Chapter 12
Momentum, Impulse, and Collisions

Two galaxies collide, leading to millions and millions of stars interacting. Can we say anything general about such a collision? For example, if you know the galaxies velocities after the collision, what can you learn about their velocities before the collision?

We have started to study the consequences of Newton's laws of motion. Our first discovery is the conservation of mechanical energy. For an object subject only to conservative forces, the mechanical energy is conserved. Energy conservation provides a useful tool when solving problems in mechanics: we can relate the velocity and position of an object without having to find the position as a function of time. This is particularly useful when the interactions are complicated, and we have limited knowledge about the forces between objects. Actually, it allows us to reason about the behavior of a system without having force models, as long as we know that the forces are conservative.

Conservation of mechanical energy: The conservation of mechanical energy is an example of a conservation law, which we found by integrating Newton's second law along the path:

$$\int_{t_0}^{t_1} \mathbf{F}^{\rm net} \cdot \mathbf{v} dt = \Delta K \; . \tag{12.1}$$

For a one-dimensional motion where the net force only depends on the position, x, we get:

$$\int_{x_0}^{x_1} F_x^{\rm net} dx = \Delta K = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 \; . \tag{12.2}$$

If we could calculate the integral on the left, we could find the velocity as a function of position without finding the complete motion.

A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_12

Conservation of momentuum: This techinque is sometimes called an integration method. To get energy conservation we integrated Newton's second law over space. But we can also integrate Newton's second law over time:

$$\int_{t_0}^{t_1} \mathbf{F}^{\text{net}} dt = \int_{t_0}^{t_1} m\mathbf{a} dt = m\mathbf{v}_1 - m\mathbf{v}_0 \ . \tag{12.3}$$

If the integral on the left is zero, that is, if the net force is zero, then we find that $m\mathbf{v}$ does not change. This gives us another conservation law: Conservation of momentum, $m\mathbf{v}$. But how can that be useful? Didn't we already know from Newton's first law that if the net external force is zero, the velocity does not change? It turns out that conservation of momentum is not that useful for a single object, but it is very useful for systems consisting of *several objects*. For systems of several objects, we will demonstrate that the total momentum is conserved if there are no external forces acting on the system. It does not matter what internal forces are acting, the total momentum is conserved at all times as long as there are no external forces.

Collisions: Conservation of momentum is particularly useful for collisions. During a collision between two objects, the interactions between the objects can be very complicated, and may consist of both conservative and non-conservative forces, but as long as the objects are not affected by any external forces, their total momentum is conserved. We use this to find the velocities of each object after a collision from the velocities before a collision, without finding the motion of each object. Conservation of momentum is more general than the conservation of energy, since it is valid for any internal force, and not only for conservative forces.

Overview: In order to introduce these concepts, we will start by introducing tranlational momentum, $\mathbf{p} = m\mathbf{v}$. We reformulate Newton's second law using momentum, and find that the integral of the net forces acting on an object corresponds to the change of momentum. We will then use these concepts to address systems with several objects, with a particular focus on collisions.

12.1 Motivating Example—Meteor Impact

You are now an expert solver of mechanics problems: Given a set of force models, you can find the motion of an object from Newton's second law of motion using analytical or numerical tools. However, in some cases we may not know (or care to model) the detailed forces acting between two objects, but you still would like to know what happens to the objects. For example, you may observe a large meteor head directly towards a small planet. You know the masses and velocities of both

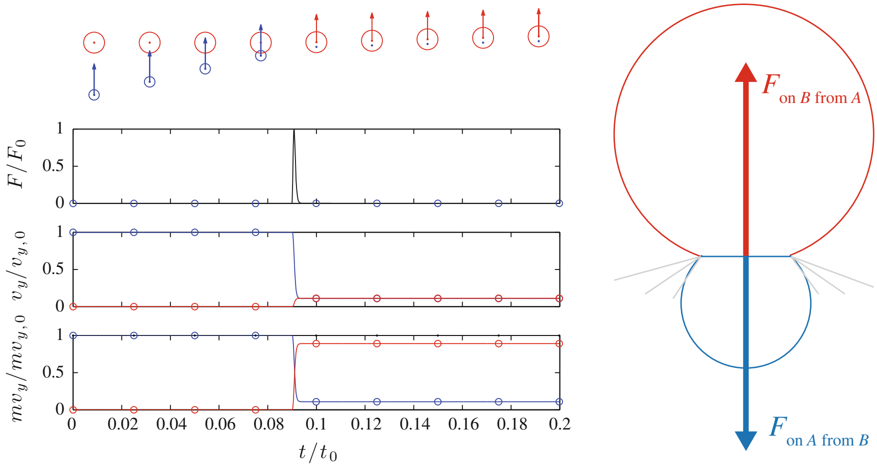

Fig. 12.1 Illustration a collision between a meteor and a planet

objects at some time before the collision, and want to know the velocity of the planet after the hit. Can you do that without having a detailed model for the collision?[1]

Identify: The meteor impact is illustrated in Fig. 12.1. We describe the meteor (object A) with its position $\mathbf{r}_A(t)$ and its mass m_A, and the motion of the planet (object B) with its position $\mathbf{r}_B(t)$ and its mass m_B. At some time t_0 before the impact you know the velocities of both objects, $\mathbf{v}_A(t_0) = \mathbf{v}_{A,0}$ and $\mathbf{v}_B(t_0) = \mathbf{v}_{B,0}$.

