Chapter S-7
Solutions for Chapter 7

S-7.1 Coupled *RLC* Oscillators (1)

a) Assuming the two currents I_1 and I_2 to flow clockwise, and applying Kirchhoff's mesh rule to the two loops of the circuit, we have

$$L\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = 0\,, \qquad L\frac{\mathrm{d}I_2}{\mathrm{d}t} + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0\,, \tag{S-7.1}$$

where Q_1 is the charge of the left capacitor, Q_2 the charge of the right capacitor, and Q_0 the charge of capacitor C_0. Charge conservation in the two loops implies

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = I_1\,, \qquad \frac{\mathrm{d}Q_2}{\mathrm{d}t} = I_2\,, \tag{S-7.2}$$

while Kirchhoff's junction rule, applied either to junction A or to junction B, leads to

$$\frac{\mathrm{d}Q_0}{\mathrm{d}t} = I_1 - I_2\,. \tag{S-7.3}$$

Differentiating (S-7.1), substituting (S-7.2) and (S-7.3), and dividing by L, we obtain

$$\begin{aligned}
\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} &= -\frac{1}{LC_1}I_1 - \frac{1}{LC_0}(I_1 - I_2), \quad \text{or} \quad \frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} = -\omega_1^2 I_1 - \omega_0^2(I_1 - I_2)\\
\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} &= -\frac{1}{LC_1}I_2 - \frac{1}{LC_0}(I_2 - I_1), \quad \text{or} \quad \frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} = -\omega_1^2 I_2 - \omega_0^2(I_2 - I_1)\,,
\end{aligned} \tag{S-7.4}$$

where we have introduced the quantities $\omega_0 = 1/\sqrt{LC_0}$ and $\omega_1 = 1/\sqrt{LC_1}$. By substituting $I_1 = A_1\mathrm{e}^{-i\omega t}$ and $I_2 = A_2\mathrm{e}^{-i\omega t}$ from (7.2) into (S-7.4), we obtain

$$\begin{aligned}
(\omega_1^2 + \omega_0^2 - \omega^2)\,A_1 - \omega_0^2\,A_2 &= 0\\
-\omega_0^2\,A_1 + (\omega_1^2 + \omega_0^2 - \omega^2)\,A_2 &= 0\,.
\end{aligned} \tag{S-7.5}$$

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Non-trivial solutions for this system exist only if the determinant

$$D = D(\omega) = (\omega_1^2 + \omega_0^2 - \omega^2)^2 - \omega_0^4 = (\omega_1^2 - \omega^2)(\omega_1^2 + 2\omega_0^2 - \omega^2) \qquad \text{(S-7.6)}$$

equals zero. Thus, the frequencies of the normal modes of the circuit are the roots of the equation $D(\omega) = 0$, i.e.,

$$\omega = \omega_1 \equiv \Omega_+ , \qquad \omega = \sqrt{\omega_1^2 + 2\omega_0^2} \equiv \Omega_- . \qquad \text{(S-7.7)}$$

Substituting these values for ω into (S-7.5) we obtain that $A_1 = A_2$, i.e., $I_1(t) = I_2(t)$, for the mode of frequency Ω_+, and that $A_1 = -A_2$, i.e., $I_1(t) = -I_2(t)$, for the mode of frequency Ω_-.

The normal modes of this simple case, with only two degrees of freedom, can also be evaluated, more simply, by taking the sum and the difference of (S-7.4), obtaining the harmonic oscillator equations

$$\frac{d^2 I_\pm}{dt^2} = -\Omega_\pm^2 I_\pm . \qquad \text{(S-7.8)}$$

for the variables $I_\pm \equiv I_1 \pm I_2$ The currents in the two meshes are $I_1 = (I_+ + I_-)/2$ and $I_2 = (I_+ - I_-)/2$, respectively.

When the circuit is in the mode of frequency Ω_+, no current flows through the AB branch (capacitor C_0), where the two currents cancel out because $I_1 = I_2$. Frequency Ω_+ is simply the resonant frequency of a single-loop LC circuit of inductance L and capacitance C_1, i.e., the frequency at which the impedance of the loop is zero

$$Z_{LC}(\omega) = Z_L(\omega) + Z_{C_1}(\omega) = -i\omega L - \frac{1}{i\omega C_1} = 0 . \qquad \text{(S-7.9)}$$

Since $Z_{LC}(\Omega_+) = 0$, the current flows "freely" through each loop.

For the mode of frequency Ω_-, we have $I_1 = -I_2$, and a current $2I_1$ flows through the AB branch. The effective impedance of the circuit is the series of $Z_{C_0} = (i\omega C_0)^{-1}$ with the parallel of the two impedances Z_{LC},

$$Z = Z_{C_0} + \frac{Z_{LC} Z_{LC}}{Z_{LC} + Z_{LC}} = Z_0 + \frac{Z_{LC}}{2} = -\frac{1}{i\omega C_0} - \frac{1}{2}\left(i\omega L + \frac{1}{i\omega C_1}\right) , \qquad \text{(S-7.10)}$$

which vanishes if

$$\omega^2 = \frac{1}{L}\left(\frac{2}{C_1} + \frac{1}{C_0}\right) = \Omega_-^2 . \qquad \text{(S-7.11)}$$

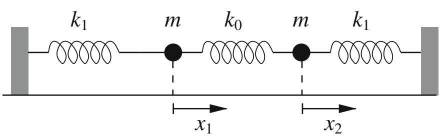

Fig. S-7.1

The circuit is equivalent to the two coupled identical harmonic oscillators of Fig. S-7.1. Each oscillator comprises a mass m, connected to a fixed wall by a spring of Hooke's constant $k_1 = m\omega_1^2$.

The two masses are connected to each other by a third spring of Hooke's constant $k_0 = m\omega_0^2$. We assume that all springs have their respective rest lengths when the two masses are at their equilibrium positions. The equations of motions for the two masses are

$$m\frac{d^2x_1}{dt^2} = -k_1x_1 - k_0(x_1 - x_2), \quad \text{or} \quad \frac{d^2x_1}{dt^2} = -\omega_1^2x_1 - \omega_0^2(x_1 - x_2)$$
$$m\frac{d^2x_2}{dt^2} = -k_1x_2 - k_0(x_2 - x_1), \quad \text{or} \quad \frac{d^2x_2}{dt^2} = -\omega_1^2x_2 - \omega_0^2(x_2 - x_1), \quad \text{(S-7.12)}$$

where x_1 and x_2 are the displacements of the two masses from their equilibrium positions. Equations (S-7.12) for x_1 and x_2 are formally equivalent to equations (S-7.4) for I_1 and I_2, and thus have the same solutions. For the mode at frequency Ω_+, the two masses oscillate in phase ($x_1 = x_2$), central spring (k_0) has always its rest length, and does not exert forces on the two masses. Thus, frequency Ω_+ is the characteristic frequency each single harmonic oscillator. For the mode at frequency Ω_-, we have $x_1 = -x_2$ and the two masses oscillate with opposite phases.
b) The presence of a nonzero resistance R in series with each inductor changes Equation (S-7.1) into

$$L\frac{dI_1}{dt} + RI_1 + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = 0\,, \qquad L\frac{dI_2}{dt} + RI_2 + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0\,, \quad \text{(S-7.13)}$$

By differentiating the equations and proceeding as for (S-7.8) we obtain

$$\frac{d^2I_\pm}{dt^2} = -\Omega_\pm^2 I_\pm - \gamma\frac{dI_\pm}{dt}\,, \quad \text{(S-7.14)}$$

with $\gamma = R/L$. These are the equations of two damped oscillators. The amplitudes of the normal modes vary in time as $\exp(-i\Omega_\pm t - \gamma t)$, decaying with a time constant $\tau = \gamma^{-1}$. The damping rate of the normal modes can also be found by looking for solutions in the form $I_{1,2} = A_{1,2}\,e^{-i\omega t}$, but allowing $A_{1,2}$ and ω to have imaginary parts. For the equivalent mechanical system, the same equations are obtained by inserting frictional forces $f_i = -m\gamma dx_i/dt$ in the equations of motion (S-7.12).
c) Inserting the voltage source, Equations (S-7.13) are modified as follows:

$$L\frac{dI_1}{dt} + RI_1 + \frac{Q_1}{C_1} + \frac{Q_0}{C_0} = V_0\,e^{-i\omega t}\,, \quad L\frac{dI_2}{dt} + RI_2 + \frac{Q_2}{C_1} - \frac{Q_0}{C_0} = 0\,, \quad \text{(S-7.15)}$$

and, by proceeding as for (S-7.8) and (S-7.14), we have

$$\frac{d^2I_\pm}{dt^2} = -\Omega_\pm^2 I_\pm - \gamma\frac{dI_\pm}{dt} - \frac{i\omega V_0}{L}e^{-i\omega t}\,, \quad \text{(S-7.16)}$$

which are the equations of two forced oscillators with a driving term $-(i\omega V_0/L)e^{-i\omega t}$. Resonances are observed when $\omega = \Omega_+$ and for $\omega = \Omega_+$, i.e., when the driving frequency equals one of the frequencies of the normal modes.

