Chapter 1
Introduction

1.1 Concept

The Finite Element Analysis (FEA) method, originally introduced by Turner et al. (1956), is a powerful computational technique for approximate solutions to a variety of "real-world" engineering problems having complex domains subjected to general boundary conditions. FEA has become an essential step in the design or modeling of a physical phenomenon in various engineering disciplines. A physical phenomenon usually occurs in a continuum of matter (solid, liquid, or gas) involving several field variables. The field variables vary from point to point, thus possessing an infinite number of solutions in the domain. Within the scope of this book, a continuum with a known boundary is called a domain.

The basis of FEA relies on the decomposition of the domain into a finite number of subdomains (elements) for which the systematic approximate solution is constructed by applying the variational or weighted residual methods. In effect, FEA reduces the problem to that of a finite number of unknowns by dividing the domain into elements and by expressing the unknown field variable in terms of the assumed approximating functions within each element. These functions (also called interpolation functions) are defined in terms of the values of the field variables at specific points, referred to as nodes. Nodes are usually located along the element boundaries, and they connect adjacent elements.

The ability to discretize the irregular domains with finite elements makes the method a valuable and practical analysis tool for the solution of boundary, initial, and eigenvalue problems arising in various engineering disciplines. Since its inception, many technical papers and books have appeared on the development and application of FEA. The books by Desai and Abel (1971), Oden (1972), Gallagher (1975), Huebner (1975), Bathe and Wilson (1976), Ziekiewicz (1977), Cook (1981), and Bathe (1996) have influenced the current state of FEA. Representative common engineering problems and their corresponding FEA discretizations are illustrated in Fig. 1.1.
The finite element analysis method requires the following major steps:

- Discretization of the domain into a finite number of subdomains (elements).
- Selection of interpolation functions.

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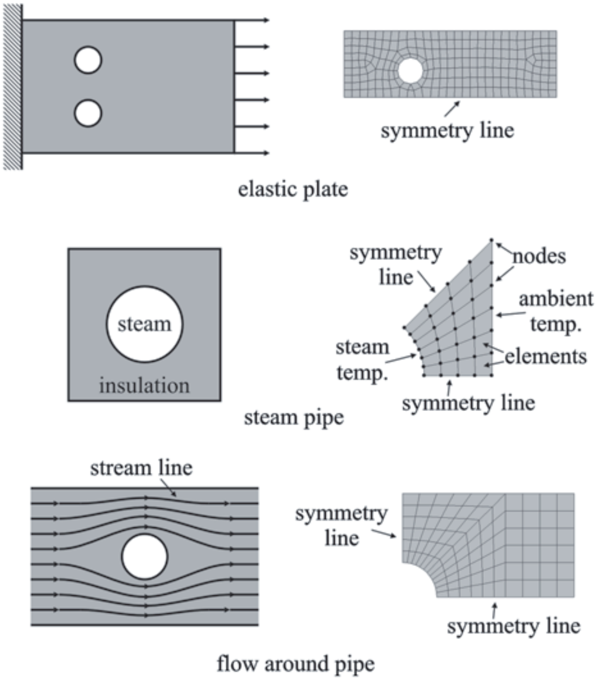

Fig. 1.1 FEA representation of practical engineering problems

- Development of the element matrix for the subdomain (element).
- Assembly of the element matrices for each subdomain to obtain the global matrix for the entire domain.
- Imposition of the boundary conditions.
- Solution of equations.
- Additional computations (if desired).

There are three main approaches to constructing an approximate solution based on the concept of FEA:

Direct Approach This approach is used for relatively simple problems, and it usually serves as a means to explain the concept of FEA and its important steps (discussed in Sect. 1.4).

Weighted Residuals This is a versatile method, allowing the application of FEA to problems whose functionals cannot be constructed. This approach directly utilizes the governing differential equations, such as those of heat transfer and fluid mechanics (discussed in Sect. 6.1).

Variational Approach This approach relies on the calculus of variations, which involves extremizing a functional. This functional corresponds to the potential energy in structural mechanics (discussed in Sect. 6.2).

