

# Chapter S-8

## Solutions for Chapter 8

### S-8.1 Poynting Vector(s) in an Ohmic Wire

For symmetry reasons, the magnetic field is azimuthal and depends only on the radial coordinate  $r$ . Applying Ampère’s law to a circular path of radius  $r < a$  around the wire axis yields

$$2\pi rB = \frac{4\pi}{c}(\pi r^2 J), \tag{S-8.1}$$

which leads to

$$\mathbf{B} = \frac{2\pi}{c} r\sigma E \hat{\phi}, \tag{S-8.2}$$

where  $\hat{\phi}$  is the azimuthal unit vector. Thus, the Poynting vector at a distance  $r$  from the axis is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\mathbf{r} \frac{\sigma}{2} E^2. \tag{S-8.3}$$

The energy flux  $\Phi_S \equiv \Phi(\mathbf{S})$  through the surface of a cylinder of radius  $r < a$  and length  $h$  and coaxial to the wire is thus

$$\begin{aligned} \Phi_S &= \oint \mathbf{S} \cdot d\mathbf{A} = -2\pi r h S(r) \\ &= -\pi r^2 h \sigma E^2, \end{aligned} \tag{S-8.4}$$

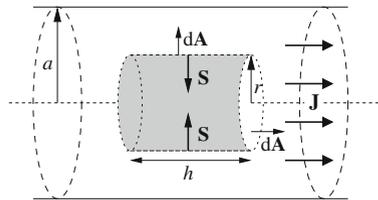


Fig. S-8.1

where  $d\mathbf{A}$  is the vector surface element of the cylinder. The energy flows inwards, and is entirely dissipated into Joule heating inside the cylinder volume, as we can check by calculating

$$W = \int \mathbf{J} \cdot \mathbf{E} dV = \pi r^2 h J E = \pi r^2 h \sigma E^2, \tag{S-8.5}$$

where the integral is extended to the volume the cylinder. The equality  $W = -\dot{\Phi}_S$  satisfies Poynting's theorem since there is no variation in time of the EM energy.

Note that, in the approximation of an infinitely long wire, the electric field is uniform also for  $r > a$  (in the case of a finite wire of length  $2h \gg a$ , this is a good approximation in the central region for  $r \ll h$ , see Problem 4.9), while the magnetic field  $B = 2\pi J a^2 / rc$ . Within this approximation,  $\mathbf{S} = -(a^2 \sigma E^2 / 2r) \hat{\mathbf{r}}$  for  $r > a$ , so that the energy flux is independent of  $r$  and it is still equal to minus the total dissipated power:

$$\Phi_S = -2\pi r h S(r) = -\pi a^2 h \sigma E^2 \quad (r > a). \quad (\text{S-8.6})$$

b) We must show that  $\nabla \cdot (\mathbf{S} - \mathbf{S}') = 0$ , i.e., that  $\mathbf{S} - \mathbf{S}' = \nabla \times \mathbf{f}$ , where  $\mathbf{f}$  is a vector function of the coordinates. Let us substitute  $\mathbf{E} = -\nabla\varphi$  into (8.7)

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\frac{c}{4\pi} \nabla\varphi \times \mathbf{B}. \quad (\text{S-8.7})$$

Now from the vector identity

$$\nabla \times (\varphi \mathbf{B}) = \nabla\varphi \times \mathbf{B} + \varphi \nabla \times \mathbf{B} = \nabla\varphi \times \mathbf{B} + \varphi \left( \frac{4\pi}{c} \mathbf{J} \right) \quad (\text{S-8.8})$$

we obtain

$$\nabla\varphi \times \mathbf{B} = \nabla \times (\varphi \mathbf{B}) - \varphi \left( \frac{4\pi}{c} \mathbf{J} \right), \quad (\text{S-8.9})$$

which can be substituted into (S-8.7), leading to

$$\mathbf{S} = \varphi \mathbf{J} - \frac{c}{4\pi} \nabla \times (\varphi \mathbf{B}). \quad (\text{S-8.10})$$

Thus, we are free to redefine the Poynting vector as

$$\mathbf{S}' = \varphi \mathbf{J}, \quad (\text{S-8.11})$$

since

$$\mathbf{S} - \mathbf{S}' = \nabla \times \left( -\frac{c}{4\pi} \varphi \mathbf{B} \right). \quad (\text{S-8.12})$$

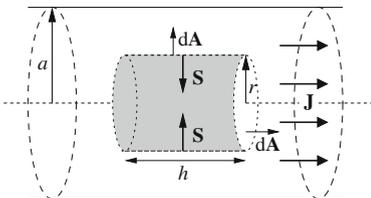


Fig. S-8.2

We can show that  $\mathbf{S}'$  is equivalent to  $\mathbf{S}$  by computing its flux through the same cylindrical surface as above. Since  $\mathbf{S}'$  is parallel to the wire axis, only the two base surfaces contribute to the flux  $\Phi_{S'} \equiv \Phi(\mathbf{S}')$

$$\begin{aligned} \Phi_{S'} &= \pi a^2 [-S(z) + S(z+h)] \\ &= \pi a^2 J[\varphi(z+h) - \varphi(z)]. \end{aligned} \tag{S-8.13}$$

Since  $\varphi = -Ez$ , we finally obtain

$$\Phi_{S'} = \pi a^2 [-S(z) + S(z+h)] = -\pi a^2 h J E, \tag{S-8.14}$$

which gives again minus the total dissipated power.

### S-8.2 Poynting Vector(s) in a Capacitor

a) The magnetic field has azimuthal symmetry, i.e.,  $\mathbf{B} = B(r)\hat{\phi}$ , and can be evaluated from the equation  $c\nabla \times \mathbf{B} = \partial_t \mathbf{E}$ , which, with our assumption  $E = E_0 t/\tau$ , leads to

$$B(r) = \frac{r}{2c} \partial_t E = \frac{r}{2c\tau} E_0. \tag{S-8.15}$$

b) The corresponding Poynting vector  $\mathbf{S}$  is

$$\mathbf{S} = \frac{c}{4\pi} E \hat{\mathbf{z}} \times (B \hat{\phi}) = -\frac{r}{8\pi} (E \partial_t E) (\hat{\mathbf{z}} \times \hat{\phi}) = -\frac{1}{2} \partial_t \left( \frac{E^2}{8\pi} \right) \mathbf{r}. \tag{S-8.16}$$

We evaluate the flux of  $\mathbf{S}$  through the smallest closed cylindrical surface enclosing our capacitor, shown in Fig. S-8.3. Since  $\mathbf{S}$  is radial, only the lateral surface of the cylinder contributes to the flux, and we have

$$\Phi(\mathbf{S}) = -2\pi a h S(a) = -\pi a^2 h \partial_t \left( \frac{E^2}{8\pi} \right). \tag{S-8.17}$$

Quantity  $(E^2 + B^2)/8\pi$  is the energy density associated to the EM field, and, since in our case  $B$  does not depend on time, is also the total EM energy density inside the capacitor. Thus,  $\Phi(\mathbf{S})$  equals minus the time derivative of the energy stored in the capacitor. For a general dependence of  $E_z(t)$  on time,  $\mathbf{B}$  is also time-dependent, and the flux of  $\mathbf{S}$  equals the time derivative of the electrostatic energy to the first order, within the slowly varying current approximation.

