Chapter S-6
Solutions for Chapter 6

S-6.1 A Square Wave Generator

a) The motion is periodic, and we choose the origin of time, $t = 0$, at an instant when the coil surface is completely in the $x \geqslant 0$ half of the xy plane. With this choice, the flux of the magnetic field through the coil, $\Phi(t)$, increases with time when $2n\pi < \omega t < (2n+1)\pi$, with n any integer, and equals $\Phi(t) = B(\omega t \bmod 2\pi)\, a^2/2$. Here, ($x$ mod y) stands for the remainder of the division of x by y with an integer quotient. When $(2n+1)\pi < \omega t < (2n+2)\pi$, the flux decreases with time and equals $\Phi(t) = B\,[2\pi - (\omega t \bmod 2\pi)]\, a^2/2$. The electromotive force in the coil, $\mathcal{E}(t)$, is thus

$$\mathcal{E}(t) = -b_{\mathrm{m}} \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -b_{\mathrm{m}} \frac{Ba^2\omega}{2} \operatorname{sign}[\pi - (\omega t \bmod 2\pi)]\,, \qquad \text{(S-6.1)}$$

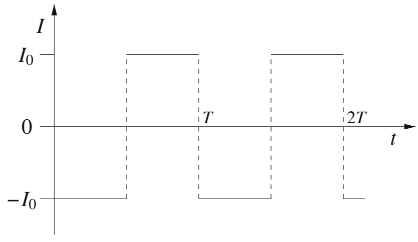

Fig. S-6.1

where $\operatorname{sign}(x) = x/|x|$ is the sign function. Thus, $\mathcal{E}$ reverses its sign whenever $\omega t = n\pi$, with n any integer. The current circulating in the coil is $I = \mathcal{E}/R$. As shown in Fig. S-6.1, I (as well as $\mathcal{E}$) is a square wave of period $T = 2\pi/\omega$, and amplitude

$$I_0 = \frac{\mathcal{E}_0}{R} = b_{\mathrm{m}} \frac{B\omega a^2}{2R}\,. \qquad \text{(S-6.2)}$$

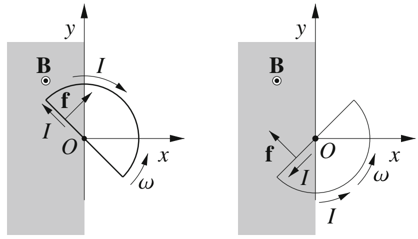

Fig. S-6.2

b) The external torque applied to the coil in order to keep its angular velocity constant must balance the torque exerted by the magnetic forces. The magnetic force on a current-carrying circuit element $\mathrm{d}\boldsymbol{\ell}$ is $\mathrm{d}\mathbf{f} = b_{\mathrm{m}} I\, \mathrm{d}\boldsymbol{\ell} \times \mathbf{B}$, and is different from zero

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only in the $x<0$ half plane. The corresponding torque $\mathrm{d}\boldsymbol{\tau} = \mathbf{r}\times\mathrm{d}\mathbf{f} = b_\mathrm{m}^2 I\mathbf{r}\times(\mathrm{d}\boldsymbol{\ell}\times\mathbf{B})$, where $\mathbf{r}$ is the distance of the coil element $\mathrm{d}\boldsymbol{\ell}$ from the z axis, is always equal to zero on the circumference arc of the coil because the three vectors of the triple product are mutually perpendicular here. Thus, $\mathrm{d}\boldsymbol{\tau}$ is different from zero only on the half of the straight part of the coil inside the magnetic field, where $\mathrm{d}\boldsymbol{\ell} = \mathrm{d}\mathbf{r}$. Here we have $\mathrm{d}\boldsymbol{\tau} = -b_\mathrm{m}^2 I_0 rB\,\mathrm{d}r\,\hat{\mathbf{z}}$, as shown in Fig. S-6.2, and the total torque on the coil, $\boldsymbol{\tau}$, is

$$\boldsymbol{\tau} = \int \mathrm{d}\boldsymbol{\tau} = -b_\mathrm{m}^2\,\omega\,\frac{B^2a^2}{2R}\,\hat{\mathbf{z}}\int_0^a r\,\mathrm{d}r = -b_\mathrm{m}^2\,\omega\,\frac{B^2a^4}{4R}\,\hat{\mathbf{z}}, \tag{S-6.3}$$

corresponding to a power dissipation

$$P_\mathrm{diss} = -\boldsymbol{\tau}\cdot\boldsymbol{\omega} = b_\mathrm{m}^2\,\omega^2\,\frac{B^2a^4}{4R} = RI_0^2\,, \tag{S-6.4}$$

that equals the power dissipated by Joule heating. The power dissipation is constant in time, neglecting the "abrupt" transient phases at $t = n\pi/\omega$, where I instantly changes sign. Thus, the external torque must provide the power dissipated by Joule heating.

c) If we take the coil self-inductance L into account, the equation for the current in the coil becomes

$$\mathcal{E}(t) - L\frac{\mathrm{d}I}{\mathrm{d}t} = RI\,, \tag{S-6.5}$$

where $\mathcal{E}(t)$ is the electromotive force (S-6.1), due to the flux change of the external field only. However "small" L may be, its contribution is not negligible because, if I were an ideal square wave, its derivative $\mathrm{d}I/\mathrm{d}t$ would diverge whenever $t = n\pi/\omega$ (instantaneous transition between $-I_0$ and $+I_0$). The general solution of (S-6.5) is, taking into account that $\mathcal{E}(t)$ is constant over each half-period,

$$I = \frac{\mathcal{E}}{R} + K\mathrm{e}^{-t/t_0}\,, \tag{S-6.6}$$

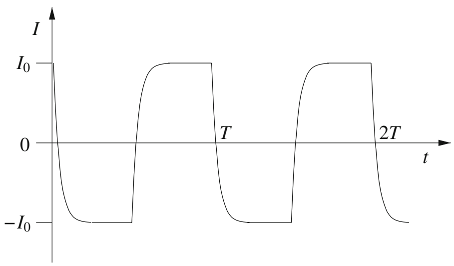

Fig. S-6.3

where $t_0 = L/R$ is the characteristic time of the loop, and K is a constant to be determined from the initial conditions. If L is small enough, we can assume that at time $t = 0^-$ we have $\mathcal{E}(t) = +\mathcal{E}_0$ and $I(t) = +I_0$. At time $t = 0$, $\mathcal{E}(t)$ switches instantaneously from $+\mathcal{E}_0$ to $-\mathcal{E}_0$, and the constant K is determined by the initial condition $I(0) = I(0^-) = I_0 = \mathcal{E}_0/R$, leading to $K = 2\mathcal{E}_0/R$. Thus, for $0 < t < \pi/\omega$,

$$I(t) = I_0\,(2e^{-t/t_0} - 1)\,. \tag{S-6.7}$$

At $t = (\pi/\omega)^-$ we have $\mathcal{E}(t) = -\mathcal{E}_0$ and we can assume that $I(t) = -I_0$. At $t = \pi/\omega$ $\mathcal{E}(t)$ switches instantaneously from $-\mathcal{E}_0$ to $+\mathcal{E}_0$, and, for $\pi/\omega < t < 2\pi/\omega$, we have

$$I(t) = -I_0\,(2e^{-(t-\pi/\omega)/t_0} - 1)\,, \tag{S-6.8}$$

and so on for the successive periods.

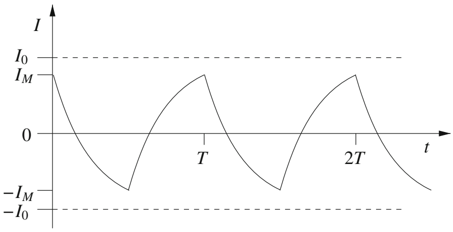

Fig. S-6.4

The self-inductance of the coil prevents the current from switching instantaneously between $+I_0$ and $-I_0$: the change occurs following an exponential with characteristic time $t_0 = L/R$.

The behavior described by (S-6.7) and (S-6.8) is valid only if $t_0 \ll T = 2\pi/\omega$, as in Fig. S-6.3, representing the case of $t_0 = 0.04\,T$. If t_0 is not negligible with respect to T, the current oscillates between two values $+I_M$ and $-I_M$, with $I_M < I_0$. Let us consider the time interval $0 \le t \le \pi/\omega$. We must have $I(0) = I_M$ and $I(\pi/\omega) = -I_M$. Replacing I by (S-6.6), we obtain

$$I_M = I_0\,\frac{1 - e^{-T/2t_0}}{1 + e^{-T/2t_0}}\,. \tag{S-6.9}$$

The plot of $I(t)$ can no longer be approximated by a square wave, as shown in Fig. S-6.4 for the case $t_0 = 0.25\,T$.

S-6.2 A Coil Moving in an Inhomogeneous Magnetic Field

a) With our assumptions, the flux of the magnetic field through the coil can be approximated as

$$\Phi_{\mathbf{B}}(t) = \Phi_{\mathbf{B}}[z(t)] \simeq \pi a^2 B_0\,\frac{z(t)}{L} = \pi a^2 B_0\,\frac{z_0 + vt}{L}\,, \tag{S-6.10}$$

where z_0 is the position of the center of the coil at $t = 0$. The rate of change of this magnetic flux is associated to an electromotive force $\mathcal{E}$, and to a current $I = \mathcal{E}/R$ circulating in the coil

$$\mathcal{E} = RI = -b_{\mathrm{m}}\frac{d\Phi}{dt} = -b_{\mathrm{m}}\,\pi a^2 B_0\,\frac{v}{L}\,. \tag{S-6.11}$$

b) The power dissipated by Joule heating is

$$P = RI^2 = \frac{\mathcal{E}^2}{R} = b_{\mathrm{m}}^2\,\frac{(\pi a^2 B_0\,v)^2}{L^2 R}\,. \tag{S-6.12}$$

Thus, in order to keep the coil in motion at constant speed, one must exert an external force $\mathbf{f}_{\text{ext}}$ on the coil, whose work compensates the dissipated power. We have

$$\mathbf{f}_{\text{ext}} \cdot \boldsymbol{v} = P = b_{\text{m}}^2 \frac{(\pi a^2 B_0 v)^2}{L^2 R}, \tag{S-6.13}$$

and the coil is submitted to a frictional force proportional to its velocity

$$\mathbf{f}_{\text{frict}} = -\mathbf{f}_{\text{ext}} = -b_{\text{m}}^2 \frac{(\pi a^2 B_0)^2}{L^2 R} \boldsymbol{v} . \tag{S-6.14}$$

c) The force $\mathbf{f}_{\text{frict}}$ is actually the net force obtained by integrating the force $\mathrm{d}\mathbf{f}_{\text{frict}} = b_{\text{m}} I \,\mathrm{d}\boldsymbol{\ell} \times \mathbf{B}$ acting on each coil element $\mathrm{d}\boldsymbol{\ell}$:

$$\mathbf{f}_{\text{frict}} = b_{\text{m}} I \oint_{\text{coil}} \mathrm{d}\boldsymbol{\ell} \times \mathbf{B} . \tag{S-6.15}$$

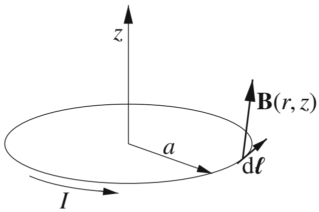

Fig. S-6.5

The contribution of the z component of $\mathbf{B}$ is a radial force tending to shrink the coil if $\partial_t \Phi > 0$, or to widen it if $\partial_t \Phi < 0$, according to Lenz's law; the case represented in Fig. S-6.5 corresponds to the latter case. Thus $\mathbf{f}_{\text{ext}}$, directed along z, is due only to the radial component B_r of $\mathbf{B}$. The component B_r is not given by the problem, but, as we saw at answer a) of Problem 5.4, it can be evaluated by applying Gauss's law to a closed cylindrical surface of radius r and height Δz. According to (S-5.25)

$$B_r \simeq -\frac{B_0}{2L} r, \quad \text{thus} \quad \mathrm{d}\mathbf{f}_{\text{frict}} = b_{\text{m}} I \,\mathrm{d}\ell \frac{B_0 a}{2L}, \tag{S-6.16}$$

and by substituting (S-6.11) and integrating over the coil we obtain

$$\begin{aligned} \mathbf{f}_{\text{frict}} &= \hat{\mathbf{z}} b_{\text{m}} I \oint_{\text{coil}} \mathrm{d}\ell \frac{B_0 a}{2L} = -\hat{\mathbf{z}} b_{\text{m}} \left(b_{\text{m}} \pi a^2 B_0 \frac{v}{L}\right)\left(2\pi a \frac{B_0 a}{2L}\right) \\ &= -\hat{\mathbf{z}} b_{\text{m}}^2 \frac{(\pi a^2 B_0)^2}{L^2 R} v, \end{aligned} \tag{S-6.17}$$

in agreement with (S-6.14).

S-6.3 A Circuit with "Free-Falling" Parts

a) We choose the x axis oriented downwards, with the origin at the location of the upper horizontal bar, as in Fig. S-6.6. The current I in the rectangular circuit is

$$I = \frac{\mathcal{E}}{R} = -b_\mathrm{m}\,\frac{1}{R}\frac{\mathrm{d}\Phi(\mathbf{B})}{\mathrm{d}t} = -b_\mathrm{m}\,\frac{Ba}{R}\frac{\mathrm{d}x}{\mathrm{d}t} = -b_\mathrm{m}\,\frac{Bav}{R}\,, \tag{S-6.18}$$

where x is the position of the falling bar, and $v = \dot{x}$ its velocity. The velocity is positive, and the current I is negative, i.e., it circulates clockwise, in agreement with Lenz's law. The magnetic force on the falling bar is $\mathbf{f}_B = b_\mathrm{m}BaI\,\hat{\mathbf{x}}$, antiparallel to the gravitational force $m\mathbf{g}$, and the equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg + b_\mathrm{m}BaI = mg - b_\mathrm{m}^2\,\frac{(Ba)^2}{R}\,v. \tag{S-6.19}$$

The solution of (S-6.19), with the initial condition $v(0) = 0$, is

$$v(t) = v_t\left(1 - \mathrm{e}^{-t/\tau}\right) \tag{S-6.20}$$

where

$$\tau = \frac{mR}{(b_\mathrm{m}Ba)^2} \quad \text{and} \quad v_t = g\tau = \frac{mRg}{(b_\mathrm{m}Ba)^2}. \tag{S-6.21}$$

As $t \to \infty$, the falling bar approaches the terminal velocity v_t.

b) When $v = v_t$, the power dissipated in the circuit by Joule heating is

$$P_J = RI_t^2 = \frac{(b_\mathrm{m}Bav_t)^2}{R} = \left(\frac{mg}{b_\mathrm{m}Ba}\right)^2 R, \tag{S-6.22}$$

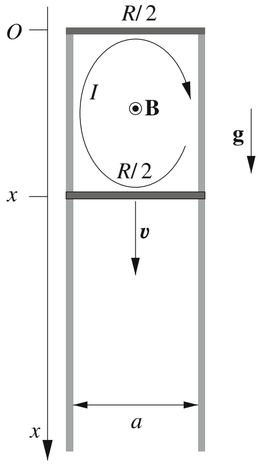

Fig. S-6.6

where $I_t = -b_\mathrm{m}Bav_t/R$ is the "terminal current". On the other hand, the work done by the force of gravity per unit time is

$$P_G = m\mathbf{g}\cdot\mathbf{v}_t = mg\frac{mgR}{(b_\mathrm{m}Ba)^2} = P_J, \tag{S-6.23}$$

in agreement with energy conservation for the bar moving at constant velocity.

c) When both horizontal bars are falling, we denote by x_1 the position of the upper bar, and by x_2 the position of the lower bar, as in Fig. S-6.7, with $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$. The current I circulating in the circuit is

$$\begin{aligned} I &= \frac{\mathcal{E}}{R} = -b_\mathrm{m}\,\frac{1}{R}\frac{\mathrm{d}\Phi(\mathbf{B})}{\mathrm{d}t} = -b_\mathrm{m}\,\frac{Ba}{R}\frac{\mathrm{d}}{\mathrm{d}t}(x_2 - x_1) \\ &= -b_\mathrm{m}\,\frac{Ba}{R}\,(v_2 - v_1), \end{aligned} \tag{S-6.24}$$

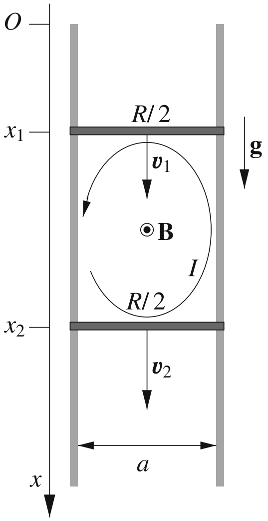

Fig. S-6.7

circulating counterclockwise ($I > 0$) if $v_1 > v_2$, and clockwise ($I < 0$) if $v_1 < v_2$. The magnetic forces acting on the two falling bars are $\mathbf{f}_{B1} = -b_\mathrm{m} BaI\,\hat{\mathbf{x}}$ and $\mathbf{f}_{B2} = b_\mathrm{m} BaI\,\hat{\mathbf{x}}$, respectively. Independently of the sign of I, we have $\mathbf{f}_{B1} = -\mathbf{f}_{B_2}$, so that the net magnetic force on the system comprising the two falling bars is zero. The equations of motion are thus

$$m\frac{\mathrm{d}v_1}{\mathrm{d}t} = mg + b_\mathrm{m}^2\frac{(Ba)^2}{R}(v_2 - v_1) \tag{S-6.25}$$

$$m\frac{\mathrm{d}v_2}{\mathrm{d}t} = mg - b_\mathrm{m}^2\frac{(Ba)^2}{R}(v_2 - v_1), \tag{S-6.26}$$

with the initial conditions $v_1(0) = v_0$ and $v_2(0) = 0$. The sum of equations (S-6.25) and (S-6.26) is

$$\frac{\mathrm{d}}{\mathrm{d}t}(v_1 + v_2) = 2g, \quad \text{with solution} \quad \frac{v_1 + v_2}{2} = \frac{v_0}{2} + gt, \tag{S-6.27}$$

meaning that the center of mass of the two horizontal bars follows a free fall, independent of the magnetic field **B**. On the other hand, the difference of equations (S-6.25) and (S-6.26) is

$$\frac{\mathrm{d}}{\mathrm{d}t}(v_1 - v_2) = -\frac{2}{\tau}(v_1 - v_2), \quad \text{with solution} \quad v_1 - v_2 = v_0\,\mathrm{e}^{-2t/\tau}, \tag{S-6.28}$$

where $\tau = mR/(b_\mathrm{m} Ba)^2$. For the velocities of the two horizontal bars we obtain

$$v_1 = \frac{v_0}{2}\left(1 + \mathrm{e}^{-2t/\tau}\right) + gt, \quad v_2 = \frac{v_0}{2}\left(1 - \mathrm{e}^{-2t/\tau}\right) + gt. \tag{S-6.29}$$

At the steady state limit ($t \gg \tau$) we have

$$\lim_{t\to\infty} v_1 = \lim_{t\to\infty} v_2 = \frac{v_0}{2} + gt \quad \text{and} \quad \lim_{t\to\infty} I = 0, \tag{S-6.30}$$

since, for $v_1 = v_2$, the flux of **B** through the loop is constant.

