[1 - \cos\left(\frac{b_m q B}{m} t\right)\right],\end{aligned}\tag{S-5.46}$$

and the observed motion is a cycloid, as shown in Fig. S-5.9 for a positive charge.

b) From the results of point **a)**, we know that the motion of the electron will be a cycloid starting from the negative plate, and reaching a maximum distance $2r = 2mE/(b_m^2 q B^2)$ from it, where $E = V/h$. The condition for the electron not reaching the positive plate is thus

$$\frac{2mE}{b_m^2 q B^2} = \frac{2mV}{b_m^2 h q B^2} < h, \quad \text{corresponding to} \quad B > \frac{1}{b_m h} \sqrt{\frac{2m_e V}{e}}, \quad (\text{S-5.47})$$

where m_e is the electron mass, and e the absolute value of the electron charge.

S-5.7 Cylindrical Conductor with an Off-Center Cavity

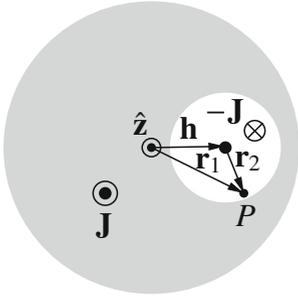


Fig. S-5.10

According to the superposition principle, a current density \mathbf{J} flowing uniformly through the cross section of the conductor in the positive z direction is equivalent to a uniform current density \mathbf{J} , flowing through the whole circular section of radius a , superposed to a current density $-\mathbf{J}$, flowing in the negative z direction through the the cavity.

The magnetic field generated by an infinite, straight wire of radius a and uniformly distributed current density $\mathbf{J} = J\hat{\mathbf{z}}$ has azimuthal symmetry. Using a cylindrical coordinate system (r, ϕ, z) with the z axis coinciding with the axis of the wire, the magnetic field $\mathbf{B} = B_\phi(r)\hat{\boldsymbol{\phi}}$ can be evaluated using

(5.3): the line integral calculated over the circle C of radius r is

$$2\pi r B_\phi(r) = 4\pi k_m \oint_C \mathbf{J} \cdot d\mathbf{S} = 4\pi k_m J \times \begin{cases} \pi r^2, & r < a, \\ \pi a^2, & r > a. \end{cases} \quad (\text{S-5.48})$$

We thus obtain

$$B_\phi(r) = \begin{cases} 2\pi k_m J r, & r < a, \\ \frac{2\pi k_m J a^2}{r}, & r > a. \end{cases} \quad (\text{S-5.49})$$

It is possible, and useful for the following, to write the above expressions in a compact vectorial form. Since $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\phi}}$ we have for the field $\mathbf{B}_w = \mathbf{B}_w(\mathbf{r}; a)$ of the infinite wire or radius a at a distance \mathbf{r} from the axis

$$\mathbf{B}_w(\mathbf{r}; a) = \begin{cases} 2k_m \mathbf{r} \times \mathbf{J}, & r < a, \\ \frac{2k_m a^2 \mathbf{r} \times \mathbf{J}}{r^2}, & r > a. \end{cases} \quad (\text{S-5.50})$$

Coming back to the cylindrical conductor with a cavity, the magnetic field in a point P is the sum of the field generated by a wire of radius a with current \mathbf{J} and a wire of radius b with current $-\mathbf{J}$, with the distance between the axes of the two wires equal to \mathbf{h} . Let \mathbf{r}_1 and \mathbf{r}_2 be the distance of P from the axes of the first and the second

wire, respectively, we have $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{h}$. We thus have $\mathbf{B}(P) = \mathbf{B}_w(\mathbf{r}_1; a) + \mathbf{B}_w(\mathbf{r}_2; b)$. In particular, inside the cavity we have $r_1 < a$ and $r_2 < b$, thus

$$\mathbf{B}(P) = 2k_m \mathbf{r}_1 \times \mathbf{J} + 2k_m \mathbf{r}_2 \times (-\mathbf{J}) = 2k_m (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{J} = 2k_m \mathbf{h} \times \mathbf{J}, \quad (\text{S-5.51})$$

which is a constant vector. Thus, inside the cavity the magnetic field is uniform and perpendicular to both \mathbf{J} and \mathbf{h} .