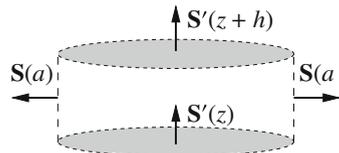


Fig. S-8.3

c) The electric potential is  $\varphi = -Ez$ . By substituting  $\mathbf{E} = -\nabla\varphi$  into (8.7) we obtain

$$\mathbf{S} = \frac{c}{4\pi} (-\nabla\varphi) \times \mathbf{B} = -\frac{c}{4\pi} [\nabla \times (\varphi\mathbf{B}) - \varphi\nabla \times \mathbf{B}]. \tag{S-8.18}$$

Thus, the vector

$$\mathbf{S}' \equiv \frac{c}{4\pi} \nabla \times \mathbf{B} = \frac{1}{4\pi} \varphi \partial_t \mathbf{E} = \mathbf{S} + \frac{c}{4\pi} \nabla \times (\varphi \mathbf{B}) \quad (\text{S-8.19})$$

equals  $\mathbf{S}$  plus the curl of a vector function, and is thus another suitable Poynting vector. Since  $\mathbf{S}'$  is perpendicular to the capacitor plates, its flux through our closed cylindrical surface is (see Fig. S-8.3)

$$\Phi(\mathbf{S}') = \pi a^2 [S'(z+h) - S'(z)] = -\pi a^2 h \left( \frac{E \partial_t E}{4\pi} \right) = -\pi a^2 h \partial_t \left( \frac{E^2}{8\pi} \right), \quad (\text{S-8.20})$$

in agreement with (S-8.17).

### S-8.3 Poynting's Theorem in a Solenoid

**a)** We take a cylindrical coordinate system with the  $z$  axis along the solenoid axis. Inside an infinite solenoid the magnetic field is uniform and equals  $\mathbf{B} = B \hat{\mathbf{z}} = (4\pi/c)nI \hat{\mathbf{z}}$ . According to Faraday's law of induction, the rate of change of  $\mathbf{B} = \mathbf{B}(t)$ , due to the time dependence of  $I = I(t)$ , generates an electric field  $\mathbf{E}$  associated to the induced electromotive force. For symmetry reasons, the field lines of  $\mathbf{E}$  are circles coaxial to the solenoid, i.e., we have  $\mathbf{E} = E(r) \hat{\boldsymbol{\phi}}$ . Applying Faraday's law to a circle of radius  $r < a$ , coaxial to the solenoid, we have

$$2\pi r E(r) = -\pi r^2 \frac{1}{c} \partial_t B = -\pi r^2 \frac{4\pi n I_0}{c^2 \tau}, \quad (\text{S-8.21})$$

from which  $E(r) = -2\pi n I_0 r / (c^2 \tau)$ .

**b)** The Poynting vector inside the solenoid ( $r < a$ ) is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = -\frac{2\pi (nI_0)^2 r t}{(c\tau)^2} (\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}}) = -\frac{2\pi (nI_0)^2 t}{(c\tau)^2} \mathbf{r}. \quad (\text{S-8.22})$$

Thus, the flux of  $\mathbf{S} = \mathbf{S}(r)$  through the surface of a closed cylinder of radius  $r$  and height  $h$  is nonzero only through the lateral surface, and we have

$$\Phi(\mathbf{S}) = 2\pi r h \mathbf{S} \cdot \hat{\mathbf{r}} = -\left( \frac{2\pi n I_0 r}{c\tau} \right)^2 h t. \quad (\text{S-8.23})$$

The magnetic energy enclosed by the cylinder surface is

$$U_M = u_M V = \frac{B^2}{8\pi} \pi r^2 h = 2\pi^2 r^2 h \left( \frac{nI_0 t}{c\tau} \right)^2, \quad (\text{S-8.24})$$

where  $V$  is the volume of the cylinder, thus

$$\frac{dU_M}{dt} = 4\pi^2 r^2 h t \left( \frac{nI_0}{c\tau} \right)^2 = -\Phi(\mathbf{S}), \quad (\text{S-8.25})$$

according to Poynting's theorem, since the electric field is constant in time, and  $\mathbf{J} \cdot \mathbf{E} = 0$  for  $r < a$ , i.e., inside the solenoid.

c) Outside the solenoid ( $r > a$ ) we have  $\mathbf{B} = 0$ . Correspondingly, also  $\mathbf{S} = 0$  and  $\Phi(\mathbf{S}) = 0$ . The rate of change of the magnetic energy is given by (S-8.25) with  $r = a$ , and must equal the volume integral of  $\mathbf{J} \cdot \mathbf{E}$ , which is the work done by the induced field on the current flowing in the coils (notice that this is different from the electric field driving the current and causing Joule heating in the coils, see Problem 13.18). In our representation, the current is distributed on the surface  $r = a$ , thus  $\mathbf{J} d^3 r$  is replaced by  $nI dS = nI a d\phi dz$  in the integral, and  $E$  is evaluated at  $r = a$ . We thus obtain

$$\begin{aligned} \int_V \mathbf{J} \cdot \mathbf{E} d^3 r &= \int_S nI E(a) dS = -2\pi a h \left( nI_0 \frac{t}{\tau} \right) \left( \frac{2\pi n I_0 a}{c^2 \tau} \right) \\ &= -4\pi a^2 h t \left( \frac{nI_0}{c\tau} \right)^2 = - \left. \frac{dU_M}{dt} \right|_{r=R}. \end{aligned} \quad (\text{S-8.26})$$

## S-8.4 Poynting Vector in a Capacitor with Moving Plates

a) We use a cylindrical coordinate system  $(r, \phi, z)$ , with the  $z$  axis along the symmetry axis of the capacitor, and the origin on the fixed plate. Thus, within the limits of our approximations, the electric field is uniform and parallel to  $\hat{\mathbf{z}}$  inside the capacitor, whose capacitance is

$$C = \frac{\pi a^2}{4\pi h(t)} = \frac{a^2}{4(h_0 + vt)}. \quad (\text{S-8.27})$$

In the case of the isolated plates the charge is constant and equal to  $Q_0$ , while the voltage between the plates  $V$  and the electric field  $E$  between the plates are, respectively,

$$V = \frac{Q_0}{C} = Q_0 \frac{4(h_0 + vt)}{a^2}, \quad E = \frac{V}{h} = \frac{4Q_0}{a^2}. \quad (\text{S-8.28})$$

In the case of constant voltage between the plates,  $V = V_0$ , the charge  $Q$  of the capacitor and the electric field  $E$  are, respectively.

$$Q = CV_0 = V_0 \frac{a^2}{4(h_0 + vt)}, \quad E = \frac{V_0}{h_0 + vt}. \quad (\text{S-8.29})$$

In the case of constant charge, the electrostatic force between the plates  $F_{\text{es}}$  is also constant and equals

$$F_{\text{es}} = -Q \frac{E}{2} = -\frac{2Q_0^2}{a^2}, \quad (\text{S-8.30})$$

while in the case of constant voltage we have

$$F_{\text{es}} = -Q \frac{E}{2} = -V_0^2 \frac{a^2}{8(h_0 + vt)^2}, \quad (\text{S-8.31})$$

in both cases the minus signs means that the force is attractive. In both cases the applied external force  $\mathbf{F}_{\text{mech}}$  must cancel the electrostatic force, i.e., we must have  $F_{\text{mech}} = -F_{\text{es}}$ , for the plates to move at constant velocity.