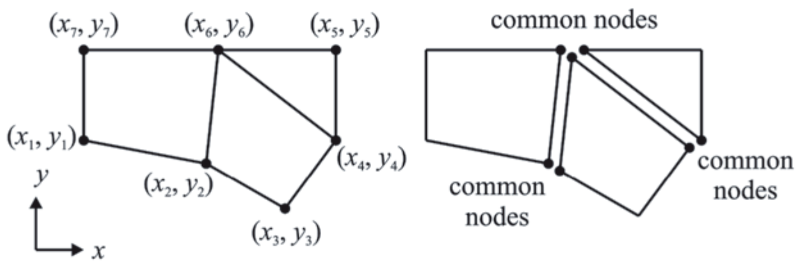

Fig. 1.2 Division of a domain into subdomains (elements)

In matrix notation, the global system of equations can be cast into

$$\mathbf{Ku} = \mathbf{F} \tag{1.1}$$

where $\mathbf{K}$ is the system stiffness matrix, $\mathbf{u}$ is the vector of unknowns, and $\mathbf{F}$ is the force vector. Depending on the nature of the problem, $\mathbf{K}$ may be dependent on $\mathbf{u}$, i.e., $\mathbf{K} = \mathbf{K}(\mathbf{u})$ and $\mathbf{F}$ may be time dependent, i.e., $\mathbf{F} = \mathbf{F}(t)$.

1.2 Nodes

As shown in Fig. 1.2, the transformation of the practical engineering problem to a mathematical representation is achieved by discretizing the domain of interest into elements (subdomains). These elements are connected to each other by their "common" nodes. A node specifies the coordinate location in space where degrees of freedom and actions of the physical problem exist. The nodal unknown(s) in the matrix system of equations represents one (or more) of the primary field variables. Nodal variables assigned to an element are called the degrees of freedom of the element.

The common nodes shown in Fig. 1.2 provide continuity for the nodal variables (degrees of freedom). Degrees of freedom (DOF) of a node are dictated by the physical nature of the problem and the element type. Table 1.1 presents the DOF and corresponding "forces" used in FEA for different physical problems.

1.3 Elements

Depending on the geometry and the physical nature of the problem, the domain of interest can be discretized by employing line, area, or volume elements. Some of the common elements in FEA are shown in Fig. 1.3. Each element, identified by an element number, is defined by a specific sequence of global node numbers. The

Table 1.1 Degrees of freedom and force vectors in FEA for different engineering disciplines

Discipline	DOF	Force vector
Structural/solids	Displacement	Mechanical forces
Heat conduction	Temperature	Heat flux
Acoustic fluid	Displacement potential	Particle velocity
Potential flow	Pressure	Particle velocity
General flows	Velocity	Fluxes
Electrostatics	Electric potential	Charge density
Magnetostatics	Magnetic potential	Magnetic intensity

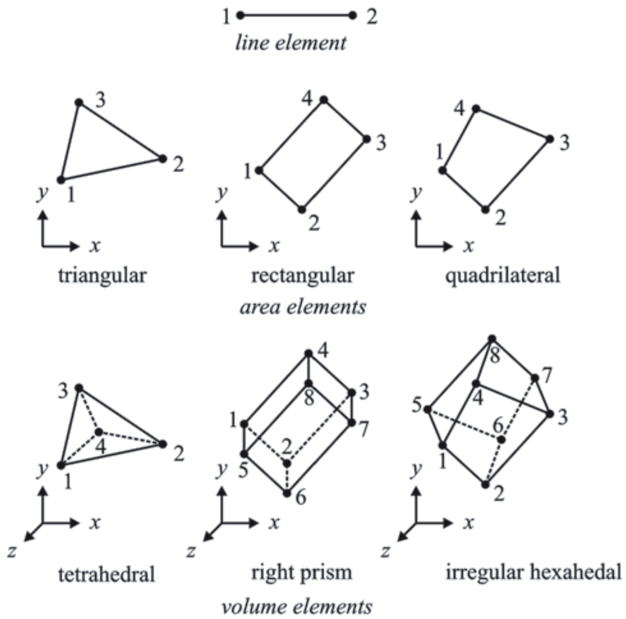

Fig. 1.3 Description of line, area, and volume elements with node numbers at the element level

specific sequence (usually counterclockwise) is based on the node numbering at the element level. The node numbering sequence for the elements shown in Fig. 1.4 are presented in Table 1.2.