S-5.8 Conducting Cylinder in a Magnetic Field

a) We use a cylindrical coordinate system (r, ϕ, z) , with the z axis along the cylinder axis. The centrifugal force, \mathbf{F}_c , and the magnetic force, \mathbf{F}_m , are both directed along $\hat{\mathbf{r}}$ and depend on r only:

$$\mathbf{F}_c = m_e \omega^2 \mathbf{r}, \quad \mathbf{F}_m = -e\mathbf{v} \times \mathbf{B}_0 = -e\omega B_0 \mathbf{r}, \quad \frac{|\mathbf{F}_c|}{|\mathbf{F}_m|} = \frac{m_e \omega}{eB_0} \simeq 7.2 \times 10^{-5}. \quad (\text{S-5.52})$$

The magnetic force is dominant, and we shall neglect the centrifugal force in the following.

b) In static conditions the magnetic force must be compensated by an electric field \mathbf{E}

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}_0 = -\omega B_0 \mathbf{r}. \quad (\text{S-5.53})$$

The existence of this electric field implies a uniform charge density

$$\rho = \frac{1}{4\pi k_e} \nabla \cdot \mathbf{E} = \frac{E(r)}{2\pi k_e r} = -\frac{\omega B_0}{2\pi k_e}. \quad (\text{S-5.54})$$

Since the cylinder carries no net charge, its lateral surface must have a charge density

$$\sigma = -\frac{\pi a^2 h \rho}{2\pi a h} = -\frac{a\rho}{2} = \frac{\omega a B_0}{4\pi k_e}. \quad (\text{S-5.55})$$

c) The volume charge density ρ is associated to a volume rotational current density $\mathbf{J}(r)$ due to the cylinder rotation

$$\mathbf{J}(r) = \rho \omega r \hat{\phi} = -\frac{\omega^2 r B_0}{2\pi k_e} \hat{\phi}. \quad (\text{S-5.56})$$

The contribution of $\mathbf{J}(r)$ to the magnetic field on the cylinder axis, B_J , can be evaluated by dividing the

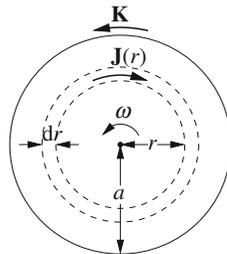


Fig. S-5.11

cylinder into infinitesimal coaxial cylindrical shells between r and $r + dr$. Each shell is equivalent to a solenoid of radius r and product $nI = J(r)dr$, contributing $dB_J = 4\pi k_m J(r)dr$ to the field at its inside. The total contribution of $J(r)$ at a distance r from the axis is thus

$$\begin{aligned} B_J(r) &= 4\pi k_m \int_r^a J(r') dr' = -4\pi k_m \frac{\omega^2 B_0}{2\pi k_e} \int_r^a r' dr' \\ &= -4\pi k_m \frac{\omega^2 B_0}{2\pi k_e} \left[\frac{r'^2}{2} \right]_r^a = -\frac{k_m}{k_e} \omega^2 B_0 (a^2 - r^2). \end{aligned} \quad (\text{S-5.57})$$

Now we must add the contribution B_K of the surface current density $K = \sigma\omega a$

$$B_K = 4\pi k_m \sigma \omega a = 4\pi k_m \omega a \frac{\omega a B_0}{4\pi k_e} = \frac{k_m}{k_e} \omega^2 a^2 B_0 \quad (\text{S-5.58})$$

and the total magnetic field $B_1(r)$ due to the rotational currents is

$$B_1(r) = B_J(r) + B_K = \frac{k_m}{k_e} \omega^2 B_0 r^2, \quad (\text{S-5.59})$$

which is zero on the axis and reaches its maximum value at $r = a^-$. We thus have

$$\frac{B_1(a^-)}{B_0} = \frac{k_m}{k_e} \omega^2 a^2 = \frac{\omega^2 a^2}{c^2} \approx (2.1 \times 10^{-7})^2 \ll 1. \quad (\text{S-5.60})$$