S-6.4 The Tethered Satellite

a) To within our approximations, we can assume that the magnetic field is constant over the satellite orbit, and equal to the field at the Earth's equator, $B_\mathrm{eq} \simeq 3.2 \times 10^{-5}$ T. The field is parallel to the axis of the satellite orbit, and constant over the tether length. The electromotive force $\mathcal{E}$ on the tether equals the line integral of the magnetic force along the wire,

$$\mathcal{E} = b_\mathrm{m} \int_\mathrm{tether} \mathrm{d}\boldsymbol{\ell} \cdot \boldsymbol{v}(r) \times \mathbf{B}_\mathrm{eq} = b_\mathrm{m} \int_{R_\oplus + h}^{R_\oplus + h - \ell} \mathrm{d}r\,\omega r\, B_\mathrm{eq}\,, \tag{S-6.31}$$

where $\omega = v/r$ is the angular velocity of the satellite. To within our approximations we can also assume that also $v(r) \simeq v(R_\oplus) \simeq 8\,000\,\mathrm{m/s}$ is constant over the wire length, and obtain

$$\mathcal{E} \simeq b_\mathrm{m} v \ell B_\mathrm{eq} = \begin{cases} 8\,000 \times 1\,000 \times 3.2 \times 10^{-5} \simeq 250\,\mathrm{V}\,, & \text{SI,} \\ \dfrac{1}{c} \times 8 \times 10^5 \times 10^5 \times 0.32 \simeq 0.85\,\mathrm{statV}\,, & \text{Gaussian.} \end{cases} \tag{S-6.32}$$

b) Neglecting the resistance of the ionosphere, the current I circulating in the wire, and the corresponding power dissipated by Joule heating P_diss are, respectively,

$$I = \frac{\mathcal{E}}{R} = b_\mathrm{m} \frac{v \ell B_\mathrm{eq}}{R}, \quad \text{and} \quad P_\mathrm{diss} = RI^2 = b_\mathrm{m}^2 \frac{v^2 \ell^2 B_\mathrm{eq}^2}{R}\,. \tag{S-6.33}$$

The power dissipated in the tether by Joule heating must equal minus the work done by the magnetic force on the wire. This can be easily verified, since the magnetic force acting on the wire is

$$\mathbf{F} = b_\mathrm{m} I \ell\, \hat{\mathbf{r}} \times \mathbf{B}_\mathrm{eq} = -b_\mathrm{m}^2 \frac{B_\mathrm{eq}^2 \ell^2}{R} \boldsymbol{v}\,, \tag{S-6.34}$$

and the corresponding work rate is

$$P = \mathbf{F} \cdot \boldsymbol{v} = -b_\mathrm{m}^2 \frac{B_\mathrm{eq}^2 \ell^2}{R} v^2 = -P_\mathrm{diss}\,. \tag{S-6.35}$$

If we assume that the tether is a copper wire (conductivity $\sigma \simeq 10^7\ \Omega^{-1}\mathrm{m}^{-1}$ SI, $\sigma \simeq 9 \times 10^{16}\,\mathrm{s}^{-1}$ Gaussian) of cross section A=1 cm^2, the magnitude of the magnetic drag force on the system is

$$F_\mathrm{drag} = b_\mathrm{m}^2 \frac{B_\mathrm{eq}^2 \ell^2}{\ell/(\sigma A)} v = \begin{cases} \dfrac{(3.2 \times 10^{-5})^2 \times 1000}{1/(10^7 \times 10^{-4})} \times 8000 \simeq 8.2\,\mathrm{N}\,, & \text{SI} \\ \dfrac{1}{c^2} \dfrac{(0.32)^2 \times 10^5}{1/(9 \times 10^{16})} \times 8 \times 10^5 \simeq 8.2 \times 10^5\,\mathrm{dyn}\,, & \text{Gaussian.} \end{cases} \tag{S-6.36}$$

This problem gives an elementary description of the principle of the "Tethered Satellite System", investigated in some Space Shuttle missions as a possible generator of electric power for orbiting systems.

S-6.5 Eddy Currents in a Solenoid

a) The time-dependent current in the solenoid generates a time-dependent magnetic field which, in turn, induces a time-dependent contribution to the electric field. The induced electric field is associated to a displacement current density, and, in a conductor, also to a conduction current density $\mathbf{J} = \sigma\mathbf{E}$. Both current densities, in turn, affect the magnetic field. According to our symmetry assumptions, in a cylindrical coordinate system (r, ϕ, z), with the solenoid axis as z axis, the only non-zero component of the magnetic field is B_z, and the only non-zero component of the electric field is E_ϕ, therefore the only non-zero component of the conduction current density is J_ϕ. Both B_z and E_ϕ depend only on r. In principle we must solve (6.1) and (6.5) which in cylindrical coordinates yield (see Table A.1 of the Appendix),

$$\frac{1}{r}\partial_r(rE_\phi) = -b_\mathrm{m}\,\partial_t B_z\,, \tag{S-6.37}$$

$$-\partial_r B_z = 4\pi k_\mathrm{m}\,\sigma E_\phi + \frac{k_\mathrm{m}}{k_\mathrm{e}}\,\partial_t E_\phi\,. \tag{S-6.38}$$

Finding the complete solution to (S-6.37) and (S-6.38) is possible but somewhat involved. However, if the angular frequency ω of the driving current is low enough, the *slowly varying current approximation* (SVCA) provides a sufficiently accurate solution of the problem.

In the SVCA, we start by calculating $\mathbf{B}$ as in the static case. Neglecting boundary effects, a DC current I would generate a uniform magnetic field $\mathbf{B} = \hat{\mathbf{z}}4\pi k_\mathrm{m}\mu_\mathrm{r} nI$ inside our solenoid, and $\mathbf{B} \equiv 0$ outside. Thus, inside the solenoid, we would have $\mathbf{B} = \hat{\mathbf{z}}\mu_0\mu_\mathrm{r} nI$ in SI units, and $\mathbf{B} = \hat{\mathbf{z}}4\pi\mu_\mathrm{r} nI/c$ in Gaussian units. If we replace I by $I_0\cos\omega t$ we obtain

$$\mathbf{B}^{(0)} = \hat{\mathbf{z}}4\pi k_\mathrm{m}\,\mu_\mathrm{r} nI_0\cos\omega t\,, \tag{S-6.39}$$

which we assume as our zeroth-order approximation for the field inside the solenoid. In the next step of SVCA, we evaluate the first order correction by calculating the electric field $\mathbf{E}^{(1)}$ induced by (S-6.39), and its associated current densities. These current densities, in turn, contribute to the first order correction to the magnetic field. *A posteriori*, our procedure will be justified if the first order correction to the magnetic field, $\mathbf{B}^{(1)}$, is much smaller than $\mathbf{B}^{(0)}$. And so on for the successive correction orders. For additional simplicity, we neglect the displacement current, i.e., the last term on the right-hand side of (S-6.38), although its inclusion would not be difficult.

Using (6.1) and the symmetry assumptions, the first-order electric field $\mathbf{E}^{(1)}(r) = \hat{\boldsymbol{\phi}}E^{(1)}(r)$ can be found from its path integral over the circumference of radius r,

$$\oint \mathbf{E}^{(1)}\cdot \mathrm{d}\boldsymbol{\ell} = 2\pi r E^{(1)}(r) = -b_\mathrm{m}\pi r^2\partial_t B^{(0)}\,, \tag{S-6.40}$$

which yields

$$\mathbf{E}^{(1)}(r) = k_\mathrm{m} b_\mathrm{m}\, 2\pi\mu_\mathrm{r} r n I_0\, \omega \sin\omega t\, \hat{\boldsymbol{\phi}}$$
$$= \begin{cases} \dfrac{\mu_0\mu_\mathrm{r}}{2} r n I_0 \omega \sin\omega t \hat{\boldsymbol{\phi}}, & \text{SI,} \\ \dfrac{1}{c^2} 2\pi\mu_\mathrm{r} r n I_0\, \omega \sin\omega t\, \hat{\boldsymbol{\phi}}, & \text{Gaussian.} \end{cases} \quad \text{(S-6.41)}$$

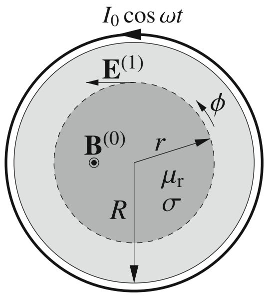

Fig. S-6.8

Notice that the induced electric field also generates an electromotive force $\mathcal{E}^{(1)}$ in the solenoid coils. We assume the generator producing the current $I(t) = I_0 \cos\omega t$ to be an ideal one, which maintains the same current against any effect occurring in the circuit (the appearance of $\mathcal{E}^{(1)}$ will require extra work to maintain the current).

b) Due to the conductivity σ of the solenoid core, the electric field $\mathbf{E}^{(1)}(r)$ originates an azimuthal current density $\mathbf{J}^{(1)}(r) = \sigma\mathbf{E}^{(1)}(r)$ (eddy currents) in the material. The corresponding Joule dissipation heats up the material. The energy turned into heat per unit volume at each instant t is

$$\mathbf{J}^{(1)}(r)\cdot\mathbf{E}^{(1)}(r) = \sigma\left[E^{(1)}(r)\right]^2 = \sigma\,(k_\mathrm{m} b_\mathrm{m}\, 2\pi\mu_\mathrm{r} r n I_0\, \omega \sin\omega t)^2\,, \quad \text{(S-6.42)}$$

with a time average

$$\left\langle \mathbf{J}^{(1)}(r)\cdot\mathbf{E}^{(1)}(r)\right\rangle = 2\sigma(k_\mathrm{m} b_\mathrm{m}\, \pi\mu_\mathrm{r} r n I_0\, \omega)^2\,. \quad \text{(S-6.43)}$$

The total dissipated power is found by integrating (S-6.43) over the volume of the cylindrical core

$$P_d = \int_{\text{cylinder}} \left\langle \mathbf{J}^{(1)}(r)\cdot\mathbf{E}^{(1)}(r)\right\rangle \mathrm{d}^3x = 2\sigma\,(k_\mathrm{m} b_\mathrm{m}\, \pi\mu_\mathrm{r} n I_0\, \omega)^2 \int_0^R r^2 \ell\, 2\pi r \mathrm{d}r$$
$$= \sigma\pi\ell\left(k_\mathrm{m} b_\mathrm{m}\, \pi\mu_\mathrm{r} n I_0\, \omega R^2\right)^2 = \begin{cases} \dfrac{\sigma\mu_0^2\mu_\mathrm{r}^2}{16}\pi n^2 I_0^2 \omega^2 \ell R^4\,, & \text{SI,} \\ \dfrac{1}{c^4}\pi^3\sigma\mu_\mathrm{r}^2 n^2 I_0^2 \omega^2 \ell R^4\,, & \text{Gaussian.} \end{cases} \quad \text{(S-6.44)}$$

c) The induced current density $\mathbf{J}^{(1)}(r) = \sigma\mathbf{E}^{(1)}(r)$ generates a magnetic field $\mathbf{B}^{(1)}(r)$ in the cylindrical volume enclosed by the surface of radius r. Each infinitesimal cylindrical shell between r and $r + \mathrm{d}r$ of Fig. S-6.9 behaves like a solenoid of radius r, generating a magnetic field whose value is obtained by replacing the product nI by the product $J^{(1)}(r)\,\mathrm{d}r$. Thus, the contribution to the magnetic field in r of the infinitesimal shell is

$$\mathrm{d}\mathbf{B}^{(1)}_{\text{int}}(r) = \hat{\mathbf{z}}\, 4\pi k_\mathrm{m}\mu_\mathrm{r}\sigma E^{(1)}(r)\,. \quad \text{(S-6.45)}$$

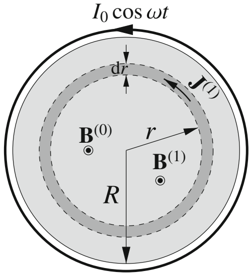

Fig. S-6.9

Also all infinitesimal cylindrical shells between r' and $r'+\mathrm{d}r'$, with $r<r'<R$, contribute to the field in r, and the resulting first-order correction to the field in r is

$$\mathbf{B}^{(1)}(r) = \int_r^R \mathrm{d}\mathbf{B}^{(1)}_{\mathrm{int}}(r) = 4\pi k_\mathrm{m}\mu_\mathrm{r}\sigma \int_r^R E^{(1)}(r')\,\mathrm{d}r' \,. \tag{S-6.46}$$

If we replace (S-6.41) into (S-6.46) we obtain

$$\begin{aligned}\mathbf{B}^{(1)}(r) &= 8\pi^2 k_\mathrm{m}^2 b_\mathrm{m}\mu_\mathrm{r}^2\sigma n I_0\omega\sin\omega t\int_r^R r'\,\mathrm{d}r' = \hat{\mathbf{z}}4\pi^2k_\mathrm{m}^2 b_\mathrm{m}\mu_\mathrm{r}^2\sigma n I_0\,(R^2-r^2)\,\omega\sin\omega t\\ &= \begin{cases}\hat{\mathbf{z}}\,\dfrac{1}{4}\,\mu_0^2\mu_\mathrm{r}^2\sigma n I_0\,(R^2-r^2)\,\omega\sin\omega t\,, & \text{SI,}\\ \hat{\mathbf{z}}\,\dfrac{4\pi^2}{c^3}\,\mu_\mathrm{r}^2\sigma n I_0\,(R^2-r^2)\,\omega\sin\omega t\,, & \text{Gaussian.}\end{cases}\end{aligned} \tag{S-6.47}$$

Thus, $\mathbf{B}^{(1)}(r)$ is maximum for $r=0$, where all infinitesimal cylindrical shells contribute, and zero for $r=R$. Our treatment is justified if $B^{(1)}(0) \ll B^{(0)}$ for all $r<R$ and for all t, i.e., if

$$\frac{\left\langle B^{(1)}(0)\right\rangle}{\left\langle B^{(0)}\right\rangle} = \pi k_\mathrm{m} b_\mathrm{m}\mu_\mathrm{r}\sigma\omega R^2 \ll 1\,, \tag{S-6.48}$$

where the angle brackets denote the average over time. This gives the condition on ω

$$\omega \ll \frac{1}{\pi k_\mathrm{m} b_\mathrm{m}\mu_\mathrm{r}\sigma R^2} = \begin{cases}\dfrac{4}{\mu_0\mu_\mathrm{r}\sigma R^2}\,, & \text{SI,}\\ \dfrac{c^2}{\pi\mu_\mathrm{r}\sigma R^2}\,, & \text{Gaussian.}\end{cases} \tag{S-6.49}$$

Thus, for materials with a high value of the product $\mu_\mathrm{r}\sigma$, the frequency must be very low. For instance, iron has a relative magnetic permeability $\mu_\mathrm{r}\simeq 5\,000$, and a conductivity $\sigma\simeq 10^7\ \Omega^{-1}\mathrm{m}^{-1}$ in SI units. Assuming a solenoid with $R=1$ cm, we obtain the following condition on the frequency ν of the driving current

$$\nu = \frac{\omega}{2\pi} \ll \frac{4}{8\pi^2\times10^{-7}\times5\times10^3\times10^7\times10^{-4}} \simeq 0.10\,\mathrm{Hz}\,, \tag{S-6.50}$$

which is a very low value. Iron is a good material as the core of an electromagnet, due to its high magnetic permeability, but a poor material as the core of a transformer or of an inductor, due to its high conductivity, which gives origin to high eddy-current losses. On the other hand, manganese-zinc ferrite (a ceramic compound containing iron oxides combined with zinc and manganese compounds) also has a relative magnetic permeability $\mu_\mathrm{r}\simeq 5000$, but a much lower conductivity,

$\sigma \simeq 5\ \Omega^{-1}\mathrm{m}^{-1}$. The condition on the frequency of the driving current is thus

$$\nu \ll \frac{4}{8\pi^2 \times 10^{-7} \times 5 \times 10^3 \times 5 \times 10^{-4}} \simeq 2 \times 10^5\,\mathrm{Hz}, \tag{S-6.51}$$

and ferrite is used in electronics industry to make cores for inductors and transformers, and in various microwave components.