**b)** The electrostatic energy can be written as

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} CV^2, \quad (\text{S-8.32})$$

so that at constant charge we have

$$U = Q_0^2 \frac{2(h_0 + vt)}{a^2}, \quad \frac{dU}{dt} = \frac{2vQ_0^2}{a^2} > 0, \quad (\text{S-8.33})$$

while at constant voltage we have

$$U = V_0^2 \frac{a^2}{8(h_0 + vt)}, \quad \frac{dU}{dt} = -\frac{a^2 v V_0^2}{8(h_0 + vt)^2} < 0. \quad (\text{S-8.34})$$

**c)** At constant charge, the electric field  $\mathbf{E} = E_0 \hat{\mathbf{z}}$  is also constant, with  $E_0 = Q_0/(\pi a^2)$ , therefore the displacement current density  $\mathbf{J}_D = \partial_t \mathbf{E}/c$  is zero. Also the conduction current density  $\mathbf{J}_C$  is zero between the plates (actually, there is a conduction current localized on the moving plate, we shall come back to this point below), so that also the magnetic field  $\mathbf{B}$  is zero between the plates.

At constant voltage, the electric field is  $\mathbf{E} = \hat{\mathbf{z}} V_0/h(t) = \hat{\mathbf{z}} V_0/(h_0 + vt)$ , implying the presence of a displacement current along  $\hat{\mathbf{z}}$ . The magnetic field can be calculated by taking the path integral of  $\mathbf{B}$  over a circumference of radius  $r < a$  coaxial with, and located between, the plates, which equals the flux of the displacement current through the enclosed circle. Due to the cylindrical symmetry of the system, the only nonzero component of  $\mathbf{B}$  is azimuthal,  $\mathbf{B} = B(r, t)\hat{\boldsymbol{\phi}}$ , and calculating its path integral over the circle of radius  $r$  corresponding to a field line we have for  $B = B(r, t)$

$$2\pi r B = -\frac{\pi r^2}{c} \partial_t E = -\frac{\pi r^2}{c} \frac{V_0 v}{(h_0 + vt)^2}, \quad (\text{S-8.35})$$

so that

$$B = -\frac{r}{2c} \frac{V_0 v}{(h_0 + vt)^2}. \quad (\text{S-8.36})$$

**d)** At constant charge we have  $\mathbf{B} = 0$ , and the Poynting vector  $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$  is also zero. In this case, (S-8.30) and (S-8.33) tell us that the rate of work done against the electric force  $W_{\text{mech}}$

$$W_{\text{mech}} = \mathbf{F}_{\text{mech}} \cdot \mathbf{v} = -\mathbf{F}_{\text{es}} \cdot \mathbf{v} = \frac{2Q_0^2 v}{a^2} \quad (\text{S-8.37})$$

equals the rate of change the electrostatic energy  $dU/dt$ . This rate of work must also equal minus the integral of  $\mathbf{J} \cdot \mathbf{E}$  over the whole space, according to Poynting's theorem. We verify this at the end of this answer.

At constant voltage, the Poynting vector is radial,  $\mathbf{S} = S \hat{\mathbf{r}}$ , and, according to (S-8.29) and (S-8.36), we have

$$S = -\frac{c}{4\pi} E_z B_\phi = \frac{V_0^2 v r}{8\pi (h_0 + vt)^3}. \quad (\text{S-8.38})$$

Evaluating the flux of  $\mathbf{S}$  through the minimum closed surface enclosing the capacitor, of lateral surface  $2\pi a(h_0 + vt)$ , we obtain

$$\Phi_S = 2\pi a(h_0 + vt) \frac{V_0^2 v a}{8\pi (h_0 + vt)^3} = \frac{a^2 V_0^2 v}{4(h_0 + vt)^2}. \quad (\text{S-8.39})$$

Through (S-8.31) and (S-8.34) we can verify that

$$-\dot{\Phi}_S = \frac{dU}{dt} + Fv. \quad (\text{S-8.40})$$

Note also that, in this case,  $\Phi_S$  equals the power *absorbed* by the voltage source. In fact, the current flowing through the circuit is

$$I = \frac{dQ}{dt} = -\frac{a^2 v}{4(h_0 + vt)^2} V_0, \quad (\text{S-8.41})$$

where we have inserted the first of (S-8.29), corresponding to a power absorption by the source

$$W = -V_0 I = \frac{a^2 v}{4(h_0 + vt)^2} V_0^2 = \Phi_S. \quad (\text{S-8.42})$$

We avoided so far to discuss the role of the conduction current circulating in the plates [the following discussion will require some familiarity with the distributions  $\delta(x)$  and  $\Theta(x)$ , where  $\Theta(x)$  is the Heaviside step function, defined by  $\Theta(x) = 1$  for  $x > 0$  and  $\Theta(x) = 0$  for  $x < 0$ ; notice that  $d\Theta(x)/dx = \delta(x)$ ]. Let us consider the constant charge case. Since the upper plate has a charge  $Q_0$  distributed on the surface  $z = -h_0 + vt$  and moves with velocity  $v$ , there is actually a current density

$$\mathbf{J}_C = \frac{Q_0}{\pi a^2} v \delta(z - h_0 - vt) \hat{\mathbf{z}}. \quad (\text{S-8.43})$$