S-5.9 Rotating Cylindrical Capacitor

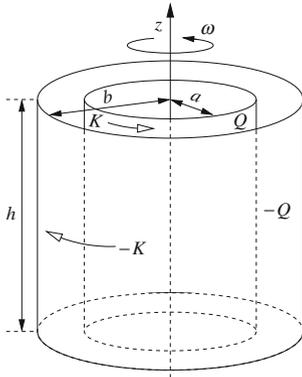


Fig. S-5.12

a) We use cylindrical coordinates (r, ϕ, z) with the z axis coinciding with capacitor axis. We assume $\omega = \omega \hat{z}$, with $\omega = 2\pi/T > 0$. The surface currents due to the capacitor rotation are thus

$$K = \sigma v = \frac{Q}{2\pi ah} \omega a = \frac{Q}{hT}, \quad (\text{S-5.61})$$

where $\sigma = Q/(2\pi ah)$ is the surface charge density on the inner shell, and $-K$ on the outer shell, independently of a and b . Thus the two cylindrical shells are equivalent to two solenoids with nI products $nI = \pm K$, respectively. The outer shell gives origin to a magnetic field $\mathbf{B}_b = -4\pi k_m K \hat{z}$ in the region $r < b$, and to no field in the region $r > b$. The inner shell

gives origin to a field $\mathbf{B}_a = -B_b$ in the region $r < a$, and to no field in the region $r > b$. The total field $\mathbf{B} = \mathbf{B}_a + \mathbf{B}_b$ is thus

$$\mathbf{B} = \begin{cases} -4\pi k_m K \hat{\mathbf{z}} = -4\pi k_m \frac{Q}{hT} \hat{\mathbf{z}}, & a < r < b, \\ 0, & r < a, r > b. \end{cases} \quad (\text{S-5.62})$$

b) The electric field is zero for $r < a$ and $r > b$, while it is $\mathbf{E}(r) = \hat{\mathbf{r}} 2k_e Q/(hr)$ for $a < r < b$, and the force between the two shells is attractive. The electrostatic force per unit area on, for instance, the external shell is thus

$$\mathbf{f}_s^{(e)} = \sigma_b \frac{\mathbf{E}(b)}{2} = -\hat{\mathbf{r}} \frac{Q}{2\pi b h} \frac{2k_e Q}{hb} = -\hat{\mathbf{r}} \frac{k_e Q^2}{\pi b^2 h^2}, \quad (\text{S-5.63})$$

where σ_b is the surface charge density on the shell. The magnetic force per unit area on the same shell is

$$\mathbf{f}_s^{(m)} = \sigma_b \mathbf{v} \times \mathbf{B} = \hat{\mathbf{r}} \frac{Q}{2\pi b h} \frac{2\pi b}{T} 4\pi k_m \frac{Q}{hT} = 4\pi k_m \frac{Q^2}{h^2 T^2}, \quad (\text{S-5.64})$$

directed opposite to the electrostatic force. The ratio $f_s^{(m)}/f_s^{(e)}$ on the outer shell is

$$\frac{f_s^{(m)}}{f_s^{(e)}} = 4\pi k_m \frac{Q^2}{h^2 T^2} \frac{\pi b^2 h^2}{k_e Q^2} = \frac{k_m}{k_e} \left(\frac{2\pi b}{T} \right)^2 = \frac{v_b^2}{c^2} \quad (\text{S-5.65})$$

where $v_b = 2\pi b/T$ is the tangential velocity of the outer shell. The ratio (S-5.65) is thus negligibly small in all practical cases.

S-5.10 Magnetized Spheres

a) The quickest way to obtain the solution is to exploit the analogy of the magnetostatic equations $\nabla \times \mathbf{H} = 0$, $\nabla \cdot \mathbf{B} = 0$ with the electrostatic ones $\nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{D} = 0$ (see also Problem 5.3), along with the definitions (3.4) and (5.19). The spatial distribution of \mathbf{M} is the same as that of \mathbf{P} in Problem (3.3), and the boundary conditions for \mathbf{H} are the same as for \mathbf{E} . Thus from (S-3.13) we immediately obtain that inside the sphere ($r < R$) the field \mathbf{H} is uniform with constant value $\mathbf{H}^{(\text{int})}$, given by

$$\mathbf{H}^{(\text{int})} = \begin{cases} -\frac{\mathbf{M}}{3}, & \text{SI,} \\ -\frac{4\pi\mathbf{M}}{3}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.66})$$