S-7.2 Coupled *RLC* Oscillators (2)

a) Proceeding as in Solution S-7.1, we assume I_1 and I_2 to flow clockwise. Applying Kirchhoff's mesh rule to both meshes of the circuit we obtain

$$L\frac{\mathrm{d}I_1}{\mathrm{d}t} + \frac{Q_1}{C} + L_0\left(\frac{\mathrm{d}I_1}{\mathrm{d}t} - \frac{\mathrm{d}I_2}{\mathrm{d}t}\right) = 0,$$
$$L\frac{\mathrm{d}I_2}{\mathrm{d}t} + \frac{Q_2}{C} - L_0\left(\frac{\mathrm{d}I_1}{\mathrm{d}t} - \frac{\mathrm{d}I_2}{\mathrm{d}t}\right) = 0\,, \qquad \text{(S-7.17)}$$

again with $I_1 = \mathrm{d}Q_1/\mathrm{d}t$ and $I_2 = \mathrm{d}Q_2/\mathrm{d}t$. Differentiating (S-7.17) with respect to t we obtain

$$(L + L_0)\frac{\mathrm{d}^2I_1}{\mathrm{d}t^2} + \frac{I_1}{C} - L_0\frac{\mathrm{d}^2I_2}{\mathrm{d}t^2} = 0$$
$$(L + L_0)\frac{\mathrm{d}^2I_2}{\mathrm{d}t^2} + \frac{I_2}{C} - L_0\frac{\mathrm{d}^2I_1}{\mathrm{d}t^2} = 0\,. \qquad \text{(S-7.18)}$$

The sum and difference of the two equations of (S-7.18) give the following equations for the new variables $I_\pm \equiv I_1 \pm I_2$

$$\frac{\mathrm{d}^2I_+}{\mathrm{d}t^2} = -\frac{I_+}{LC} \equiv -\Omega_+^2 I_+\,, \qquad \frac{\mathrm{d}^2I_-}{\mathrm{d}t^2} = -\frac{I_-}{(L + 2L_0)C} \equiv -\Omega_-^2 I_-\,, \qquad \text{(S-7.19)}$$

which show that $I_\pm$ are the normal oscillation modes of the circuit, and $\Omega_\pm$ the corresponding frequencies.

b) Inserting $R \neq 0$, (S-7.17) turn into

$$L\frac{\mathrm{d}I_1}{\mathrm{d}t} + RI_1 + \frac{Q_1}{C} + L_0\left(\frac{\mathrm{d}I_1}{\mathrm{d}t} - \frac{\mathrm{d}I_2}{\mathrm{d}t}\right) + R(I_1 - I_2) = 0$$
$$L\frac{\mathrm{d}I_2}{\mathrm{d}t} + RI_2 + \frac{Q_2}{C} - L_0\left(\frac{\mathrm{d}I_1}{\mathrm{d}t} - \frac{\mathrm{d}I_2}{\mathrm{d}t}\right) - R(I_1 - I_2) = 0\,. \qquad \text{(S-7.20)}$$

Performing again the sum and difference of the two equations we obtain

$$\frac{\mathrm{d}^2I_+}{\mathrm{d}t^2} = -\gamma_+\frac{\mathrm{d}I_+}{\mathrm{d}t} - \Omega_+^2 I_+\,, \qquad \frac{\mathrm{d}^2I_-}{\mathrm{d}t^2} = -\gamma_-\frac{\mathrm{d}I_-}{\mathrm{d}t} - \Omega_-^2 I_-\,, \qquad \text{(S-7.21)}$$

with $\gamma_+ = R/L$, and $\gamma_- = 3R/(L + 2L_0)$. These are the equations for two damped oscillators, with different damping rates $\gamma_\pm$.

S-7.3 Coupled *RLC* Oscillators (3)

a) Let us denote by Q_1 and Q_2 the charges of the capacitors on the AB and on the DE branches, respectively. According to Kirchhoff's mesh rule we have, for the three meshes of the circuit,

$$L\frac{\mathrm{d}I_1}{\mathrm{d}t} = -\frac{Q_1}{C}\,, \quad L\frac{\mathrm{d}I_2}{\mathrm{d}t} = \frac{Q_1}{C} - \frac{Q_2}{C}\,, \quad L\frac{\mathrm{d}I_3}{\mathrm{d}t} = \frac{Q_2}{C}\,, \tag{S-7.22}$$

and, according to Kirchhoff's junction rule applied to the A and D junctions,

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = I_1 - I_2\,, \quad \frac{\mathrm{d}Q_2}{\mathrm{d}t} = I_2 - I_3\,. \tag{S-7.23}$$

Differentiating Equations (S-7.22) with respect to t, and substituting $\mathrm{d}Q_1/\mathrm{d}t$ and $\mathrm{d}Q_2/\mathrm{d}t$ from (S-7.23), we obtain

$$\begin{aligned}
\frac{\mathrm{d}^2 I_1}{\mathrm{d}t^2} &= \frac{1}{LC}(-I_1 + I_2)\,,\\
\frac{\mathrm{d}^2 I_2}{\mathrm{d}t^2} &= \frac{1}{LC}(I_1 - 2I_2 + I_3)\,,\\
\frac{\mathrm{d}^2 I_3}{\mathrm{d}t^2} &= \frac{1}{LC}(I_2 - I_3)\,.
\end{aligned} \tag{S-7.24}$$

Mathematically, the circuit is equivalent to a mechanical system comprising three identical masses m, coupled by two identical springs of Hooke's constant k, as shown in Fig. S-7.2. If we denote by x_1, x_2, and x_3 the displacement of each mass from its rest position, the equations of motion for the three masses are

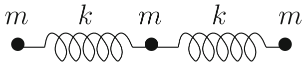

Fig. S-7.2

$$\begin{aligned}
\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= \frac{k}{m}(x_2 - x_1)\,,\\
\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= -\frac{k}{m}(x_2 - x_1) + \frac{k}{m}(x_2 - x_3)\,,\\
\frac{\mathrm{d}^2 x_3}{\mathrm{d}t^2} &= -\frac{k}{m}(x_2 - x_3)\,,
\end{aligned} \tag{S-7.25}$$

which are identical to (S-7.25), after substituting $I_j \to x_j$, with $j = 1, 2, 3$, and $1/(LC) \to k/m$.

b) The frequencies of the normal modes can be found by looking for solutions of (S-7.25) in the form

$$I_j(t) = A_j\,\mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-7.26}$$

After substituting (S-7.26) and $\omega_0^2 = 1/(LC)$ into (S-7.25), and dividing by the common exponential factor, we obtain the system of linear equations in matrix form

$$\begin{pmatrix} (\omega_0^2 - \omega^2) & -\omega_0^2 & 0 \\ -\omega_0^2 & (2\omega_0^2 - \omega^2) & -\omega_0^2 \\ 0 & -\omega_0^2 & (\omega_0^2 - \omega^2) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0\,, \tag{S-7.27}$$

which has non-trivial solutions only if the determinant of the matrix is zero, i.e., if

$$\left(\omega_0^2-\omega^2\right)\left[\left(2\omega_0^2-\omega^2\right)\left(\omega_0^2-\omega^2\right)-\omega_0^4\right]-\omega_0^4\left(\omega_0^2-\omega^2\right)=0\,. \tag{S-7.28}$$

Equation (S-7.28) is a cubic equation in ω^2, in the following we shall consider only the corresponding nonnegative values of ω. A first solution is $\omega=\omega_0=\Omega_1$. If we substitute $\omega=\Omega_1$ into (S-7.27) we obtain $A_1=-A_3$, and $A_2=0$, corresponding to zero current in the central mesh, and I_1 and I_2 oscillating with opposite phases. For the mechanical system of Fig. S-7.2, this solution corresponds to the central mass at rest, while the left and right masses oscillate with opposite phases.

Dividing (S-7.28) by $(\omega_0^2-\omega^2)$ we obtain the equation

$$-3\omega_0^2\omega^2+\omega^4=0\,, \tag{S-7.29}$$

which has the two solutions $\omega=\sqrt{3}\omega_0=\Omega_2$ and $\omega=0=\Omega_3$. The mode of zero frequency (Ω_3) corresponds to a DC current $I=I_1=I_2=I_3$ flowing freely through the inductors, while I_1 and I_2 cancel out in branch AB, and I_2 and I_3 cancel out in branch DE. For the mechanical system, this solution correspond to a pure translational motions of the three masses.

Substituting Ω_2 into (S-7.27) we obtain

$$A_2=-2A_1\,,\quad A_3=A_1\,, \tag{S-7.30}$$

i.e., I_1 and I_3 have the same amplitude and oscillate in phase, while I_2 oscillates with double amplitude and opposite phase. The two external masses of Fig. S-7.2 oscillate in phase, at constant distance from each other, while the central mass oscillates with opposite phase and double amplitude, so that the center of mass is at rest.

The three quantities

$$\mathcal{J}_0=I_1+I_2+I_3\,,\quad \mathcal{J}_1=I_1-I_3\,,\quad \mathcal{J}_2=I_1-2I_2+I_3\,, \tag{S-7.31}$$

corresponding to the three normal modes of the circuits, oscillate at the frequencies $\Omega_0=0$, Ω_1, and Ω_2, respectively.

c) Taking the finite resistances into account, (S-7.22) become

$$\frac{\mathrm{d}I_1}{\mathrm{d}t}+RI_1=-\frac{Q_1}{C}\,,\quad \frac{\mathrm{d}I_2}{\mathrm{d}t}+RI_2=\frac{Q_1}{C}-\frac{Q_2}{C}\,,\quad \frac{\mathrm{d}I_3}{\mathrm{d}t}+RI_3=\frac{Q_2}{C}\,, \tag{S-7.32}$$

which give for the normal modes

$$\begin{aligned}
&\frac{\mathrm{d}^2\mathcal{J}_0}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}\mathcal{J}_0}{\mathrm{d}t}=0\,,\\
&\frac{\mathrm{d}^2\mathcal{J}_1}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}t}+\Omega_1^2\mathcal{J}_1=0\,,\\
&\frac{\mathrm{d}^2\mathcal{J}_2}{\mathrm{d}t^2}+\frac{R}{L}\frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}t}+\Omega_2^2\mathcal{J}_2=0\,.
\end{aligned} \tag{S-7.33}$$

The solution for $\mathcal{J}_0$ describes a non-oscillating, exponentially decreasing current $\mathcal{J}_0 = C_0\,\mathrm{e}^{-\gamma t}$, with decay rate $\gamma = R/L$. The last two equations describe damped oscillating currents $\mathcal{J}_{1,2} = C_{1,2}\exp(-\mathrm{i}\tilde{\Omega}_{1,2}t - \gamma t)$, with

$$\tilde{\Omega}_{1,2} = \sqrt{\omega_{1,2}^2 - \frac{\gamma^2}{4}}\,, \tag{S-7.34}$$

where we have assumed $\Omega_{1,2} > \gamma/2$.

