

Chapter S-9

Solutions for Chapter 9

S-9.1 The Fields of a Current-Carrying Wire

a) In the S reference frame the wire generates an azimuthal magnetic field $\mathbf{B} = B_\phi(r)\hat{\phi}$. In cylindrical coordinates we have $B_\phi = B_\phi(r) = (2I/rc)$. The Lorentz force on the charge q is

$$\mathbf{F} = q \frac{\mathbf{v}}{c} \times \mathbf{B} = \hat{\mathbf{r}} F_r = -\hat{\mathbf{r}} q B_\phi(r) \frac{v}{c} = -\hat{\mathbf{r}} \frac{2qIv}{rc^2}. \quad (\text{S-9.1})$$

The S' frame moves with velocity v with respect to S . Applying the Lorentz transformations, in S' the force on q is $\mathbf{F}' = \hat{\mathbf{r}} F'_r = \hat{\mathbf{r}} \gamma F_r$ (where $\gamma = 1/\sqrt{1-\beta^2}$, and $\beta = v/c$ with $v = |\mathbf{v}|$), see Fig. S-9.1. Since q is at rest in S' , the force \mathbf{F}' is due to the electric field \mathbf{E}' only, with $\mathbf{E}' = \hat{\mathbf{r}} E'_r = \hat{\mathbf{r}} F'_r/q$. This corresponds to the transformation $\mathbf{E}'_\perp = \hat{\mathbf{r}} E'_r = -\hat{\mathbf{r}} \gamma \beta B_\phi$ or, in vector form,

$$\mathbf{E}'_\perp = \gamma(\boldsymbol{\beta} \times \mathbf{B}), \quad (\text{S-9.2})$$

where the subscript “ \perp ” refers to the direction perpendicular to v . At the limit $|v| \ll c$ (for which $\mathbf{F} = \mathbf{F}'$) we get $\mathbf{E}'_\perp \simeq \boldsymbol{\beta} \times \mathbf{B}$, which is

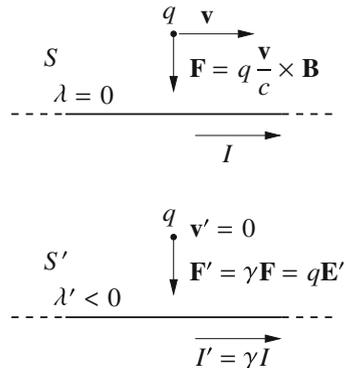


Fig. S-9.1

correct up to first order in $\beta = v/c$, and may be called the “Galilei” transformation of the field. The electric field¹

$$E'_r = -\gamma\beta B_\phi(r') = -2\gamma\beta I/(r'c) \quad (\text{S-9.3})$$

is generated by a uniform linear charge density $\lambda' = -\beta\gamma I/c$ on the wire, as can be easily verified by applying Gauss’s law. Thus *the wire is negatively charged in S'* .²

Since the force is purely magnetic in S and purely electric in S' , at this point we cannot say much about the magnetic field in S' .

b) We know that $J = (\rho c, \mathbf{J})$ is a four-vector. The cross-section W of the wire is invariant for a Lorentz boost along the wire axis, thus the linear charge density $\lambda = W\rho$ and the electric current $\mathbf{I} = W\mathbf{J}$ transform like ρ and \mathbf{J} . Therefore the linear charge density of the wire in S' is

$$\lambda' = \gamma\left(\lambda - \beta\frac{I}{c}\right) = -\gamma\beta\frac{I}{c}, \quad (\text{S-9.4})$$

which, according to Gauss’s law, generates the radial electric field $E'_r = 2\lambda'/r$, in agreement with our result of point (a). We also obtain the current intensity in S' ,

$$I' = \gamma(I - \beta c\lambda) = \gamma I, \quad (\text{S-9.5})$$

which generates the magnetic field $B'_\phi = 2I'/(r'c) = \gamma B_\phi$.

The same results can be obtained through the transformation of the four-potential (ϕ, \mathbf{A}) . In S , we have obviously $\phi = 0$, since there is no net charge, while the vector potential \mathbf{A} satisfies the equation

$$\nabla^2 \mathbf{A} = -\frac{4\pi}{c}\mathbf{J} \quad (\text{S-9.6})$$

¹In general, the complete transformation is $\mathbf{E}'_\perp(\mathbf{r}', t) = \gamma\beta \times \mathbf{B}[\mathbf{r}(\mathbf{r}', t'), t(\mathbf{r}', t')]$, where $\mathbf{r} = \mathbf{r}(\mathbf{r}', t')$ and $t = t(\mathbf{r}', t')$, according to the Lorentz transformations of the coordinates. Since in cylindrical coordinates B_ϕ depends on r only, and for the coordinates in the plane transverse to the boost velocity $\mathbf{r}'_\perp = \mathbf{r}_\perp$, in the present case we have the trivial transformation $r' = r$.

²It might seem that the law of charge conservation is violated in the transformation from S to S' . Actually, this is a consequence of the somewhat “pathological” nature of currents which are not closed in a loop, as in the case of an infinite wire. In fact, strictly speaking, the infinite current-carrying wire is not a steady system, since charges of opposite sign are accumulating at the two “ends” of the wire, i.e., at $z = \pm\infty$. If we introduce “return” currents to close the loop in S , e.g., if we assume the wire to be the inner conductor of a coaxial cable, or if we add a second wire carrying the current $-I$ at some distance, we find that the return currents would appear as opposite charge densities in S' , as required by charge conservation.

Thus, \mathbf{A} is parallel to the wire and its only non-zero component is A_z , which can be evaluated from the equation

$$\nabla^2 A_z = -\frac{4\pi}{c} I \delta(r). \quad (\text{S-9.7})$$

This is mathematically identical to the Poisson equation for the electrostatic potential of a uniformly charged wire, thus the solution is

$$A_z = -\frac{2I}{c} \ln\left(\frac{r}{a}\right), \quad (\text{S-9.8})$$

where a is an arbitrary constant. It is straightforward to verify that $B_\phi = -\partial_r A_z$.

The scalar potential in S' is

$$\phi' = \gamma(\phi - \beta A_z) = -\gamma \beta A_z = -\frac{2\gamma\beta I}{c} \ln\left(\frac{r}{a}\right) = -2\lambda' \ln\left(\frac{r}{a}\right), \quad (\text{S-9.9})$$

where $\lambda' = -\beta\gamma I/c$. The electric field is evaluated from $\mathbf{E}' = -\nabla\phi'$, obtaining the same result of point **a**). For the vector potential in S' , trivially $A'_z = \gamma(A_z - \beta\phi) = \gamma A_z$ from which we get $B'_\phi = \gamma B_\phi$ again.