It is also instructive to compare the energy dissipated per cycle, $U_{\mathrm{diss}} = (2\pi/\omega)P_{\mathrm{diss}}$, to the total magnetic energy stored in the solenoid,

$$U_{\mathrm{M}} = \left\langle \frac{b_{\mathrm{m}}\left(B^{(0)}\right)^2}{2k_{\mathrm{m}}\mu_{\mathrm{r}}} \right\rangle \pi R^2 \ell = \begin{cases} \left\langle \dfrac{\left(B^{(0)}\right)^2}{2\mu_0\mu_{\mathrm{r}}} \right\rangle \pi R^2 \ell\,, & \text{SI,} \\ \left\langle \dfrac{\left(B^{(0)}\right)^2}{8\pi\mu_{\mathrm{r}}} \right\rangle \pi R^2 \ell\,, & \text{Gaussian.} \end{cases} \tag{S-6.52}$$

The ratio is

$$\frac{U_{\mathrm{diss}}}{U_{\mathrm{M}}} \simeq \frac{\pi}{4} k_{\mathrm{m}} b_{\mathrm{m}} \mu_{\mathrm{r}} \sigma \omega R^2. \tag{S-6.53}$$

Thus, the condition (S-6.49) is also equivalent to the requirement that the energy loss per cycle due to Joule heating is small compared to the total stored magnetic energy.

S-6.6 Feynman's "Paradox"

a) The mutual inductance M between the charged ring and the superconducting ring is, assuming $a \ll R$ (see Problem 6.12),

$$M = 4\pi k_{\mathrm{m}} b_{\mathrm{m}} \frac{\pi a^2}{2R}\,. \tag{S-6.54}$$

Thus, when a current $I(t)$ is circulating in the smaller ring of radius a, the magnetic flux through the charged ring is

$$\Phi_I = M I(t) = 4\pi k_{\mathrm{m}} b_{\mathrm{m}} \frac{\pi a^2}{2R} I(t)\,. \tag{S-6.55}$$

If Φ_I is time-dependent, it gives origin to an induced electric field $\mathbf{E}_I$, whose line-integral around the charged ring is

$$\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -b_{\mathrm{m}} \frac{\mathrm{d}\Phi_I}{\mathrm{d}t} = -4\pi k_{\mathrm{m}} b_{\mathrm{m}}^2 \frac{\pi a^2}{2R}\, \partial_t I(t)\,. \tag{S-6.56}$$

Due to the symmetry of our problem, field $\mathbf{E}_I$ is azimuthal on the xy plane, and independent of ϕ. Its magnitude on the charged ring is thus

$$E_I = \frac{1}{2\pi R}\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R^2}\,\partial_t I(t)\,, \tag{S-6.57}$$

and the force exerted on an infinitesimal element $\mathrm{d}\ell$ of the charged ring is

$$\mathrm{d}\mathbf{f} = \mathbf{E}_I \lambda\,\mathrm{d}\ell = -\hat{\boldsymbol{\phi}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R^2}\,\lambda\,\mathrm{d}\ell\,\partial_t I(t)\,, \tag{S-6.58}$$

corresponding to a torque $\mathrm{d}\boldsymbol{\tau}$ about the center of the ring

$$\mathrm{d}\boldsymbol{\tau} = \mathbf{r}\times\mathrm{d}\mathbf{f} = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\,\lambda\,\mathrm{d}\ell\,\partial_t I(t)\,. \tag{S-6.59}$$

The total torque on the charged ring is thus

$$\boldsymbol{\tau} = \int \mathrm{d}\boldsymbol{\tau} = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\,\lambda\, 2\pi R\,\partial_t I(t) = -\hat{\mathbf{z}}\, k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\, Q\,\partial_t I(t)\,, \tag{S-6.60}$$

where $Q = 2\pi R\lambda$ is the total charge of the ring. The equation of motion for the charged ring is thus

$$mR^2 \frac{\mathrm{d}\omega}{\mathrm{d}t} = \boldsymbol{\tau} = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{R}\, Q\,\partial_t I(t)\,, \tag{S-6.61}$$

where mR^2 is the moment of inertia of the ring. The solution for $\omega(t)$ is

$$\omega(t) = -k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q \int_0^t \partial_t I(t')\,\mathrm{d}t' = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q\,[I_0 - I(t)]\,, \tag{S-6.62}$$

and the final angular velocity is

$$\omega_\mathrm{f} = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2}{mR^3}\, Q I_0 = \begin{cases} \dfrac{\mu_0 a^2 Q}{4mR^3}\, I_0\,, & \text{SI,} \\[2ex] \dfrac{\pi a^2 Q}{c^3 mR^3}\, I_0\,, & \text{Gaussian,} \end{cases} \tag{S-6.63}$$

corresponding to a final angular momentum

$$L_\mathrm{f} = mR^2\omega_\mathrm{f} = k_\mathrm{m} b_\mathrm{m}^2 \frac{\pi a^2 Q}{R}\, I_0 = \begin{cases} \dfrac{\mu_0 a^2 Q}{R}\, I_0\,, & \text{SI,} \\[2ex] \dfrac{\pi a^2 Q}{c^3 R}\, I_0\,, & \text{Gaussian,} \end{cases} \tag{S-6.64}$$

independent of the mass m of the ring.

b) The rotating charged ring is equivalent to a circular loop carrying a current $I_{\mathrm{rot}} = Q\omega/2\pi$. Thus, after the current in the small ring is switched off, there is still a magnetic field due to the rotation of the charged ring. The final magnetic field at the center of the rings is

$$\begin{aligned} \mathbf{B}_{\mathrm{c}} &= \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}}{2}\,\frac{I_{\mathrm{rot}}}{R} = \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}}{4\pi}\,\frac{Q\omega_{\mathrm{f}}}{R} \\ &= \hat{\mathbf{z}}\,\frac{k_{\mathrm{m}}^2 b_{\mathrm{m}}^2 a^2 Q^2}{4mR^4}\,I_0 = \begin{cases} \hat{\mathbf{z}}\,\dfrac{\mu_0^2 a^2 Q^2}{64\pi^2 mR^4}\,I_0\,, & \text{SI,} \\ \hat{\mathbf{z}}\,\dfrac{a^2 Q^2}{4c^4 mR^4}\,I_0\,, & \text{Gaussian,} \end{cases} \end{aligned} \tag{S-6.65}$$

parallel to the initial field $\mathbf{B}_0 = \hat{\mathbf{z}}\,k_{\mathrm{m}} I_0/(2a)$, in agreement with Lenz's law. We further have

$$\pi a^2 B_c = M I_{\mathrm{rot}}\,, \tag{S-6.66}$$

where M is the mutual inductance of the rings (S-6.54).

c) As seen above at point **b)**, the rotating charged ring generates a magnetic field all over the space. This field modifies the magnetic flux through the rotating ring itself, giving origin to self-induction. Let $\mathcal{L}$ be the "self-inductance" of the rotating ring. The magnetic flux generated by the rotating ring through itself is

$$\Phi_{\mathrm{rot}} = \frac{1}{b_{\mathrm{m}}}\,\mathcal{L} I_{\mathrm{rot}} = \frac{1}{b_{\mathrm{m}}}\,\mathcal{L}\,\frac{Q\omega}{2\pi}\,. \tag{S-6.67}$$

Correspondingly, (S-6.56) for the line integral of the electric field around the charged ring is modified as follows:

$$\oint \mathbf{E}_I \cdot \mathrm{d}\boldsymbol{\ell} = -b_{\mathrm{m}}\left(\frac{\mathrm{d}\Phi_I}{\mathrm{d}t} + \frac{\mathrm{d}\Phi_{\mathrm{rot}}}{\mathrm{d}t}\right) = -\frac{4\pi^2 k_{\mathrm{m}} b_{\mathrm{m}}^2 a^2}{2R}\,\partial_t I - \mathcal{L}\,\frac{Q}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\,. \tag{S-6.68}$$

The torque on the ring becomes

$$\boldsymbol{\tau} = -\hat{\mathbf{z}}\left(\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I + \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\right), \tag{S-6.69}$$

and the equation of motion (S-6.61) becomes

$$mR^2\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I - \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\,\frac{\mathrm{d}\omega}{\mathrm{d}t}\,,$$

or

$$\left(mR^2 + \mathcal{L}\,\frac{Q^2 a^2}{2\pi}\right)\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{k_{\mathrm{m}} b_{\mathrm{m}}^2 \pi a^2 Q}{R}\,\partial_t I\,, \tag{S-6.70}$$

which is equivalent to (S-6.61) if we replace the mass of the charged ring by an effective value

$$m_{\text{eff}} = m + \mathcal{L}\,\frac{Q^2 a^2}{2\pi R^2}\,. \tag{S-6.71}$$

Thus we obtain for the dependence of ω on $I(t)$

$$\omega(t) = k_{\text{m}} b_{\text{m}}^2\,\frac{\pi a^2 Q}{m_{\text{eff}} R^3}\,[I_0 - I(t)]\,, \tag{S-6.72}$$

and for its final value

$$\omega_{\text{f}} = k_{\text{m}} b_{\text{m}}^2\,\frac{\pi a^2 Q}{m_{\text{eff}} R^3}\,I_0\,, \tag{S-6.73}$$

corresponding to a final angular momentum

$$\begin{aligned} L_{\text{f}} = mR^2\omega_{\text{f}} &= k_{\text{m}} b_{\text{m}}^2\,\frac{\pi a^2 Q}{R + \mathcal{L}a^2Q^2/(2\pi m R)}\,I_0 \\ &= \begin{cases} \dfrac{\mu_0}{4\pi}\,\dfrac{\pi a^2 Q}{R + \mathcal{L}a^2Q^2/(2\pi m R)}\,I_0\,, & \text{SI,} \\ \dfrac{1}{c^3}\,\dfrac{\pi a^2 Q}{R + \mathcal{L}a^2Q^2/(2\pi m R)}\,I_0\,, & \text{Gaussian.} \end{cases} \end{aligned} \tag{S-6.74}$$

The final magnetic flux through the charged ring is

$$\Phi_{\text{f}} = \frac{1}{b_{\text{m}}}\,\mathcal{L}\,\frac{Q\omega_{\text{f}}}{2\pi} = k_{\text{m}} b_{\text{m}}\,\frac{\mathcal{L}a^2Q^2}{2mR^3 + \mathcal{L}Q^2a^2R/\pi}\,I_0\,, \tag{S-6.75}$$

and the approximations of point a) are valid only if

$$\Phi_{\text{f}} \ll \Phi_0 = 4\pi k_{\text{m}} b_{\text{m}}\,\frac{\pi a^2}{2R}\,I_0\,, \quad \text{or} \quad \frac{\mathcal{L}Q^2}{4\pi^2 m R^2 + 2\pi\mathcal{L}Q^2 a^2} \ll 1\,. \tag{S-6.76}$$

S-6.7 Induced Electric Currents in the Ocean

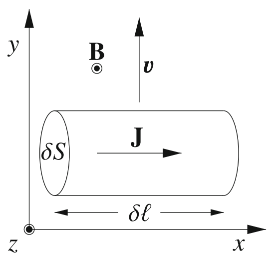

Fig. S-6.10

a) We choose a Cartesian coordinate system with the y axis parallel to the velocity $\boldsymbol{v}$ of the fluid and the z axis parallel to the magnetic field, as shown in Fig. S-6.10. Due to the motion of the fluid, the charge carriers (mainly the Na^+ and Cl^- ions of the dissolved salt) are subject to a force per unit charge equal to $b_{\text{m}}\,\boldsymbol{v}\times\mathbf{B}$, parallel to the x axis. This is equivalent to an electric field $\mathbf{E}_{\text{eq}} \equiv b_{\text{m}}\,\boldsymbol{v}\times\mathbf{B}$. The induced current density is thus

$$\mathbf{J} = \sigma \mathbf{E}_{\text{eq}} = b_{\text{m}} \sigma \boldsymbol{v} \times \mathbf{B}. \tag{S-6.77}$$

b) Inserting the typical values given in the text into (S-6.77) we obtain

$$J \simeq \begin{cases} 4 \times 1 \times 5 \times 10^{-5} = 2 \times 10^{-4}\ \text{A/m}^2, & \text{SI,} \\ 3.6 \times 10^{10} \times \dfrac{100}{c} \times 0.5 = 60\,\text{statA/cm}^2, & \text{Gaussian.} \end{cases} \tag{S-6.78}$$

c) We evaluate the force on a fluid element of cylindrical shape, with area of the bases δS and height $|\delta \boldsymbol{\ell}|$, where $\boldsymbol{\ell}$ is parallel to $\mathbf{J}$ and to the x axis. The current intensity in the cylinder is $I = J\delta S$, and the force acting on it is thus $\delta \mathbf{F} = b_{\text{m}} I \delta \boldsymbol{\ell} \times \mathbf{B} = -b_{\text{m}} BJ\,\delta S\,\delta \ell\,\hat{\mathbf{y}} = -b_{\text{m}} BJ\,\delta \mathcal{V}\,\hat{\mathbf{y}}$, where $\delta \mathcal{V}$ is the volume of the cylinder. The mass of the cylinder is $\delta m = \rho\,\delta \mathcal{V}$, with $\rho = 10^3\,\text{kg/m}^3$ ($1\ \text{g/cm}^3$ in Gaussian units), for water. Both $\boldsymbol{v}$ and $\delta \mathbf{F}$ are parallel to the y direction, and the equation of motion can be written in scalar form

$$\delta m \frac{\mathrm{d}v}{\mathrm{d}t} = \delta F. \tag{S-6.79}$$

Replacing the values of δm and δF we obtain

$$\begin{aligned} \rho\,\delta \mathcal{V} \frac{\mathrm{d}v}{\mathrm{d}t} &= -b_{\text{m}} BJ\,\delta \mathcal{V}, \\ \rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -b_{\text{m}}^2 B^2 \sigma v, \end{aligned} \tag{S-6.80}$$

where we have divided both sides by $\delta \mathcal{V}$ and replaced J by its expression (S-6.77). The solution is a decreasing exponential $v = v_0 \mathrm{e}^{-t/\tau}$ with a time constant

$$\tau = \frac{\rho}{\sigma b_{\text{m}}^2 B^2} \simeq 10^{11}\ \text{s} \simeq 3 \times 10^3\ \text{yr}. \tag{S-6.81}$$

S-6.8 A Magnetized Sphere as Unipolar Motor

a) We recall from Problem 5.10 that the magnetic field inside a uniformly magnetized sphere is uniform and equals

$$\mathbf{B} = \frac{8\pi}{3} \frac{k_{\text{m}}}{b_{\text{m}}} \mathbf{M} = \begin{cases} \dfrac{2\mu_0}{3} \mathbf{M}, & \text{SI,} \\ \dfrac{8\pi}{3} \mathbf{M}, & \text{Gaussian.} \end{cases} \tag{S-6.82}$$

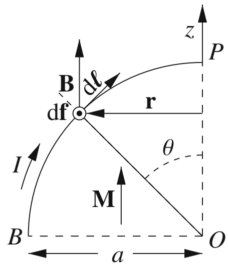

Fig. S-6.11

Outside of the sphere we have the same magnetic field that would be generated by a magnetic dipole of moment $\mathbf{m} = \mathbf{M}4\pi a^3/3$, located at the center of the sphere. When an electric current I flows in the circuit, the magnetic force on an element $\mathrm{d}\boldsymbol{\ell}$ of the "meridian" wire BP is $\mathrm{d}\mathbf{f} = I\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}$, directed out of paper in Fig. S-6.11. Since the component of $\mathbf{B}$ perpendicular to $\mathrm{d}\boldsymbol{\ell}$ is continuous across the surface of the sphere, there is no ambiguity. The torque $\mathrm{d}\boldsymbol{\tau}$ on the wire element $\mathrm{d}\boldsymbol{\ell}$ is

$$\begin{aligned}\mathrm{d}\boldsymbol{\tau} &= \mathbf{r}\times\mathrm{d}\mathbf{f} = I\mathbf{r}\times(\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}) = \hat{\mathbf{z}}Ia\sin\theta\, a\,\mathrm{d}\theta\, B\cos\theta \\ &= \hat{\mathbf{z}}Ia^2B\cos\theta\,\sin\theta\,\mathrm{d}\theta,\end{aligned} \tag{S-6.83}$$

where $\mathbf{r}$ is the distance of $\mathrm{d}\boldsymbol{\ell}$ form the rotation axis of the sphere ($r = a\sin\theta$), and we have used $a\,\mathrm{d}\theta = \mathrm{d}\ell$. The total torque on the meridian wire BP is thus

$$\boldsymbol{\tau} = \int \mathrm{d}\boldsymbol{\tau} = \hat{\mathbf{z}}Ia^2B\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta = \hat{\mathbf{z}}\frac{1}{2}Ia^2B, \tag{S-6.84}$$

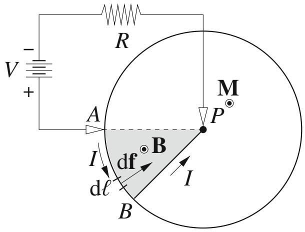

Fig. S-6.12

while the torque on the current-carrying portion AB of the "equatorial" wire is zero, because the magnetic force is radial, as shown in Fig. S-6.12. Thus, (S-6.84) is the total torque on the sphere.

b) When the sphere rotates, the total electromotive force $\mathcal{E}_{\mathrm{tot}}$ in the circuit is the sum of the electromotive force of the voltage source and the electromotive force $\mathcal{E}_{\mathrm{rot}}$ due to the rotation of the of the wires

$$\mathcal{E}_{\mathrm{tot}} = V + \mathcal{E}_{\mathrm{rot}} = V - b_{\mathrm{m}}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{S-6.85}$$

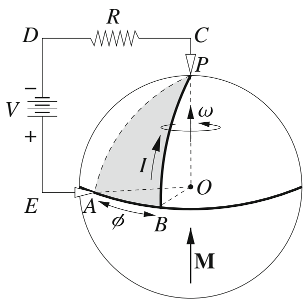

Fig. S-6.13

where Φ is the flux of the magnetic field through any surface bounded by the closed path $ABPCDEA$ in Fig. S-6.13. Lines PC, CD, DE and EA are coplanar lines, lying on a plane containing also the rotation axis OP of the sphere and the meridian arc PA, while AB is an equatorial arc, and BP a meridian arc, both lying on the surface of the sphere. The flux of $\mathbf{B}$ through any surface bounded by the closed path $ABPCDEA$ is the same, because $\nabla\cdot\mathbf{B} = 0$. For simplicity, we choose a surface comprising two parts:

1. the planar surface $PCDEA$, its perimeter being closed by the arc AP, through which the flux is zero, and
2. the spherical polar triangle PAB shaded in Fig. S-6.13.