S-7.4 The *LC* Ladder Network

a) Let Q_n be the charge on the nth capacitor. Kirchhoff's junction rule at junction D of Fig. 7.4 implies

$$\frac{\mathrm{d}Q_n}{\mathrm{d}t} = I_{n-1} - I_n\,, \tag{S-7.35}$$

while Kirchhoff's mesh rule applied to mesh $DEFG$ implies

$$\frac{Q_n}{C} - \frac{Q_{n+1}}{C} = L\frac{\mathrm{d}I_n}{\mathrm{d}t}\,. \tag{S-7.36}$$

Now we differentiate (S-7.36) with respect to time, and insert (S-7.35) for the derivatives of Q_n, obtaining

$$\frac{\mathrm{d}^2 I_n}{\mathrm{d}t^2} = \omega_0^2\,(I_{n-1} - 2I_n + I_{n+1})\,, \quad \text{where} \quad \omega_0^2 = \frac{1}{LC}\,. \tag{S-7.37}$$

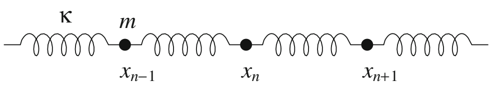

Fig. S-7.3

The equivalent mechanical system is a linear sequence of N identical masses m, each pair of consecutive masses being bound to each other by a spring of Hooke's constant κ (we use the Greek letter κ here because we shall need the letter k for the wavevector later on), as shown in Fig. S-7.3. We denote by x_n the displacement of each mass from its equilibrium position, i.e., its position when all springs have their rest length. Thus, the equation of motion of the nth mass is

$$m\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = -\kappa\,(x_n - x_{n-1}) + \kappa\,(x_{n+1} - x_n)\,, \tag{S-7.38}$$

which, divided by m, and after introducing $\omega_0^2 = \kappa/m$ becomes

$$\frac{d^2x_n}{dt^2} = \omega_0^2 \, (x_{n-1} - 2x_n + x_{n+1}) \,, \tag{S-7.39}$$

mathematically equivalent to (S-7.37). This equation can be generalized to the case of a mechanical system where transverse displacements are allowed, in addition to the longitudinal displacements. If the masses can move in three dimensions, and we denote by $\mathbf{r}_n$ the displacement of the nth mass from its equilibrium position, the equation of motion is written

$$\frac{d^2\mathbf{r}_n}{dt^2} = \omega_0^2 \, (\mathbf{r}_{n-1} - 2\mathbf{r}_n + \mathbf{r}_{n+1}) \,,$$

which is separable into three one-dimensional equations, each identical to (S-7.37).
b) First, we note that, without loss of generality, we can assume the wavevector k appearing in Equation (7.3) to be positive ($k > 0$), so that (7.3) represents a wave traveling from left to right. Changing the sign of k simply gives a wave of the same frequency propagating in the opposite direction, whose dispersion relation is the same as for the forward-propagating wave, because of the inversion symmetry of the problem.

Inserting (7.3) in (S-7.37), and dividing both sides by $C\mathrm{e}^{-\mathrm{i}\omega t}$ we obtain

$$-\,\omega^2\,\mathrm{e}^{\mathrm{i}kna} = \omega_0^2 \left[\mathrm{e}^{\mathrm{i}k(n+1)a} - 2\mathrm{e}^{\mathrm{i}kna} + \mathrm{e}^{\mathrm{i}k(n-1)a}\right] , \tag{S-7.40}$$

where, again, we have substituted $\omega_0^2 = 1/LC$. Dividing both sides by $\mathrm{e}^{\mathrm{i}kna}$ we obtain

$$\omega^2 = \omega_0^2 \left(2 - \mathrm{e}^{\mathrm{i}ka} - \mathrm{e}^{-\mathrm{i}ka}\right) = 2\omega_0^2\,(1 - \cos ka) = 4\omega_0^2 \sin^2(ka/2) \,, \tag{S-7.41}$$

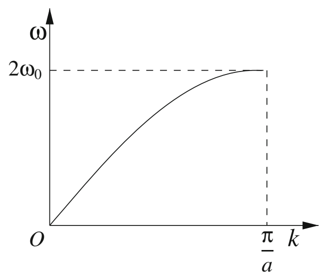

Fig. S-7.4

or, performing the square root,

$$\omega = 2\omega_0 \left|\sin\left(\frac{ka}{2}\right)\right| \,. \tag{S-7.42}$$

The dispersion relation (S-7.42) is shown in Fig. S-7.4 for $0 < k \leq \pi/a$, this range being sufficient to describe all waves propagating in the system. In fact, although (S-7.42) seems to imply that $\omega(k)$ is a periodic function of k, with period $2\pi/a$, the wavevectors k and $k' = k + 2\pi s/a$, with s any integer, actually represent the same wave, since

$$\mathrm{e}^{\mathrm{i}k'na} = \mathrm{e}^{\mathrm{i}(k+2\pi s/a)na} = \mathrm{e}^{\mathrm{i}kna}\mathrm{e}^{2\pi\mathrm{i}sn} = \mathrm{e}^{\mathrm{i}kna} \,, \tag{S-7.43}$$

sn being an integer. This is why it is sufficient to consider the range $0 < k \leq \pi/a$.

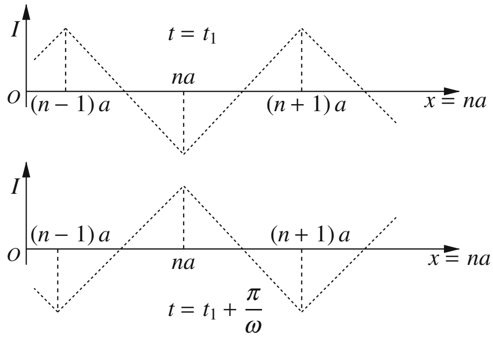

Fig. S-7.5

The existence of a maximum wave vector and of a cut-off frequency is related to the discrete periodic nature of the network, which imposes a minimum sampling rate a. The value $k_{\max} = \pi/a$ corresponds to $\lambda_{\min} = 2\pi/k_{\max} = 2a$, and waves with a smaller wavelength cannot exist. In these waves, the current intensity value is repeated every two meshes of the network, as shown in Fig. S-7.5. A wave with a smaller period cannot exist because of the geometry of the network. One can also note that the direction of wave propagation cannot be determined by observing the wave profile a two instants t_1 and $t_1 + \pi/\omega$ (half a period later, upper and lower parts of Fig. S-7.5). This is consistent with the group velocity $v_g(k_{\max}) = (\partial_k\omega)(k_{\max}) = 0$. The maximum wavevector corresponds to a high cut-off frequency $\omega_{\max} = 2\omega_0$. Since higher frequencies cannot be transmitted, the *LC* network is a low-pass filter.

c) The general monochromatic solution of frequency ω is a standing wave, i.e., the sum of two waves, one propagating from left to right and the other form right to left

$$I_n(t) = A\mathrm{e}^{\mathrm{i}kna-\mathrm{i}\omega t} + B\mathrm{e}^{-\mathrm{i}kna-\mathrm{i}\omega t}\,, \qquad \text{(S-7.44)}$$

where ω and k are related by the dispersion relation (S-7.42). Because of our boundary conditions we must have

$$x_0(t) = 0 \Rightarrow A + B = 0\,; \qquad x_N(t) = 0 \Rightarrow A\mathrm{e}^{\mathrm{i}kNa} + B\mathrm{e}^{-\mathrm{i}kNa} = 0. \qquad \text{(S-7.45)}$$

This gives the condition $\mathrm{e}^{\mathrm{i}kNa} - \mathrm{e}^{-\mathrm{i}kNa} = 2\mathrm{i}\sin(kNa) = 0$, i.e., $k = \pi l/Na$ with $l = 1, 2, 3, \ldots, N-1, N$. We have N allowed wavevectors k_l and frequencies $\omega_l = \omega(k_l)$. Note that $k_{\min} = \pi/Na$ corresponds to $\lambda_{\max} = 2\pi/k_{\min} = 2Na$, this is a standing wave of wavelength twice the length of the system.

d) We obtain the limit to a continuous by letting $a \to 0$ and $n \to \infty$ with $na = x =$ constant so that

$$\lim_{a\to 0}\frac{I_{n+1}(t) - 2I_n(t) + I_{n-1}(t)}{a^2} = \lim_{a\to 0}\frac{I(x+a,t) - 2I(x,t) + I(x-a,t)}{a^2} = \partial_x^2 I(x,t)\,. \qquad \text{(S-7.46)}$$

At this limit we can define a *capacity per unit length* C_ℓ, and an *inductance per unit length* L_ℓ, of the circuit, such that the capacitance and inductance of a circuit segment of length Δx are, respectively,

$$C = C_\ell \Delta x \quad \text{and} \quad L = L_\ell \Delta x. \tag{S-7.47}$$

If we further introduce the quantity

$$v = \sqrt{\frac{1}{L_\ell C_\ell}}\,, \tag{S-7.48}$$

which has the dimensions of a velocity, (S-7.37) is written for the continuous system

$$\begin{aligned}\partial_t^2 I(x,t) &= \lim_{a\to 0} \frac{v^2}{a^2}\left[I(x+a,t) - 2I(x,t) + I(x-a,t)\right] \\ &= v^2\,\partial_x^2 I(x,t)\,.\end{aligned} \tag{S-7.49}$$

This is the equation for a wave propagating with velocity v, independent of the wave frequency ω. At the limit of a continuous system there is no dispersion. This is the case of ideal transmission lines, like parallel wires and coaxial cables with no resistance. See Prob. 7.6 for the case of a realistic transmission line with resistive losses where, however, dispersion can be eliminated.