1.4 Direct Approach

Although the direct approach is suitable for simple problems, it involves each fundamental step of a typical finite element analysis. Therefore, this approach is demonstrated by considering a linear spring system and heat flow in a one-dimensional (1-D) domain.

Table 1.2 Description of numbering at the element level

Element Number	Node 1	Node 2	Node 3	Node 4
1	1	2	6	7
2	3	4	6	2
3	4	5	6	

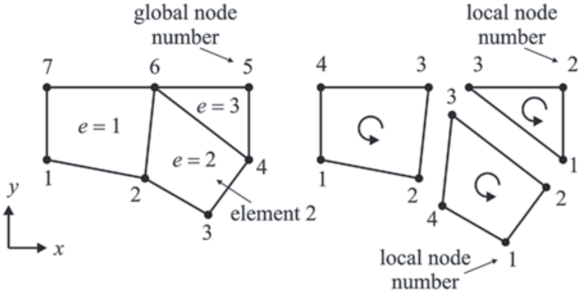

Fig. 1.4 Discretization of a domain: element and node numbering

Fig. 1.5 Free-body diagram of a linear spring element

1.4.1 Linear Spring

As shown in Fig. 1.5, a linear spring with stiffness k has two nodes. Each node is subjected to axial loads of f_1 and f_2, resulting in displacements of u_1 and u_2 in their defined positive directions.

Subjected to these nodal forces, the resulting deformation of the spring becomes

$$u = u_1 - u_2 \tag{1.2}$$

which is related to the force acting on the spring by

$$f_1 = ku = k(u_1 - u_2) \tag{1.3}$$

The equilibrium of forces requires that

$$f_2 = -f_1 \tag{1.4}$$

which yields

$$f_2 = k(u_2 - u_1) \tag{1.5}$$

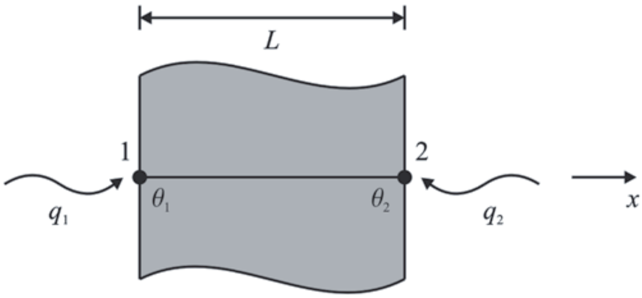

Fig. 1.6 One-dimensional heat flow

Combining Eq. (1.3) and (1.5) and rewriting the resulting equations in matrix form yield

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \text{ or } \mathbf{k}^{(e)}\mathbf{u}^{(e)} = \mathbf{f}^{(e)} \tag{1.6}$$

in which $\mathbf{u}^{(e)}$ is the vector of nodal unknowns representing displacement and $\mathbf{k}^{(e)}$ and $\mathbf{f}^{(e)}$ are referred to as the element characteristic (stiffness) matrix and element right-hand-side (force) vector, respectively. The superscript (e) denotes the element numbered as 'e'.

The stiffness matrix can be expressed in indicial form as $k_{ij}^{(e)}$

$$\mathbf{k}^{(e)} \sim k_{ij}^{(e)} \tag{1.7}$$

where the subscripts i and j $(i, j = 1, 2)$ are the row and the column numbers. The coefficients, $k_{ij}^{(e)}$, may be interpreted as the force required at node i to produce a unit displacement at node j while all the other nodes are fixed.