Model: You know you can find the motion of both objects by applying Newton's second law to get an equation of motion. If we assume that the only interactions are between the planet and the meteor, Newton's second law for objects A and B are:

$$\begin{aligned} \mathbf{F}_A^{\text{net}} &= \mathbf{F}_{\text{from } B \text{ on } A} = m_A \mathbf{a}_A \\ \mathbf{F}_B^{\text{net}} &= \mathbf{F}_{\text{from } A \text{ on } B} = m_B \mathbf{a}_B \end{aligned} \; . \tag{12.4}$$

Unfortunately, we do not have a simple force model for the force from the meteor on the planet during the collision. How can we then solve the problem?

Using Newton's third law: Fortunately, we can use a common trick: We know that the force from the meteor on the planet is the reaction force to the force from the planet on the meteor: Newton's third law tells us that:

$$\mathbf{F}_{\text{from } A \text{ on } B} = -\mathbf{F}_{\text{from } B \text{ on } A} = \mathbf{F} \; . \tag{12.5}$$

[1] A model for such a collision would be very complicated, as there are many different processes involved.

We can therefore rewrite Newton's second law in (12.4) to be:

$$m_A \mathbf{a}_A = -\mathbf{F}$$
$$m_B \mathbf{a}_B = \mathbf{F} \tag{12.6}$$

which is valid as long as there are no other forces acting on the meteor or the planet. Now, we see that we can get rid of the unknown force, **F**, altogether by adding the two equations, getting:

$$m_A \mathbf{a}_A + m_B \mathbf{a}_B = -\mathbf{F} + \mathbf{F} = 0 \; . \tag{12.7}$$

Integration method: Now, we can integrate this equation, from the initial time t_0, where we know the velocities, to the

$$\int_{t_0}^{t_1} m_A \mathbf{a}_A + m_B \mathbf{a}_B dt = 0 \; , \tag{12.8}$$

$$m_A \int_{t_0}^{t_1} \mathbf{a}_A dt + m_B \int_{t_0}^{t_1} \mathbf{a}_B dt = 0 \; , \tag{12.9}$$

$$m_A \left(\mathbf{v}_A(t_1) - \mathbf{v}_A(t_0)\right) + m_B \left(\mathbf{v}_B(t_1) - \mathbf{v}_B(t_0)\right) = 0 \; . \tag{12.10}$$

Let us now group the quantities relating to t_1 on the left side, and the quantities relating to t_0 on the right side:

$$m_A \mathbf{v}_A(t_1) + m_B \mathbf{v}_B(t_1) = m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \; . \tag{12.11}$$

This looks like what we have previously found for energy: It is a conservation law. But for what? For the quantity:

$$\mathbf{P} = m_A \mathbf{v}_A + m_B \mathbf{v}_B \; , \tag{12.12}$$

which we call the *total momentum* of the system consisting of the planet and the meteor.

Solve: How can we use (12.11) to find the velocity of the planet and the meteor after the collision? First, we notice that (12.11) actually is valid for all times, also at any time during the collision. But it is not sufficient to find the velocities of each object after the collision. We only know what the awkward sum, $m_A \mathbf{v}_A + m_B \mathbf{v}_B$ is after the collision. We do not know how this sum is distributed between the two objects. But since this is meteor impact we know something else, we know that the two objects move as one object after the collision: The meteor and the planet have the same velocity afterwards:

$$\mathbf{v}_A(t_1) = \mathbf{v}_B(t_1) = \mathbf{v}_1 \; . \tag{12.13}$$

Now, we can determine the velocity after the collision. Starting from (12.11) we have:

$$
\begin{aligned}
m_A \mathbf{v}_A(t_1) + m_B \mathbf{v}_B(t_1) &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
m_A \mathbf{v}_1 + m_B \mathbf{v}_1 &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
(m_A + m_B)\, \mathbf{v}_1 &= m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0) \\
\mathbf{v}_1 &= \frac{m_A \mathbf{v}_A(t_0) + m_B \mathbf{v}_B(t_0)}{m_A + m_B}
\end{aligned}
\tag{12.14}
$$

We have found the velocity of the planet and the meteor after the collision, without solving the equations of motion!

Discussion: The method we applied here is very similar to the energy conservation method, but it is based on a different conservation law. But notice that we could not have found the velocity after the collision, if we did not know that the objects were moving with the same velocity after the collision. Since the equation we have used here, conservation of momentum, only provides one equation, we cannot use it to find two velocities. We need more equations. We therefore need to know something more about the collision. For example, we may use that the energy is conserved during the collision, or that we know how much energy is lost during the collision.