These results are in agreement with the explicit formulas for the transformation of the EM field (9.3), which, in our case, lead to $\mathbf{E}' = \gamma\boldsymbol{\beta} \times \mathbf{B}$ and $\mathbf{B}' = \gamma\mathbf{B}$.

c) Let us first consider the linear charge densities of both ions ($\lambda_i = Ze n_i W$) and electrons ($\lambda_e = -en_e W$) in S , where \ni and n_e are the ion and electron volume densities, respectively. Since there is no net charge on the wire in S , we have $\lambda_i = -\lambda_e$.

Let us evaluate the charge densities λ'_i and λ'_e in S' from relativistic kinematics. In S , a wire segment of length ΔL carries an ion charge $\Delta Q = \lambda_i \Delta L$. In S' , the segment has the same charge as in S (the charge is a Lorentz invariant), but the length undergoes a Lorentz contraction, $\Delta L' = \Delta L/\gamma$. Thus we have a higher charge density $\lambda'_i = \Delta Q/\Delta L' = \gamma\lambda_i$. This is a quite general result: in a frame where a fluid moves at velocity v , the fluid has a higher density (by a factor γ) than in its rest frame.

On the other hand, the electrons are *not* at rest in S : they move along the wire with a velocity $v_e < 0$ such that $I = -en_e v_e W = \lambda_e v_e = -\lambda_i v_e$. Thus, their density is already *higher* by a factor $\gamma_e = 1/\sqrt{1-v_e^2/c^2}$ than the density λ_{e0} in the rest frame of the electrons: we have $\lambda_{e0} = \lambda_e/\gamma_e$. In S' , the electrons drift with a velocity v'_e

$$v'_e = \frac{v_e - v}{1 - v_e v/c^2}, \quad (\text{S-9.10})$$

according to Lorentz transformations. Thus, the electron density in S' is

$$\lambda'_e = \gamma'_e \lambda_{e0} = \frac{\gamma'_e}{\gamma_e} \lambda_e, \quad (\text{S-9.11})$$

where $\gamma'_e = 1/\sqrt{1 - v_e'^2/c^2}$. The expression for γ' can be put in a more convenient form by some algebra:

$$\begin{aligned} \gamma'_e &= \frac{1}{\sqrt{1 - \frac{(v_e - v)^2}{c^2 \left(1 - \frac{v_e v}{c^2}\right)^2}}} = \sqrt{\frac{\left(1 - \frac{v_e v}{c^2}\right)^2}{\left(1 - \frac{v_e v}{c^2}\right)^2 - \frac{(v_e - v)^2}{c^2}}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \frac{1}{\sqrt{1 - 2\frac{v_e v}{c^2} + \frac{v_e^2 v^2}{c^4} - \frac{v_e^2}{c^2} + 2\frac{v_e v}{c^2} - \frac{v^2}{c^2}}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \frac{1}{\sqrt{\left(1 - \frac{v_e^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}} \\ &= \left(1 - \frac{v_e v}{c^2}\right) \gamma_e \gamma. \end{aligned} \quad (\text{S-9.12})$$

We thus obtain for the *total* charge density in S'

$$\begin{aligned} \lambda' &= \lambda'_i + \lambda'_e = \lambda_i \left(\gamma - \frac{\gamma'_e}{\gamma_e}\right) = \lambda_i \gamma \left(1 - 1 + \frac{v_e v}{c^2}\right) = \lambda_i \gamma \frac{v_e v}{c^2} \\ &= -\gamma v \frac{I}{c^2}, \end{aligned} \quad (\text{S-9.13})$$

as previously found on the basis of Lorentz transformations for the forces, charge and current densities, and EM fields.

It might be interesting to remark that there is an issue of charge conservation already in the S frame. The wire is electrically neutral, thus its ion and electron charge densities are exactly equal and opposite when it is disconnected from any voltage or current source, and in the absence of external fields. Now assume that we drive a steady current I through the wire, keeping the conduction electrons in motion with a velocity v_e along the wire axis. If the wire is still electrically neutral, as we assumed, the absolute values of the charge densities of ions and electrons must still be equal and opposite. However, while the charge density of the ions, at rest, has not changed, the charge density of the moving electrons undergoes a “relativistic increase” by a factor γ_e . If the total charge density does not change (the wire must

still be neutral), some electrons must have left the wire.³ We can explain where the missing electrons have gone only by recalling that the wire is not “open”, but must be part of a closed current loop, with specific boundary conditions and how the circuit is closed.

S-9.2 The Fields of a Plane Capacitor

a) We choose a Cartesian coordinate system with the y axis perpendicular to the plates, so that the lower plate is at $y = 0$ and the upper plate at $y = h$, and the x axis parallel to \mathbf{v} , so that $\mathbf{v} = \beta c \hat{\mathbf{x}}$. The only non-vanishing component of the EM field in S is $E_y = 4\pi\sigma$. By applying a Lorentz transformation we find for the fields in S'

$$E'_y = \gamma E_y = 4\pi\gamma\sigma, \quad B'_z = -\beta\gamma E_y = -4\pi\beta\gamma\sigma. \quad (\text{S-9.14})$$

b) In S' the electric field E'_y is generated by the surface charge densities $\pm\sigma' = \pm E'_y/4\pi = \pm\gamma\sigma$ on the capacitor plates. Similarly, the magnetic field B'_z is generated by the two surface current densities $\pm\mathbf{K}' = \pm K'_x \hat{\mathbf{x}}$ with $K'_x = cB'_z/(4\pi) = -\beta\gamma\sigma c$, flowing on the two capacitor plates.

These results are in agreement with the Lorentz transformation of the four-vector

$$K_\mu = (c\sigma, \mathbf{K}). \quad (\text{S-9.15})$$

We can check that K_μ is actually a four-vector, by imagining two volume four-current densities $J_\mu = (c\rho, \pm\mathbf{J})$ distributed over the two thin layers, $|y| < \delta/2$ and $|h-y| < \delta/2$, around the capacitor plates, such that $\sigma = \rho\delta$ and $\mathbf{K} = \mathbf{J}\delta$. Since δ is invariant for transformations with velocity parallel to \mathbf{J} , it follows that also $K_\mu \equiv J_\mu\delta$ transforms as a four-vector:

$$\sigma' = \gamma(\sigma - \beta K_x/c) = \gamma\sigma, \quad K'_x = \gamma(K_x - \beta c\sigma) = -\beta\gamma\sigma c. \quad (\text{S-9.16})$$

c) In S there is a perpendicular force per unit surface $p = \sigma E_y/2 = 2\pi\sigma^2$ on the internal surfaces of the plates, such that the plates attract each other. In S' , the force per unit surface is the sum of two terms of electrostatic and magnetic nature, respectively,

$$\begin{aligned} p' &= \frac{1}{2}\sigma' E'_y + \frac{1}{2}K'_x B'_z = 2\pi\sigma^2\gamma^2 - 2\pi\sigma^2\beta^2\gamma^2 = 2\pi\sigma^2\gamma^2(1 - \beta^2) = 2\pi\sigma^2 \\ &= p. \end{aligned} \quad (\text{S-9.17})$$

³Of course, the effect is negligibly small for ordinary conduction in metals, for which the typical electron velocities v_e are of the order of $10^{-10}c$. On the other hand, this issue is very important for relativistic hydrodynamics, i.e., for contexts where fluids move at velocities close to c .