On the other hand, the electric field between the plates may be written as

$$\mathbf{E} = -\frac{4Q_0}{a^2} [\Theta(z) - \Theta(z - h_0 - vt)] \hat{\mathbf{z}}, \quad (\text{S-8.44})$$

where  $\Theta(z)$  is the Heaviside step function, defined by  $\Theta(z) = 1$  for  $z > 0$  and  $\Theta(z) = 0$  for  $z < 0$ . This expression takes into account the fact that at each time  $t$  the field exists only in the  $0 < z < vt$  region, so it is actually a time-dependent field. Since  $d\Theta(z)/dz = \delta(z)$ , the displacement current is

$$\mathbf{J}_D = \frac{1}{c} \partial_t \mathbf{E} = -\frac{4Q_0 v}{a^2 c} \delta(z - h_0 - vt) \hat{\mathbf{z}} = -\frac{4\pi}{c} \mathbf{J}_C, \quad (\text{S-8.45})$$

so that the source term for the magnetic field  $(4\pi/c)\mathbf{J}_C + \mathbf{J}_D$  is zero. It also follows that

$$\mathbf{J}_C \cdot \mathbf{E} = -\frac{1}{4\pi} (\partial_t \mathbf{E}) \cdot \mathbf{E} = -\frac{1}{8\pi} \partial_t E^2, \quad (\text{S-8.46})$$

which ensures energy conservation, since the work done on the current equals the rate of change of the electrostatic energy. In detail, we have

$$\partial_t E^2 = (4\pi\sigma_{\text{up}})^2 \partial_t \Theta(z - h_0 - vt) = -v \left( \frac{4Q_0}{a^2} \right)^2 \delta(z - h_0 - vt), \quad (\text{S-8.47})$$

thus

$$\begin{aligned} \int \mathbf{J}_C \cdot \mathbf{E} d^3r &= \frac{v(4\pi\sigma)^2}{8\pi} \int \delta(z - h_0 - vt) d^3r \\ &= 2\pi^2 a^2 \sigma^2 v = \frac{2vQ_0^2}{a^2} = \frac{dU}{dt}. \end{aligned} \quad (\text{S-8.48})$$

### S-8.5 Radiation Pressure on a Perfect Mirror

a) We consider the case of perpendicular incidence first, and choose a Cartesian reference frame with the  $x$  axis perpendicular to the mirror surface. The incident plane wave packet has duration  $\tau$  (with  $\tau \gg 2\pi/\omega$ , the laser period), corresponding to a length  $c\tau$ , and propagates along  $\hat{\mathbf{x}}$ . We want to calculate how much momentum is transferred to an area  $A$  of the mirror surface during the reflection of the whole wave packet. The momentum transferred per unit time and area is the pressure exerted by the radiation.

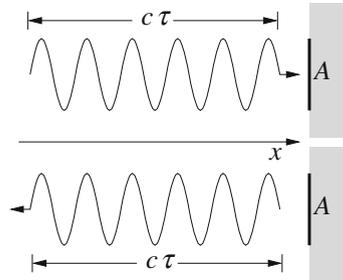


Fig. S-8.4

The momentum density of an EM field is  $\mathbf{S}/c^2$ , where  $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/4\pi$  is the Poynting vector. Thus the total momentum delivered by the incident wave packet on the area  $A$  is

$$\mathbf{p}_i = \left\langle \frac{\mathbf{S}_i}{c^2} \right\rangle c\tau A = \hat{\mathbf{x}} \frac{I}{c} \tau A \quad (\text{S-8.49})$$

where the angle brackets denote the average over one cycle,  $\mathbf{S}_i$  is the Poynting vector of the incident packet, and  $I = |\langle \mathbf{S}_i \rangle|$  is the intensity of the incident pulse (the average flux of energy per unit time and area), according to Poynting's theorem of energy conservation.

The reflected wave packet carries a total momentum, over the area  $A$ ,

$$\mathbf{p}_r = \left\langle \frac{\mathbf{S}_r}{c^2} \right\rangle c\tau A = -\hat{\mathbf{x}} \frac{I}{c} \tau A \quad (\text{S-8.50})$$

where  $\mathbf{S}_r = -\mathbf{S}_i$  is the Poynting vector of the reflected packet. The momentum transferred to the mirror over the surface area  $A$  during the time interval  $\tau$  is thus

$$\Delta \mathbf{p} = \mathbf{p}_i - \mathbf{p}_r = |\Delta \mathbf{p}| \hat{\mathbf{x}} \quad (\text{S-8.51})$$

and the corresponding pressure is

$$P_{\text{rad}} = \frac{|\Delta \mathbf{p}|}{\tau A} = 2 \frac{I}{c}. \quad (\text{S-8.52})$$

Using a similar heuristic argument, it is quite straightforward to find the radiation pressure for oblique incidence at an angle  $\theta$  from the normal to the mirror surface. In fact, in this case the momentum transferred to the mirror along the normal is

$$\Delta \mathbf{p} = \mathbf{p}_i - \mathbf{p}_r = 2 \hat{\mathbf{x}} \frac{I}{c^2} c \tau A \cos \theta = 2 \hat{\mathbf{x}} \frac{I}{c} \tau A \cos \theta, \quad (\text{S-8.53})$$

and the area of incidence is now  $A/\cos \theta$ . Thus

$$P_{\text{rad}} = |\mathbf{p}_i - \mathbf{p}_r| \frac{\cos \theta}{\tau A} = 2 \frac{I}{c} \tau A \cos \theta \frac{\cos \theta}{\tau A} = 2 \frac{I}{c} \cos^2 \theta. \quad (\text{S-8.54})$$

**b)** The mechanical force on a closed system of charges, currents and fields is given by the following integral over the volume of the system

$$\mathbf{F}_{\text{mech}} = \frac{d\mathbf{p}_{\text{mech}}}{dt} = \int_V \left( \rho \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \right) d^3r. \quad (\text{S-8.55})$$

From now on, we shall consider the case of perpendicular incidence only, and leave the case of oblique incidence as a further exercise for the reader. In the present case,  $\rho = 0$  everywhere and only the magnetic term contributes. Thus, in plane geometry the time-averaged force on a planar surface of area  $A$  is

$$\langle F_{\text{mech}} \rangle = \int_0^{+\infty} \left\langle \frac{1}{c} \mathbf{J} \times \mathbf{B} \right\rangle A dx \quad (\text{S-8.56})$$

and is directed along  $\hat{\mathbf{x}}$  for symmetry reasons.

The current in a perfect mirror is localized on the surface, where the magnetic field is discontinuous. Here we assume that the wave fields  $\mathbf{B}$  and  $\mathbf{E}$  are parallel to  $\hat{\mathbf{z}}$  and  $\hat{\mathbf{y}}$ , respectively. Let  $\mathbf{E}_i(x, t) = \hat{\mathbf{y}} E_i \cos(kx - \omega t)$  be the incident electric field. The total field  $\mathbf{E}(x, t)$  is the sum of  $\mathbf{E}_i$  and the field  $\mathbf{E}_r(x, t) = -\hat{\mathbf{y}} E_i \cos(-kx - \omega t)$  of the reflected wave, so that  $\mathbf{E}(0, t) = 0$ . Thus the total fields for  $x < 0$  have the form of standing waves

$$E_y(x, t) = 2E_i \sin(kx) \sin \omega t, \quad (\text{S-8.57})$$

$$B_z(x, t) = 2E_i \cos(kx) \cos \omega t. \quad (\text{S-8.58})$$

The discontinuity of  $B_z$  leads to a surface current  $J_y = K_y \delta(x)$  where

$$K_y = -\frac{c}{4\pi} [B_z(0^+, t) - B_z(0^-, t)] = \frac{c}{4\pi} B_z(0^-, t) = \frac{c}{2\pi} E_i \cos \omega t, \quad (\text{S-8.59})$$