The flux through PAB can be easily calculated remembering that the flux of **B** through any closed surface is zero. Consider the closed surface formed by PAB and the three circular sectors OAP, OBP and OAB. The flux through OAP and OBP is zero, thus the flux $\Phi_{\rm PAB}$ through PAB and the flux $\Phi_{\rm OAB}$ through OAB must be equal ($\Phi_{\rm OAB}$ must be taken with the minus sign when evaluating its contribution to the flux through the total closed surface, since the magnetic field *enters* through OAB and exits through PAB), and we have

$$\Phi_{\rm PAB} = \Phi_{\rm OAB} = \frac{1}{2} B a^2 \phi \,, \tag{S-6.86}$$

where ϕ is the angle $\widehat{AOB}$. We thus have

$$\mathcal{E}_{\rm tot} = V - b_{\rm m} \frac{{\rm d}\Phi}{{\rm d}t} = V - b_{\rm m} \frac{Ba^2}{2} \frac{{\rm d}\phi}{{\rm d}t} = V - b_{\rm m} \frac{Ba^2}{2} \omega \,, \tag{S-6.87}$$

and the current flowing in the circuit is

$$I = \frac{\mathcal{E}_{\rm tot}}{R} = \frac{1}{R} \left(V - b_{\rm m} \frac{Ba^2}{2} \omega \right) . \tag{S-6.88}$$

The torque on the sphere is zero when $I = 0$, thus the terminal angular velocity of the sphere is

$$\omega_{\rm t} = \frac{2V}{b_{\rm m} B a^2} = \begin{cases} \dfrac{2V}{Ba^2} \,, & \text{SI,} \\ \dfrac{2V}{Ba^2} c \,, & \text{Gaussian,} \end{cases} \tag{S-6.89}$$

independent of the moment of inertia of the sphere $\mathcal{I}$ and of the resistance R of the circuit. The equation of motion for the sphere is

$$\mathcal{I} \frac{{\rm d}\omega}{{\rm d}t} = \tau = \frac{1}{2} B a^2 I = \frac{Ba^2}{2R} \left(V - b_{\rm m} \frac{Ba^2}{2} \omega \right) , \tag{S-6.90}$$

which, using (S-6.89), can be rewritten as

$$\mathcal{I} \frac{{\rm d}\omega}{{\rm d}t} = -b_{\rm m} \frac{(Ba^2)^2}{4\mathcal{I}R} (\omega - \frac{2V}{b_{\rm m} B a^2}) \equiv = -\frac{1}{\tau} (\omega - \omega_{\rm t}) \,, \tag{S-6.91}$$

where

$$\tau = \frac{4\mathcal{I}R}{b_{\rm m} (Ba^2)^2} \,. \tag{S-6.92}$$

Assuming that the sphere is at rest at $t = 0$, the solution is

$$\omega(t) = \omega_{\rm t} \left(1 - {\rm e}^{-t/\tau} \right) . \tag{S-6.93}$$

S-6.9 Induction Heating

a) When the displacement current is neglected, (6.5) can be written as

$$\nabla\times\mathbf{B} = 4\pi k_\mathrm{m}(\mathbf{J}_\mathrm{f}+\mathbf{J}_\mathrm{m}) = 4\pi k_\mathrm{m}\,\mu_\mathrm{r}\,\mathbf{J}_\mathrm{f}\,, \tag{S-6.94}$$

where $\mathbf{J}_\mathrm{f}$ is the free current density and $\mathbf{J}_\mathrm{m}$ is the magnetization current density.
Now, using (A.12),

$$\nabla\times(\nabla\times\mathbf{B}) = -\nabla^2\mathbf{B}+\nabla(\nabla\cdot\mathbf{B}) = 4\pi k_\mathrm{m}\mu_\mathrm{r}\,\nabla\times\mathbf{J}_\mathrm{f}\,, \tag{S-6.95}$$

and recalling that $\nabla\cdot\mathbf{B}=0$ and $\mathbf{J}_\mathrm{f}=\sigma\mathbf{E}$ we obtain

$$-\nabla^2\mathbf{B} = 4\pi k_\mathrm{m}\mu_\mathrm{r}\sigma\,\nabla\times\mathbf{E}\,. \tag{S-6.96}$$

Finally, using $\nabla\times\mathbf{E} = -b_\mathrm{m}\,\partial_t\mathbf{B}$ we have

$$\partial_t\mathbf{B} = (4\pi k_\mathrm{m}b_\mathrm{m}\,\mu_\mathrm{r}\sigma)^{-1}\nabla^2\mathbf{B} \equiv \alpha\,\nabla^2\mathbf{B}\,, \tag{S-6.97}$$

where

$$\alpha = \frac{1}{4\pi k_\mathrm{m}b_\mathrm{m}\,\mu_\mathrm{r}\sigma} = \begin{cases} \dfrac{1}{\mu_0\mu_\mathrm{r}\sigma}\,, & \text{SI,} \\[2ex] \dfrac{c^2}{4\pi\mu_\mathrm{r}\sigma}\,, & \text{Gaussian.} \end{cases} \tag{S-6.98}$$

b) The tangential component of the auxiliary vector **H** must be continuous through the $x=0$ plane, thus, the tangential component of $\mathbf{B}/\mu_\mathrm{r}$ must be continuous. In the vacuum half-space ($x<0$) we have $\mathbf{B} = \hat{\mathbf{y}}\,B_0\cos(\omega t)$, correspondingly, the field at $x=0^+$ (just inside our medium) is

$$\mathbf{B}(0^+,t) = \hat{\mathbf{y}}\,\mu_\mathrm{r}B_0\cos(\omega t)\,. \tag{S-6.99}$$

In one dimension, (6.6) is rewritten

$$\partial_t B = \alpha\partial_x^2 B\,, \tag{S-6.100}$$

and, as an educated guess, we look for a solution of the form $B(x,t) = \mathrm{Re}\left[\tilde{B}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\right]$. The differential equation for the time-independent function $\tilde{B}(x)$ is

$$-\mathrm{i}\omega\tilde{B} = \alpha\partial_x^2\tilde{B}\,, \tag{S-6.101}$$

and we look for an exponential solution of the form $\tilde{B}(x) = \tilde{B}(0)\,\mathrm{e}^{\gamma x}$, with $\tilde{B}(0)$ and γ two constants to be determined. The boundary condition gives $\tilde{B}(0)=\mu_\mathrm{r}B_0$, and, by substituting into (S-6.101), we have

$$-\mathrm{i}\omega\mu_\mathrm{r}B_0\,\mathrm{e}^{\gamma x} = \alpha\gamma^2\mu_\mathrm{r}B_0\,\mathrm{e}^{\gamma x}\,, \tag{S-6.102}$$

which leads to $\alpha\gamma^2 = -\mathrm{i}\omega$, so that

$$\gamma = \pm\sqrt{-\mathrm{i}\,\frac{\omega}{\alpha}} = \pm\frac{1-\mathrm{i}}{\sqrt{2}}\sqrt{\frac{\omega}{\alpha}} = \pm(1-\mathrm{i})\,\frac{1}{\ell_\mathrm{s}} \tag{S-6.103}$$

where $(1-\mathrm{i})/\sqrt{2} = \sqrt{-\mathrm{i}}$, and the quantity

$$\ell_\mathrm{s} = \sqrt{\frac{2\alpha}{\omega}} = \sqrt{\frac{2}{4\pi k_\mathrm{m} b_\mathrm{m}\mu_\mathrm{r}\sigma\omega}} = \begin{cases} \sqrt{\dfrac{2}{\mu_0\mu_\mathrm{r}\sigma\omega}}\,, & \text{SI,} \\ \sqrt{\dfrac{c^2}{2\pi\mu_\mathrm{r}\sigma\omega}}\,, & \text{Gaussian,} \end{cases} \tag{S-6.104}$$

which has the dimension of a length, is called the (*resistive*) *skin depth*. We disregard the positive value of γ, which would lead to a magnetic field exponentially increasing with distance into the material, and obtain

$$\mathbf{B} = \hat{\mathbf{y}}\,\mathrm{Re}\left[\mu_\mathrm{r}B_0\,\mathrm{e}^{-(1-i)x/\ell_s-\mathrm{i}\omega t}\right] = \hat{\mathbf{y}}\,\mu_\mathrm{r}B_0\,\mathrm{e}^{-x/\ell_s}\cos\left(\frac{x}{\ell_s}-\omega t\right). \tag{S-6.105}$$

Thus the magnetic field decreases exponentially with distance into the material, with a decay length ℓ_s. A slab of our material can be considered as semi-infinite if its depth is much larger than ℓ_s.

c) The electric field $\mathbf{E}(x)$ inside the material can be evaluated from $\nabla\times\mathbf{E} = -b_\mathrm{m}\,\partial_t\mathbf{B}$. Assuming $\mathbf{E}(x,t) = \mathrm{Re}\left[\tilde{\mathbf{E}}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\right]$ we have

$$\begin{aligned} (\nabla\times\mathbf{E})_y &= -\partial_x\,\mathrm{Re}\left[\tilde{E}_z(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\right], \\ \partial_t B &= \mathrm{Re}\left[-\mathrm{i}\omega\mu_\mathrm{r}B_0\,\mathrm{e}^{-(1-\mathrm{i})x/\ell_s-i\omega t}\right], \end{aligned} \tag{S-6.106}$$

thus $\mathbf{E}(x,t) = \hat{\mathbf{z}}\,\mathrm{Re}\left[\tilde{E}_z(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\right]$, with $\partial_x\tilde{E}z = -\mathrm{i}\,b_\mathrm{m}\omega\mu_\mathrm{r}B_0\,\mathrm{e}^{-(1-\mathrm{i})x/\ell_s}$. Integrating with respect to x we obtain

$$\tilde{E}_z = \frac{\mathrm{i}}{1-\mathrm{i}}\,b_\mathrm{m}\omega\mu_\mathrm{r}\ell_\mathrm{s}B_0\,\mathrm{e}^{-(1-\mathrm{i})x/\ell_s} = -\frac{1-\mathrm{i}}{2}\,b_\mathrm{m}\omega\mu_\mathrm{r}\ell_\mathrm{s}B_0\,\mathrm{e}^{-(1-\mathrm{i})x/\ell_s}. \tag{S-6.107}$$

The dissipated power per unit volume, *due to the free currents only*, is thus

$$\begin{aligned} \langle\mathbf{J}_\mathrm{f}\cdot\mathbf{E}\rangle &= \frac{\sigma}{2}\left|\tilde{E}_z\right|^2 = \frac{\sigma}{4}\,b_\mathrm{m}^2\mu_\mathrm{r}^2\omega^2\ell_\mathrm{s}^2B_0^2\mathrm{e}^{-2x/\ell_\mathrm{s}} = \frac{\sigma}{4}\,b_\mathrm{m}^2\,\frac{2\mu_\mathrm{r}^2\omega^2B_0^2}{4\pi k_\mathrm{m}b_\mathrm{m}\mu_\mathrm{r}\sigma\omega}\,\mathrm{e}^{-2x/\ell_\mathrm{s}} \\ &= b_\mathrm{m}\,\frac{\mu_\mathrm{r}\omega B_0^2}{8\pi k_\mathrm{m}}\,\mathrm{e}^{-2x/\ell_\mathrm{s}} = \begin{cases} \dfrac{\mu_\mathrm{r}\omega B_0^2}{2\mu_0}\,\mathrm{e}^{-2x/\ell_\mathrm{s}}\,, & \text{SI,} \\ \dfrac{\mu_\mathrm{r}\omega B_0^2}{32\pi^2}\,\mathrm{e}^{-2x/\ell_\mathrm{s}}\,, & \text{Gaussian,} \end{cases} \end{aligned} \tag{S-6.108}$$

where we have substituted (S-6.104) for ℓ_s in the fraction. The total dissipated power per unit surface of the slab is

$$\int_0^\infty \langle \mathbf{J}_f \cdot \mathbf{E} \rangle \, \mathrm{d}x = b_m \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s = \frac{B_0^2}{16\pi} \sqrt{\frac{b_m \mu_r \omega}{2k_m \sigma}} \, . \tag{S-6.109}$$

One might wonder if there is also a contribution of the magnetization volume and surface current densities, $\mathbf{J}_m$ and $\mathbf{K}_m$, to the dissipated power. In the presence of the magnetic field (S-6.105), our medium of relative magnetic permeability μ_r acquires a magnetization $\mathbf{M}$

$$\mathbf{M} = \frac{b_m}{4\pi k_m} \frac{\mu_r - 1}{\mu_r} \mathbf{B} = \hat{\mathbf{y}} \frac{b_m}{4\pi k_m} (\mu_r - 1) \, \mathrm{Re}\left[B_0 \, \mathrm{e}^{-(1-\mathrm{i})x/\ell_s - i\omega t} \right] , \tag{S-6.110}$$

which corresponds to

$$\mathbf{J}_m = \frac{1}{b_m} \boldsymbol{\nabla} \times \mathbf{M} \, . \tag{S-6.111}$$

Taking the symmetry of the problem into account, and introducing the complex amplitudes $\tilde{\mathbf{J}}_m$ and $\tilde{J}_{m\,z}$ such that $\mathbf{J}_m = \mathrm{Re}\left(\tilde{\mathbf{J}}_m \, \mathrm{e}^{-\mathrm{i}\omega t}\right) = \hat{\mathbf{z}} \, \mathrm{Re}\left(\tilde{J}_{m\,z} \mathrm{e}^{-\mathrm{i}\omega t}\right)$, we have

$$\tilde{J}_{m\,z} = \frac{\mu_r - 1}{4\pi k_m \mu_r} \partial_x \tilde{B} = -\frac{\mu_r - 1}{4\pi k_m} \frac{1-\mathrm{i}}{\ell_s} B_0 \, \mathrm{e}^{-(1-\mathrm{i})x/\ell_s} \, . \tag{S-6.112}$$

The corresponding power per unit volume is

$$\langle \mathbf{J}_m \cdot \mathbf{E} \rangle = \frac{1}{2} \mathrm{Re}\left(\tilde{J}_{m\,z} \tilde{E}_z^* \right) = b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{8\pi k_m} \mathrm{e}^{-2x/\ell_s} = (\mu_r - 1) \langle \mathbf{J}_f \cdot \mathbf{E} \rangle \, , \tag{S-6.113}$$

and the total power per unit surface is

$$\int_0^\infty \langle \mathbf{J}_m \cdot \mathbf{E} \rangle \, \mathrm{d}x = (\mu_r - 1) \int_0^\infty \langle \mathbf{J}_f \cdot \mathbf{E} \rangle \, \mathrm{d}x = b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s \, . \tag{S-6.114}$$

However, we also have a surface magnetization current density $\mathbf{K}_m$ flowing on the $x = 0$ plane, given by

$$\mathbf{K}_m = \frac{1}{b_m} \mathbf{M}(0^+) \times \hat{\mathbf{n}} = \hat{\mathbf{z}} \frac{\mu_r - 1}{4\pi k_m} \mathrm{Re}\left(B_0 \, \mathrm{e}^{-\mathrm{i}\omega t} \right) = \hat{\mathbf{z}} \, K_{m\,z} \cos(\omega t) \, , \tag{S-6.115}$$

where $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$ is the outward-pointing unit vector on the $x = 0$ boundary plane. This surface current density corresponds to a power per unit surface

$$\langle \mathbf{K}_m(t) \cdot \mathbf{E}(0,t) \rangle = \frac{1}{2} \mathrm{Re}\left[K_{m\,z} \tilde{E}_z(0) \right] = -b_m (\mu_r - 1) \frac{\mu_r \omega B_0^2}{16\pi k_m} \ell_s \, , \tag{S-6.116}$$

which cancels out the contribution (S-6.114). Thus, the total dissipated power in the medium is due to the free current only, and given by (S-6.109). Note that the parallel component of the electric field must be continuous at the boundary between two media, so that $E_z(0,t)$ appearing in (S-6.116) is a well defined quantity.