The invariance of p is also proven from the equivalent expression

$$p' = \frac{1}{8\pi} E_y'^2 - \frac{1}{8\pi} B_z'^2 = \frac{1}{8\pi} (\mathbf{E}'^2 - \mathbf{B}'^2), \quad (\text{S-9.18})$$

which is a Lorentz invariant.

In S , the total force is $F = pA$. In S' , due to the Lorentz contraction of lengths, $A' = (L/\gamma)L = A/\gamma$, so that $F' = p'A' = pA/\gamma = F/\gamma$.

S-9.3 The Fields of a Solenoid

a) We choose a Cartesian reference frame with the solenoid axis as z axis, and the x axis such that $\mathbf{v} = v_x \hat{\mathbf{x}}$. In addition, we shall also use a cylindrical reference frame sharing the z axis with the Cartesian frame, and with the azimuthal coordinate ϕ such that the $\phi = 0$ plane coincides with the xz plane. In S , the magnetic field inside the solenoid is longitudinal and uniform, $\mathbf{B} = B\hat{\mathbf{z}}$, with $B = 4\pi nI/c$, and the force on q is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}/c = -q\beta B \hat{\mathbf{y}}$.

In the S' frame the charge q is at rest, thus the force on it must be due to an electric field only. According to the Lorentz transformations of the fields we have

$$\begin{aligned} E_x' &= E_x = 0, & B_x' &= B_x = 0, \\ E_y' &= \gamma(E_y - \beta B_z) = -\gamma\beta B, & B_y' &= \gamma(B_y + \beta E_z) = 0, \\ E_z' &= \gamma(E_z + \beta B_y) = 0, & B_z' &= \gamma(B_z - \beta E_y) = \gamma B, \end{aligned} \quad (\text{S-9.19})$$

and the force on q is thus $\mathbf{F}' = qE_y' \hat{\mathbf{y}} = -q\gamma\beta B_z \hat{\mathbf{y}} = \gamma\mathbf{F}$.

b) Since we are assuming $\beta \ll 1$, we have $\gamma = 1/\sqrt{1-\beta^2} = 1 + \beta^2/2 + \dots \approx 1$ up to the first order in β , and we can neglect the relativistic contraction of lengths. Thus the cross-section of the solenoid remains circular in S' to within our approximations. The electric field outside the solenoid is zero (we discuss this point further below), thus the electric field component perpendicular to the solenoid winding surface is discontinuous, implying the presence of surface charge density σ' . We have from Gauss's theorem

$$\sigma' = \frac{E'_\perp}{4\pi} = \frac{E'_y}{4\pi} \sin\phi = -\beta \frac{B}{4\pi} \sin\phi = -\beta n \frac{I}{c} \sin\phi, \quad (\text{S-9.20})$$

where the subscript \perp means perpendicular to the solenoid winding surface.

This result is in agreement with the transformation laws for the four-vector $K_\mu = (c\sigma, \mathbf{K})$, where \mathbf{K} is the surface current density on the walls of the solenoid (see Problem 9.2). In S we have $\mathbf{K} = nI\hat{\boldsymbol{\phi}} = nI(-\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi)$, and in S'

$$\sigma' = \gamma \left(\sigma - \beta \frac{K_x}{c} \right) \approx -\beta \frac{K_x}{c} = \beta n \frac{I}{c} \sin\phi. \quad (\text{S-9.21})$$

A surface charge density varying as $\sin\phi$ on the lateral surface of an infinite cylinder generates a uniform electrostatic field inside the cylinder, as seen in the solution of Problem 3.11. But there we also saw that surface charge density generates a “two-dimensional dipole” field *outside* the cylinder. This might seem in contradiction with the fact that, since the external EM field is zero in the S frame, it must be zero in S' as well. But there are *not only static fields* in S' , because the transverse motion of the solenoid generates a time-dependent magnetic field, which, in turn, is related to a non-conservative electric field and to boundary conditions which are different from the static case.

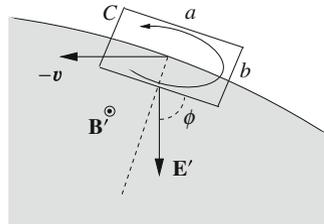


Fig. S-9.2

We start by noting that an electric field that is uniform (and nonzero) inside the solenoid, and zero outside, is not conservative. Let us choose a rectangular path C of sides a and b crossing the solenoid winding as in Fig. S-9.2. The path C is at rest in S' , while the solenoid moves toward the left with velocity $-v$. At $t = 0$ the upper side of length a is tangent to the winding at its central point, and a is sufficiently small for the enclosed winding arc to be well approximated by a straight line segment. We also have $b \ll a$. The field \mathbf{E}' is not conservative because the line integral of \mathbf{E}' along C does not vanish:

$$\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = E'_\parallel a = E'_y a \cos\phi. \tag{S-9.22}$$

This is consistent with the fact that the flux of \mathbf{B}' through the rectangle enclosed by the path C is time-dependent. The winding arc enclosed by the rectangle moves towards the lower side of length a with velocity $v \cos\phi$, and the flux of \mathbf{B}' through the rectangle is

$$\Phi_C(\mathbf{B}') = \int_C \mathbf{B}' \cdot d\mathbf{S} = B' a [b - (v \cos\phi) t], \tag{S-9.23}$$

corresponding to a line integral

$$-\frac{1}{c} \frac{d\Phi_C(\mathbf{B}')}{dt} = \frac{v}{c} B' a \cos\phi = E'_y a \cos\phi, \tag{S-9.24}$$

in agreement with (S-9.22).