where we have used Stokes' theorem and  $B_z(0^+, t) = 0$ . The force per unit surface, i.e., the pressure, is given by the surface current multiplied by the mean value of the field across the current layer (the argument is identical to the one used for calculating the electrostatic pressure on a surface charge layer in electrostatics):

$$\begin{aligned} P_{\text{rad}} &= \left\langle K_y \frac{1}{2c} [B_z(0^+, t) + B_z(0^-, t)] \right\rangle = \frac{c}{8\pi} \langle B_z^2(0^-, t) \rangle \\ &= \frac{c}{8\pi} \left\langle \frac{1}{2} 4 |E_i|^2 \right\rangle = 2 \frac{I}{c}, \end{aligned} \quad (\text{S-8.60})$$

since  $B_z(0^-, t) = 2E_i \cos \omega t$ , and  $I = (c/4\pi) (|E_i|^2/2)$ . This is equivalent to evaluate the integral in (S-8.56) as

$$\begin{aligned} \int_{0^-}^{+\infty} J_y B_z dx &= - \int_{0^-}^{+\infty} \left( \frac{c}{4\pi} \partial_x B_z - \frac{1}{4\pi} \partial_t E_y \right) B_z dx = -\frac{c}{4\pi} \int_{0^-}^{+\infty} \frac{1}{2} \partial_x B_z^2 dx \\ &= \frac{c}{8\pi} B_z^2(0^-, t), \end{aligned} \quad (\text{S-8.61})$$

where we used the fact that  $E_y = 0$  and  $\partial_t E_y = 0$  for  $x \geq 0^-$ .

**c)** The momentum conservation theorem (8.8) states that, for a closed system of charges, currents and EM fields bounded by a closed surface  $\mathcal{S}$ , the following balance equation holds:

$$\frac{d}{dt} (\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{EM}})_i = \oint_{\mathcal{S}} \sum_j \mathbb{T}_{ij} \hat{n}_j d^2 r, \quad (\text{S-8.62})$$

where  $i, j = x, y, z$ ,  $\mathbb{T}_{ij}$  is the Maxwell stress tensor, and  $\hat{n}_j$  is the  $j$  component of the outward-pointing unit vector locally normal to  $\mathcal{S}$ . Thus, the integral on the right-hand side is the outward the flux of the vector  $\mathbb{T} \cdot \hat{\mathbf{n}}$  through  $\mathcal{S}$ . In (S-8.62),  $\mathbf{p}_{\text{mech}}$  is the mechanical momentum of the system, while the momentum associated to the EM field is

$$\mathbf{p}_{\text{EM}} = \int_V \mathbf{g} d^3 r, \quad (\text{S-8.63})$$

where  $\mathbf{g} = \mathbf{S}/c^2$  is the momentum density (8.9), and the integral is evaluated over the volume bounded by  $\mathcal{S}$ .

In our case, we take the front surface  $A$  of the mirror and close it by adding a surface extending deep into the mirror, where the fields are zero. Thus, the amount of EM momentum which flows into the mirror (and “transformed” into mechanical momentum) is given by the integral

$$\int_A \sum_j T_{ij} \hat{n}_j d^2r = A \sum_j T_{ij}(0^-, t) \hat{n}_j. \quad (\text{S-8.64})$$

The radiation pressure on the mirror is the time-averaged momentum flow per unit area,

$$P_{\text{rad}} = \left\langle \sum_j T_{1j}(0^-, t) \hat{n}_j \right\rangle = -\langle T_{11}(0^-, t) \rangle, \quad (\text{S-8.65})$$

since, in our case,  $\hat{\mathbf{n}} = (-1, 0, 0)$ . Thus we actually need to evaluate  $T_{11}(0, t)$  only:

$$T_{11}(0, t) = -\frac{1}{8\pi} B_z^2(0^-, t). \quad (\text{S-8.66})$$

The radiation pressure is thus

$$P_{\text{rad}} = -\langle T_{11}(0, t) \rangle = \frac{1}{8\pi} \langle B_z^2(0^-, t) \rangle = \frac{1}{4\pi} |E_1|^2 = 2 \frac{I}{c}. \quad (\text{S-8.67})$$

## S-8.6 Poynting Vector for a Gaussian Light Beam

a) The divergence of the electric field in vacuum is zero. With our geometry, this means that, since we have assumed  $E_y = 0$ , we have

$$0 = \nabla \cdot \mathbf{E} = \partial_x E_x + \partial_z E_z. \quad (\text{S-8.68})$$

From (S-8.68) and (8.14) we obtain

$$\partial_z E_z = -\partial_x E_x = -2E_0 x e^{-r^2/r_0^2} \cos(kz - \omega t), \quad (\text{S-8.69})$$

where the divergence is calculated in the generic point  $(x, y, z)$ , and we have used  $r^2 = x^2 + y^2$ . Integrating with respect to  $z$ , we have

$$E_z = -E_0 \frac{2x}{kr_0^2} e^{-r^2/r_0^2} \sin(kz - \omega t). \quad (\text{S-8.70})$$

Analogously, we obtain for the longitudinal component of  $\mathbf{B}$

$$B_z = -B_0 \frac{2y}{kr_0^2} e^{-r^2/r_0^2} \sin(kz - \omega t). \quad (\text{S-8.71})$$

Let us verify if these fields are consistent with Maxwell's equations. First, we check if  $\partial_t E_z = c(\nabla \times \mathbf{B})_z = c\partial_x B_y$  holds. We have

$$\partial_x B_y = -B_0 \frac{2x}{r_0^2} e^{-r^2/r_0^2} \cos(kz - \omega t), \quad (\text{S-8.72})$$

$$\partial_t E_z = -E_0 \omega \frac{2x}{kr_0^2} e^{-r^2/r_0^2} \cos(kz - \omega t), \quad (\text{S-8.73})$$

which implies  $B_0 = (\omega/kc)E_0 = E_0$ . Analogously we can check that  $\partial_t B_z = -c(\nabla \times \mathbf{E})_z$ .