S-6.10 A Magnetized Cylinder as DC Generator

a) We can consider the magnetic field as due to the azimuthal magnetization surface current density $\mathbf{K}_m$, flowing on the lateral surface of the cylinder. We have $\mathbf{K}_\mathrm{m} = \mathbf{M} \times \hat{\mathbf{n}}/b_\mathrm{m}$, where $\hat{\mathbf{n}}$ is the outward unit vector normal to the surface. Thus, the magnetized cylinder is equivalent to a solenoid of the same sizes, with n turns per unit length, current I per turn, and the product $nI = K_\mathrm{m}$. Far from the two bases we have an approximately uniform field $\mathbf{B}_0$, independent of the radius and height of the cylinder,

$$\mathbf{B}_0 \simeq 4\pi k_\mathrm{m} K_\mathrm{m}\,\hat{\mathbf{z}} = 4\pi \frac{k_\mathrm{m}}{b_\mathrm{m}}\,\mathbf{M} = \begin{cases} \mu_0 \mathbf{M}, & \text{SI}, \\ 4\pi \mathbf{M}, & \text{Gaussian}. \end{cases} \tag{S-6.117}$$

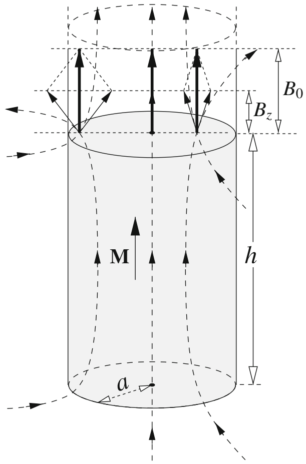

Fig. S-6.14

The field at, for instance, the upper base, can be evaluated by considering an "extended" cylinder, obtained by joining an identical, coaxial cylinder, at the base we are considering, as shown in Fig. S-6.14. The total field at the base is now due to both cylinders, and, being far from both bases of the extended cylinder, its value is $\mathbf{B}_0 \simeq 4\pi(k_\mathrm{m}/b_\mathrm{m})\mathbf{M}$. Both cylinders contribute to this field, and, for symmetry reasons, the z components B_z of both contributions are equal, while the radial components cancel each other. The dashed lines of Fig. S-6.14 represent three **B** field lines for each cylinder, one along the axis and two off-axis. Thus, the z component of the field generated by the single cylinder at its base is

$$B_z = 2\pi \frac{k_\mathrm{m}}{b_\mathrm{m}} M = \frac{B_0}{2}\,. \tag{S-6.118}$$

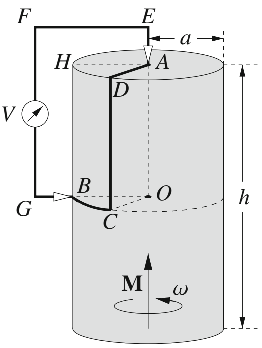

Fig. S-6.15

b) We apply Faraday's law of induction to the flux of the magnetic field through the closed path *AEFGBCDA*, represented by the thick line in Fig. S-6.15. Points *A*, *E*, *F*, *G*, and *B* are fixed in the laboratory frame, while points *C* and *D* rotate with the magnetized cylinder. We have

$$\mathcal{E} = -b_{\mathrm{m}}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{S-6.119}$$

where $\mathcal{E}$ is the electromotive force around the closed path, measured by the voltmeter V, and Φ is the flux of the magnetic field through any surface bounded by the closed path. We choose a surface consisting of three parts:

1. the plane surface bounded by the path *AEFGBHA*, fixed in the laboratory frame, through which the flux of **B** is zero;
2. the surface bounded by the path *BCDHB*, lying on the lateral surface of the cylinder; and
3. the circular sector *AHD* on the upper base, where points *A* and *H* are fixed, while point *D* is rotating.

The flux of **B** through the two surfaces *BCDH* and *AHD* can be calculated analogously to the flux through the polar spherical triangle *PAB* of Fig. S-6.13, Problem 6.8. We consider the closed surface comprising, in addition to *BCDH* and *AHD*, the circular sector *OBC* and the two rectangles *COAD* and *BOAH*. The flux must be zero through the total closed surface, and is zero through the two rectangles because **B** is parallel to their surfaces. Thus we have

$$\Phi_{AHD} + \Phi_{BCDH} + \Phi_{OBC} = 0, \tag{S-6.120}$$

and

$$\Phi_{AHD} + \Phi_{BCDH} = -\Phi_{OBC} = \frac{1}{2}B_0 a^2\phi, \tag{S-6.121}$$

where ϕ is the angle $\widehat{BOC} = \widehat{HAD}$, and the sign accounts for the fact that the magnetic field is entering the closed surface through *OBC*. The electromotive force is

$$\mathcal{E} = -b_{\mathrm{m}}\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -b_{\mathrm{m}}\frac{1}{2}B_0 a^2\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2\pi k_{\mathrm{m}} M a^2\omega = \begin{cases} \dfrac{\mu_0}{2}Ma^2\omega, & \text{SI}, \\ \dfrac{2\pi}{c}Ma^2\omega, & \text{Gaussian}. \end{cases} \tag{S-6.122}$$

The same result can be obtained by evaluating the electromotive force $\mathcal{E}$ as the integral of $b_{\mathrm{m}}(\boldsymbol{v}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{\ell}$ along the path *AOB*

$$\begin{aligned}\mathcal{E} &= b_\mathrm{m}\int_A^O (\boldsymbol{v}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{\ell} + b_\mathrm{m}\int_O^B (\boldsymbol{v}\times\mathbf{B})\cdot\mathrm{d}\boldsymbol{\ell} = b_\mathrm{m}\int_O^a \omega r B_0\,\mathrm{d}r \\ &= b_\mathrm{m}\frac{1}{2}B_0 a^2\omega = 2\pi k_\mathrm{m} M a^2\omega\,, \end{aligned} \tag{S-6.123}$$

since $\boldsymbol{v} = 0$ along the path AO, which lies on the rotation axis of the cylider.

S-6.11 The Faraday Disk and a Self-sustained Dynamo

a) The magnetic force on the each charge carrier of the rotating disk is $q b_\mathrm{m}\boldsymbol{v}\times\mathbf{B}$, where q is the charge of the carrier ($-e$ for the electrons), and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ is the velocity of a charge-carrier at a distance $\mathbf{r}$ from the rotation axis, at rest relative to the disk. At equilibrium, carriers must be at rest relative to the disk, and the magnetic force must be compensated by a static electric field $\mathbf{E}$ such that $\mathbf{E}+b_\mathrm{m}\boldsymbol{v}\times\mathbf{B}=0$. This corresponds to an electric potential drop V between the center and the circumference of the disk

$$V = \varphi(a)-\varphi(0) = -\int_0^a \mathbf{E}\cdot\mathrm{d}\mathbf{r} = b_\mathrm{m}\int_0^a \omega r B\,\mathrm{d}r = b_\mathrm{m}\,\omega B\frac{a^2}{2}\,. \tag{S-6.124}$$

The rotating disk is thus a voltage source, known as a *Faraday disk*.

b) In the presence of the brush contacts at points O and A of Fig. 6.9, the electromotive force $\mathcal{E}$ of he circuit equals the voltage drop V of (S-6.124). The total current I circulating in the circuit is thus

$$I = \frac{\mathcal{E}}{R} = b_\mathrm{m}\frac{\omega B a^2}{2R}\,. \tag{S-6.125}$$

The power dissipated in the circuit by Joule heating is $P_\mathrm{d} = I^2R = \mathcal{E}^2/R$, and there must an external a torque $\boldsymbol{\tau}_\mathrm{ext}$ providing a mechanical power $P_\mathrm{m} = \boldsymbol{\tau}_\mathrm{ext}\cdot\boldsymbol{\omega} = P_\mathrm{d}$ in order to maintain a rotation at constant angular velocity. Thus,

$$\boldsymbol{\tau}_\mathrm{ext} = \hat{\mathbf{z}}\, b_\mathrm{m}^2\,\frac{\omega B^2 a^4}{4R}\,. \tag{S-6.126}$$

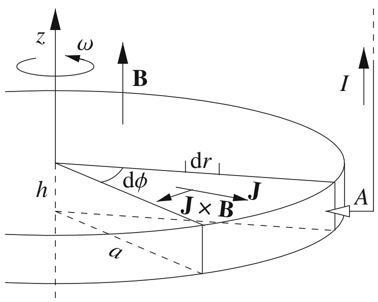

Fig. S-6.16

Alternatively, the external torque must compensate the torque of the magnetic forces on the disk. Since the current exits the disk through the brush contact A, it is difficult to make assumptions on the symmetry of the current density distribution. However, the problem can be tackled as follows. The torque on an infinitesimal volume element,

$r\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}z$ in cylindrical coordinates (Fig. S-6.16), is $\mathrm{d}\boldsymbol{\tau} = b_\mathrm{m}\,\mathbf{r}\times(\mathbf{J}\times\mathbf{B})\,r\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}z$, and the total magnetic torque on the disk is obtained by integrating $\mathrm{d}\boldsymbol{\tau}$ over the disk volume

$$\boldsymbol{\tau}_\mathrm{B} = b_\mathrm{m}\int_0^a \mathrm{d}r\int_0^h \mathrm{d}z\int_0^{2\pi} r\,\mathrm{d}\phi\,\mathbf{r}\times(\mathbf{J}\times\mathbf{B})\,. \tag{S-6.127}$$

The vector triple product in (S-6.127) can be rewritten

$$\mathbf{r}\times(\mathbf{J}\times\mathbf{B}) = \mathbf{J}\,(\mathbf{r}\cdot\mathbf{B}) - \mathbf{B}\,(\mathbf{r}\cdot\mathbf{J}) = -\hat{\mathbf{z}}\,BrJ_r\,, \tag{S-6.128}$$

since $(\mathbf{r}\cdot\mathbf{B}) = 0$ because $\mathbf{r}$ and $\mathbf{B}$ are orthogonal to each other, and J_r is the r component of $\mathbf{J}$. We further have

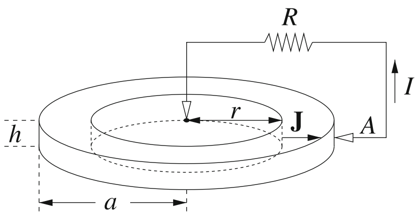

Fig. S-6.17

$$\int_0^h \mathrm{d}z\int_0^{2\pi} r\,\mathrm{d}\phi\,J_r = I\,, \tag{S-6.129}$$

independently of r, since the double integral is the flux of $\mathbf{J}$ through a lateral cylindrical surface of radius r and height h, as shown in Fig. S-6.17. Thus we have for the torque exerted by the magnetic forces on the disk

$$\boldsymbol{\tau}_\mathrm{B} = -\hat{\mathbf{z}}\,b_\mathrm{m}\,BI\int_0^a r\,\mathrm{d}r\,, \tag{S-6.130}$$

and finally, substituting (S-6.125) for I,

$$\boldsymbol{\tau}_\mathrm{b} = -\hat{\mathbf{z}}\,b_\mathrm{m}\,BI\frac{a^2}{2} = -\hat{\mathbf{z}}\,b_\mathrm{m}^2\,\frac{\omega B^2 a^4}{4R} = -\boldsymbol{\tau}_\mathrm{ext}\,. \tag{S-6.131}$$

c) If the disk acts as the current source for the solenoid we must have

$$B = 4\pi k_\mathrm{m}\,nI = 4\pi k_\mathrm{m} b_\mathrm{m}\,n\frac{\omega Ba^2}{2R}\,, \tag{S-6.132}$$

from which we find that the frequency must be a function of the circuit parameters

$$\omega = \frac{2R}{4\pi k_\mathrm{m} b_\mathrm{m}\,na^2} = \begin{cases}\dfrac{2R}{\mu_0 na^2}\,, & \text{SI,}\\[2ex] \dfrac{Rc^2}{2\pi na^2}\,, & \text{Gaussian,}\end{cases} \tag{S-6.133}$$

independently of the intensity of the magnetic field B.

S-6.12 Mutual Induction Between Circular Loops

a) We can assume the magnetic field generated by the current I circulating in loop B to be uniform and equal to $B_0\,\hat{\mathbf{z}} = \hat{\mathbf{z}}\,2\pi k_\mathrm{m}\,I/b$ all over the surface of loop A, since $a \ll b$. The angle between the axis of loop A and the z axis is $\theta = \omega t$, and the flux of the magnetic field through the surface of loop A is

$$\Phi = B_0\,\pi a^2 \cos\omega t = \frac{2\pi k_\mathrm{m} I}{b}\pi a^2 \cos\omega t = \frac{2\pi^2 a^2 k_\mathrm{m} I}{b}\cos\omega t\,. \tag{S-6.134}$$

Thus, according to Faraday's law of induction, there is an induced electromotive force $\mathcal{E}$ on loop A

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{2\pi^2 a^2 k_\mathrm{m} I}{b}\,\omega \sin\omega t\,, \tag{S-6.135}$$

and the current circulating in loop A is

$$I_A = \frac{2\pi^2 a^2 k_\mathrm{m} I}{Rb}\,\omega \sin\omega t\,. \tag{S-6.136}$$

b) The power dissipated into Joule heating is

$$P_\mathrm{diss} = R I_A^2 = \frac{4\pi^4 a^4 \omega^2 k_\mathrm{m}^2 I^2}{Rb^2}\sin^2\omega t\,. \tag{S-6.137}$$

c) The torque acting on loop A is $\tau = \mathbf{m} \times \mathbf{B}_0$, where $\mathbf{m} = \hat{\mathbf{n}}\,I_A \pi a^2$ is the magnetic moment of loop A, and $\hat{\mathbf{n}}$ is the unit vector perpendicular to its surface, directed so that its tip sees I_A circulating counterclockwise. Thus

$$\boldsymbol{\tau} = \frac{2\pi^2 a^2 k_\mathrm{m} I}{Rb}\,\omega \sin\omega t\,\pi a^2\,\frac{2\pi k_\mathrm{m} I}{b}\sin\omega t = \frac{4\pi^4 a^4 \omega k_\mathrm{m}^2 I^2}{Rb^2}\,sin^2\omega t\,, \tag{S-6.138}$$

and the corresponding mechanical power is

$$P_\mathrm{mech} = \boldsymbol{\tau}\cdot\boldsymbol{\omega} = \frac{4\pi^4 a^4 \omega^2 k_\mathrm{m}^2 I^2}{Rb^2}\,sin^2\omega t = P_\mathrm{diss}\,, \tag{S-6.139}$$

and all the mechanical power needed to keep loop A rotating at constant angular velocity is turned into Joule heating.
d) The flux through the surface of loop B of the magnetic field generated by the current I circulating in loop A is

$$\Phi_B = M_{AB} I\,, \tag{S-6.140}$$

where M_{AB} is the coefficient of mutual induction between loop A and loop B. We know that $M_{AB} = M_{BA}$, and from (S-6.134) we have

$$M_{AB} = M_{BA} = \frac{2\pi^2 a^2 k_\mathrm{m}}{b} \cos\omega t\,, \tag{S-6.141}$$

thus

$$\Phi_B = \frac{2\pi^2 a^2 k_\mathrm{m} I}{b} \cos\omega t\,, \tag{S-6.142}$$

and

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{2\pi^2 a^2 k_\mathrm{m} I}{b}\,\omega \sin\omega t\,, \tag{S-6.143}$$

as (S-6.134) and (S-6.135) .

S-6.13 Mutual Induction between a Solenoid and a Loop

a) Neglecting boundary effects, the magnetic field inside the solenoid is uniform, parallel to the solenoid axis z, and equal to

$$\mathbf{B} = 4\pi k_\mathrm{m} n I\,\hat{\mathbf{z}}\,. \tag{S-6.144}$$

Thus, its flux through the surface **S** of the rotating coil is

$$\Phi_a(t) = \mathbf{B}\cdot\mathbf{S}(t) = 4\pi k_\mathrm{m} n I \pi a^2 \cos\omega t = 4\pi^2 a^2 k_\mathrm{m} n I \cos\omega t = M_\mathrm{sl}(t)\,I\,, \tag{S-6.145}$$

where

$$M_\mathrm{sl}(t) = 4\pi^2 a^2 k_\mathrm{m} n \cos\omega t \tag{S-6.146}$$

is the coefficient of mutual inductance between solenoid and loop, time dependent because the loop is rotating. The coefficient of mutual inductance is symmetric, $M_\mathrm{sl} = M_\mathrm{ls}$, i.e., the inductance by the solenoid on the loop equals the inductance by the loop on the solenoid, we shall use this property for the answer to point **c)**.

b) The electromotive force acquired by the loop equals the rate of change of the magnetic flux through it,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 4\pi^2 a^2 k_\mathrm{m} n I \omega \sin\omega t\,, \tag{S-6.147}$$

and the current circulating in the loop is

$$I_a = \frac{4\pi^2 a^2 k_\mathrm{m} n I \omega}{R} \sin\omega t\,. \tag{S-6.148}$$

The loop dissipates a power P_{diss} due to Joule heating

$$P_{\text{diss}} = RI_a^2 = \frac{(4\pi^2 a^2 k_{\text{m}} nI\omega)^2}{R} \sin^2 \omega t\,. \tag{S-6.149}$$

This power must be provided by the work of the torque $\boldsymbol{\tau}$ applied to the loop in order to keep it in rotation at constant angular velocity. The time-averaged power is

$$\langle P_{\text{diss}} \rangle = \frac{(4\pi^2 a^2 k_{\text{m}} nI\omega)^2}{2R}\,, \tag{S-6.150}$$

since $\left\langle \sin^2 \omega t \right\rangle = 1/2$.

c) The magnetic field generated by a magnetic dipole **m** is identical to the field generated by a current-carrying loop of radius a and current I_{l} such that $b_{\text{m}}\pi a^2 I_{\text{l}} = m$, at distances $r \gg a$ from the center of the loop. The result of point **a)** is valid, in particular, in the case $a \ll b$. In this case we can replace the magnetic dipole by a loop, and use the symmetry property of the mutual-inductance coefficient. The flux Φ_{s} generated by the dipole through the solenoid is thus

$$\Phi_{\text{s}} = M_{\text{ls}}(t)\, I_{\text{l}} = 4\pi^2 a^2 k_{\text{m}} n I_{\text{l}} \cos\omega t = 4\pi \frac{k_{\text{m}}}{b_{\text{m}}} n\, m \cos\omega t\,. \tag{S-6.151}$$