S-7.5 The *CL* Ladder Network

a) We have the same electric potential on the lower horizontal branch of each mesh, and we assume it to be zero. The voltage drop across the nth capacitor is

$$V_{n-1} - V_n = \frac{Q_n}{C}. \tag{S-7.50}$$

The current in the nth inductor is $I_n - I_{n+1}$, corresponding to a voltage drop across the inductor $L\,(\mathrm{d}I_n/\mathrm{d}t - \mathrm{d}I_{n+1}/\mathrm{d}t)$. Thus we have

$$V_{n-1} = L\left(\frac{\mathrm{d}I_{n-1}}{\mathrm{d}t} - \frac{\mathrm{d}I_n}{\mathrm{d}t}\right), \quad V_n = L\left(\frac{\mathrm{d}I_n}{\mathrm{d}t} - \frac{\mathrm{d}I_{n+1}}{\mathrm{d}t}\right), \tag{S-7.51}$$

which, inserted into (S-7.50), give

$$L\left(\frac{\mathrm{d}I_{n-1}}{\mathrm{d}t} - 2\frac{\mathrm{d}I_n}{\mathrm{d}t} + \frac{\mathrm{d}I_{n+1}}{\mathrm{d}t}\right) = \frac{Q_n}{C}\,. \tag{S-7.52}$$

Differentiating (S-7.52) with respect to time, and using $\mathrm{d}Q_n/\mathrm{d}t = I_n$, we obtain

$$L\left(\frac{\mathrm{d}^2 I_{n-1}}{\mathrm{d}t^2} - 2\frac{\mathrm{d}^2 I_n}{\mathrm{d}t^2} + \frac{\mathrm{d}^2 I_{n+1}}{\mathrm{d}t^2}\right) = \frac{I_n}{C}\,, \tag{S-7.53}$$

which is (7.4).

b) By substituting $I_n = A\mathrm{e}^{\mathrm{i}kna-\mathrm{i}\omega t}$ and $I_{n\pm1} = A\mathrm{e}^{\mathrm{i}k(n\pm1)a-\mathrm{i}\omega t}$ into (S-7.53), defining $\omega_0^2 = (LC)^{-1}$, and dividing both sides by $LA\,\mathrm{e}^{\mathrm{i}kna-\mathrm{i}\omega t}$, we obtain

$$-\,\omega^2\left(\mathrm{e}^{\mathrm{i}ka} - 2 + \mathrm{e}^{-\mathrm{i}ka}\right) = \omega_0^2\,. \tag{S-7.54}$$

The left-hand side can be rewritten

$$\begin{aligned}
-\omega^2\left(\mathrm{e}^{\mathrm{i}ka} - 2 + \mathrm{e}^{-\mathrm{i}ka}\right) &= -\omega^2\,[2\cos(ka) - 2] = -2\omega^2\,[\cos(ka) - 1] \\
&= -2\omega^2\left[\cos(ka) - \cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right)\right] \\
&= -2\omega^2\left[\cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right) - \cos^2\left(\frac{ka}{2}\right) - \sin^2\left(\frac{ka}{2}\right)\right] \\
&= 4\omega^2\sin^2\left(\frac{ka}{2}\right)\,.
\end{aligned} \tag{S-7.55}$$

Substituting into (S-7.54) we have

$$\omega^2 = \frac{\omega_0^2}{4\sin^2(ka/2)}\,, \tag{S-7.56}$$

or

$$\omega = \frac{\omega_0}{2|\sin(ka/2)|}\,. \tag{S-7.57}$$

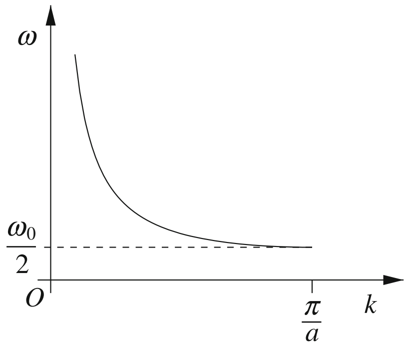

Fig. S-7.6

Fig. S-7.6 shows the plot of the dispersion relation. Compare this behavior with the dispersion relation shown in Fig. S-7.4 for an LC network, where capacitors and inductors are swapped with respect to the present case (Problem 7.4). In the LC network $2\omega_0$ is an upper cut-off frequency. Here, in the CL network, we have a lower cut-off frequency $\omega_0/2$, and the CL ladder network acts as a low-pass filter.

S-7.6 A non-dispersive transmission line

a) The voltage drop from x to $x + \mathrm{d}x$ is

$$V(x,t) - V(x + \mathrm{d}x, t) = \partial_t I(x,t)L + I(x,t)R\,, \tag{S-7.58}$$

which yields, after replacing R by $R_\ell \mathrm{d}x$ and L by $L_\ell \mathrm{d}x$,

$$\partial_x V = -L_\ell \partial_t I - R_\ell I\,. \tag{S-7.59}$$

The charge associated to the capacitance per unit length is $Q = Q(x,t) = CV(x,t)$, and charge conservation yields

$$\partial_t Q(x,t) = I(x-dx,t) - I(x,t) - I_L(x,t) , \tag{S-7.60}$$

with the leakage current given by

$$I_L = I_L(x,t) = V(x,t)/R_p = V(x,t)\, G_\ell \mathrm{d}x . \tag{S-7.61}$$

We thus obtain, by eliminating Q and replacing C by $C_\ell \mathrm{d}x$,

$$C_\ell \partial_t V = -\partial_x I - G_\ell V . \tag{S-7.62}$$

Now we eliminate V by calculating

$$\begin{aligned} \partial_x^2 I &= -C_\ell \partial_t \partial_x V - G_\ell \partial_x V \\ &= +L_\ell C_\ell \partial_t^2 I + C_\ell R_\ell \partial_t I + G_\ell L_\ell \partial_t I + G_\ell R_\ell I , \end{aligned} \tag{S-7.63}$$

which yields Eq. (7.6).
b) By substituting (7.7) in (7.6) we obtain

$$-k^2 + \frac{\omega^2}{v_0^2} = -\mathrm{i}\omega(R_\ell C_\ell + L_\ell G_\ell) + R_\ell G_\ell , \tag{S-7.64}$$

where $v_0^2 = (L_\ell C_\ell)^{-1}$. Thus, the wavevector k is a complex number. Writing $k = k_r + \mathrm{i}k_i$ we obtain

$$k_r^2 - k_i^2 = \frac{\omega^2}{v_0^2} - R_\ell G_\ell , \tag{S-7.65}$$

$$2k_r k_i = \omega(R_\ell C_\ell + L_\ell G_\ell) . \tag{S-7.66}$$

The wave is thus evanescent,

$$I(x,t) = I_0 \mathrm{e}^{-k_i x} \mathrm{e}^{\mathrm{i}k_r x - \mathrm{i}\omega t} , \tag{S-7.67}$$

where the acceptable values for k_i are positive. Since in general $k_r = k_r(\omega)$ if $R_\ell \neq 0$ or $G_\ell \neq 0$, resistive effects make the line to be dispersive, so that a wavepacket is distorted along its propagation.
c) If we assume that $k_i^2 = R_\ell G_\ell$ in (S-7.65), then $k_r = \omega/v_0$, which means that the propagation is non-dispersive: the phase velocity $v_p = \omega/k_r = v_0$ is independent of frequency. In addition, since k_i does not depend on ω, the evanescence length k_i^{-1} is also frequency-independent. By substituting $k_i = \sqrt{R_\ell G_\ell}$ and $k_r = \omega(L_\ell C_\ell)^{1/2}$ in (S-7.66) we obtain the condition

$$2\sqrt{R_\ell G_\ell}\,\sqrt{L_\ell C_\ell} = R_\ell C_\ell + L_\ell G_\ell . \tag{S-7.68}$$

Squaring both sides and rearranging the terms yields $(R_\ell C_\ell - L_\ell G_\ell)^2 = 0$, which leads to the simple, equivalent condition

$$R_\ell C_\ell = L_\ell G_\ell \ . \tag{S-7.69}$$

This is the condition for a non-dispersive or *distortionless* transmission line due to O. Heaviside.