**b)** The Poynting vector is

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{c}{4\pi} (\hat{\mathbf{x}}E_z B_y - \hat{\mathbf{y}}E_x B_z + \hat{\mathbf{z}}E_x B_y), \quad (\text{S-8.74})$$

and its components averaged over one cycle are

$$\langle S_x \rangle = \frac{c}{4\pi} \frac{2x}{kr_0^2} E_0^2 e^{-2r^2/r_0^2} \langle \sin(kz - \omega t) \cos(kz - \omega t) \rangle = 0, \quad (\text{S-8.75})$$

$$\langle S_y \rangle = -\frac{c}{4\pi} \frac{2y}{kr_0^2} E_0^2 e^{-2r^2/r_0^2} \langle \cos(kz - \omega t) \sin(kz - \omega t) \rangle = 0, \quad (\text{S-8.76})$$

$$\langle S_z \rangle = \frac{c}{4\pi} E_0^2 e^{-2r^2/r_0^2} \langle \cos^2(kz - \omega t) \rangle = \frac{c}{8\pi} E_0^2 e^{-2r^2/r_0^2}. \quad (\text{S-8.77})$$

Thus, we can define the local intensity and the total power of the beam as

$$I(r) = \langle S_z \rangle, \quad P = \int_0^\infty I(r) 2\pi r dr. \quad (\text{S-8.78})$$

**c)** We have

$$\nabla^2 E_x = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_x}{\partial r} \right) - \frac{\partial^2 E_x}{\partial z^2} = \left[ \frac{4}{r_0^2} \left( \frac{r^2}{r_0^2} - 1 \right) - k^2 \right] E_x \quad (\text{S-8.79})$$

(see Table A.1 for the Laplacian operator in cylindrical coordinates; notice that here the fields are independent of  $\phi$ ). We can easily check that  $(\nabla^2 + \omega^2/c^2)E_x \neq 0$ ; the “extra” terms being of the order of  $\sim 1/(kr_0)^2$ . Thus we expect our approximate expressions for the fields to be accurate as long as  $r_0 \gg 1/k = \lambda/2\pi$ , i.e., if the beam is much wider than one wavelength.

It is known that a beam with finite width actually undergoes diffraction. The width of a Gaussian beam doubles after a typical distance, called Rayleigh length,

$r_R = kr_0^2$ . This corresponds to an aperture angle

$$\theta_d \simeq \frac{r_0}{z_R} = \frac{1}{kr_0} \simeq \frac{\lambda}{r_0}. \quad (\text{S-8.80})$$

It might be interesting to notice that this result may be inferred from the values for the longitudinal field components obtained at point **a**). In fact, the beam may be obtained as a linear superposition of plane waves of the same frequency but different wavevectors. For the plane wave, the electric and magnetic field are perpendicular to the wavevector  $\mathbf{k}$ . Thus, the typical ratio  $E_z/E_y \sim 2/(kr_0)$  (at  $r = r_0$ ) also corresponds to a typical value  $k_x/k_z \sim 2/(kr_0)$ , which should determine the typical angular spread of the wavevector spectrum, hence the spreading angle of the beam.

### S-8.7 Intensity and Angular Momentum of a Light Beam

**a)** First, we define the shorthand symbols  $C = \cos(kz - \omega t)$ ,  $S = \sin(kz - \omega t)$ , and  $E'_0 = \partial_r E_0(r)$ , that we shall use throughout the solution of the problem. We have for the intensity of the beam

$$\begin{aligned} I(r) \equiv S_z &= \frac{c}{4\pi} (E_x B_y - E_y B_x) = \frac{c}{4\pi} E_0^2(r) [CC - (-SS)] \\ &= \frac{c}{4\pi} E_0^2(r) [C^2 + S^2] = \frac{c}{4\pi} E_0^2(r). \end{aligned} \quad (\text{S-8.81})$$

**b)** The divergence of the fields in vacuum must be zero. For the electric field we have

$$0 = \nabla \cdot \mathbf{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z, \quad (\text{S-8.82})$$

thus

$$\partial_z E_z = -\partial_x E_x - \partial_y E_y = -\frac{x}{r} E'_0(r) C + \frac{y}{r} E'_0(r) S,$$

and, integrating with respect to  $z$ ,

$$E_z = -\frac{1}{kr} E'_0(r) [xS + yC]. \quad (\text{S-8.83})$$

Analogously, we can evaluate  $B_z$ :

$$\begin{aligned} \partial_z B_z &= -\partial_x B_x - \partial_y B_y = -\frac{x}{rc} E'_0(r) S - \frac{y}{rc} E'_0(r) C, \\ B_z &= +\frac{1}{krc} E'_0(r) [xC - yS]. \end{aligned} \quad (\text{S-8.84})$$

**c)** The  $x$  and  $y$  components of the Poynting vector are

$$\begin{aligned}
S_x &= \frac{c}{4\pi} (E_y B_z - E_z B_y) \\
&= \frac{c}{4\pi} \left\{ (-E_0 S) \left[ \frac{E'_0}{krc} (xC - yS) \right] - \left[ -\frac{E'_0}{kr} (xS + yC) \right] \frac{1}{c} E_0 C \right\} \\
&= \frac{c}{4\pi} \frac{E_0 E'_0}{kr} (-xSC + yS^2 + xSC + yC^2) = \frac{c}{4\pi} E_0 E'_0 \frac{y}{kr}. \quad (\text{S-8.85})
\end{aligned}$$

$$\begin{aligned}
S_y &= \frac{c}{4\pi} (E_z B_x - E_x B_z) \\
&= \frac{c}{4\pi} \left\{ \left[ -\frac{E'_0}{kr} (xS + yC) \right] \frac{1}{c} E_0 S - (E_0 C) \left[ \frac{E'_0}{krc} (xC - yS) \right] \right\} \\
&= \frac{c}{4\pi} \frac{E_0 E'_0}{kr} (-xS^2 - yCS - xC^2 + yCS) = -\frac{c}{4\pi} E_0 E'_0 \frac{x}{kr} \quad (\text{S-8.86})
\end{aligned}$$

Since we have

$$\frac{c}{4\pi} E_0(r) E'_0(r) = \frac{c}{8\pi} \partial_r E_0^2(r) = \frac{1}{2} \partial_r I(r), \quad (\text{S-8.87})$$

the Poynting vector can be written

$$\mathbf{S} = \left( \frac{y}{2kr} \partial_r I(r), -\frac{x}{2kr} \partial_r I(r), I(r) \right). \quad (\text{S-8.88})$$

Assuming a Gaussian beam, we have  $E_0(r) \propto e^{-r^2/r_0^2}$ , and  $S_{x,y} \propto S_z/(kr_0) \propto \theta_d S_z$ , with  $\theta_d$  the diffraction angle of (S-8.80).

**d)** We have

$$\begin{aligned}
\ell_z &= \frac{1}{c^2} (xS_y - yS_x) = \frac{-x^2 - y^2}{2krc^2} \partial_r I(r) = -\frac{r}{2kc^2} \partial_r I(r) \\
&= -\frac{r}{2c\omega} \partial_r I(r). \quad (\text{S-8.89})
\end{aligned}$$

We eventually obtain the total angular momentum by integrating the above expression by parts,

$$\begin{aligned}
L_z &= \int_0^\infty \ell_z(r) 2\pi r \, dr = - \int_0^\infty \frac{r}{2c\omega} \partial_r I(r) 2\pi r \, dr \\
&= \frac{1}{c\omega} \int_0^\infty I(r) 2\pi r \, dr = \frac{W}{c\omega}. \quad (\text{S-8.90})
\end{aligned}$$

### S-8.8 Feynman's Paradox solved

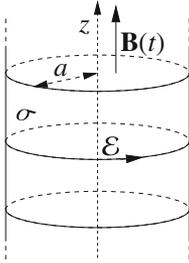


Fig. S-8.5

**a)** We use a cylindrical coordinate system  $(r, \phi, z)$  with the cylinder axis as  $z$  axis. The induced electric field  $\mathbf{E}_{\text{ind}}$  has azimuthal symmetry, i.e.,  $\mathbf{E}_{\text{ind}} = E_\phi(r, t)\boldsymbol{\phi}$ , and can be obtained from Faraday's law by equating its line integral over the circumference of radius  $r$  to the temporal derivative of the magnetic field flux through the circle:

$$E_\phi = -\frac{r}{2c} \partial_t B_{\text{ext}}(t). \quad (\text{S-8.91})$$

(We assumed the slowly varying current approximation, whose validity is ensured by the  $t_f \gg a/c$  condition.)