S-6.14 Skin Effect and Eddy Inductance in an Ohmic Wire

Assuming a very long, straight cylindrical wire, the problem has cylindrical symmetry. We choose a cylindrical coordinate system (r,ϕ,z) with the z axis along the axis of the wire, and expect that the electric field inside the wire can be written as

$$\mathbf{E} = \hat{\mathbf{z}}\, E(r,t) = \hat{\mathbf{z}}\, \mathrm{Re}\left[\tilde{E}(r)\, \mathrm{e}^{\mathrm{i}\omega t}\right], \tag{S-6.152}$$

where $\tilde{E}(r)$ is the static complex amplitude associated to the electric field. We start from the two Maxwell equations

$$\nabla \times \mathbf{E} = -b_{\text{m}}\, \partial_t \mathbf{B}\,, \qquad \nabla \times \mathbf{B} = 4\pi k_{\text{m}}\, \mathbf{J} + \frac{1}{b_{\text{m}} c^2}\, \partial_t \mathbf{E}\,, \tag{S-6.153}$$

where we have assumed $\varepsilon_{\text{r}} = 1$ and $\mu_r = 1$ inside copper. If we substitute $\mathbf{J} = \sigma \mathbf{E}$ into the second of (S-6.153) we obtain

$$\nabla\times\mathbf{B} = 4\pi k_\mathrm{m}\sigma\mathbf{E} + \frac{1}{b_\mathrm{m}c^2}\,\partial_t\mathbf{E} = 4\pi k_\mathrm{m}\sigma\mathbf{E} + \hat{\mathbf{z}}\,\frac{\omega}{b_\mathrm{m}c^2}\,\mathrm{Re}\left[\mathrm{i}\tilde{E}(r)\,\mathrm{e}^{\mathrm{i}\omega t}\right]$$
$$= \begin{cases} \mu_0\sigma\mathbf{E} + \hat{\mathbf{z}}\,\dfrac{\omega}{c^2}\,\mathrm{Re}\left[\mathrm{i}\tilde{E}(r)\,\mathrm{e}^{\mathrm{i}\omega t}\right], & \text{SI,} \\ \dfrac{4\pi\sigma}{c}\,\mathbf{E} + \hat{\mathbf{z}}\,\dfrac{\omega}{c}\,\mathrm{Re}\left[\mathrm{i}\tilde{E}(r)\mathrm{e}^{\mathrm{i}\omega t}\right], & \text{Gaussian.} \end{cases} \tag{S-6.154}$$

In SI units, the conductivity of copper is $\sigma = 5.96\times10^7\,\Omega^{-1}\mathrm{m}^{-1}$, and the product $\mu_0\sigma c^2$ is

$$\mu_0\,\sigma c^2 = 6.77\times10^{18}\,\mathrm{s}^{-1}\,. \tag{S-6.155}$$

Alternatively, in Gaussian units, the conductivity of copper is $\sigma = 5.39\times10^{17}\,\mathrm{s}^{-1}$ and the product $4\pi\sigma$ is $6.77\times10^{18}\,\mathrm{s}^{-1}$. Thus the displacement current is negligible compared to the conduction current $\mathbf{J}$ for frequencies $\nu = \omega/(2\pi) \ll 10^{18}\,\mathrm{Hz}$, i.e., up to the ultraviolet. In other words, the displacement current can be neglected compared to the conduction current for all practical purposes in good conductors, and we can rewrite the second of (S-6.153) simply as $\nabla\times\mathbf{B} = 4\pi k_\mathrm{m}\,\sigma\mathbf{E}$. Evaluating the curl of both sides of the first of (S-6.153) we have

$$\nabla\times(\nabla\times\mathbf{E}) = -b_\mathrm{m}\partial_t(\nabla\times\mathbf{B}) = -4\pi k_\mathrm{m}b_\mathrm{m}\sigma\,\partial_t\mathbf{E}\,, \tag{S-6.156}$$

which, remembering that

$$\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}\,, \tag{S-6.157}$$

and assuming $\nabla\cdot\mathbf{E} = 0$, turns into a diffusion equation for the electric field $\mathbf{E}$

$$\nabla^2\mathbf{E} = 4\pi k_\mathrm{m}b_\mathrm{m}\sigma\,\partial_t\mathbf{E}\,. \tag{S-6.158}$$

Introducing our assumption (S-6.152), we have the following equation in cylindrical coordinates for the complex amplitude $\tilde{E}(r)$,

$$\nabla^2\tilde{E}(r) = \frac{1}{r}\,\partial_r[r\,\partial_r\tilde{E}(r)] = \mathrm{i}\omega\,4\pi k_\mathrm{m}b_\mathrm{m}\,\sigma\,\tilde{E}(r) \tag{S-6.159}$$

or

$$\frac{1}{r}\,\partial_r[r\,\partial_r\tilde{E}(r)] = \mathrm{i}\,\frac{2}{\delta^2}\,\tilde{E}(r)\,, \tag{S-6.160}$$

where we have introduced the skin depth

$$\delta = \sqrt{\frac{1}{2\pi\,k_\mathrm{m}b_\mathrm{m}\sigma\omega}} = \begin{cases} \sqrt{\dfrac{2}{\mu_0\sigma\omega}}\,, & \text{SI,} \\ \dfrac{c}{\sqrt{2\pi\sigma\omega}}\,, & \text{Gaussian.} \end{cases} \tag{S-6.161}$$

Equation (S-6.160), multiplied by r^2, is Bessel's differential equation with $n = 0$. However, in this context, we prefer to find the approximate solutions for the two limiting cases $\delta \gg r_0$ and $\delta \ll r_0$, where r_0 is the radius of the wire. For the weak skin effect, i.e., for $\delta \gg r_0$, we write the solution of (S-6.160) as a Taylor series

$$\tilde{E}(r) = E_0 \sum_{n=0}^{\infty} a_n \left(\frac{r}{\delta}\right)^n \tag{S-6.162}$$

which, substituted into the left-hand side of (S-6.160) gives

$$\begin{aligned}\frac{1}{r}\,\partial_r[r\,\partial_r \tilde{E}(r)] &= \frac{1}{r}\partial_r\left[rE_0 \sum_{n=0}^{\infty} \frac{a_n n\, r^{n-1}}{\delta^n}\right] = \frac{1}{r} E_0\, \partial_r \sum_{n=0}^{\infty} \frac{a_n n r^n}{\delta^n} \\ &= \frac{1}{r} E_0 \sum_{n=0}^{\infty} \frac{a_n n^2 r^{n-1}}{\delta^n} = E_0 \sum_{n=0}^{\infty} \frac{a_n n^2 r^{n-2}}{\delta^n}\,,\end{aligned} \tag{S-6.163}$$

while the right-hand side is

$$\mathrm{i}\,\frac{2}{\delta^2}\,\tilde{E}(r) = 2\mathrm{i}\,E_0 \sum_{n=0}^{\infty} \frac{a_n r^n}{\delta^{n+2}}\,. \tag{S-6.164}$$

Comparing the coefficients of the same powers of r in (S-6.163) and (S-6.164) we obtain the recurrence relation

$$a_{n+2} = \frac{2\mathrm{i}}{(n+2)^2}\,a_n\,, \tag{S-6.165}$$

which leads to

$$a_{2n} = \frac{\mathrm{i}^n}{2^n (n!)^2} \quad \text{and} \quad a_{2n+1} = 0\,, \tag{S-6.166}$$

for all $n \geq 0$ and $n \in \mathbb{N}$. We thus have

$$\begin{aligned}\tilde{E}(r) &= E_0 \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{2^n (n!)^2} \left(\frac{r}{\delta}\right)^{2n} \\ &= E_0\left[1 + \frac{\mathrm{i}}{2}\frac{r^2}{\delta^2} - \frac{1}{16}\frac{r^4}{\delta^4} - \frac{\mathrm{i}}{48}\frac{r^6}{\delta^6} + \cdots + \frac{\mathrm{i}^n}{2^n (n!)^2}\frac{r^{2n}}{\delta^{2n}} + \ldots\right].\end{aligned} \tag{S-6.167}$$

The complex amplitude I associated to the total current through the wire is

$$
\begin{aligned}
I &= \int_0^{r_0} J 2\pi r \, \mathrm{d}r = 2\pi\sigma \int_0^{r_0} \tilde{E}(r)\, r \, \mathrm{d}r \\
&= 2\pi\sigma E_0 \int_0^{r_0} \left[1 + \frac{\mathrm{i}}{2}\frac{r^2}{\delta^2} - \frac{1}{16}\frac{r^4}{\delta^4} - \frac{\mathrm{i}}{48}\frac{r^6}{\delta^6} + \dots + \frac{\mathrm{i}^n}{2^n (n!)^2}\frac{r^{2n}}{\delta^{2n}} + \dots \right] r \, \mathrm{d}r \\
&= 2\pi\sigma E_0 \left[\frac{r_0^2}{2} + \frac{\mathrm{i}}{8}\frac{r_0^4}{\delta^2} - \frac{1}{96}\frac{r_0^6}{\delta^4} - \frac{\mathrm{i}}{2304}\frac{r_0^8}{\delta^6} + \dots + \frac{\mathrm{i}^n}{2^n (n!)^2 (2n+2)}\frac{r_0^{2n+2}}{\delta^{2n}} \right] \\
&= \pi r_0^2 \sigma E_0 \left[1 + \frac{\mathrm{i}}{4}\frac{r_0^2}{\delta^2} - \frac{1}{48}\frac{r_0^4}{\delta^4} - \frac{\mathrm{i}}{1152}\frac{r_0^6}{\delta^6} + \dots + \frac{\mathrm{i}^n}{2^n (n+1)! n!}\frac{r_0^{2n}}{\delta^{2n}} + \dots \right]. \qquad \text{(S-6.168)}
\end{aligned}
$$

We can define the impedance per unit length of the wire, $Z_\ell = R_\ell + \mathrm{i}\omega L_\ell$ (where R_ℓ is the resistance per unit length, and L_ℓ the self-inductance per unit length), as the ratio of the electric field at the wire surface to the total current through the wire, i.e., as

$$
\begin{aligned}
Z_\ell = \frac{1}{\pi r_0^2 \sigma} & \underbrace{\left[1 + \frac{\mathrm{i}}{2}\left(\frac{r_0}{\delta}\right)^2 - \frac{1}{16}\left(\frac{r_0}{\delta}\right)^4 - \frac{\mathrm{i}}{48}\left(\frac{r_0}{\delta}\right)^6 + \dots \right]}_{A} \\
& \times \underbrace{\left[1 + \frac{\mathrm{i}}{4}\left(\frac{r_0}{\delta}\right)^2 - \frac{1}{48}\left(\frac{r_0}{\delta}\right)^4 - \frac{\mathrm{i}}{1152}\left(\frac{r_0}{\delta}\right)^6 + \dots \right]^{-1}}_{B^{-1}}, \qquad \text{(S-6.169)}
\end{aligned}
$$

where A and B are Taylor expansions in even powers of $r_0/\delta \ll 1$, which we have truncated at the 6th order. The first four expansion coefficients of B^{-1}, i.e., 1, b_1, b_2, and b_3,

$$
B^{-1} = \left[1 + b_1\left(\frac{r_0}{\delta}\right)^2 + b_2\left(\frac{r_0}{\delta}\right)^4 + b_3\left(\frac{r_0}{\delta}\right)^6 + \dots \right], \qquad \text{(S-6.170)}
$$

can be evaluated by requiring that the product BB^{-1} equals 1 with a remainder of the order of $(r_0/\delta)^8$, i.e.,

$$
\begin{aligned}
1 = BB^{-1} \simeq & \left[1 + \frac{\mathrm{i}}{4}\left(\frac{r_0}{\delta}\right)^2 - \frac{1}{48}\left(\frac{r_0}{\delta}\right)^4 - \frac{\mathrm{i}}{1152}\left(\frac{r_0}{\delta}\right)^6 + \dots \right] \\
& \times \left[1 + b_1\left(\frac{r_0}{\delta}\right)^2 + b_2\left(\frac{r_0}{\delta}\right)^4 + b_3\left(\frac{r_0}{\delta}\right)^6 + \dots \right] \qquad \text{(S-6.171)}
\end{aligned}
$$

leading to

$$
b_1 = -\frac{\mathrm{i}}{4}, \quad b_2 = -\frac{1}{24}, \quad \text{and} \quad b_3 = \frac{7\mathrm{i}}{1152} \qquad \text{(S-6.172)}
$$

Thus we have for Z_ℓ

$$\begin{aligned} Z_\ell &\simeq \frac{1}{\pi r_0^2\sigma}\left[1+\frac{\mathrm{i}}{2}\left(\frac{r_0}{\delta}\right)^2-\frac{1}{16}\left(\frac{r_0}{\delta}\right)^4-\frac{\mathrm{i}}{48}\left(\frac{r_0}{\delta}\right)^6+\dots\right] \\ &\quad\times\left[1-\frac{\mathrm{i}}{4}\left(\frac{r_0}{\delta}\right)^2-\frac{1}{24}\left(\frac{r_0}{\delta}\right)^4+\frac{7\mathrm{i}}{1152}\left(\frac{r_0}{\delta}\right)^6+\dots\right] \\ &\simeq \frac{1}{\pi r_0^2\sigma}\left[1+\frac{\mathrm{i}}{4}\left(\frac{r_0}{\delta}\right)^2+\frac{1}{48}\left(\frac{r_0}{\delta}\right)^4-\mathrm{i}\,\frac{23}{1152}\left(\frac{r_0}{\delta}\right)^6+\dots\right]. \end{aligned} \tag{S-6.173}$$

The zeroth-order term of the expansion,

$$R_\ell^{(0)} = \frac{1}{\pi r_0^2\sigma}\,, \tag{S-6.174}$$

is simply the direct-current resistance per unit length of the wire. The third term

$$R_\ell^{(1)} = \frac{r_0^2}{48\pi\sigma\delta^4} = \frac{k_\mathrm{m}^2 b_\mathrm{m}^2 \pi r_0^2\sigma\omega^2}{12} = \begin{cases} \dfrac{\mu_0^2\pi r_0^2\sigma\omega^2}{192}\,, & \text{SI,} \\ \dfrac{\pi r_0^2\sigma\omega^2}{12c^4}\,, & \text{Gaussian} \end{cases} \tag{S-6.175}$$

is the lowest order contribution of the weak skin effect to the resistance increase. The second-order term of the expansion can be interpreted as

$$\frac{\mathrm{i}}{4\pi r_0^2\sigma}\left(\frac{r_0}{\delta}\right)^2 = \mathrm{i}\,\omega L_\ell^{(0)}\,, \tag{S-6.176}$$

leading to

$$L_\ell^{(0)} = \frac{1}{4\pi\sigma\omega r_0^2}\left(\frac{r_0}{\delta}\right)^2 = \frac{1}{2}k_\mathrm{m} b_\mathrm{m} = \begin{cases} \dfrac{\mu_0}{8\pi}\,, & \text{SI,} \\ \dfrac{1}{2c^2}\,, & \text{Gaussian,} \end{cases} \tag{S-6.177}$$

which is the DC self-inductance per unit length of a straight cylindrical wire, while the sixth-order term is the lowest order contribution of the weak skin-effect to the self-inductance of the cylindrical wire. Thus, at the low-frequency limit, the current depends on the radial coordinate, but no true skin effect is observed. According to (S-6.168), the current is actually stronger on the axis of the wire than at its surface.

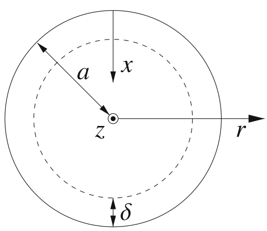

Fig. S-6.18

Things are different at the high-frequency limit. As the frequency increases, the skin depth becomes progressively smaller. When $\delta \ll a$, the fields will be varying in space on a distance much smaller than the wire radius, so that we expect the effect of the curvature to be negligible. For a strong skin effect, i.e., for $\delta \ll a$, the electric field is significantly different from zero only close to the wire surface. Thus, we introduce the variable $x = a - r$, shown in Fig. S-6.18, and assume $r \simeq a$ in (S-6.160). Using $\partial_r = -\partial_x$ we get

$$\partial_x^2 \tilde{E} = \mathrm{i}\,\frac{2}{\delta^2}\,\tilde{E}\,. \tag{S-6.178}$$

Substituting $\tilde{E} = E_0\,\mathrm{e}^{\alpha x}$, we have

$$\alpha = \pm\sqrt{\mathrm{i}\,\frac{2}{\delta^2}} = \pm\frac{1+\mathrm{i}}{\delta}\,, \tag{S-6.179}$$

and the solution corresponding to a field decreasing for increasing x (increasing depth into the wire) is

$$\tilde{E} \simeq E_0\,\mathrm{e}^{-x/\delta}\,\mathrm{e}^{-\mathrm{i}x/\delta} = E_0\,\mathrm{e}^{-(a-r)/\delta}\,\mathrm{e}^{-\mathrm{i}(a-r)/\delta}\,, \tag{S-6.180}$$

where $E_0\,\mathrm{e}^{\mathrm{i}\omega t}$ is the electric field at the wire surface. The complex amplitude corresponding to the total current current through the wire is thus

$$\begin{aligned} I &= \int_0^a J\,2\pi r\,\mathrm{d}r = 2\pi\sigma\int_0^a \tilde{E}(r)\,r\,\mathrm{d}r = 2\pi\sigma E_0\int_0^a \mathrm{e}^{-(a-r)/\delta}\,\mathrm{e}^{-i(a-r)/\delta}\,\mathrm{e}^{\mathrm{i}\omega t}\,r\,\mathrm{d}r \\ &= 2\pi\sigma E_0 \mathrm{e}^{-a(1+\mathrm{i})/\delta+\mathrm{i}\omega t}\int_0^a \mathrm{e}^{r(1+i)/\delta}\,r\,\mathrm{d}r\,. \end{aligned} \tag{S-6.181}$$