S-9.4 The Four-Potential of a Plane Wave

a) The fields of the plane wave may be written in complex notation as

$$\mathbf{E} = \hat{\mathbf{y}} E_0 e^{ikx - i\omega t}, \quad \mathbf{B} = \hat{\mathbf{z}} B_0 e^{ikx - i\omega t}, \quad (\text{S-9.25})$$

with $E_0 = B_0$. A vector potential of the form $\mathbf{A} = \hat{\mathbf{y}} A_0 e^{ikx - i\omega t}$ generates an electric field along $\hat{\mathbf{y}}$ and a magnetic field along $\hat{\mathbf{z}}$ given by

$$\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A} - \nabla \varphi, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (\text{S-9.26})$$

In the absence of electric charges we have $\varphi \equiv 0$, and we obtain from (S-9.26)

$$A_0 = -\frac{ic}{\omega} E_0, \quad A_0 = -\frac{i}{k} B_0, \quad (\text{S-9.27})$$

which are equivalent since $\omega = kc$. The vector potential $\mathbf{A} = \hat{\mathbf{y}} A_0 e^{ikx - i\omega t}$ obviously satisfies the wave equation in vacuum, and also respects the Lorenz gauge condition.

b) The Lorentz transformations from S to S' give $\omega' = \gamma\omega$ (transverse Doppler effect), and $k'_x = k_x = k$, $k'_y = -\omega'v/c^2 = -\gamma\beta k_x$. The nonzero components of the fields in S' are $E'_y = E_y$, $E'_x = \gamma\beta B_z$, and $B'_z = \gamma B_z$. We may thus write

$$\mathbf{E}' = (\hat{\mathbf{x}}\gamma\beta + \hat{\mathbf{y}})E_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad \mathbf{B}' = \hat{\mathbf{z}}\gamma B_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad (\text{S-9.28})$$

The polarization is linear and directed along the unit vector $\boldsymbol{\varepsilon} = \beta\hat{\mathbf{x}} + \hat{\mathbf{y}}/\gamma$.

c) Assuming $\varphi' = 0$, we have in S'

$$\mathbf{E}' = -\frac{1}{c} \partial'_t \mathbf{A}', \quad \mathbf{B}' = \nabla' \times \mathbf{A}', \quad (\text{S-9.29})$$

which are both satisfied if we choose

$$\mathbf{A}' = -\frac{c}{\omega'} \mathbf{E}' = \left(\hat{\mathbf{x}}\beta + \hat{\mathbf{y}}\frac{1}{\gamma} \right) A_0 e^{i(k'_x x' + k'_y y' - \omega' t')}, \quad (\text{S-9.30})$$

being $\omega' = \gamma\omega$.

d) The Lorentz transformation from S to S' for the four-potential $A_\mu = (\varphi, \mathbf{A}) = (0, 0, A_y, 0)$ gives

$$\bar{A}'_\mu = (-\gamma\beta A_y, 0, \gamma A_y, 0) \equiv (\bar{\varphi}', 0, \bar{A}'_y, 0). \quad (\text{S-9.31})$$

The fields derived from this four-potential are

$$\bar{E}'_x = -\partial'_x \bar{\varphi}' = -ik'_x \bar{\varphi}' = ikc\gamma\beta A_y = \gamma\beta(i\omega A_y) = \gamma\beta E_y = E'_x, \quad (\text{S-9.32})$$

$$\begin{aligned} \bar{E}'_y &= -\frac{1}{c}\partial'_t \bar{A}'_y - \partial'_y \bar{\varphi}' = i\frac{\omega'}{c}\bar{A}'_y - ik'_y \bar{\varphi}' = i\left(\gamma\frac{\omega}{c}\right)\gamma A_y - i(-\gamma\beta k)(-\gamma\beta c)A_y \\ &= i\frac{\omega}{c}\gamma^2(1-\beta^2)A_y = E_y = E'_y, \end{aligned} \quad (\text{S-9.33})$$

$$\bar{B}'_z = \partial'_x \bar{A}'_y = ik'_x \bar{A}'_y = ik\gamma A_y = \gamma B_z = B'_z, \quad (\text{S-9.34})$$

in agreement with the results of point **c**).

e) The expressions $A'_\mu = (0, \mathbf{A}')$ and $\bar{A}'_\mu = (\bar{\varphi}', \bar{\mathbf{A}}')$ are two possible choices for the four-potential. Thus they must differ at most by a gauge transformation, i.e., there must be a scalar function $f = f(x', t')$ such that

$$\mathbf{A}' = \bar{\mathbf{A}}' + \nabla' f, \quad \varphi' = \bar{\varphi}' - \frac{1}{c}\partial'_t f. \quad (\text{S-9.35})$$

Since $\varphi' = 0$ we find $\partial'_t f/c = \bar{\varphi}'$, i.e.,

$$f = \frac{c}{\omega'} \bar{\varphi}' = -\frac{ic}{\omega'} \gamma\beta A_y, \quad (\text{S-9.36})$$

Now, since

$$\nabla' f = (\hat{\mathbf{x}} ik'_x + \hat{\mathbf{y}} ik'_y) f = (\hat{\mathbf{x}}\beta - \hat{\mathbf{y}}\gamma\beta^2) A_y, \quad (\text{S-9.37})$$

we also have that

$$\bar{\mathbf{A}}' + \nabla' f = \left[\hat{\mathbf{x}}\beta + \hat{\mathbf{y}}(\gamma - \gamma\beta^2) \right] A_y = \left(\hat{\mathbf{x}}\beta + \hat{\mathbf{y}}\frac{1}{\gamma} \right) A_y = \mathbf{A}'. \quad (\text{S-9.38})$$

S-9.5 The Force on a Magnetic Monopole

a) In the reference frame S' , where the magnetic monopole is at rest ($\mathbf{v}' = 0$), the magnetic field is

$$\mathbf{B}' = -\gamma \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (\text{S-9.39})$$

thus the force on the monopole is $\mathbf{F}' = q_m \mathbf{B}'$. On the other hand we must have $\mathbf{F}' = \gamma \mathbf{F}$, since \mathbf{F} is perpendicular to \mathbf{v} , so that in the laboratory frame S we have

$$\mathbf{F} = \frac{q_m}{\gamma} \mathbf{B}' = -q_m \frac{\mathbf{v}}{c} \times \mathbf{E}, \quad (\text{S-9.40})$$

which proves (9.8).

b) The equation of motion for a magnetic monopole in the presence of a uniform electric field $\mathbf{E} = \hat{\mathbf{z}}E$ alone is identical to the equation of motion at for a an electric charge in the presence of a uniform magnetic field $\mathbf{B} = \hat{\mathbf{z}}B$, after replacing $-q_m \mathbf{E}$ by $q \mathbf{B}$. The solution is a helicoidal motion, with a constant drift velocity parallel to \mathbf{E} , and a constant angular velocity $\omega_m = \hat{\mathbf{z}} q_m E / mc$. (Notice that, for a magnetic monopole, the angular velocity vector is parallel to \mathbf{E} , while it is antiparallel to \mathbf{B} in the case of an electric charge.)