1.4.2 Heat Flow

Uniform heat flow through the thickness of a domain whose in-plane dimensions are long in comparison to its thickness can be considered as a one-dimensional analysis. The cross section of such a domain is shown in Fig. 1.6. In accordance with Fourier's Law, the rate of heat flow per unit area in the x-direction can be written as

$$q = -kA\frac{d\theta}{dx} \tag{1.8}$$

where A is the area normal to the heat flow, θ is the temperature, and k is the coefficient of thermal conductivity. For constant k, Eq. (1.8) can be rewritten as

$$q = -kA\frac{\Delta\theta}{L} \tag{1.9}$$

in which $\Delta\theta = \theta_2 - \theta_1$ denotes the temperature drop across the thickness denoted by L of the domain.

As illustrated in Fig. 1.6, the nodal flux (heat flow entering a node) at Node 1 becomes

$$q_1 = \frac{kA}{L}(\theta_1 - \theta_2) \tag{1.10}$$

The balance of the heat flux requires that

$$q_2 = -q_1 \tag{1.11}$$

which yields

$$q_2 = -\frac{kA}{L}(\theta_1 - \theta_2) \tag{1.12}$$

Combining Eq. (1.10) and (1.12) and rewriting the resulting equations in matrix form yield

$$\frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \text{ or } \mathbf{k}^{(e)}\boldsymbol{\theta}^{(e)} = \mathbf{q}^{(e)} \tag{1.13}$$

in which $\boldsymbol{\theta}^{(e)}$ is the vector of nodal unknowns representing temperature and $\mathbf{k}^{(e)}$ and $\mathbf{q}^{(e)}$ are referred to as the element characteristic matrix and element right-hand-side vector, respectively.

1.4.3 Assembly of the Global System of Equations

Modeling an engineering problem with finite elements requires the assembly of element characteristic (stiffness) matrices and element right-hand-side (force) vectors, leading to the global system of equations

$$\mathbf{Ku} = \mathbf{F} \tag{1.14}$$

in which $\mathbf{K}$ is the assembly of element characteristic matrices, referred to as the global system matrix and $\mathbf{F}$ is the assembly of element right-hand-side vectors, referred to as the global right-hand-side (force) vector. The vector of nodal unknowns is represented by $\mathbf{u}$.

The global system matrix, $\mathbf{K}$, can be obtained from the "expanded" element coefficient matrices, $\mathbf{k}^{(e)}$, by summation in the form

$$\mathbf{K} = \sum_{e=1}^{E} \mathbf{k}^{(e)} \tag{1.15}$$

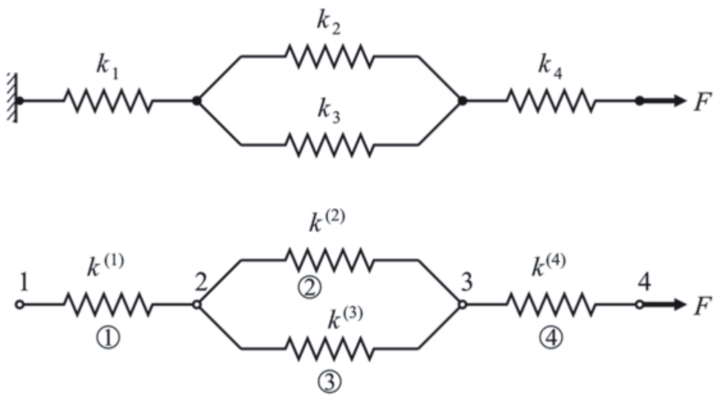

Fig. 1.7 System of linear springs (*top*) and corresponding FEA model (*bottom*)

in which the parameter E denotes the total number of elements. The "expanded" element characteristic matrices are the same size as the global system matrix but have rows and columns of zeros corresponding to the nodes not associated with element (e). The size of the global system matrix is dictated by the highest number among the global node numbers.

Similarly, the global right-hand-side vector, $\mathbf{F}$, can be obtained from the "expanded" element coefficient vectors, $\mathbf{f}^{(e)}$, by summation in the form

$$\mathbf{F} = \sum_{e=1}^{E} \mathbf{f}^{(e)} \tag{1.16}$$

The "expanded" element right-hand-side vectors are the same size as the global right-hand-side vector but have rows of zeros corresponding to the nodes not associated with element (e). The size of the global right-hand-side vector is also dictated by the highest number among the global node numbers.