On an infinitesimal surface element of the cylindrical surface  $dS = a d\phi dz$  the induced electric field exerts a force

$$d\mathbf{f} = \boldsymbol{\phi} df = \boldsymbol{\phi} \sigma E_\phi(r=a) dS = -\boldsymbol{\phi} \sigma \frac{a}{2c} \partial_t B_{\text{ext}}(t) dS, \quad (\text{S-8.92})$$

where  $\sigma = Q/(2\pi a)$  is the surface charge density. The corresponding mechanical torque is  $d\boldsymbol{\tau} = \hat{\mathbf{z}} a d\mathbf{f}$ . By integrating over the whole surface of the cylinder we obtain for the total torque

$$\boldsymbol{\tau} = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t). \quad (\text{S-8.93})$$

The equation of motion for the rotation of the cylinder is

$$\mathcal{I} \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau} = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t), \quad (\text{S-8.94})$$

with solution (the total time derivative being trivially equivalent to the partial derivative when applied to  $\mathbf{B}_{\text{ext}}(t)$ )

$$\boldsymbol{\omega}(t) = -\frac{\pi a^3 h}{\mathcal{I} c} \sigma [\mathbf{B}_{\text{ext}}(t) - \mathbf{B}_{\text{ext}}(0)] = -\frac{a^2 Q}{2\mathcal{I} c} [\mathbf{B}_{\text{ext}}(t) - \mathbf{B}_{\text{ext}}(0)]. \quad (\text{S-8.95})$$

The angular momentum is  $L_c(t) = \mathcal{I} \boldsymbol{\omega}(t)$ . The final values depend only on the initial value of  $\mathbf{B}_{\text{ext}}$  and not on its temporal profile,

$$\boldsymbol{\omega}(t_f) = -\frac{a^2}{2\mathcal{I} c} Q \mathbf{B}_0. \quad (\text{S-8.96})$$

b) The rotation of the charged cylinder leads to a surface current  $\mathbf{K}$  at  $r = a$ ,

$$\mathbf{K} = \sigma \mathbf{v} = \sigma a \omega \hat{\phi}. \quad (\text{S-8.97})$$

This current generates a uniform magnetic field  $\mathbf{B}_{\text{ind}}$  inside the long cylinder (equivalent to a solenoid where  $nI = K$ ),

$$\mathbf{B}_{\text{ind}} = \frac{4\pi}{c} K \hat{\mathbf{z}} = \frac{4\pi}{c} \sigma a \omega. \quad (\text{S-8.98})$$

We now proceed as in point a) but adding the induced field  $\mathbf{B}_{\text{ind}}$  to the external field  $\mathbf{B}_{\text{ext}}$ :

$$\begin{aligned} I \frac{d\omega}{dt} = \tau &= -\frac{\pi a^3 h \sigma}{c} \partial_t [\mathbf{B}_{\text{ext}}(t) + \mathbf{B}_{\text{ind}}(t)] \\ &= -\frac{a^2 Q}{2c} \partial_t \left[ \mathbf{B}_{\text{ext}}(t) + \frac{4\pi}{c} \sigma a \omega \right] \\ &= -\frac{a^2 Q}{2c} \partial_t \mathbf{B}_{\text{ext}}(t) - \frac{a^2 Q^2}{hc^2} \frac{d\omega}{dt}, \end{aligned} \quad (\text{S-8.99})$$

which can be rewritten as

$$I' \frac{d\omega}{dt} = \tau = -\frac{\pi a^3 h \sigma}{c} \partial_t \mathbf{B}_{\text{ext}}(t), \quad (\text{S-8.100})$$

$$I' = I + \frac{a^2 Q^2}{hc^2}. \quad (\text{S-8.101})$$

Equation (S-8.100) is identical to (S-8.94) but for the replacement  $I \rightarrow I'$ , which means that the effects of the rotation-induced magnetic field  $\mathbf{B}_{\text{ind}}$  are equivalent to an additional inertia of the cylinder. The final velocity becomes

$$\omega'(t_f) = -\frac{a^2}{2I'c} Q \mathbf{B}_0. \quad (\text{S-8.102})$$

Notice that the total magnetic field does not vanish inside the cylinder at  $t = t_f$ , being equal to the induced field

$$\mathbf{B}_{\text{tot}}(t_f) = \mathbf{B}_{\text{ind}}(t_f) = \frac{4\pi}{c} \sigma a \omega'(t_f). \quad (\text{S-8.103})$$

c) For a magnetic field  $\mathbf{B} = B_z \hat{\mathbf{z}}$  and a configuration with cylindrical symmetry the density of EM angular momentum (8.18) becomes

$$\boldsymbol{\ell} \equiv \mathbf{r} \times \mathbf{g} = -\frac{1}{4\pi} r E_r B_z \hat{\mathbf{z}}. \quad (\text{S-8.104})$$

The contribution of the induced electric field  $E_\phi$  vanishes in the vector product. However, the angular momentum is not zero because of the radial electrostatic field inside the cylinder, which is easily found from Gauss's theorem:

$$E_r(r) = \begin{cases} \frac{2\lambda}{r} = -\frac{2Q}{hr} & (r < a) \\ 0 & (r > a) \end{cases} . \quad (\text{S-8.105})$$

Thus,  $\ell \neq 0$  inside the cylinder ( $r < a$ ). The total EM angular momentum is thus

$$\mathbf{L}_{\text{EM}} = \frac{1}{4\pi} B_z \int_0^a r \frac{2\lambda}{r} 2\pi r h dr = \frac{1}{2} B_z \lambda h a^2 \hat{\mathbf{z}} = \frac{Qa^2}{2} \mathbf{B} . \quad (\text{S-8.106})$$

Notice that  $\mathbf{B}$  represents the total field inside the cylinder and that the equation for  $\mathbf{L}_{\text{EM}}$  is valid at any time. Now, (S-8.95) can be rewritten (using the total field) as

$$\mathcal{I}\omega(t) + \frac{a^2 Q}{2c} \mathbf{B}_{\text{ext}}(t) = \frac{a^2 Q}{2c} \mathbf{B}_{\text{ext}}(0) , \quad (\text{S-8.107})$$

which is equivalent to

$$\mathbf{L}_c(t) + \mathbf{L}_{\text{EM}}(t) = \mathbf{L}_{\text{EM}}(0) , \quad (\text{S-8.108})$$

thus showing that the total angular momentum of the system is conserved, since  $\mathbf{L}_c(0) = 0$ . The “paradox” thus consists in ignoring that a static EM field can contain a finite angular momentum. Similar considerations hold for Problem 6.6 where, however, the EM angular momentum is more difficult to calculate.<sup>1</sup>