If the input current at one side of the line, say at $x = 0$, is

$$I(0,t) = I_0(t) = \int \tilde{I}_0(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega \ , \tag{S-7.70}$$

where $\tilde{I}_0(\omega)$ is the Fourier transform, then the current along the line will be given by

$$\begin{aligned} I(x,t) &= \int \tilde{I}_0(\omega) \mathrm{e}^{\mathrm{i}k_r x - \mathrm{i}\omega t} \mathrm{e}^{-k_i x} \mathrm{d}\omega = \mathrm{e}^{-k_i x} \int \tilde{I}_0(\omega) \mathrm{e}^{-\mathrm{i}\omega(t - x/v_0)} \mathrm{d}\omega \\ &= \mathrm{e}^{-k_i x} I_0(t - x/v_0) \ , \end{aligned} \tag{S-7.71}$$

since k_i is independent on ω. This is equivalent to state that the general solution of (7.6) with the condition (S-7.69) has the form (7.8) with $\boldsymbol{v} = v_0$ and $\kappa = k_i$.

The same conclusion may be obtained by direct substitution of (7.8) into Eq. (7.6). The partial derivatives are given by

$$\begin{aligned} \partial_t I &= -\boldsymbol{v} \mathrm{e}^{-\kappa x} f'(x - \boldsymbol{v}t) \ , \\ \partial_t^2 I &= \boldsymbol{v}^2 \mathrm{e}^{-\kappa x} f''(x - \boldsymbol{v}t) \ , \\ \partial_x I &= -\kappa \mathrm{e}^{-\kappa x} f(x - \boldsymbol{v}t) + \mathrm{e}^{-\kappa x} f'(x - \boldsymbol{v}t) \ , \\ \partial_x^2 I &= \kappa^2 \mathrm{e}^{-\kappa x} f(x - \boldsymbol{v}t) - 2\kappa \mathrm{e}^{-\kappa x} f'(x - \boldsymbol{v}t) + \mathrm{e}^{-\kappa x} f''(x - \boldsymbol{v}t) \ , \end{aligned} \tag{S-7.72}$$

where $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$ and $f''(x) = \mathrm{d}^2 f(x)/\mathrm{d}x^2$. Thus Eq. (7.6) becomes

$$(\kappa^2 - R_\ell G_\ell) f + (\boldsymbol{v}(R_\ell C_\ell + L_\ell G_\ell) - 2\kappa) f' + (1 - L_\ell C_\ell \boldsymbol{v}^2) f'' = 0 \ . \tag{S-7.73}$$

For this equation to be true for arbitrary f, the coefficients of f, f' and f'' must be all zero. Thus

$$\kappa^2 = R_\ell G_\ell \ , \qquad 2\kappa = \boldsymbol{v}(R_\ell C_\ell + L_\ell G_\ell) \ , \qquad \boldsymbol{v}^2 = (L_\ell C_\ell)^{-1} \ , \tag{S-7.74}$$

which bring again the conditions on the line parameters found above.

S-7.7 An “Alternate” *LC* Ladder Network

a) Let Q_n be the charge of the capacitor at the right of mesh n. Applying Kirchhoff’s mesh rule to the even and odd meshes of the ladder network we have, respectively,

$$-\frac{Q_{2n-1}}{C} + L_2 \frac{\mathrm{d}I_{2n}}{\mathrm{d}t} + \frac{Q_{2n}}{C} = 0 \, , \qquad -\frac{Q_{2n}}{C} + L_1 \frac{\mathrm{d}I_{2n+1}}{\mathrm{d}t} + \frac{Q_{2n+1}}{C} = 0 \, , \tag{S-7.75}$$

while Kirchhoff’s junction rule gives

$$\frac{dQ_{2n-1}}{dt} = I_{2n-1} - I_{2n}\,, \quad \frac{dQ_{2n}}{dt} = I_{2n} - I_{2n+1}\,. \tag{S-7.76}$$

Differentiating (S-7.75) with respect to time, and inserting (S-7.76), we obtain

$$\begin{aligned} L_2\frac{d^2I_{2n}}{dt^2} &= \frac{1}{C}\left(\frac{dQ_{2n-1}}{dt} - \frac{d^2Q_{2n}}{dt^2}\right) = \frac{1}{C}\left(I_{2n-1} - 2I_{2n} + I_{2n+1}\right) \\ L_1\frac{d^2I_{2n+1}}{dt^2} &= \frac{1}{C}\left(\frac{dQ_{2n}}{dt} - \frac{d^2Q_{2n+1}}{dt^2}\right) = \frac{1}{C}\left(I_{2n} - 2I_{2n+1} + I_{2n+2}\right)\,, \end{aligned} \tag{S-7.77}$$

identical to (7.9).

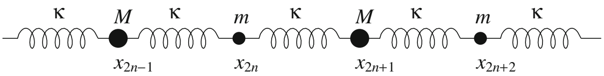

Fig. S-7.7

A mechanical equivalent to our network is the one-dimensional sequence of masses and springs shown in Fig. S-7.7, where the masses have, alternately, the values M and m, while all springs are identical, with Hooke's constant κ. If we denote by x_{2n+1} the positions of the odd masses M, and by and x_{2n} the positions of even masses m, the equations of motion for the system are

$$\begin{aligned} m\frac{d^2x_{2n}}{dt^2} &= -\kappa(x_{2n} - x_{2n+1}) + \kappa(x_{2n-1} - x_{2n}) = \kappa(x_{2n-1} - 2x_{2n} + x_{2n+1}) \\ M\frac{d^2x_{2n+1}}{dt^2} &= -\kappa(x_{2n+1} - x_{2n+2}) + \kappa(x_{2n} - x_{2n+1}) = \kappa(x_{2n} - 2x_{2n+1} + x_{2n+2}), \end{aligned} \tag{S-7.78}$$

which, after the substitutions $m \to L_2$, $M \to L_1$, $x \to I$, and $\kappa \to 1/C$, are identical to (S-7.77).

b) Substituting (7.10) into (S-7.77), and dividing both sides by $e^{-i\omega t}$, we obtain

$$\begin{aligned} -\omega^2 L_2 I_e\, e^{i(2n)\,ka} &= \frac{1}{C}\left(I_o\, e^{i(2n-1)\,ka} - 2I_e\, e^{i(2n)\,ka} + I_o\, e^{i(2n+1)\,ka}\right) \\ -\omega^2 L_2 I_o\, e^{i(2n+1)\,ka} &= \frac{1}{C}\left(I_e\, e^{i(2n)\,ka} - 2I_o\, e^{i(2n+1)\,ka} + I_e\, e^{i(2n+2)\,ka}\right)\,. \end{aligned} \tag{S-7.79}$$

Now we define the two angular frequencies $\omega_o = 1/\sqrt{L_1C}$ and $\omega_e = 1/\sqrt{L_2C}$, and divide (S-7.79) by $e^{i(2n)\,ka}$, obtaining

$$\begin{aligned} (2\omega_e^2 - \omega^2)\, I_e - 2\omega_e^2 \cos(ka)\, I_o &= 0 \\ 2\omega_o^2 \cos(ka)\, I_e - (2\omega_o^2 - \omega^2)\, I_o &= 0\,. \end{aligned} \tag{S-7.80}$$

This system of linear equations has non-trivial solutions if and only if its determinant is zero, i.e., if

$$(2\omega_e^2 - \omega^2)(2\omega_o^2 - \omega^2) - 4\omega_o^2\omega_e^2\cos^2(ka) = 0\,, \tag{S-7.81}$$

the solution of this quadratic equation in ω^2 is

$$\omega^2 = \omega_e^2 + \omega_o^2 \pm \sqrt{\left(\omega_e^2 + \omega_o^2\right)^2 - 4\omega_o^2\omega_e^2\sin^2(ka)}\,. \tag{S-7.82}$$

Both solutions are physically acceptable: the system allows for *two* types of propagating waves, described by two different dispersion relations.

At the limit $L_2 \ll L_1$ (or $m \ll M$, for the equivalent mechanical system) we have $\omega_o^2 \ll \omega_e^2$, and (S-7.82) can be approximated as

$$\omega^2 \simeq \omega_e^2 + \omega_o^2 \pm \omega_e^2\sqrt{1 + 2\frac{\omega_o^2}{\omega_e^2} - 4\frac{\omega_o^2}{\omega_e^2}\sin^2(ka)} \tag{S-7.83}$$

where we have disregarded the fourth-order term ω_o^4/ω_e^4 inside the square root. If we further use the approximation $\sqrt{1+x} \simeq 1 + x/2$, valid for $x \ll 1$, (S-7.83) becomes

$$\omega^2 \simeq \omega_e^2 + \omega_o^2 \pm \omega_e^2\left\{1 + \frac{\omega_o^2}{\omega_e^2}\left[1 - 2\sin^2(ka)\right]\right\}, \tag{S-7.84}$$

corresponding to the two dispersion relations

$$\omega \simeq \begin{cases} \sqrt{2(\omega_e^2 + \omega_o^2) - 2\omega_o^2\sin^2(ka)} \\ \sqrt{2}\,\omega_o\sin(ka) \end{cases} \tag{S-7.85}$$

The lower branch can propagate for frequencies between 0 and $\omega_1 = \sqrt{2}\omega_o$, while the upper branch lies between $\omega_2 = \omega_e\sqrt{2(1 - \omega_o^2/\omega_e^2)}$ and $\omega_3 = \sqrt{2}\omega_e$. Thus, there is a gap of "forbidden" frequencies between ω_1 and ω_2. Figure S-7.8 shows the exact solution (continuous lines), and the approximate solution (dashed lines), still in good agreement, for $\omega_o^2/\omega_e^2 = 0.25$.