Remembering that

$$\int x\,\mathrm{e}^{ax}\mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right), \tag{S-6.182}$$

and neglecting terms in δ^2, we obtain finally

$$I = \pi a\delta\sigma(1-\mathrm{i})\,E_0\,. \tag{S-6.183}$$

The impedance per unit length of the wire, Z_ℓ, can again be defined as

$$Z_\ell = R_\ell + \mathrm{i}X_\ell = \frac{E_0}{I} = \frac{1}{\pi a\delta\sigma(1-\mathrm{i})} = \frac{1}{2\pi a\delta\sigma} + \frac{\mathrm{i}}{2\pi a\delta\sigma}\,, \tag{S-6.184}$$

so that the magnitudes of the resistance per unit length R_ℓ, and of the reactance per unit length X_ℓ, are equal at the high frequency limit:

$$R_\ell = X_\ell = \frac{1}{2\pi a\delta\sigma}\,. \tag{S-6.185}$$

The value of R_ℓ shows that the current actually flows through a thin annulus close to the surface (the "skin" of the wire), of width δ and approximate area $2\pi a\delta$. The reactance per unit length can be considered as due to a self-inductance per unit length L_ℓ, according to $X_\ell = \omega L_\ell$, with

$$L_\ell = \frac{1}{2\pi a\delta\sigma\omega} = \sqrt{\frac{k_\mathrm{m} b_\mathrm{m}}{2\pi a^2\sigma\omega}} = \begin{cases} \sqrt{\dfrac{\mu_0}{8\pi^2 a^2\sigma\omega}}, & \text{SI,} \\ \sqrt{\dfrac{1}{2\pi c^2 a^2\sigma\omega}}, & \text{Gaussian.} \end{cases} \tag{S-6.186}$$

S-6.15 Magnetic Pressure and Pinch Effect for a Surface Current

a) We use a cylindrical coordinate system (r,ϕ,z), with the cylinder axis as z axis. The field lines of $\mathbf{B}$ are circles around the z axis because of symmetry. Thus, $B_\phi(r)$ is the only nonzero component of $\mathbf{B}$. According to Ampère's law we have

$$B_\phi(r) = \begin{cases} 0, & r < a, \\ 2k_\mathrm{m}\dfrac{I}{r} = 4\pi k_\mathrm{m} K\dfrac{a}{r}, & r > a. \end{cases} \tag{S-6.187}$$

b) First approach (heuristic). The current $\mathrm{d}I$ flowing in an infinitesimal surface strip parallel to z, of width $a\,\mathrm{d}\phi$, is $\mathrm{d}I = Ka\,\mathrm{d}\phi$. The force $\mathrm{d}\mathbf{f}$ exerted by an azimuthal magnetic field $\mathbf{B} \equiv (0, B_\phi, 0)$ on an infinitesimal strip portion of length $\mathrm{d}z$ is

$$\mathrm{d}\mathbf{f} = b_\mathrm{m}\,\mathrm{d}z\,\mathrm{d}I\,\hat{\mathbf{z}}\times\mathbf{B} = -b_\mathrm{m} K a B_\phi\,\mathrm{d}\phi\,\mathrm{d}z\,\hat{\mathbf{r}}, \tag{S-6.188}$$

directed towards the axis, i.e., so to shrink the conducting surface (pinch effect). However, here we must remember that $B_\phi(r)$ is discontinuous at the cylinder surface, being zero inside. Therefore, we replace the value of B_ϕ in (S-6.188) by its "average" value $B_\phi^\mathrm{aver} = [B_\phi(a^+) - B_\phi(a^-)]/2 = 2\pi k_\mathrm{m} K$ (the point is the same as for the calculation of electrostatic pressure on a surface charge layer). Thus, the absolute value of the force acting on an infinitesimal area $\mathrm{d}S = a\,\mathrm{d}\phi\,\mathrm{d}z$ is

$$|\mathrm{d}f| = 2\pi k_\mathrm{m} b_\mathrm{m} K^2\,\mathrm{d}S = \begin{cases} \dfrac{\mu_0}{2} K^2\,\mathrm{d}S, & \text{SI} \\ \dfrac{2\pi}{c^2} K^2\,\mathrm{d}S, & \text{Gaussian,} \end{cases} \tag{S-6.189}$$

and the magnetic pressure on the surface is

$$P = \frac{|\mathrm{d}f|}{\mathrm{d}S} = 2\pi k_\mathrm{m} b_\mathrm{m} K^2 = k_\mathrm{m} b_\mathrm{m} \frac{I^2}{2\pi a^2} = \begin{cases} \dfrac{\mu_0 I^2}{2(2\pi a)^2}, & \text{SI} \\ \dfrac{1}{c^2}\dfrac{I^2}{2\pi a^2}, & \text{Gaussian.} \end{cases} \tag{S-6.190}$$

Second method (rigorous). The magnetic force per infinitesimal volume $\mathrm{d}^3 r$, where a current density $\mathbf{J}$ is flowing in the presence of a magnetic field $\mathbf{B}$, is

$$\mathrm{d}^3\mathbf{f} = b_\mathrm{m}\,\mathbf{J}\times\mathbf{B}\,\mathrm{d}^3 r\,. \tag{S-6.191}$$

Due to the symmetry of our problem, the term $(\mathbf{B}\cdot\nabla)\mathbf{B}$ appearing in (6.7) is

$$(\mathbf{B}\cdot\nabla)\mathbf{B} = \left(B_\phi \frac{1}{r}\partial_\phi\right)\mathbf{B} = 0\,, \tag{S-6.192}$$

where we have used the gradient components in cylindrical coordinates of Table A.1, and the fact that the only nonzero component of $\mathbf{B}$, i.e., B_ϕ, is independent of ϕ. The infinitesimal volume element in cylindrical coordinates is $\mathrm{d}^3 r = r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z$, thus

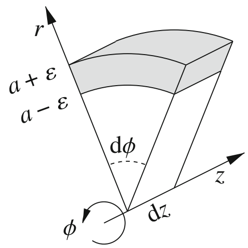

Fig. S-6.19

$$\mathrm{d}^3\mathbf{f} = -\hat{\mathbf{r}}\,\frac{b_\mathrm{m}}{8\pi k_\mathrm{m}}\left[\partial_r B_\phi^2(r)\right] r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z\,. \tag{S-6.193}$$

Now we integrate (S-6.193) with respect to $\mathrm{d}r$ between $r = a-\varepsilon$ and $a+\varepsilon$, obtaining the force $\mathrm{d}^2\mathbf{f}$ acting on the small shaded volume of Fig. S-6.19, delimited by the two cylindrical surfaces $r = a-\varepsilon$ and $r = a+\varepsilon$, with infinitesimal azimuthal aperture $\mathrm{d}\phi$, and longitudinal length $\mathrm{d}z$. Integrating by parts we have

$$\int_{a-\varepsilon}^{a+\varepsilon}\left[\partial_r B_\phi^2(r)\right] r\,\mathrm{d}r = \left[r B_\phi^2(r)\right]_{a-\varepsilon}^{a+\varepsilon} - \int_{a-\varepsilon}^{a+\varepsilon} B_\phi^2(r)\,\mathrm{d}r\,, \tag{S-6.194}$$

At the limit $\varepsilon \to 0$, the first term on the right-hand side equals $B_\phi^2(a^+)$, because $B_\phi^2(r) = 0$ for $r < a$. At the same limit $\varepsilon \to 0$, the integral on the right-hand side approaches zero because, according to the mean-value theorem, it equals $2\varepsilon B_\phi^2(\bar{r})$, with $\bar{r}$ some value in the range $(a-\varepsilon, a+\varepsilon)$. We thus have

$$\mathrm{d}^2\mathbf{f} = -\hat{\mathbf{r}}\,\frac{b_\mathrm{m}}{8\pi k_\mathrm{m}}\,B_\phi^2(a^+)\,a\,\mathrm{d}\phi\,\mathrm{d}z\,. \tag{S-6.195}$$

where $a\,\mathrm{d}\phi\,\mathrm{d}z$ is the infinitesimal surface element on which $\mathrm{d}^2\mathbf{f}$ is acting. The pressure is thus

$$P = \frac{b_\mathrm{m}}{8\pi k_\mathrm{m}} B_\phi^2(a^+) = \frac{b_\mathrm{m}}{8\pi k_\mathrm{m}} (4\pi k_\mathrm{m} K)^2 = 2\pi b_\mathrm{m} k_\mathrm{m} K^2, \tag{S-6.196}$$

in agreement with (S-6.190). Now we prove (6.7):

$$\begin{aligned} 4\pi k_\mathrm{m}(\mathbf{J}\times\mathbf{B})_i &= [(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}]_i = \varepsilon_{ijk}\left(\varepsilon_{jlm}\frac{\partial B_m}{\partial x_l}\right)B_k = \varepsilon_{ijk}\varepsilon_{jlm}\frac{\partial B_m}{\partial x_l}B_k \\ &= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})\,B_k\frac{\partial B_m}{\partial x_l} = B_k\frac{\partial B_i}{\partial x_k} - B_k\frac{\partial B_k}{\partial x_i} \\ &= (\mathbf{B}\times\boldsymbol{\nabla})B_i - \frac{1}{2}\frac{\partial(B_kB_k)}{\partial x_i} = (\mathbf{B}\times\boldsymbol{\nabla})B_i - \frac{1}{2}\nabla_i B^2\,, \end{aligned} \tag{S-6.197}$$

where the subscripts i,j,k,l,m range from 1 to 3, and $x_{1,2,3} = x,y,z$, respectively. The symbol ε_{ijk} is the Levi-Civita symbol, defined by $\varepsilon_{ijk} = 1$ if (i,j,k) is a cyclic permutation of $(1,2,3)$, $\varepsilon_{i,j,k} = -1$ if (i,j,k) is an anticyclic permutation of $(1,2,3)$, and $\varepsilon_{i,j,k} = 0$ if at least two of the subscripts (i,j,k) are equal.

c) The magnetic energy ΔU_M stored in the infinite layer between z and $z+\Delta z$ equals the volume integral

$$\begin{aligned} \Delta U_\mathrm{M} &= \int_\mathrm{layer} u_\mathrm{M}\,\mathrm{d}^3r = \int_\mathrm{layer}\frac{b_\mathrm{m}}{8\pi k_\mathrm{m}}B^2(r)\,\mathrm{d}^3r \\ &= 2\pi\Delta z\int_a^\infty \frac{b_\mathrm{m}}{8\pi k_\mathrm{m}}B_\phi^2(r)\,r\,\mathrm{d}r\,, \end{aligned} \tag{S-6.198}$$

which, involving the integral $\int_a^\infty r^{-1}\mathrm{d}r$, is infinite. However, if the radius of the cylinder increases by $\mathrm{d}a$, the integrand does not change for $r > a+\mathrm{d}a$, while the integration (Fig. S-6.20) volume decreases. Correspondingly, the (infinite) value of the integral decreases by the finite value

$$\mathrm{d}(\Delta U_\mathrm{M}) = -\Delta z\,\frac{b_\mathrm{m}}{8\pi k_\mathrm{m}}B_\phi^2(a^+)\,2\pi a\,\mathrm{d}a\,. \tag{S-6.199}$$

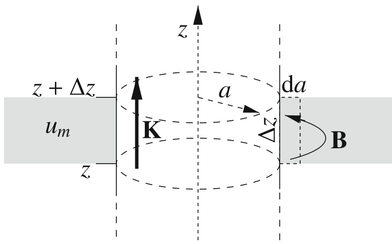

Fig. S-6.20

Thus, an expansion of the current carrying surface leads to a *decrease* of the magnetic energy. If the system were isolated, the force $\mathrm{d}\mathbf{f}$ acting on the surface element $\mathrm{d}S = a\,\mathrm{d}\phi\,\mathrm{d}z$ would be directed radially outwards, leading to an expansion of the cylinder. However, the system is *not* isolated, because a current source is required to keep the current surface density $\mathbf{K}$ constant. An increase of the radius $\mathrm{d}a$ leads to a decrease of the magnetic flux in the layer equal to $\mathrm{d}(\Delta\Phi) = B_\phi(a^+)\,\Delta z\,\mathrm{d}a$ (see figure), which, in turn, implies the appearance of an electromotive force $\Delta\mathcal{E}$. In fact, in order to keep **K** constant during the time interval $\mathrm{d}t$ in which the cylinder radius increases by $\mathrm{d}a$, the source must provide to the layer the energy $\mathrm{d}(\Delta U_{\mathrm{source}})$, that compensates the work $\mathrm{d}(\Delta W) = \Delta\mathcal{E}I\,\mathrm{d}t$ done by the electromotive force $\Delta\mathcal{E} = -b_{\mathrm{m}}\,\mathrm{d}(\Delta\Phi)/\mathrm{d}t$, so that

$$\begin{aligned}\mathrm{d}(\Delta U_{\mathrm{source}}) &= -b_{\mathrm{m}} I\,\mathrm{d}(\Delta\Phi) = 2\pi b_{\mathrm{m}} a K B_\phi(a^+)\,\Delta z\,\mathrm{d}a \\ &= b_{\mathrm{m}}\,2\pi a\,\frac{1}{4\pi k_{\mathrm{m}}}\,B_\phi^2(a^+)\,\Delta z\,\mathrm{d}a = -2\mathrm{d}(\Delta U_{\mathrm{m}})\,. \end{aligned} \tag{S-6.200}$$

Thus, the total energy balance for the layer is given by

$$\mathrm{d}(\Delta U_{\mathrm{tot}}) = \mathrm{d}(\Delta U_{\mathrm{source}}) + \mathrm{d}(\Delta U_m) = -\mathrm{d}(\Delta U_m)\,, \tag{S-6.201}$$

and the force per unit surface is

$$P = -\frac{1}{2\pi a\Delta z}\,\frac{\mathrm{d}(\Delta U_{\mathrm{tot}})}{\mathrm{d}a} = +\frac{1}{2\pi a\Delta z}\,\frac{\mathrm{d}(\Delta U_m)}{\mathrm{d}a}\,, \tag{S-6.202}$$

in agreement with (S-6.190).

S-6.16 Magnetic Pressure on a Solenoid

a) The magnetic force $\mathrm{d}\mathbf{f}$ on an infinitesimal coil arc of length $\mathrm{d}\boldsymbol{\ell}$, carrying a current I, is

$$\mathrm{d}\mathbf{f} = b_{\mathrm{m}} I\,\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}\,. \tag{S-6.203}$$

Thus, the force $\mathrm{d}\mathbf{F}$ on the surface element $\mathrm{d}\mathbf{S} = \mathrm{d}\boldsymbol{\ell}\times\mathrm{d}\mathbf{z}$ of the solenoid, of width $\mathrm{d}\ell$, is

$$\mathrm{d}\mathbf{F} = b_{\mathrm{m}} IBn\,\mathrm{d}\boldsymbol{\ell}\times\mathrm{d}\mathbf{z} = b_{\mathrm{m}} IBn\,\mathrm{d}\mathbf{S}\,, \tag{S-6.204}$$

since the surface element comprises $n\mathrm{d}z$ coil arcs, each of length $\mathrm{d}\ell$. The force $\mathrm{d}\mathbf{F}$ is directed towards the exterior of the solenoid, and the solenoid tends to expand radially.

The magnetic field **B** is discontinuous at the surface of the solenoid, due to the presence of the electric current in the coils. At the limit of an infinitely long solenoid we have

$$\mathbf{B} = \mathbf{B}_0 = 4\pi k_{\mathrm{m}} nI\,\hat{\mathbf{z}} = \begin{cases} \mu_0 nI\,\hat{\mathbf{z}}\,, & \text{SI,} \\ \dfrac{4\pi}{c}\,nI\hat{\mathbf{z}}\,, & \text{Gaussian,} \end{cases} \tag{S-6.205}$$

inside, where $\hat{\mathbf{z}}$ is the unit vector along the solenoid axis, and $\mathbf{B} = 0$ outside. Thus we substitute the average value

$$\frac{B(a^+) + B(a^-)}{2} = \frac{B_0}{2} = 2\pi k_\mathrm{m}\, nI$$

for B in (S-6.204), obtaining

$$\mathrm{d}\mathbf{F} = 2\pi b_\mathrm{m} k_\mathrm{m}\, n^2 I^2 \mathrm{d}\mathbf{S}\,. \tag{S-6.206}$$

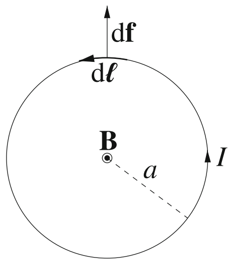

Fig. S-6.21

The pressure P on the solenoid surface is obtained by dividing $\mathrm{d}F$ by $\mathrm{d}S$, thus

$$P = \frac{\mathrm{d}F}{\mathrm{d}S} = 2\pi b_\mathrm{m} k_\mathrm{m}\, n^2 I^2 = \frac{b_\mathrm{m} B_0^2}{8\pi k_\mathrm{m}} = \begin{cases} \dfrac{\mu_0}{2} n^2 I^2 = \dfrac{B_0^2}{2\mu_0}\,, & \text{SI,} \\[2ex] 2\pi n^2 I^2 = \dfrac{B_0^2}{8\pi}\,, & \text{Gaussian.} \end{cases} \tag{S-6.207}$$

b) The magnetic energy of the solenoid can be written in terms of the magnetic energy density u_M associated to the magnetic field $\mathbf{B}_0$

$$u_\mathrm{M} = \frac{b_\mathrm{m}}{8\pi k_\mathrm{m}} B_0^2 = \begin{cases} \dfrac{B_0^2}{2\mu_0}\,, & \text{SI,} \\[2ex] \dfrac{B_0^2}{8\pi}\,, & \text{Gaussian.} \end{cases} \tag{S-6.208}$$

Neglecting the boundary effects, we obtain the total magnetic energy of the solenoid U_M by multiplying u_M by the solenoid volume

$$U_\mathrm{M} = \pi a^2 h\, u_\mathrm{M} = \frac{a^2 h b_\mathrm{m} B_0^2}{8 k_\mathrm{m}} = 2\pi^2 a^2 h b_\mathrm{m} k_\mathrm{m} n^2 I^2\,, \tag{S-6.209}$$

thus, if the solenoid radius a increases by $\mathrm{d}a$ the energy U_M increases by

$$\mathrm{d}U_\mathrm{M} = 4\pi^2 a h b_\mathrm{m} k_\mathrm{m} n^2 I^2\, \mathrm{d}a\,, \tag{S-6.210}$$

while B_0, given by (S-6.205), and thus u_M, remain constant. This implies an increase in the flux Φ of $\mathbf{B}_0$ through each coil of the solenoid

$$\mathrm{d}\Phi = 2\pi a B_0\, \mathrm{d}a = 8\pi^2 k_\mathrm{m} a\, nI\, \mathrm{d}a\,, \tag{S-6.211}$$

corresponding to a total electromotive force (the solenoid comprises hn coils)

$$\mathcal{E} = -b_\mathrm{m} \frac{\mathrm{d}\Phi}{\mathrm{d}t} = -b_\mathrm{m} k_\mathrm{m}\, 8\pi^2 a h n^2 I^2 \frac{\mathrm{d}a}{\mathrm{d}t} \tag{S-6.212}$$

that must be compensated by the current source in order to keep I constant. The work $\mathrm{d}W_{\mathrm{source}}$ done by the current source is thus

$$\mathrm{d}W_{\mathrm{source}} = -\mathcal{E}I\,\mathrm{d}t = b_{\mathrm{m}}k_{\mathrm{m}}\,8\pi^2 an^2 hI^2\,\mathrm{d}a\,. \tag{S-6.213}$$

Thus the total energy of the system solenoid+current source changes by

$$\mathrm{d}U_{\mathrm{tot}} = \mathrm{d}U_{\mathrm{M}} - \mathrm{d}W_{\mathrm{source}} = -4\pi^2 ahb_{\mathrm{m}}k_{\mathrm{m}}n^2 I^2\,\mathrm{d}a\,. \tag{S-6.214}$$

The pressure on the solenoid surface is $P = -\mathrm{d}U_{\mathrm{tot}}/\mathrm{d}V$, where $V = \pi a^2 h$ is the volume of the solenoid. Thus

$$P = -\frac{\mathrm{d}U_{\mathrm{tot}}}{\mathrm{d}V} = -\frac{1}{2\pi ah}\frac{\mathrm{d}U_{\mathrm{tot}}}{\mathrm{d}a} = 2\pi b_{\mathrm{m}}k_{\mathrm{m}}\,n^2 I^2\,, \tag{S-6.215}$$

in agreement with (S-6.207).