## S-8.9 Magnetic Monopoles

a) We build a magnetic dipole  $\mathbf{m}$  by locating two magnetic charges (magnetic monopoles)  $+q_m$  and  $-q_m$  at a distance  $\mathbf{h}$  from each other, so that  $\mathbf{m} = q_m \mathbf{h}$ . The magnetic field at distances  $r \gg h$  can be evaluated from (8.19), using the same approximations as for the field of an electric dipole, obtaining

$$\mathbf{B}_{\text{dip}} = \alpha \frac{(\mathbf{m} \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} - \mathbf{m}}{r^3} . \quad (\text{S-8.109})$$

<sup>1</sup>The present explanation of Feynman's “paradox” is taken from J. Belcher and K. McDonald ([http://cosmology.princeton.edu/~mcdonald/examples/feynman\\_cylinder.pdf](http://cosmology.princeton.edu/~mcdonald/examples/feynman_cylinder.pdf)) who further discuss subtle aspects of this problem.

On the other hand, the field of a usual magnetic dipole  $\mathbf{m} = I\mathbf{S}$ , consisting of a small circular loop of surface  $\mathbf{S}$  carrying a current  $I$ , with the head of  $\mathbf{S}$  pointing so that it “sees”  $I$  circulating counterclockwise, is

$$\mathbf{B}_{\text{dip}} = k_m \frac{(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}. \quad (\text{S-8.110})$$

Comparing the formulas, we obtain  $\alpha = k_m$ , i.e.,  $\alpha = \mu_0/4\pi = 1/4\pi\epsilon_0 c^2$  in SI units, and  $\alpha = 1/c$  in Gaussian units.

The magnetic force on an electric charge  $q_e$ , moving with velocity  $\mathbf{v}$  in the presence of a magnetic field  $\mathbf{B}$ , is  $\mathbf{f}_L = q_e b_m \mathbf{v} \times \mathbf{B}$ . The force exerted by a magnetic field  $\mathbf{B}$  on a magnetic monopole of charge  $q_m$  is  $\mathbf{f}_m = q_m \mathbf{B}$ . Thus the physical dimensions of the magnetic charge  $q_m$  are

$$[q_m] = [q_e b_m v] = \begin{cases} [q_e v], & \text{SI} \\ [q_e], & \text{Gaussian} \end{cases} \quad (\text{S-8.111})$$

i.e., the same physical dimensions as an electric charge in Gaussian units, and the dimensions of an electric charge times a velocity in SI units.

**b)** In analogy with the equation  $\nabla \cdot \mathbf{E} = 4\pi k_e \varrho_e$ , where  $\varrho_e$  is the volume density of electric charge, Maxwell’s equation  $\nabla \cdot \mathbf{B} = 0$  is modified as

$$\nabla \cdot \mathbf{B} = 4\pi k_m \varrho_m = \begin{cases} \mu_0 \varrho_m & \text{SI} \\ \frac{4\pi}{c} \varrho_m & \text{Gaussian,} \end{cases} \quad (\text{S-8.112})$$

where  $\varrho_m$  is the volume density of magnetic charge. Equation (S-8.112) can be proved by first observing that, in the presence of magnetic charges, Gauss’s law for the magnetic field is

$$\oint \mathbf{B} \cdot d\mathbf{S} = 4\pi k_m Q_m = 4\pi k_m \int \varrho_m d^3x, \quad (\text{S-8.113})$$

where the flux of  $B$  is evaluated through any closed surface, and  $Q_m$  is the net magnetic charge inside the surface, then applying the divergence theorem.

The conservation of magnetic charge is expressed by the continuity equation

$$\nabla \cdot \mathbf{J}_m = -\partial_t \varrho_m. \quad (\text{S-8.114})$$

Maxwell’s equation for  $\nabla \times \mathbf{E}$  (describing Faraday’s law of induction) must be completed in order to take the magnetic current density into account, by writing

$$\nabla \times \mathbf{E} = \eta \mathbf{J}_m - b_m \partial_t \mathbf{B}. \quad (\text{S-8.115})$$

The constant  $\eta$  can be determined, for instance, by applying the divergence operator to both sides of the equation, remembering the divergence of the curl of any vector field is always zero,

$$0 = \nabla \cdot (\nabla \times \mathbf{E}) = \eta \nabla \cdot \mathbf{J}_m - b_m \partial_t \nabla \cdot \mathbf{B} = \eta \nabla \cdot \mathbf{J}_m - 4\pi k_m \partial_t \rho_m, \quad (\text{S-8.116})$$

from which  $\eta = -4\pi k_m$  follows. We thus obtain

$$\nabla \times \mathbf{E} = -4\pi k_m \mathbf{J}_m - b_m \partial_t \mathbf{B} = \begin{cases} -\mu_0 \mathbf{J}_m - \partial_t \mathbf{B}, & \text{SI} \\ -\frac{4\pi}{c} \mathbf{J}_m - \frac{1}{c} \partial_t \mathbf{B}, & \text{Gaussian.} \end{cases} \quad (\text{S-8.117})$$

c) We choose a cylindrical reference frame  $(r, \phi, z)$  with the  $z$  axis coinciding with the axis of the beam. Because of the cylindrical symmetry of our magnetic charge distribution, the only non-zero component of the magnetic field is  $B_r$ . Applying Gauss's law to a cylindrical surface coaxial with the beam we obtain

$$B_r = \begin{cases} 2\pi k_m n q_m r, & r \leq a \\ \frac{2\pi k_m n q_m a^2}{r}, & r \geq a. \end{cases} \quad (\text{S-8.118})$$

The electric field  $\mathbf{E}$  is solenoidal and can be obtained by applying Kelvin-Stokes theorem to a circular path of radius  $r$  coaxial with the beam

$$\oint \mathbf{E} \cdot d\mathbf{l} = 2\pi r E_\phi = \int \nabla \times \mathbf{E} \cdot d\mathbf{S} = \begin{cases} -\pi r^2 4\pi k_m n q_m v, & r \leq a \\ -\pi r^2 4\pi k_m n q_m \frac{a^2}{r} v, & r \geq a, \end{cases} \quad (\text{S-8.119})$$

leading finally to

$$E_\phi = \begin{cases} 2\pi k_m n q_m v r & \text{if } r \leq a \\ \frac{2\pi k_m n q_m v a^2}{r} & \text{if } r \geq a. \end{cases} \quad (\text{S-8.120})$$

Thus, for instance for  $r \leq a$ , we have

$$E_\phi = \begin{cases} \frac{\mu_0 n q_m v r}{2}, & \text{SI} \\ \frac{2\pi n q_m v r}{c}, & \text{Gaussian.} \end{cases} \quad (\text{S-8.121})$$