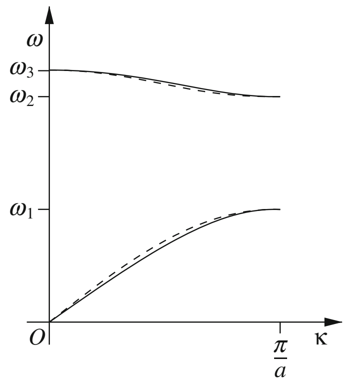

Fig. S-7.8

Of course, the two branches are present also in the case of the alternating mechanical oscillators, and provide a model for an effect known in solid state physics. The vibrations of a lattice formed by identical ions have a single branch (Problem 7.4), with a dispersion relation similar to the lower branch, which is named "acoustic branch". In an ionic crystal, formed by two ion species alternating on the sites of the lattice, we observe also the upper branch, named "optical branch".

S-7.8 Resonances in an *LC* Ladder Network

a) According to Problem 7.4, the current flowing in the nth mesh is

$$\frac{\mathrm{d}^2 I_n}{\mathrm{d}t^2} = \omega_0^2 (I_{n+1} - 2I_n + I_{n-1}) \,. \tag{S-7.86}$$

We are looking for a propagating wave solution, and define the phase

$$\phi \equiv ka \,, \tag{S-7.87}$$

where a is the length of a single mesh, to be substituted into (S-7.44), writing $I_n(t)$ as

$$I_n(t) = A\mathrm{e}^{\mathrm{i}n\phi - \mathrm{i}\omega t} \,. \tag{S-7.88}$$

Substituting (S-7.88) into (S-7.86), and dividing by $\mathrm{e}^{-\mathrm{i}\omega t}$, we get

$$-\,\omega^2 \mathrm{e}^{\mathrm{i}n\phi} = \omega_0^2 \left[\mathrm{e}^{\mathrm{i}(n+1)\phi} - 2\mathrm{e}^{\mathrm{i}n\phi} + \mathrm{e}^{\mathrm{i}(n-1)\phi}\right] , \tag{S-7.89}$$

from which we obtain the dispersion relation

$$\omega^2 = \omega_0^2 \,(2 - \mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}) = 2\omega_0^2 \,(1 - \cos\phi) = 4\omega_0^2 \sin^2(\phi/2) \,, \tag{S-7.90}$$

whose inverse is

$$\sin\left(\frac{\phi}{2}\right) = \frac{\omega}{2\omega_0} \,, \quad \text{or} \quad \phi = 2 \arcsin\left(\frac{\omega}{2\omega_0}\right) , \tag{S-7.91}$$

that shows that ϕ is a real number if $\omega < 2\omega_0$.

Due to the presence of the current source, (S-7.88) holds if the current in the 0th mesh is

$$I_0(t) = I_\mathrm{s}\mathrm{e}^{-\mathrm{i}\omega t} \,, \tag{S-7.92}$$

thus we must have $A = I_\mathrm{s}$, and the final expression for $I_n(t)$ is

$$I_n(t) = I_\mathrm{s}\mathrm{e}^{\mathrm{i}n\phi - \mathrm{i}\omega t} \,, \tag{S-7.93}$$

where ϕ is given by (S-7.91)

b) If $\omega > 2\omega_0$ the current wave cannot propagate in the ladder. We look for a solution of the form suggested by the hint. Substituting (7.12) into (S-7.86) we obtain

$$-\omega^2\alpha^{-n} = \omega_0^2\left[\alpha^{-(n+1)} - 2\alpha^{-n} + \alpha^{-(n-1)}\right], \tag{S-7.94}$$

which, multiplied by α^n/ω_0^2, turns into

$$\alpha^2 + \left(\frac{\omega^2}{\omega_0^2} - 2\right)\alpha + 1 = 0\,. \tag{S-7.95}$$

The solutions are

$$\alpha = 1 - \frac{\omega^2}{2\omega_0^2} \pm \sqrt{\left(1 - \frac{\omega^2}{2\omega_0^2}\right)^2 - 1}\,. \tag{S-7.96}$$

We must have $|\alpha| < 1$ for an infinite ladder, otherwise the current would grow indefinitely in successive meshes. Thus, we keep the solution with the plus sign, because $\omega > 2\omega_0$ implies that all solutions of (S-7.96) are negative, obtaining

$$I_n(t) = I_\text{s}(-1)^n|\alpha|^n\text{e}^{-\text{i}\omega t}, \qquad |\alpha| = \frac{\omega^2}{2\omega_0^2} - 1 - \sqrt{\left(\frac{\omega^2}{2\omega_0^2} - 1\right)^2 - 1}\,, \tag{S-7.97}$$

that we can rewrite as

$$I_n(t) = I_\text{s}\,\text{e}^{-\gamma n - \text{i}\omega t}\,, \quad \text{where} \quad \gamma = \text{i}\pi + \ln|\alpha|\,. \tag{S-7.98}$$

c) We consider the case of the propagating wave ($\omega < 2\omega_0$) first. If the ladder comprises N meshes numbered as in Fig. 7.8, the boundary condition at the right end is $I_N(t) \equiv 0$ (mesh number N does not exist!). The most general solution is the sum of two counterpropagating waves

$$I_n(t) = A\text{e}^{\text{i}n\phi - \text{i}\omega t} + B\text{e}^{-\text{i}n\phi - \text{i}\omega t}\,. \tag{S-7.99}$$

Imposing the conditions $I_0 = I_\text{s}$ and $I_N = 0$, we obtain

$$A + B = I_\text{s}, \qquad A\text{e}^{\text{i}N\phi} + B\text{e}^{-\text{i}N\phi} = 0\,, \tag{S-7.100}$$

with solutions

$$A = +\frac{\text{i}}{2}\,I_\text{s}\,\frac{\text{e}^{-\text{i}N\phi}}{\sin(N\phi)}, \qquad B = -\frac{\text{i}}{2}\,I_\text{s}\,\frac{\text{e}^{+\text{i}N\phi}}{\sin(N\phi)}, \tag{S-7.101}$$

where $\phi = \phi(\omega)$ depends on ω according to (S-7.91). We observe resonances when $\sin(N\phi) = 0$, i.e., for $\phi = m\pi/N$ with m an integer. Remembering (S-7.87)

$$N = m\frac{\pi}{\phi} = m\frac{\pi}{ka} = m\frac{\pi}{a}\frac{\lambda}{2\pi} = m\frac{1}{a}\frac{\lambda}{2}\,, \tag{S-7.102}$$

and, multiplying both sides by a

$$L = Na = m\frac{\lambda}{2}\,, \tag{S-7.103}$$

where L is the total length of the ladder network. This corresponds to the case when the frequency of the current source equals the frequency of one of the standing waves allowed in the network, i.e., when the length of the ladder network is an integer multiple of a half wavelength.

If $\omega > 2\omega_0$, the general solution is

$$I_n(t) = A\alpha_+^n \mathrm{e}^{-\mathrm{i}\omega t} + B\alpha_-^n \mathrm{e}^{-\mathrm{i}\omega t}, \tag{S-7.104}$$

where $\alpha_\pm = \alpha_\pm(\omega)$ are the two solutions of (S-7.96). Here also the case $|\alpha| > 1$ is allowed, because $|\alpha|^n$ cannot diverge if n is limited. The boundary conditions are

$$A + B = I_\mathrm{s}\,, \qquad A\alpha_+^N + B\alpha_-^{+N} = 0\,, \tag{S-7.105}$$

with solutions

$$A = +I_\mathrm{s}\frac{\alpha_-^N}{\alpha_-^N - \alpha_+^N}\,, \qquad B = -I_\mathrm{s}\frac{\alpha_+^N}{\alpha_-^N - \alpha_+^N}\,. \tag{S-7.106}$$

The A and B coefficients diverge if $\alpha_- = \alpha_+ = 1$, i.e., if $\omega = 2\omega_0$. Thus, for $\omega > 2\omega_0$ there are no resonances, but the response of the system diverges as the frequency approaches the cut-off value, i.e. as $\omega \to 2\omega_0$.

S-7.9 Cyclotron Resonances (1)

a) The rotating electric field can be written as

$$\mathbf{E} = \mathbf{E}(t) = E_0\,(\hat{\mathbf{x}}\cos\omega t \pm \hat{\mathbf{y}}\sin\omega t)\,, \tag{S-7.107}$$

where the positive (negative) sign indicates counterclockwise (clockwise) rotation. From the equation of motion

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\left(\mathbf{E} + \frac{\boldsymbol{v}}{c}\times\mathbf{B}\right)\,, \tag{S-7.108}$$

we see that $\mathrm{d}v_z/\mathrm{d}t = 0$, thus, if we assume that $v_z(0) = 0$, the motion occurs in the (x, y) plane. The equations of motion along the x and y axes are

$$\frac{dv_x}{dt} = +qv_y\frac{B_0}{mc} + \frac{qE_0}{m}\cos\omega t$$
$$\frac{dv_y}{dt} = -qv_x\frac{B_0}{mc} \pm \frac{qE_0}{m}\sin\omega t\,. \qquad \text{(S-7.109)}$$

In principle, we can differentiate both equations with respect to time, and then substitute the expressions for $dv_{x,y}/dt$, thus obtaining two uncoupled second-order differential equations for a driven harmonic oscillator.