The explicit steps in the construction of the global system matrix and the global right-hand-side-vector are explained by considering the system of linear springs shown in Fig. 1.7. Associated with element (e), the element equations for a spring given by Eq. (1.6) are rewritten as

$$\begin{bmatrix} k_{11}^{(e)} & k_{12}^{(e)} \\ k_{21}^{(e)} & k_{22}^{(e)} \end{bmatrix} \begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} f_1^{(e)} \\ f_2^{(e)} \end{Bmatrix} \tag{1.17}$$

in which $k_{11}^{(e)} = k_{22}^{(e)} = k^{(e)}$ and $k_{12}^{(e)} = k_{21}^{(e)} = -k^{(e)}$. The subscripts used in Eq. (1.17) correspond to Node 1 and Node 2, the local node numbers of element (e). The global node numbers specifying the connectivity among the elements for this system of springs is shown in Fig. 1.7, and the connectivity information is tabulated in Table 1.3.

Table 1.3 Table of connectivity

Element number	Local node numbering	Global node numbering
1	1	1
	2	2
2	1	2
	2	3
3	1	2
	2	3
4	1	3
	2	4

In accordance with Eq. (1.15), the size of the global system matrix is (4×4) and the specific contribution from each element is captured as

$$\text{Element 1:}\ \begin{array}{c}\begin{array}{cc}\boxed{1} & \boxed{2}\end{array}\\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix}\end{array}\begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \Rightarrow \begin{array}{c}\begin{array}{cccc}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{array}\\ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array} \equiv \mathbf{k}^{(1)} \quad (1.18)$$

$$\text{Element 2:}\ \begin{array}{c}\begin{array}{cc}\boxed{2} & \boxed{3}\end{array}\\ \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix}\end{array}\begin{array}{c}\boxed{2}\\\boxed{3}\end{array} \Rightarrow \begin{array}{c}\begin{array}{cccc}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{array}\\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array} \equiv \mathbf{k}^{(2)} \quad (1.19)$$

$$\text{Element 3:}\ \begin{array}{c}\begin{array}{cc}\boxed{2} & \boxed{3}\end{array}\\ \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix}\end{array}\begin{array}{c}\boxed{2}\\\boxed{3}\end{array} \Rightarrow \begin{array}{c}\begin{array}{cccc}\boxed{1} & \boxed{2} & \boxed{3} & \boxed{4}\end{array}\\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_{11}^{(3)} & k_{12}^{(3)} & 0 \\ 0 & k_{21}^{(3)} & k_{22}^{(3)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\end{array}\begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array} \equiv \mathbf{k}^{(3)} \quad (1.20)$$

$$\text{Element 4: } \begin{array}{c} \boxed{3} \quad \boxed{4} \\ \begin{bmatrix} k_{11}^{(4)} & k_{12}^{(4)} \\ k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{matrix} \boxed{3} \\ \boxed{4} \end{matrix} \end{array} \Rightarrow \begin{array}{c} \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \boxed{4} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11}^{(4)} & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \end{array} \equiv \mathbf{k}^{(4)} \tag{1.21}$$

Performing their assembly leads to

$$\mathbf{K} = \sum_{e=1}^{4} \mathbf{k}^{(e)} = \mathbf{k}^{(1)} + \mathbf{k}^{(2)} + \mathbf{k}^{(3)} + \mathbf{k}^{(4)} \tag{1.22}$$

or

$$\mathbf{K} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & \left(k_{22}^{(1)} + k_{11}^{(2)} + k_{11}^{(3)}\right) & \left(k_{12}^{(2)} + k_{12}^{(3)}\right) & 0 \\ 0 & \left(k_{21}^{(2)} + k_{21}^{(3)}\right) & \left(k_{22}^{(2)} + k_{22}^{(3)} + k_{11}^{(4)}\right) & k_{12}^{(4)} \\ 0 & 0 & k_{21}^{(4)} & k_{22}^{(4)} \end{bmatrix} \tag{1.23}$$