S-6.17 A Homopolar Motor

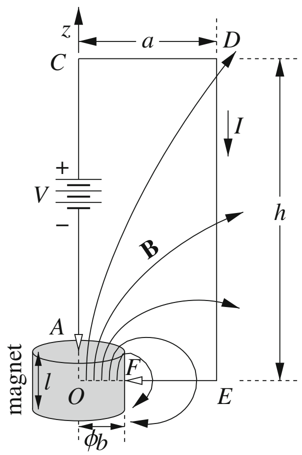

Fig. S-6.22

The motor is schematized in the diagram of Fig. S-6.22, that displays only one "half" of the circuit because of the symmetry of the problem. We use cylindrical coordinates (r,ϕ,z) with the origin O at the center of the cylindrical magnet, of radius b and length l. The z axis coincides with the axes of the magnet and of the cell, which here is represented by the voltage source V. The circuit $ACDEF$ is closed by brush contacts (white arrows in the figure) to the magnet at points $A \equiv (0,\phi,l/2)$ and $F \equiv (b,\phi,0)$, so that the current I can flow through the conducting magnet. The circuit is free to rotate around the z axis. Let $a > b$ and h be the horizontal and vertical sizes of the circuit, respectively. We denote by $\mathbf{B} = \mathbf{B}(r,\phi,z)$ the magnetic field generated by the magnet, independent of ϕ, and with $B_\phi \equiv 0$. Some field lines of $\mathbf{B}$ are sketched in Fig. S-6.22. The magnetic field on the $z = 0$ plane is parallel to the z axis, directed upwards for $r < b$, and downwards for $r > b$. For simplicity, we approximate $\mathbf{B}(r,\phi,0) = B_0\,\hat{\mathbf{z}}$ for $r < b$, with B_0 independent of r, even if this approximation is valid only for $l \gg b$.

The voltage source drives a current I through the circuit. When the circuit is at rest we simply have $I = V/R$, but, when the circuit rotates, we must take into account the motion of the circuit in the presence of the magnetic field. Since the magnetic field $\mathbf{B}$ lies on the plane of the circuit, the force $\mathrm{d}\mathbf{f} = I\,\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}$ on an infinitesimal segment of the circuit $\mathrm{d}\boldsymbol{\ell}$ is perpendicular to the plane of the circuit (out of paper in the case represented in Fig. S-6.23). The corresponding infinitesimal torque relative to the z axis is thus

$$\mathrm{d}\boldsymbol{\tau} = \mathbf{r}\times\mathrm{d}\mathbf{f} = b_\mathrm{m} I\,\mathbf{r}\times(\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}), \tag{S-6.216}$$

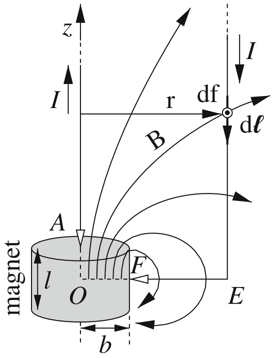

Fig. S-6.23

where $\mathbf{r}$ is the distance of $\mathrm{d}\boldsymbol{\ell}$ from the z axis. The torque $\mathrm{d}\boldsymbol{\tau}$ is always parallel (or antiparallel) to $\hat{\mathbf{z}}$, independently of the circuit element $\mathrm{d}\boldsymbol{\ell}$ we are considering. For the vector product $\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}$ we have

$$\begin{aligned}\mathrm{d}\boldsymbol{\ell}\times\mathbf{B} &= -\hat{\boldsymbol{\phi}}\,B\,\mathrm{d}\ell\sin\theta = -\hat{\boldsymbol{\phi}}\,B\,\mathrm{d}\ell\cos\psi\\ &= -\hat{\boldsymbol{\phi}}\,\mathbf{B}\cdot\hat{\mathbf{n}}\,\mathrm{d}\ell,\end{aligned} \tag{S-6.217}$$

where θ is the angle between $\mathrm{d}\boldsymbol{\ell}$ and $\mathbf{B}$, $\hat{\mathbf{n}}$ is the unit vector perpendicular to $\mathrm{d}\boldsymbol{\ell}$, and $\psi = \theta - \pi/2$ is the angle between $\mathbf{B}$ and $\hat{\mathbf{n}}$, as shown in Fig. S-6.24. Since $\hat{\mathbf{r}}$ is perpendicular to $\hat{\boldsymbol{\phi}}$ (unit vectors of the corresponding cylindrical coordinates), we have for the total torque acting on the circuit

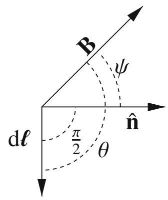

Fig. S-6.24

$$\boldsymbol{\tau} = b_\mathrm{m} I\int_A^F \mathbf{r}\times(\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}) = -\hat{\mathbf{z}}\,b_\mathrm{m}\,I\int_A^F \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell. \tag{S-6.218}$$

The last integral of (S-6.218) can calculated, within our approximations, if we first demonstrate that the line integral of $\mathbf{B}\cdot\hat{\mathbf{n}}\,r$ around the closed path $OCDEO$ of Fig. S-6.22 is zero, i.e., that

$$\oint \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = \int_A^F \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell + \int_F^O \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell + \int_O^A \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = 0. \tag{S-6.219}$$

First, we note that the integral along the whole $\overline{OC}$ path is zero, both because r is zero, and because $\mathbf{B}$ is parallel to $\mathrm{d}\boldsymbol{\ell}$, thus perpendicular to $\hat{\mathbf{n}}$. Thus, the integral of (S-6.219) becomes

$$\oint \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = \int_C^D \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}r - \int_D^E \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}z - \int_E^O \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}r$$
$$= \int_C^D \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}r + \int_E^D \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}z + \int_O^E \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}r, \qquad \text{(S-6.220)}$$

since $\mathrm{d}\ell = \mathrm{d}r$ along $\overline{CD}$, $\mathrm{d}\ell = -\mathrm{d}z$ along $\overline{DE}$, and $\mathrm{d}\ell = -\mathrm{d}r$ along $\overline{EO}$.

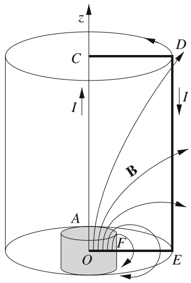

Fig. S-6.25

As a next step, we generate a cylinder by rotating the *CDEO* path around the z axis, as in Fig. S-6.25. The outgoing flux of the magnetic field **B** through the total surface of the cylinder is

$$\int_{\substack{\text{upper}\\\text{base}}} \mathbf{B}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{\substack{\text{lateral}\\\text{surface}}} \mathbf{B}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{\substack{\text{lower}\\\text{base}}} \mathbf{B}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0, \qquad \text{(S-6.221)}$$

since $\nabla\cdot\mathbf{B} = 0$. Equation (S-6.221) can be rewritten

$$0 = \int_0^a \mathbf{B}(r,\phi,h)\cdot\hat{\mathbf{n}}\,2\pi r\,\mathrm{d}r + \int_0^h \mathbf{B}(a,\phi,z)\cdot\hat{\mathbf{n}}\,2\pi r\,\mathrm{d}z$$
$$+ \int_0^a \mathbf{B}(r,\phi,0)\cdot\hat{\mathbf{n}}\,2\pi r\,\mathrm{d}r = 2\pi \oint \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell, \qquad \text{(S-6.222)}$$

which demonstrates (S-6.219). For the last integral appearing in (S-6.218) we thus have

$$\int_A^F \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = -\int_F^O \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = \int_0^b B_0\,r\,\mathrm{d}r = \frac{B_0\,b^2}{2}, \qquad \text{(S-6.223)}$$

where we have remembered that the line integrals are zero on the z axis, that $\mathrm{d}\ell = -\mathrm{d}r$ on the $\overline{FO}$ line, and that, within our approximations, $\mathbf{B}\cdot\hat{\mathbf{n}} = -B_0$, independently of r, on the $\overline{FO}$ line. The torque on the rotating circuit is

$$\boldsymbol{\tau} = -\hat{\mathbf{z}}\,b_\mathrm{m}\,I\int_A^F \mathbf{B}\cdot\hat{\mathbf{n}}\,r\,\mathrm{d}\ell = -\hat{\mathbf{z}}\,b_\mathrm{m}\,I\frac{1}{R}\left(V + b_\mathrm{m}\,\omega\frac{B_0\,b^2}{2}\right). \qquad \text{(S-6.224)}$$

This is why sliding contacts are needed in a homopolar motor. If the line segment $\overline{FO}$ were rotating with the rest of the circuit, the total torque on the complete circuit around the z axis would be zero, because the torque acting on $\overline{FO}$ would compensate the torque on the rest of the circuit.

If we denote by $\mathcal{I}$ the moment of inertia of the rotating circuit and for the moment, neglect frictional effects, the equation of motion is

$$\mathcal{I}\,\frac{\mathrm{d}\omega}{\mathrm{d}t} = \tau = -b_\mathrm{m}\, I\,\frac{B_0\, b^2}{2} - \eta\omega\,, \tag{S-6.225}$$

where we have assumed the presence of a frictional torque $\boldsymbol{\tau}_\mathrm{fr} = -\eta\boldsymbol{\omega}$ proportional to the angular velocity. The current I is determined by the voltage source and by the electromotive force $\mathcal{E}$, due to the rotation of the circuit in the presence of the magnetic field **B**,

$$\begin{aligned}\mathcal{E} &= b_\mathrm{m}\int_C^F (\boldsymbol{\omega}\times\mathbf{r})\times\mathbf{B}\cdot\mathrm{d}\boldsymbol{\ell} = b_\mathrm{m}\int_C^F \omega r\,\hat{\boldsymbol{\phi}}\times\mathbf{B}\cdot\mathrm{d}\boldsymbol{\ell} = -b_\mathrm{m}\,\omega\int_C^F r\,\hat{\boldsymbol{\phi}}\cdot\mathrm{d}\boldsymbol{\ell}\times\mathbf{B}\\ &= b_\mathrm{m}\,\omega\int_C^F r\,\mathbf{B}\cdot\hat{\mathbf{n}}\,\mathrm{d}\ell = b_\mathrm{m}\,\omega\,\frac{B_0\, b^2}{2}\,,\end{aligned} \tag{S-6.226}$$

where we have used (S-6.217) and (S-6.223) in the last two steps. The current is thus

$$I = \frac{1}{R}\left(V + b_\mathrm{m}\,\omega\frac{B_0\, b^2}{2}\right), \tag{S-6.227}$$

and the equation of motion is

$$\begin{aligned}\mathcal{I}\,\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -b_\mathrm{m}\frac{1}{R}\left(V + b_\mathrm{m}\,\omega\frac{B_0\, b^2}{2}\right)\frac{B_0\, b^2}{2} - \eta\omega\\ &= -b_\mathrm{m}\,\frac{V B_0\, b^2}{2R} - \omega\left(b_\mathrm{m}^2\,\frac{B_0^2\, b^4}{4R} + \eta\right),\end{aligned} \tag{S-6.228}$$

with solution

$$\omega = -\frac{2b_\mathrm{m} V B_0\, b^2}{b_\mathrm{m}^2 B_0^2\, b^4 + 4R\eta}\left(1 - \mathrm{e}^{-t/T}\right), \quad \text{where} \quad T = \frac{4R\mathcal{I}}{b_\mathrm{m}^2 B_0^2\, b^4 + 4R\eta}\,. \tag{S-6.229}$$

If we assume negligible frictional torque, i.e., $\eta \ll b_\mathrm{m}^2 B_0^2 b^4/(4R)$, (S-6.229) reduces to

$$\omega = -\frac{2V}{b_\mathrm{m} B_0\, b^2}\left(1 - \mathrm{e}^{-t/T}\right), \quad \text{where} \quad T = \frac{4R\mathcal{I}}{b_\mathrm{m}^2 B_0^2\, b^4}\,, \tag{S-6.230}$$

however, inserting "reasonable values" into (S-6.230), such as $V = 1.5\,\mathrm{V}$, $B_0 = 100\,\mathrm{Gauss} = 10^{-2}\,\mathrm{T}$ and $b = 0.5\,\mathrm{cm}$ we obtain for the steady state solution

$$\omega_0 = -\frac{2V}{b_\mathrm{m} B_0\, b^2} \simeq -1\,200\,\mathrm{rad/s}\,, \quad \text{i.e.,} \quad \nu_0 \simeq 190\,\mathrm{s}^{-1}\,, \tag{S-6.231}$$

which is indeed a very fast rotation! In the absence of friction, the steady state is reached when $V + \mathcal{E} = 0$, so that $I = 0$ and there is no torque acting on the circuit. The final steady-state kinetic energy of the rotating circuit in these conditions is

$$K_{\text{ss}} = \frac{1}{2}\mathcal{I}\omega_0^2 = \frac{1}{2}\mathcal{I}\frac{4V^2}{b_\text{m}^2 B_0^2 b^4} = \frac{2V^2\mathcal{I}}{b_\text{m}^2 B_0^2 b^4}\,. \tag{S-6.232}$$

The current flowing in the circuit is

$$I(t) = \frac{1}{R}\left(V + \frac{b_\text{m} B_0 b^2 \omega}{2}\right) = \frac{V}{R}\,\text{e}^{-t/T}\,, \tag{S-6.233}$$

and the total energy provided by the voltage source is

$$U = \int_0^\infty VI\,\text{d}t = \frac{V^2}{R}\int_0^\infty \text{e}^{-t/T}\,\text{d}t = \frac{V^2}{R}T = \frac{4V^2\mathcal{I}}{b_\text{m}^2 B_0^2 b^4} = 2K_{\text{ss}}\,, \tag{S-6.234}$$

or twice the final kinetic energy. An amount equal to K_{ss} is dissipated into Joule heat. More realistically, we must take the frictional torque into account. For instance, the steady-state angular velocity is reduced by a factor 10 if we assume $4R\eta = 9\,b_\text{m}\,B_0\,b^2$. This, assuming $R = 1\,\Omega$, means

$$\eta \simeq 6\times 10^{-5}\,\text{Nms}\,. \tag{S-6.235}$$

In the presence of friction the steady-state angular velocity is

$$\omega_\text{f} = -\frac{2b_\text{m} V B_0\, b^2}{b_\text{m}^2 B_0^2\, b^4 + 4R\eta}\,, \tag{S-6.236}$$

and the power dissipated by friction is

$$P_{\text{fr}} = \tau_{\text{fr}}\,\omega_\text{f} = \eta\omega_\text{f}^2 = \eta\left(\frac{2b_\text{m} V B_0\, b^2}{b_\text{m}^2 B_0^2\, b^4 + 4R\eta}\right)^2. \tag{S-6.237}$$

The voltage source drives a current

$$I_\text{f} = \frac{V}{R}\left(1 - \frac{b_\text{m}^2 B_0^2 b^4}{b_\text{m}^2 B_0^2 b^4 + 4R\eta}\right) = \frac{4V\eta}{b_\text{m}^2 B_0^2 b^4 + 4R\eta}\,, \tag{S-6.238}$$

and provides a power

$$P_{\text{source}} = V I_{\text{f}} = \frac{4V^2\eta}{b_{\text{m}}^2 B_0^2 b^4 + 4R\eta} \,. \qquad \text{(S-6.239)}$$

The power dissipated into Joule heat is

$$P_{\text{J}} = R I_{\text{f}}^2 = R\left(\frac{4V\eta}{b_{\text{m}}^2 B_0^2 b^4 + 4R\eta}\right)^2 , \qquad \text{(S-6.240)}$$

and we can easily check that

$$P_{\text{J}} + P_{\text{fr}} = P_{\text{source}} \,. \qquad \text{(S-6.241)}$$