But we prefer a a more "elegant" approach, introducing the complex variable $\zeta = v_x + iv_y$. The velocity is thus represented by a complex vector in the $(\mathrm{Re}\zeta, \mathrm{Im}\zeta)$ plane. Adding the second of (S-7.109), multiplied by i, to the first, we obtain

$$\frac{d\zeta}{dt} = -i\omega_c\,\zeta + \frac{qE_0}{m}\,e^{\pm i\omega t}, \qquad \text{(S-7.110)}$$

where $\omega_c = qB_0/mc$ is the cyclotron (or Larmor) frequency. The solution of the associated homogeneous equation is

$$\zeta(t) = A\,e^{-i\omega_c t}\,, \qquad \text{(S-7.111)}$$

where A is an arbitrary complex constant. Equation (S-7.111) describes the motion in the absence of the electric field, when the velocity rotates clockwise with frequency ω_c in the ζ plane. We then search for a particular integral of the inhomogeneous equation in the form

$$\zeta = \zeta_0\,e^{\pm i\omega t}\,,$$

and find, by direct substitution,

$$\zeta_0 = -i\,\frac{qE_0}{m\,(\omega_c \pm \omega)}\,. \qquad \text{(S-7.112)}$$

Thus the general solution of (S-7.110) is

$$\zeta(t) = A\,e^{-i\omega_c t} - i\,\frac{qE_0}{m\,(\omega_c \pm \omega)}\,e^{\pm i\omega t}\,. \qquad \text{(S-7.113)}$$

Assuming $\omega_c > 0$, we observe a resonance at $\omega = \omega_c$ only if the field rotates clockwise. In this case the electric field accelerates the particle along the direction of its "natural" motion.

b) At resonance ($\omega = \omega_c$), we search for a non-periodic solution of the form

$$\zeta(t) = \zeta_R(t)\,e^{-i\omega t}\,, \qquad \text{(S-7.114)}$$

which, substituted into (S-7.110), gives

$$\left(\frac{\mathrm{d}\zeta_\mathrm{R}}{\mathrm{d}t}\right) \mathrm{e}^{-\mathrm{i}\omega t} - \mathrm{i}\omega\, \zeta_\mathrm{R}\, \mathrm{e}^{-\mathrm{i}\omega t} = -\mathrm{i}\omega_\mathrm{c}\, \zeta_\mathrm{R}\, \mathrm{e}^{-\mathrm{i}\omega t} + \frac{qE_0}{m}\, \mathrm{e}^{-\mathrm{i}\omega t}\,, \tag{S-7.115}$$

and, since $\omega_\mathrm{c} = \omega$,

$$\left(\frac{\mathrm{d}\zeta_\mathrm{R}}{\mathrm{d}t}\right) = \frac{qE_0}{m}\,. \tag{S-7.116}$$

The solution of (S-7.116) is

$$\zeta_\mathrm{R}(t) = \zeta(0) + \frac{qE_0}{m}\, t\,, \tag{S-7.117}$$

which gives

$$\zeta(t) = \left[\zeta(0) + \frac{qE_0}{m}\, t\right] \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-7.118}$$

The trajectory is a spiral, with the radial velocity increasing linearly with time.
c) Introducing a viscous force $\mathbf{f}_v = -m\gamma\boldsymbol{v}$ we obtain the following equation for ζ

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{c}\, \zeta - \gamma\zeta + \frac{qE_0}{m}\, \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-7.119}$$

The solution has the form of (S-7.113) with ω_c replaced by $(\omega_\mathrm{c} - \mathrm{i}\gamma)$,

$$\zeta = -\mathrm{i}\, \frac{qE_0}{m(\omega_\mathrm{c} - \omega - \mathrm{i}\gamma)}\, \mathrm{e}^{-\mathrm{i}\omega t} + A\, \mathrm{e}^{-\mathrm{i}\omega_\mathrm{c} t - \gamma t}\,, \tag{S-7.120}$$

where the second term undergoes an exponential decay, and any memory of the initial conditions is lost after a transient phase, while the periodic part of the solution does not diverge at resonance, due to the presence of $\mathrm{i}\gamma$ in the denominator. Thus, the steady-state solution at resonance is

$$\zeta_\mathrm{R} = \frac{qE_0}{m\gamma}\, \mathrm{e}^{-\mathrm{i}\omega t}\,. \tag{S-7.121}$$

The average dissipated power is the time average of the instantaneous dissipated power over a period

$$P = \langle \mathbf{f} \cdot \boldsymbol{v} \rangle = \langle q\mathbf{E} \cdot \boldsymbol{v} \rangle. \tag{S-7.122}$$

The components of the particle velocity in the steady state are

$$\begin{aligned} v_x &= \mathrm{Re}(\zeta) = \frac{qE_0\gamma}{m\left[(\omega_\mathrm{c} - \omega)^2 + \gamma^2\right]} \cos\omega t - \frac{qE_0\,(\omega_\mathrm{c} - \omega)}{m\left[(\omega_\mathrm{c} - \omega)^2 + \gamma^2\right]} \sin\omega t\,, \\ v_y &= \mathrm{Im}(\zeta) = -\frac{qE_0\gamma}{m\left[(\omega_\mathrm{c} - \omega)^2 + \gamma^2\right]} \sin\omega t - \frac{qE_0\,(\omega_\mathrm{c} - \omega)}{m\left[(\omega_\mathrm{c} - \omega)^2 + \gamma^2\right]} \cos\omega t\,. \end{aligned} \tag{S-7.123}$$

Thus, inserting (S-7.123) and the relations

$$E_x = E_0 \cos\omega t\,, \quad E_y = -E_0 \sin\omega t\,, \quad \langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2}\,,$$
$$\langle\cos\omega t \sin\omega t\rangle = 0\,, \qquad \text{(S-7.124)}$$

into (S-7.122), we obtain for the average dissipated power

$$P = \frac{q^2 E_0^2 \gamma}{m\left[(\omega_c - \omega)^2 + \gamma^2\right]}\,. \qquad \text{(S-7.125)}$$

At resonance we have

$$P = \frac{q^2 E_0^2}{m\gamma}\,. \qquad \text{(S-7.126)}$$

S-7.10 Cyclotron Resonances (2)

a) The equations of motion are

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = +\omega_c v_y + \frac{qE_0}{m}\cos\omega t\,, \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega_c v_x\,, \qquad \text{(S-7.127)}$$

where $\omega_c = qB_0/m$. By differentiating (S-7.127) with respect to time, and substituting the values for $\dot v_x$ and $\dot v_y$ from (S-7.127) itself, we obtain the two equations

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = +\omega_c \frac{\mathrm{d}v_y}{\mathrm{d}t} - \frac{qE_0\omega}{m}\sin\omega t = -\omega_c^2 v_x - \frac{qE_0\omega}{m}\sin\omega t\,,$$
$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_c \frac{\mathrm{d}v_x}{\mathrm{d}t} = -\omega_c^2 v_y - \frac{qE_0\omega_c}{m}\cos\omega t\,, \qquad \text{(S-7.128)}$$

each of which describes the velocity of a driven harmonic oscillator. The steady state solutions are

$$v_x = \frac{qE_0\omega}{m(\omega^2 - \omega_c^2)}\sin\omega t\,, \qquad v_y = \frac{qE_0\omega_c}{m(\omega^2 - \omega_c^2)}\cos\omega t\,. \qquad \text{(S-7.129)}$$

We observe a resonance if $\omega = |\omega_c|$, independently on the signs of q and B_0. With respect to Problem 7.9, where a rotating electric field was assumed, here a resonance is always found because the linearly oscillating electric field can be decomposed into two counter-rotating fields of the same amplitude, of which one will excite the resonance.

b) In the presence of a frictional force $\mathbf{f} = -m\gamma\boldsymbol{v}$ the equations of motion become

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = +\omega_\mathrm{c} v_y - \gamma v_x + \frac{qE_0}{m}\cos\omega t\,, \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega_\mathrm{c} v_x - \gamma v_y\,, \tag{S-7.130}$$

and cannot be uncoupled by the procedure of point a). Analogously to Problem 7.9, we introduce the complex quantity $\zeta = v_x + \mathrm{i}v_y$, obtaining the single equation

$$\frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{c}\zeta - \gamma\zeta + \frac{qE_0}{2m}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right), \tag{S-7.131}$$

where we have used Euler's formula for the cosine. Differently from Problem 7.9, now we search for a steady-state solution of the form

$$\zeta = A\mathrm{e}^{-\mathrm{i}\omega t} + B\mathrm{e}^{\mathrm{i}\omega t}\,, \tag{S-7.132}$$

where A and B are two complex constants to be determined. By direct substitution into (S-7.131) we have

$$-\mathrm{i}\omega A\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{i}\omega B\mathrm{e}^{\mathrm{i}\omega t} = -(\mathrm{i}\omega_\mathrm{c} + \gamma)A\mathrm{e}^{-\mathrm{i}\omega t} - (\mathrm{i}\omega_\mathrm{c} + \gamma)B\mathrm{e}^{\mathrm{i}\omega t} + \frac{qE_0}{2m}\left(\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{e}^{-\mathrm{i}\omega t}\right),$$

which is separable into two equations relative, respectively, to the terms rotating clockwise and counterclockwise in the complex plane

$$-\,\mathrm{i}\omega A = -(\mathrm{i}\omega_\mathrm{c} + \gamma)A + \frac{qE_0}{2m}\,, \quad \mathrm{i}\omega B = -(\mathrm{i}\omega_\mathrm{c} + \gamma)B + \frac{qE_0}{2m}\,. \tag{S-7.133}$$