In accordance with Eq. (1.16), the size of the global right-hand-side vector is (4×1) and the specific contribution from each element is captured as

$$\text{Element 1: } \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \end{matrix} \Rightarrow \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \equiv \mathbf{f}^{(1)} \tag{1.24}$$

$$\text{Element 2: } \begin{Bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \end{matrix} \Rightarrow \begin{Bmatrix} 0 \\ f_1^{(2)} \\ f_2^{(2)} \\ 0 \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \equiv \mathbf{f}^{(2)} \tag{1.25}$$

$$\text{Element 3: } \begin{Bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{Bmatrix} \begin{matrix} \boxed{2} \\ \boxed{3} \end{matrix} \Rightarrow \begin{Bmatrix} 0 \\ f_1^{(3)} \\ f_2^{(3)} \\ 0 \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \equiv \mathbf{f}^{(3)} \tag{1.26}$$

$$\text{Element 4:} \quad \begin{Bmatrix} f_1^{(4)} \\ f_2^{(4)} \end{Bmatrix} \begin{matrix} \boxed{3} \\ \boxed{4} \end{matrix} \Rightarrow \begin{Bmatrix} 0 \\ 0 \\ f_1^{(4)} \\ f_2^{(4)} \end{Bmatrix} \begin{matrix} \boxed{1} \\ \boxed{2} \\ \boxed{3} \\ \boxed{4} \end{matrix} \equiv \mathbf{f}^{(4)} \tag{1.27}$$

Similarly, performing their assembly leads to

$$\mathbf{F} = \sum_{e=1}^{4} \mathbf{f}^{(e)} = \mathbf{f}^{(1)} + \mathbf{f}^{(2)} + \mathbf{f}^{(3)} + \mathbf{f}^{(4)} \tag{1.28}$$

or

$$\mathbf{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} + f_1^{(3)} \\ f_2^{(2)} + f_2^{(3)} + f_1^{(4)} \\ f_2^{(4)} \end{Bmatrix} \tag{1.29}$$

Consistent with the assembly of the global system matrix and the global right-hand-side vector, the vector of unknowns, $\mathbf{u}$, becomes

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} = u_1^{(2)} = u_1^{(3)} \\ u_2^{(2)} = u_2^{(3)} = u_1^{(4)} \\ u_2^{(4)} \end{Bmatrix} \tag{1.30}$$

1.4.4 Solution of the Global System of Equations

In order for the global system of equations to have a unique solution, the determinant of the global system matrix must be nonzero. However, an examination of the global system matrix reveals that one of its eigenvalues is zero, thus resulting in a zero determinant or singular matrix. Therefore, the solution is not unique. The eigenvector corresponding to the zero eigenvalue represents the translational mode, and the remaining nonzero eigenvalues represent all of the deformation modes.

For the specific values of $k_{11}^{(e)} = k_{22}^{(e)} = k^{(e)}$ and $k_{12}^{(e)} = k_{21}^{(e)} = -k^{(e)}$, the global system matrix becomes

$$\mathbf{K} = k^{(e)} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{1.31}$$

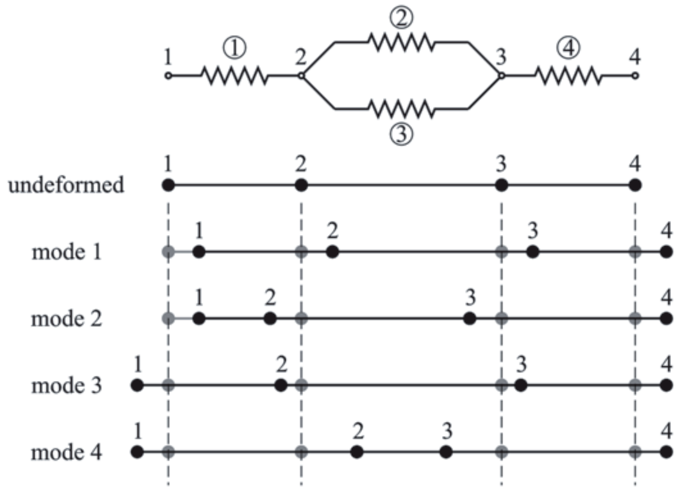

Fig. 1.8 Possible solution modes for the system of linear springs

with its eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = 3 - \sqrt{5}$, and $\lambda_4 = 3 + \sqrt{5}$. The corresponding eigenvectors are