In the case of crossed electric and magnetic fields, the condition $E > B$ ensures that there is a reference frame S' where the magnetic field vanishes. In fact, taking a Lorentz boost with $\boldsymbol{\beta} = (\mathbf{E} \times \mathbf{B}) / E^2$ we have

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) = \gamma \left(\mathbf{B} + \frac{E^2 \mathbf{B} - (\mathbf{E} \cdot \mathbf{B}) \mathbf{E}}{E^2} \right) = 0, \quad (\text{S-9.41})$$

since $\mathbf{E} \cdot \mathbf{B} = 0$. Thus, in the boosted frame there is only the electric field

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) = \gamma \left(\mathbf{E} - \frac{B^2}{E^2} \mathbf{E} \right) = \frac{\mathbf{E}}{\gamma}, \quad (\text{S-9.42})$$

since $\gamma = 1 / \sqrt{1 - \beta^2} = 1 / \sqrt{1 - B^2 / E^2}$. Thus the motion in S' is a circular orbit with angular frequency $\omega' = (q_m E / \gamma c)$. By transforming back to the laboratory frame S we add a drift velocity $-c\boldsymbol{\beta}$, and the trajectory in S is a cycloid.

S-9.6 Reflection from a Moving Mirror

a) As an ansatz, we write the total electromagnetic field as the sum of the fields of the incident wave and the fields of a reflected wave of the same frequency and polarization, but opposite direction

$$\begin{aligned} \mathbf{E}(x, t) &= \hat{\mathbf{y}} E_y(x, t), & \mathbf{B} &= \hat{\mathbf{z}} B_z(x, t), \\ E_y(x, t) &= \text{Re} \left(E_i e^{ikx - i\omega t} + E_r e^{-ikx - i\omega t} \right), \end{aligned} \quad (\text{S-9.43})$$

$$B_z(x, t) = \text{Re} \left(E_i e^{ikx - i\omega t} - E_r e^{-ikx - i\omega t} \right). \quad (\text{S-9.44})$$

The amplitude of the reflected wave E_r must be determined by the boundary condition at the mirror surface $x = 0$. We may already know that the electric field component parallel to the bounding surface between two media is continuous across the surface, i.e., that $E_{\parallel}(0^-) = E_{\parallel}(0^+)$. However, here we prefer to derive this result in detail, because this will help the discussion of the reflection at the surface of a *moving* mirror, which we shall consider in the following. Evaluating the line integral of \mathbf{E} over a closed rectangular loop across the boundary, as in Fig. S-9.3, yields

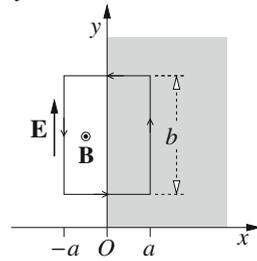


Fig. S-9.3

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{l} &= [E_y(a, t) - E_y(-a, t)]b \\ &= -\frac{1}{c} \frac{d\Phi(\mathbf{B})}{dt} = -\frac{b}{c} \int_{-a}^0 \partial_t B_z dx \\ &= \frac{i\omega b}{c} \int_{-a}^0 B_z dx = \frac{i\omega}{c} \bar{B}_z ab, \end{aligned} \tag{S-9.45}$$

where \bar{B}_z is the mean value of B_z in the $(-a, a)$ interval. If B_z is finite, the “rightmost RHS” of (S-9.45) vanishes at the limit $a \rightarrow 0$, and $E_y(0^+, t) = E_y(0^-, t)$.

For a perfect mirror we must have $E_y(0^+, t) = 0$, and the boundary condition implies that also $E_y(0^-, t) = 0$. Thus we obtain

$$E_y(0, t) = (E_i + E_r)e^{-i\omega t} = 0, \quad E_r = -E_i. \tag{S-9.46}$$

The total electric field for $x \leq 0$ is thus a standing wave

$$E_y = E_i (e^{ikx-i\omega t} - e^{-ikx-i\omega t}) = 2iE_i \sin(kx) e^{-i\omega t}, \tag{S-9.47}$$

with nodes where $\sin kx = 0$ and maximum amplitude $2E_i$. Recalling that $\omega/k = c$, the magnetic field of the wave is

$$B_z = 2E_i \cos(kx) e^{-i\omega t}. \tag{S-9.48}$$

Thus, B_z is *discontinuous* at the $x = 0$ surface. This implies the presence of a surface current density $\mathbf{K} = \hat{\mathbf{y}} K_y(t)$ at $x = 0$, corresponding to a volume current density $\mathbf{J} = \mathbf{K} \delta(x) = \hat{\mathbf{y}} K_y(t) \delta(x)$. By evaluating the line integral of \mathbf{B} over a closed path crossing the mirror surface we find the boundary condition

$$B_z(0^+, t) - B_z(0^-, t) = \frac{4\pi}{c} K_y(t), \tag{S-9.49}$$

and the surface current density on the surface of a perfect mirror is

$$K_y(t) = -\frac{c}{4\pi} B_z(0^-, t) = -\frac{cE_i}{2\pi} e^{-i\omega t}. \quad (\text{S-9.50})$$

b) Let $\beta = v/c$ (in what follows v , and, consequently, β , may have both positive or negative values, depending on whether the wave and the mirror velocity are parallel or antiparallel, respectively). We know that $(\omega/c, \mathbf{k})$ is a four-vector, and that \mathbf{k} is parallel to \mathbf{v} . Thus the frequency of the incident wave in S' is

$$\omega'_1 = \gamma(\omega - \mathbf{k} \cdot \mathbf{v}) = \gamma\omega(1 - \beta), \quad (\text{S-9.51})$$

where $k = \omega/c$ has been used. The magnitude of the incident wave vector in S' is $k'_1 = \omega'_1/c$. If $v > 0$ ($v < 0$) we have $\omega'_1 < \omega$ ($\omega'_1 > \omega$).