Using (5.19) we obtain for the magnetic field inside the sphere

$$\mathbf{B}^{(\text{int})} = \begin{cases} \frac{2\mu_0\mathbf{M}}{3}, & \text{SI,} \\ \frac{8\pi\mathbf{M}}{3}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.67})$$

Outside the sphere the field is that of a magnetic dipole $\mathbf{m} = \mathbf{M}(4\pi R^3/3)$ located at the center of the sphere ($r = 0$).

b) The rotation of the sphere with uniform surface charge $\sigma = Q/(4\pi R^2)$ generates an azimuthal surface current

$$\mathbf{K}_{\text{rot}} = \sigma v = \sigma R\omega \sin\theta \hat{\phi}, \quad (\text{S-5.68})$$

where $\theta = 0$ corresponds to the direction of \mathbf{M} . This surface current distribution is analogous to that of the magnetization current distribution (5.17) for the magnetized sphere of point **a**),

$$\mathbf{K}_m = \mathbf{M} \times \hat{\mathbf{r}} = M \sin\theta \hat{\phi}. \quad (\text{S-5.69})$$

Thus, the magnetic field generated by \mathbf{K}_{rot} is the same as that generated by \mathbf{K}_m , with the replacement $\mathbf{M} = \sigma R\omega = Q\omega/(4\pi R)$.

c) Analogously to Problem 3.4 for a dielectric sphere in an external electric field, we assume that the induced magnetization $\mathbf{M} = \chi_m \mathbf{H}$ is uniform and parallel to \mathbf{B}_0 . The total field will be the sum of the external field $\mathbf{B}_0 \equiv [\mu_0]\mathbf{H}_0$ (with $[\mu_0]$ replaced by unity for Gaussian units) and of that generated by the magnetization. Thus, inside the sphere \mathbf{H} is uniform and has the value $\mathbf{H}^{(\text{int})}$ given by

$$\mathbf{H}^{(\text{int})} = \begin{cases} \mathbf{H}_0 - \frac{\mathbf{M}}{3} = \mathbf{H}_0 - \frac{\chi_m}{3} \mathbf{H}^{(\text{int})}, & \text{SI,} \\ \mathbf{H}_0 - \frac{4\pi\mathbf{M}}{3} = \mathbf{H}_0 - \frac{4\pi\chi_m}{3} \mathbf{H}^{(\text{int})}, & \text{Gaussian.} \end{cases} \quad (\text{S-5.70})$$

Solving for $\mathbf{H}^{(\text{int})}$ and finally using $\mathbf{B}^{(\text{int})} = [\mu_0]\mu_r \mathbf{H}^{(\text{int})}$ we obtain

$$\mathbf{H}^{(\text{int})} = \frac{3}{\mu_r + 2} \mathbf{H}_0, \quad \mathbf{B}^{(\text{int})} = \frac{3\mu_r}{\mu_r + 2} \mathbf{B}_0, \quad (\text{S-5.71})$$

independently of the system of units; it may be interesting to compare the result with (S-3.21) for the dielectric sphere. The magnetization is given by $\mathbf{M} = \chi_m \mathbf{H}^{(\text{int})}$.

In the case of a perfectly diamagnetic sphere (a superconducting sphere) we have $\mu_r = 0$ and $\mathbf{B}^{(\text{int})} = 0$, and the magnetization is

$$\mathbf{M} = \frac{3\chi_m}{2} \mathbf{H}_0 = -\frac{3}{8\pi} \frac{b_m}{k_m} \mathbf{B}_0. \quad (\text{S-5.72})$$

Actually, inside the sphere the external field is completely screened by the surface currents (S-5.69) due to the magnetization (S-5.72).

It is instructive to double check the above solution by verifying the boundary conditions at the surface of the sphere, analogously to the dielectric case of Solution S-3.4. We choose a spherical coordinate system (r, θ, ϕ) with the zenith direction z parallel to the external magnetic field \mathbf{B}_0 , and the origin at the center of the sphere O , as shown in Fig. S-5.13. As an educated guess, we look for a solution where i) the magnetic field inside the sphere, $\mathbf{B}^{(\text{int})}$, is uniform and proportional to \mathbf{B}_0 , and, accordingly, ii) the magnetization \mathbf{M} of the sphere, proportional to $\mathbf{B}^{(\text{int})}$, is uniform, and iii) the total external field, $\mathbf{B}^{(\text{ext})}$, is the superposition of the applied external field \mathbf{B}_0 and of the field $\mathbf{B}^{(\text{mag})}$, generated by the sphere magnetization. Thus, $\mathbf{B}^{(\text{mag})}$ will be the field generated by a magnetic dipole $\mathbf{m} = \alpha \mathbf{B}_0$ located at the center of the sphere, with α a constant to be determined. Summing up, we are looking for a solution