The solutions for A and B are

$$\begin{aligned} A &= \frac{qE_0}{2m[i(\omega_\mathrm{c} - \omega) + \gamma]} = \frac{qE_0\gamma}{2m[(\omega_\mathrm{c} - \omega)^2 + \gamma^2]} - \mathrm{i}\,\frac{qE_0(\omega_\mathrm{c} - \omega)}{2m[(\omega_\mathrm{c} - \omega)^2 + \gamma^2]} \\ B &= \frac{qE_0}{2m[i(\omega_\mathrm{c} + \omega) + \gamma]} = \frac{qE_0\gamma}{2m[(\omega_\mathrm{c} + \omega)^2 + \gamma^2]} - \mathrm{i}\,\frac{qE_0(\omega_\mathrm{c} + \omega)}{2m[(\omega_\mathrm{c} + \omega)^2 + \gamma^2]}\,, \end{aligned} \tag{S-7.134}$$

from which we obtain the stationary-state velocity components of the particle

$$\begin{aligned} v_x &= [\mathrm{Re}(A) + \mathrm{Re}(B)]\cos\omega t + [\mathrm{Im}(A) - \mathrm{Im}(B)]\sin\omega t \\ v_y &= [\mathrm{Im}(A) + \mathrm{Im}(B)]\cos\omega t - [\mathrm{Re}(A) - \mathrm{Re}(B)]\sin\omega t\,. \end{aligned} \tag{S-7.135}$$

The average absorbed power is

$$\begin{aligned} P &= \langle q\boldsymbol{v}\cdot\mathbf{E}\rangle = \langle qv_xE_x\rangle = q[\mathrm{Re}(A) + \mathrm{Re}(B)]E_0\,\frac{1}{2} \\ &= \frac{q^2E_0^2\gamma}{4m[(\omega_\mathrm{c} - \omega)^2 + \gamma^2]} + \frac{q^2E_0^2\gamma}{4m[(\omega_\mathrm{c} + \omega)^2 + \gamma^2]}\,, \end{aligned} \tag{S-7.136}$$

since $E_x = E_0 \cos\omega t$, $\langle\cos^2\omega t\rangle = 1/2$, and $\langle\cos\omega t \sin\omega t\rangle = 0$. Thus, again, we observe a resonance at $\omega = |\omega_c|$, independently of the signs of q and B_0. Assuming $\gamma \ll \omega_c$ the power absorbed at resonance is

$$P_{\max} \simeq \frac{q^2 E_0^2}{4m\gamma}\,. \tag{S-7.137}$$

S-7.11 A Quasi-Gaussian Wave Packet

We need to evaluate the inverse transform

$$\begin{aligned} f(x) &= A\int_{-\infty}^{+\infty} \mathrm{e}^{-L^2(k-k_0)^2}\mathrm{e}^{\mathrm{i}\phi(k)}\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k \\ &\simeq \int_{-\infty}^{+\infty} \exp\Big[-L^2(k-k_0)^2 + \mathrm{i}\phi_0 + \mathrm{i}\phi_0'(k-k_0) + \\ &\quad + \frac{\mathrm{i}}{2}\phi_0''(k-k_0)^2 + \mathrm{i}(k-k_0)x + \mathrm{i}k_0 x\Big]\mathrm{d}k\,, \end{aligned} \tag{S-7.138}$$

where, for brevity, we wrote x instead of $(x - vt)$, and ϕ_0, ϕ_0', ... instead of $\phi(k_0)$, $\phi'(k_0)$, ... By using (7.1) we obtain

$$\begin{aligned} f(x) &\simeq A\mathrm{e}^{\mathrm{i}k_0 x + \mathrm{i}\phi_0}\int_{-\infty}^{+\infty} \exp\left[-L^2(k-k_0)^2\left(1 - \mathrm{i}\frac{\phi_0''}{2L^2}\right) + \mathrm{i}(k-k_0)(x+\phi_0')\right]\mathrm{d}k \\ &= C\exp\left[-\frac{(x+\phi_0')^2}{4L^2(1-\mathrm{i}\phi_0''/2L^2)}\right], \end{aligned} \tag{S-7.139}$$

where C is a constant, whose value is not relevant for our purposes. By substituting

$$\frac{1}{1-\mathrm{i}\phi_0''(2L^2)} = \frac{1+\mathrm{i}\phi_0''/(2L^2)}{1+\phi_0''^2/(4L^4)} \tag{S-7.140}$$

we obtain the wave packet profile as

$$f(x - vt) = C\exp\left\{-\frac{(x - vt + \phi_0')^2[1+\mathrm{i}\phi_0''/(2L^2)]}{L^2[1+\phi_0''^2/(4L^4)]}\right\}. \tag{S-7.141}$$

We thus see that the packet is wider than the purely Gaussian case, since $L^2[1 + \phi_0''^2/(4L^4)] > L^2$. In addition, the center of the packet is shifted from $(x - \boldsymbol{v}t)$ to $(x - \boldsymbol{v}t + \phi_0')$, and there is an aperiodic (anharmonic) modulation due to the factor $(\mathrm{i}\phi_0''/2L^2)$ in the numerator of the exponent.

S-7.12 A Wave Packet Traveling along a Weakly Dispersive Line

a) There is no dispersion if $b = 0$. In these conditions the signal propagates at velocity v keeping its shape:

$$f(x - vt) = A\mathrm{e}^{-\mathrm{i}\omega_0(t-x/v)}\mathrm{e}^{-(t-x/v)^2/\tau^2} . \tag{S-7.142}$$

b) The phase velocity and the group velocity are, by definition,

$$v_\phi = \frac{\omega}{k} = v(1 + bk) , \qquad v_g = \frac{\partial\omega}{\partial k} = v(1 + 2bk) . \tag{S-7.143}$$

We can write v_ϕ and v_g as functions of ω by first inverting (7.17), obtaining for $k = k(\omega)$

$$k = \sqrt{\frac{1}{(2b)^2} + \frac{\omega}{bv}} - \frac{1}{2b} . \tag{S-7.144}$$

Then we expand the square root to the second order in ω/v, obtaining

$$k \simeq \frac{\omega}{v} - \frac{\omega^2 b}{v^2}. \tag{S-7.145}$$

The same result can also be obtained by an iterative procedure, by inserting the first order value for k, i.e., $k = \omega/v$, into the bracket at the right hand side of (7.17). Thus, the phase and group velocities to the first order are, using (S-7.143),

$$v_{\phi 0} \simeq v + b\omega_0 , \qquad v_{g0} \simeq v + 2b\omega_0 . \tag{S-7.146}$$

c) The peak of the signal propagates at the group velocity, thus $t_x = x/v_{g0}$. The spectral width of the wave packet may be estimated as $\Delta\omega \simeq 1/\tau$, which corresponds to a spread in the propagation velocity of its Fourier components

$$\Delta v \simeq v\left(\frac{2b}{v}\Delta\omega\right) \simeq \frac{2b}{\tau} . \tag{S-7.147}$$

Thus the spread of the wave packet in time and space can be estimated as

$$\Delta t \simeq \frac{\partial t_x}{\partial v_g}\Delta v = t_x\frac{\Delta v}{v_g} , \qquad \Delta x \simeq v_g\Delta t = \frac{2bx}{v_g\tau} . \tag{S-7.148}$$

d) We approximate

$$k(\omega) \simeq k_0 + k_0'(\omega - \omega_0) + \frac{1}{2}k_0''(\omega - \omega_0)^2 , \tag{S-7.149}$$

where

$$k_0' = \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0} \simeq \frac{1}{v} - 2\frac{\omega_0 b}{v^2} \simeq \frac{1}{v_g}\,, \qquad k_0'' = \left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0} \simeq -2\frac{b}{v^2}\,. \tag{S-7.150}$$

The spectrum of the wave packet (i.e., its Fourier transform) is

$$\tilde{f}(\omega) = \sqrt{\pi}\tau A \mathrm{e}^{-[\omega-\omega_0]^2\tau^2/4}\,. \tag{S-7.151}$$

Since we are only interested in the behavior of the function, we evaluate the following integral forgetting proportionality constants,

$$\begin{aligned} f(x,t) &\sim \int \exp\left[\mathrm{i}k(\omega)x - \mathrm{i}\omega t - \frac{(\omega-\omega_0)^2\,\tau^2}{4}\right]\mathrm{d}\omega \\ &\sim \int \exp\left[\mathrm{i}k_0 x + \mathrm{i}k_0' x(\omega-\omega_0) + \mathrm{i}\frac{k_0'' x}{2}(\omega-\omega_0)^2 - \frac{(\omega-\omega_0)^2\tau^2}{4}\right]\mathrm{d}\omega \\ &\sim \exp(\mathrm{i}k_0 x - \mathrm{i}\omega_0 t)\int \exp\left[-\mathrm{i}(t - k_0' x)\,\omega' + \left(-\frac{\tau^2}{4} + \mathrm{i}\frac{k_0'' x}{2}\right)\omega'^2\right]\mathrm{d}\omega' \\ &\sim \exp\left[\mathrm{i}k_0 x - \mathrm{i}\omega_0 t - \frac{(t-k_0' x)^2}{\tau^2 - 2\mathrm{i}k_0'' x}\right]. \end{aligned} \tag{S-7.152}$$

The factor which describes the envelope of the wave packet (recalling that $k_0' = 1/v_{g0}$) is

$$\begin{aligned} \exp\left[-\frac{(t-x/v_g)^2}{\tau^2 - 2\mathrm{i}k_0'' x}\right] &= \exp\left[-(t-x/v_g)^2\frac{\tau^2 + 2\mathrm{i}k_0'' x}{\tau^4 + (2k_0'' x)^2}\right] \\ &= \exp\left\{-(t-x/v_g)^2\frac{1 + 2\mathrm{i}k_0'' x/\tau^2}{\tau^2[1 + (2k_0'' x/\tau)^2]}\right\}\,. \end{aligned} \tag{S-7.153}$$

The temporal width of the wave packet increases during the propagation as

$$\Delta t(x) = \tau\sqrt{1 + \left(\frac{2k_0'' x}{\tau}\right)^2}\,. \tag{S-7.154}$$