The Lorentz transformations give the following amplitudes for the fields in S'

$$E'_{1y} = \gamma(E_{iy} - \beta B_{iz}) = \gamma(1 - \beta)E_i, \quad (\text{S-9.52})$$

$$B'_{1z} = \gamma(B_{iz} - \beta E_{iy}) = \gamma(1 - \beta)E_i, \quad (\text{S-9.53})$$

since $B_{iz} = E_{iy}$. In the S' frame the reflected wave has frequency $\omega'_r = \omega'_1$, and field amplitudes $E'_{ry} = -E'_{1y}$, $B'_{rz} = B'_{1z}$.

c) The frequency ω_r of the reflected wave in the laboratory frame S can be evaluated by applying the inverse transformation from S' to S

$$\begin{aligned} \omega_r &= \gamma(\omega'_r + \mathbf{k}_r \cdot \mathbf{v}) = \gamma(\omega'_r - k_r v) = \gamma\omega'_r(1 - \beta) = \omega\gamma^2(1 - \beta)^2 \\ &= \omega \frac{1 - \beta}{1 + \beta}. \end{aligned} \quad (\text{S-9.54})$$

The electric and magnetic field amplitudes of the reflected wave in S' are $E'_r = -E'_1 = -\gamma(1 - \beta)E_i$ and $B'_r = B'_1 = \gamma(1 - \beta)E_i$. We thus have in S

$$E_{ry} = \gamma(E'_{ry} + \beta B'_{rz}) = -\gamma(1 - \beta)E'_1 = -\gamma^2(1 - \beta)^2 E_i = -\frac{1 - \beta}{1 + \beta} E_i, \quad (\text{S-9.55})$$

$$B_{rz} = \gamma(B'_{rz} + \beta E'_{ry}) = \gamma(1 - \beta)E'_1 = \gamma^2(1 - \beta)^2 E_i = \frac{1 - \beta}{1 + \beta} E_i. \quad (\text{S-9.56})$$

If $\beta < 0$ we have $|E_r| > |E_i|$: in S the reflected wave has a higher amplitude than the incident wave.

d) The complete expressions for the fields in S are

$$E_y(x, t) = E_i e^{ikx - i\omega t} - \frac{1 - \beta}{1 + \beta} E_i e^{-ik_r x - i\omega_r t}, \quad (\text{S-9.57})$$

$$B_z(x, t) = E_i e^{ikx - i\omega t} + \frac{1 - \beta}{1 + \beta} E_i e^{-ik_r x - i\omega_r t}, \quad (\text{S-9.58})$$

thus, also E_y has a finite value at the mirror surface $x(t) = vt$, and is therefore *discontinuous*:

$$E_y[x(t), t] = \frac{2\beta}{1 + \beta} E_i e^{-i(1 - \beta)\omega t}, \quad (\text{S-9.59})$$

$$B_z[x(t), t] = \frac{2}{1 + \beta} E_i e^{-i(1 - \beta)\omega t}. \quad (\text{S-9.60})$$

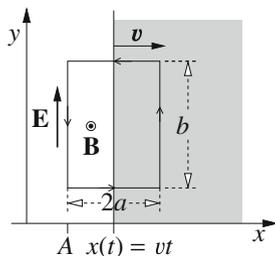


Fig. S-9.4

This can be seen by considering again the line integral of the electric field \mathbf{E} along a closed rectangular path of sides $2a$ and b , at rest in S . We assume that the left vertical side of the path is on the $x = A$ line, that at time t the mirror surface cuts the two horizontal sides, as in Fig. S-9.4, and that $a \ll \lambda$, where λ is the wavelength in S . The flux of the magnetic field through the rectangular path at time t is thus

$$\Phi(t) \simeq B_z[x(t), t][x(t) - A]b = B_z[x(t), t](vt - A)b, \quad (\text{S-9.61})$$

so that

$$-\frac{1}{c} \frac{d\Phi(t)}{dt} \simeq -\frac{1}{c} \left[\partial_t B_z[x(t), t](vt - A)b + B_z[x(t), t]vb \right]. \quad (\text{S-9.62})$$

At the limit $a \rightarrow 0$, $A \rightarrow vt$, the first term of the right-hand side vanishes, and we are left with

$$-\frac{1}{c} \frac{d\Phi(t)}{dt} \simeq -\frac{1}{c} B_z[x(t), t]vb = -B_z[x(t), t]\beta b. \quad (\text{S-9.63})$$

On the other hand, the line integral of \mathbf{E} along the closed rectangular path of Fig. S-9.4 is

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -E_y[x(t), t]b = -\frac{2\beta}{1 + \beta} b E_i e^{-i(1 - \beta)\omega t} = -B_z[x(t), t]\beta b. \quad (\text{S-9.64})$$

S-9.7 Oblique Incidence on a Moving Mirror

a) We choose a Cartesian reference frame S where \mathbf{v} is parallel to the x axis, the mirror surface lies on the yz plane and the wave vector \mathbf{k}_i of the incident wave lies in the xy plane. The Lorentz transformations to the frame S' give

$$k'_{ix} = \gamma \left(k_{ix} - \omega \frac{v}{c^2} \right) = \gamma \frac{\omega}{c} (\cos \theta_i - \beta), \quad (\text{S-9.65})$$

$$k'_{iy} = k_{iy}, \quad (\text{S-9.66})$$

$$\omega'_i = \gamma (\omega_i - k_x v) = \gamma \omega_i (1 - \beta \cos \theta_i), \quad (\text{S-9.67})$$

$$\tan \theta'_i = \frac{k'_{iy}}{k'_{ix}} = \frac{k_{iy} \tan \theta_i}{\gamma (\omega_i/c) (\cos \theta_i - \beta)} = \frac{k_{ix} \sin \theta_i}{\gamma k_{ix} (\cos \theta_i - \beta)} = \frac{\sin \theta_i}{\gamma (\cos \theta_i - \beta)}, \quad (\text{S-9.68})$$

where, as usual, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. In S' the reflection angle θ'_r equals the incidence angle θ'_i , thus

$$k'_{rx} = -k'_{ix}, \quad k'_{ry} = k'_{iy}, \quad \omega'_r = \omega'_i. \quad (\text{S-9.69})$$

b) By performing the Lorentz transformations back to the laboratory frame S we obtain

$$k_{ry} = k'_{ry} = k_{iy}, \quad (\text{S-9.70})$$

$$\begin{aligned} k_{rx} &= \gamma \left(k'_{rx} + \omega' \frac{v}{c^2} \right) = -\gamma^2 \left[k_{ix} (1 + \beta^2) - 2\omega_i \frac{\beta}{c} \right] \\ &= -2\gamma^2 \frac{\omega}{c} \left[(1 + \beta^2) \cos \theta_i - 2\beta \right], \end{aligned} \quad (\text{S-9.71})$$

$$\omega_r = \gamma (\omega'_r + k'_{rx} v) = \gamma^2 \left[\omega (1 + \beta^2) - 2k_{ix} v \right] = \gamma^2 \frac{\omega}{c} (1 + \beta^2 - 2\beta \cos \theta_i), \quad (\text{S-9.72})$$

from which

$$\tan \theta_r \equiv -\frac{k_{ry}}{k_{rx}} = \frac{\sin \theta_i}{\gamma^2 [2\beta - (1 + \beta^2) \cos \theta_i]}, \quad (\text{S-9.73})$$

For $\cos \theta_i = v/c = \beta$ the denominator of the “rightmost right-hand side” of (S-9.68) is zero, and the incidence angle θ'_i in S' is a right angle. This means that, in S' , the incident wave propagates parallel to the mirror surface, without hitting the mirror, and no reflection occurs. For incidence angles such that $\cos \theta_i > \beta$, all the above formulas are meaningless, since they would imply $k'_{ix} < 0$, i.e., that the wave is incident on the other side of the mirror.