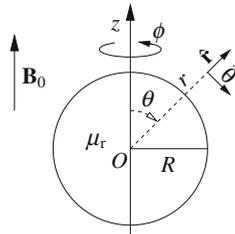


Fig. S-5.13

$$\begin{aligned} \mathbf{B}^{(\text{int})} &= \psi \mathbf{B}_0, \\ \mathbf{B}^{(\text{ext})} &= \mathbf{B}_0 + \mathbf{B}^{(\text{mag})}, \end{aligned} \quad (\text{S-5.73})$$

with ψ a further constant to be determined. $\mathbf{B}^{(\text{mag})}$ and its spherical components are

$$\mathbf{B}^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \left[\left(3 \frac{\hat{\mathbf{z}} \cdot \mathbf{r}}{r^5} \right) \mathbf{r} - \frac{\hat{\mathbf{z}}}{r^3} \right], \quad (\text{S-5.74})$$

$$B_r^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \frac{2 \cos \theta}{r^3}, \quad (\text{S-5.75})$$

$$B_\theta^{(\text{mag})} = \alpha B_0 \frac{k_m}{b_m} \frac{\sin \theta}{r^3}, \quad (\text{S-5.76})$$

$$B_\phi^{(\text{mag})} = 0, \quad (\text{S-5.77})$$

where $k_m/b_m = \mu_0/(4\pi)$ in SI units, and $k_m/b_m = 1$ in Gaussian units. The constants α and ψ are determined from the boundary conditions on \mathbf{B} and \mathbf{B}/μ_r at the surface of the sphere

$$B_\perp^{(\text{int})}(R, \theta) = B_\perp^{(\text{ext})}(R, \theta), \quad \frac{B_\parallel^{(\text{int})}(R, \theta)}{\mu_r} = B_\parallel^{(\text{ext})}(R, \theta), \quad (\text{S-5.78})$$

which lead to

$$\psi B_0 \cos \theta = B_0 \cos \theta + B_r^{(\text{mag})}(R, \theta) = B_0 \cos \theta \left(1 + \alpha \frac{k_m}{b_m} \frac{2}{R^3} \right), \quad (\text{S-5.79})$$

$$\psi \frac{B_0}{\mu_r} \sin \theta = B_0 \sin \theta - B_\theta^{(\text{mag})}(R, \theta) = B_0 \sin \theta \left(1 - \alpha \frac{k_m}{b_m} \frac{1}{R^3} \right). \quad (\text{S-5.80})$$

Dividing (S-5.79) by $B_0 \cos \theta$, and (S-5.80) by $B_0 \sin \theta$, we obtain

$$\psi = 1 + \alpha \frac{k_m}{b_m} \frac{2}{R^3}, \quad \frac{\psi}{\mu_r} = 1 - \alpha \frac{k_m}{b_m} \frac{1}{R^3}, \quad (\text{S-5.81})$$

with solutions

$$\psi = \frac{3\mu_r}{\mu_r + 2}, \quad \alpha = R^3 \frac{b_m}{k_m} \left(\frac{\mu_r - 1}{\mu_r + 2} \right) = \begin{cases} \frac{4\pi R^3}{\mu_0} \left(\frac{\mu_r - 1}{\mu_r + 2} \right), & \text{SI,} \\ R^3 \left(\frac{\mu_r - 1}{\mu_r + 2} \right), & \text{Gaussian,} \end{cases} \quad (\text{S-5.82})$$

which eventually lead to

$$\mathbf{B}^{(\text{int})} = \frac{3\mu_r}{\mu_r + 2} \mathbf{B}_0, \quad \mathbf{m} = \frac{4\pi R^3}{3} \mathbf{M} = R^3 \frac{b_m}{k_m} \left(\frac{\mu_r - 1}{\mu_r + 2} \right) \mathbf{B}_0. \quad (\text{S-5.83})$$