$$\mathbf{u}^{(1)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \mathbf{u}^{(2)} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}, \mathbf{u}^{(3)} = \left\{ \begin{bmatrix} -1 \\ 2-\sqrt{5} \\ -2+\sqrt{5} \\ 1 \end{bmatrix} \right\}, \mathbf{u}^{(4)} = \left\{ \begin{bmatrix} -1 \\ 2+\sqrt{5} \\ -2-\sqrt{5} \\ 1 \end{bmatrix} \right\} \tag{1.32}$$

Each of these eigenvectors represents a possible solution mode. The contribution of each solution mode is illustrated in Fig. 1.8.

In order for the global system of equations to have a unique solution, the global system matrix is rendered nonsingular by eliminating the zero eigenvalue. This is achieved by introducing a boundary condition so as to suppress the translational mode of the solution corresponding to the zero eigenvalue.

1.4.5 Boundary Conditions

As shown in Fig. 1.7, Node 1 is restrained from displacement. This constraint is satisfied by imposing the boundary condition of $u_1 = 0$. Either the nodal displacements, u_i, or the nodal forces, f_i, can be specified at a given node. It is physically impossible to specify both of them as known or as unknown. Therefore, the nodal force f_1 remains as one of the unknowns. The nodal displacements, u_2, u_3, and u_4 are treated as unknowns, and the corresponding nodal forces have values of $f_2 = 0$, $f_3 = 0$, and $f_4 = F$.

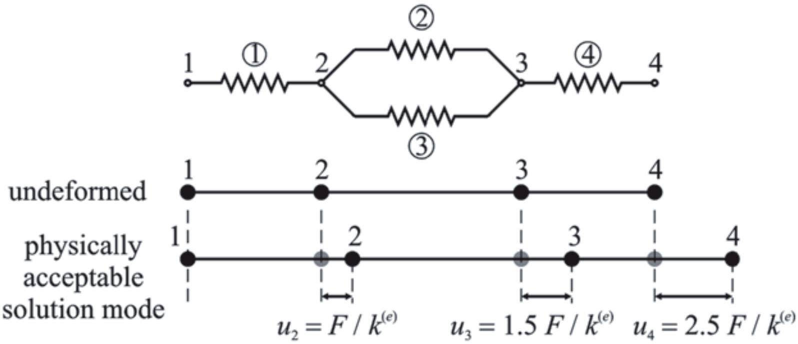

Fig. 1.9 Physically acceptable solution mode for the system of linear springs

These specified values are invoked into the global system of equations as

$$k^{(e)}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 = 0 \\ f_3 = 0 \\ f_4 = F \end{Bmatrix} \tag{1.33}$$

leading to the following equations:

$$k^{(e)}\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F \end{Bmatrix} \tag{1.34}$$

and

$$-k^{(e)}u_2 = f_1 \tag{1.35}$$

The coefficient matrix in Eq. (1.34) is no longer singular, and the solutions to these equations are obtained as

$$u_2 = \frac{F}{k^{(e)}},\ u_3 = \frac{3}{2}\frac{F}{k^{(e)}},\ u_4 = \frac{5}{2}\frac{F}{k^{(e)}} \tag{1.36}$$

and the unknown nodal force f_1 is determined as $f_1 = -F$. The final physically acceptable solution mode is shown in Fig. 1.9.

There exist systematic approaches to assemble the global coefficient matrix while invoking the specified nodal values (Bathe and Wilson 1976; Bathe 1996). The specified nodal variables are eliminated in advance from the global system of equations prior to the solution.