S-9.8 Pulse Modification by a Moving Mirror

a) The number of oscillations in the wave packet is a relativistic invariant, and the Lorentz transformations are linear in the EM fields. Thus, in the reference frame S' , where the mirror is at rest, the incident wave packet is still square and comprises the same number of oscillations. On the other hand, as already seen in Problem 9.6, the frequency ω'_i and the amplitude E'_i are

$$\omega'_i = \gamma(1-\beta)\omega_i, \quad E'_i = \gamma(1-\beta)E_i, \quad (\text{S-9.74})$$

where $\beta = v/c$. In S' , the reflected packet has the same shape, duration, and frequency of the incident packet, but opposite amplitude and direction.

$$E'_r = -E'_i, \quad \omega'_r = \omega'_i, \quad \tau'_r = \tau'_i = N \frac{2\pi}{\omega'_i} = N \frac{2\pi}{\gamma(1-\beta)\omega_i} = \frac{\tau_i}{\gamma(1-\beta)}. \quad (\text{S-9.75})$$

Back-transforming to S (see also Problem 9.6) we have

$$E_r = -\frac{1-\beta}{1+\beta}E_i, \quad \omega_r = \gamma(1-\beta)\omega'_r = \gamma^2(1-\beta^2)\omega_i = \frac{1-\beta}{1+\beta}\omega_i \quad (\text{S-9.76})$$

The duration of the reflected wave packet is thus

$$\tau_r = N \frac{2\pi}{\omega_r} = N \frac{2\pi}{\omega} \frac{1+\beta}{1-\beta} = \frac{1+\beta}{1-\beta} \tau_i. \quad (\text{S-9.77})$$

If $\beta > 0$, i.e., if the mirror velocity is parallel to the packet propagation direction, the reflected packet has a longer duration than the incident packet, while the reflected packet is shorter if the mirror velocity is antiparallel.

b) The energy per unit surface of each packet is given by its intensity I times its duration τ . The intensity is proportional to the square of the electric field amplitude, thus the relation between the reflected and incident intensities is

$$I_r = \left(\frac{1-\beta}{1+\beta} \right)^2 I_i, \quad (\text{S-9.78})$$

and the relation between the energies per unit surface of the whole reflected and incident packets is

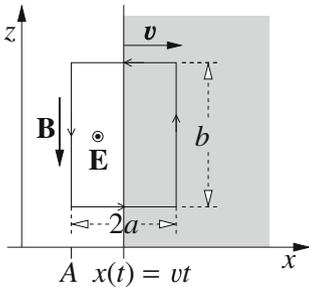
$$U_r = I_r \tau_r = \left(\frac{1-\beta}{1+\beta} \right)^2 I_i \frac{1+\beta}{1-\beta} \tau_i = \frac{1-\beta}{1+\beta} I_i \tau_i = \frac{1-\beta}{1+\beta} U_i. \quad (\text{S-9.79})$$

We see that $U_r \neq U_i$, hence some work per unit surface is needed in order to keep the mirror moving at constant velocity, namely

$$W = U_r - U_i = -\frac{2\beta}{1+\beta} U_i. \quad (\text{S-9.80})$$

Thus a mirror with $\beta < 0$, i.e., moving in the direction opposite to the incident wave packet, transfers some energy to the packet.

c) As a first step, we determine the distribution of the current density \mathbf{J} . Since all the fields are null inside the mirror, i.e., for $x > x(t) = vt$, the current must be localized on the mirror surface, $\mathbf{J}(x, t) = \mathbf{K}(t) \delta(x - vt)$. We can evaluate the surface current



evaluate the surface current density $\mathbf{K}(t)$ on the mirror surface by considering the fields close to the surface. By calculating the line integral of \mathbf{B} over a closed rectangular path, fixed in S , of sides b , parallel to \mathbf{B} and to the mirror surface, and $2a$, perpendicular to, and crossing the mirror surface, as in Fig. S-9.5, we obtain

$$\oint_{\text{path}} \mathbf{B} \cdot d\boldsymbol{\ell} = \frac{4\pi}{c} \Phi(\mathbf{J}) + \frac{1}{c} \frac{d\Phi(\mathbf{E})}{dt}, \quad (\text{S-9.81})$$

Fig. S-9.5

where $\Phi(\mathbf{J})$ and $\Phi(\mathbf{E})$ are the fluxes through the surface delimited by the path Jv and \mathbf{E} , respectively. At the limit $a \rightarrow 0$ and $A \rightarrow vt$, we have

$$\oint_{\text{path}} \mathbf{B} \cdot d\boldsymbol{\ell} \simeq B(vt)b, \quad \Phi(\mathbf{J}) = K(t)b, \quad (\text{S-9.82})$$

$$\frac{d\Phi(\mathbf{E})}{dt} \simeq \partial_t E(vt)(vt - A)b + E(vt)bv \simeq E(vt)bv. \quad (\text{S-9.83})$$

From the knowledge of E and B at the mirror surface (Prob. 9.6) we obtain

$$\begin{aligned} K(t) &= \frac{c}{4\pi} \left[B(vt) - \frac{v}{c} E(vt) \right] = \frac{c}{4\pi} (1 - \beta^2) \frac{2E_i}{1 + \beta} e^{-i(1-\beta)\omega t} \\ &= \frac{cE_i}{2\pi} (1 - \beta) e^{-i(1-\beta)\omega t}. \end{aligned} \quad (\text{S-9.84})$$

Thus, \mathbf{K} and \mathbf{E} are in phase. In order to evaluate the total mechanical work per unit surface on the mirror, we first switch back to the real quantities

$$K(t) = \frac{cE_i}{2\pi} (1 - \beta) \cos[(1 - \beta)\omega t], \quad (\text{S-9.85})$$

$$E(vt) = \frac{2\beta}{1 + \beta} E_i \cos[(1 - \beta)\omega t], \quad (\text{S-9.86})$$

and evaluate the integral over the mirror depth

$$\int_{vt}^{\infty} \mathbf{J} \cdot \mathbf{E} dx = \frac{1}{2} K(t)E(vt) = \frac{cE_i^2}{2\pi} \frac{\beta(1-\beta)}{1+\beta} \cos^2[(1-\beta)\omega t]. \quad (\text{S-9.87})$$

We have inserted the factor $1/2$ to account for the discontinuity of E at $x = vt$ (see also Prob. 2.12). Equation (S-9.87) gives the mechanical power per unit surface exerted on the mirror. To find the mechanical work, (S-9.87) must be integrated over the time interval for which $K(t) \neq 0$, i.e., for the time needed by the wave packet to undergo a complete reflection. If the front of the wave packet reaches the mirror at $t = 0$, the end of the packet will leave the mirror at $t = \tau/(1-\beta)$, which is different from the pulse duration τ because the mirror moves while the wave train is reflected. We thus need the integral

$$\int_0^{\tau/(1-\beta)} \cos^2[(1-\beta)\omega t] dt = \frac{1}{\omega(1-\beta)} \int_0^{\omega\tau} \cos^2 x dx = \frac{\pi N}{(1-\beta)\omega}, \quad (\text{S-9.88})$$

since $\omega\tau = 2\pi N$, and the integral of $\cos^2 x$ over one period equals π . We thus obtain

$$W = \int \frac{1}{2} K(t)E(vt) dt = \frac{cE_i^2}{2\pi} \frac{\beta(1-\beta)}{1+\beta} \frac{\pi N}{(1-\beta)\omega} = \frac{cE_i^2}{4\pi} \frac{\beta}{1+\beta} \tau \quad (\text{S-9.89})$$

$$= 2I_i \tau \frac{\beta}{1+\beta} = \frac{2\beta}{1+\beta} U_i, \quad (\text{S-9.90})$$

in agreement with (S-9.80).

The work W , divided by the reflection time gives, the mechanical power per unit surface

$$\mathcal{P} = W \frac{1-\beta}{\tau} = \frac{2\beta(1-\beta)}{1+\beta} I_i = \frac{2(1-\beta)}{1+\beta} I_i \frac{v}{c}, \quad (\text{S-9.91})$$

which must be equal to the the pressure exerted on the moving mirror times its velocity v . We thus obtain that the radiation pressure on a moving mirror is

$$P_{\text{rad}} = \frac{\mathcal{P}}{v} = \frac{2I_i}{c} \frac{1-\beta}{1+\beta}, \quad (\text{S-9.92})$$

a result which can also be obtained in different ways (see Problems 13.7 & 13.8).

S-9.9 Boundary Conditions on a Moving Mirror

a) We can assume the wave to be linearly polarized along y , without loss of generality. We choose the origin of the frame S' , where the mirror is at rest, so that the mirror surface is on the $x' = 0$ plane. In S' the total fields at the mirror surface are

$$\begin{aligned}\mathbf{E}'_s(t') &= \hat{\mathbf{y}}' E'_s(t') \equiv 0, \\ \mathbf{B}'_s(t') &= \hat{\mathbf{z}}' B'_s(t') e^{-i\omega'_i t'} = -\hat{\mathbf{z}}' 2E'_i e^{-i\omega'_i t'},\end{aligned}\quad (\text{S-9.93})$$

respectively, where

$$E'_i = \gamma(1-\beta)E_i, \quad \omega'_i = \gamma(1-\beta)\omega_i, \quad (\text{S-9.94})$$

are the amplitude and frequency of the incident wave in S' , as seen in Problem 9.6. Notice that \mathbf{E}' is continuous at $x' = 0$, while \mathbf{B}' is not. By transforming the field amplitudes at the mirror surface back to S we obtain

$$\begin{aligned}E_s &= \gamma(E'_s + \beta B'_s) = \gamma\beta B'_s = -2\gamma^2\beta(1-\beta)E_i, \\ B_s &= \gamma(B'_s - \beta E'_s) = \gamma B'_s = -2\gamma^2(1-\beta)E_i,\end{aligned}\quad (\text{S-9.95})$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Thus, in general, in S we have both $E_s \neq 0$ and $B_s \neq 0$, while the fields are zero inside the mirror.

b) The EM fields are related to the vector potential by

$$\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (\text{S-9.96})$$

Thus, the only nonzero component of the vector potential is A_y , and we have

$$E_s = -\frac{1}{c}\partial_t A_y, \quad B_s = \partial_x A_y. \quad (\text{S-9.97})$$

The total derivative of \mathbf{A} appearing in (9.9) can be rewritten

$$\left. \frac{d\mathbf{A}}{dt} \right|_{x=x(t)} = \left[\partial_t A_y + v\partial_x A_y \right]_{x=x(t)} = cE_s - vB_s = c(E_s - \beta B_s) = 0, \quad (\text{S-9.98})$$

according to (S-9.95). Thus the equations (S-9.93) and (S-9.95) imply $d\mathbf{A}/dt = 0$ on the mirror surface in S .

c) The total vector potential in S is the sum of the vector potentials of the incident and the reflected waves,

$$\mathbf{A}(x, t) = \hat{\mathbf{y}} \left[A_i e^{ik_i x - i\omega_i t} + A_r e^{-ik_r x - i\omega_r t} \right] = \hat{\mathbf{y}} \left[A_i e^{ik_i(x-ct)} + A_r e^{-ik_r(x+ct)} \right], \quad (\text{S-9.99})$$

where $A_i = icE_i/\omega_i$, $k_i = \omega_i/c$, and $k_r = \omega_r/c$. The boundary condition gives

$$0 = A_y(vt, t) = A_i e^{-ik_i(c-v)t} + A_r e^{-ik_r(c+v)t} . \quad (\text{S-9.100})$$

This equation is satisfied if

$$A_r = -A_i , \quad \frac{k_r}{k_i} = \frac{\omega_r}{\omega_i} = \frac{c-v}{c+v} = \frac{1-\beta}{1+\beta} . \quad (\text{S-9.101})$$

For the total electric field we find

$$\begin{aligned} E_y &= -\frac{1}{c} \partial_t A_y = i \frac{\omega_i}{c} A_i e^{ik_i x - i\omega_i t} - i \frac{\omega_r}{c} A_r e^{-ik_r x - i\omega_r t} \\ &= E_i e^{ik_i x - i\omega_i t} - \frac{1-\beta}{1+\beta} E_i e^{-ik_r x - i\omega_r t} . \end{aligned} \quad (\text{S-9.102})$$