In the remaining of this chapter we will more thoroughly introduce the concepts briefly mentioned here, and apply the concepts systematically to study two-particle and multi-particle collisions.

12.2 Translational Momentum

The **translational momentum** of an object is defined as:

$$
\mathbf{p} = m\mathbf{v} \ , \tag{12.15}
$$

The translational momentum is a property of the object that depends on both the objects inertial mass, m, and the objects velocity, $\mathbf{v}$. The *translational momentum* is often also called the *linear momentum*. We prefer the term translational momentum to discern it from rotational momentum, which we encounter when discussing rotational motion. In the following we use the short term momentum instead of translational momentum.

The translational momentum of an object is a *vector*, which is in contrast to energy, which is a *scalar*. Translational momentum is a general property that may be extended also to particles without mass. For example, photons have a translational momentum even though they have no mass.

Newton's Second Law

We have previously introduced Newton's second law of motion as:

$$\sum_j \mathbf{F}_j^{\text{ext}} = m\mathbf{a} \ . \tag{12.16}$$

However, the most fundamental, and the original, form of Newton's second law is:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{p} \ , \tag{12.17}$$

The net force acting on an object causes a change in the momentum of the object.[2]

For an object with constant mass, this formulation reduces to the original formulation:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{p} = \frac{d}{dt}(m\mathbf{v}) = \underbrace{\frac{dm}{dt}}_{=0}\mathbf{v} + m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \ . \tag{12.18}$$

This law is a fundamental principle in physics, on the same level as the energy-principle. It is the general formulation of Newton's second law, and we use the term Newton's second law for this law as well as the special case when the mass is constant.

12.3 Impulse and Change in Momentum

What is causing a change in the momentum of an object? Let us study an object that is affected by a force during a short time interval, such as a tennis ball during a serve. While the ball is in contact with the racket, the contact force $\mathbf{F}(t)$ on the ball varies as illustrated in Fig. 12.2. When the racket makes contact with the ball, the contact force is small, but it grows rapidly as the racket deforms against the ball. As the ball speeds up, the force decreases while the racket returns to its original shape, until the force reaches zero as the ball leaves the racket.

During the collision, other forces such as gravitational forces are negligible compared with the contact force from the racket. We can therefore assume that the contact force $\mathbf{F}(t)$ is approximately equal to the net force on the ball.

The change of momentum of the ball from the time t_0 before contact with the racket to the time t_1 after the ball has left the racket is

$$\Delta\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1 = \int_{t_0}^{t_1} \frac{d\mathbf{p}}{dt}\, dt = \int_{t_0}^{t_1} \mathbf{F}(t)\, dt \ . \tag{12.19}$$

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[2] This formulation of Newton's second law can also be exteded to relativistic mechanics.

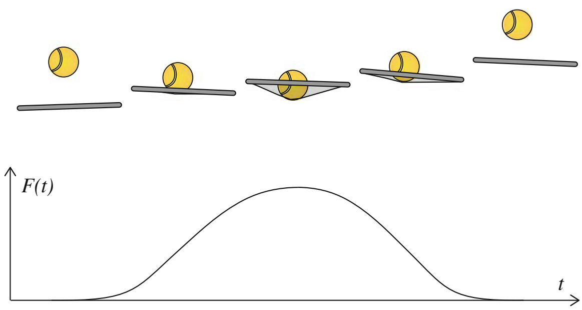

Fig. 12.2 Illustration a ball being hit by a tennis racket, showing an illustration of the collision as a function of time, and a plot of the force $F(t)$ from the racket on the ball as a function of time

The left side is the change in momentum of the ball. The right side includes both the strength of the interaction—the force **F**—and the duration of the interaction. We call this quantity the **impulse**, **J**, experienced by the object during the collision:

Impulse:

$$\mathbf{J} = \int_{t_0}^{t_1} \mathbf{F}^{\text{net}}(t)\, dt \ . \tag{12.20}$$

If the direction of the net force $\mathbf{F}(t)$ does not change during the collision, the impulse is directed in the same direction as $\mathbf{F}(t)$. In this case, the impulse, J, is the area under the curve, $F(t)$, in Fig. 12.2.

Time-Averaged Force

Unfortunately, we generally do not know the time dependency of the net force, since this requires a detailed force model for the collision, or a detailed measurement of the forces acting. Instead, we can use our knowledge of the change in momentum to determine the *average force* acting on the ball during the collision. The time-average force is defined as:

$$\mathbf{F}^{\text{net}}_{\text{avg}} = \langle \mathbf{F}^{\text{ext}}(t) \rangle = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{F}(t)\, dt \ . \tag{12.21}$$

where $\Delta t = t_1 - t_0$ is the duration of the collision. We define the start of the collision as the time t_0 when the ball comes in contact with the racket (when the contact force becomes non-zero), and the end of the collision as the time t_1 when the ball loses contact with the racket (when the contact force becomes zero). We recognize the

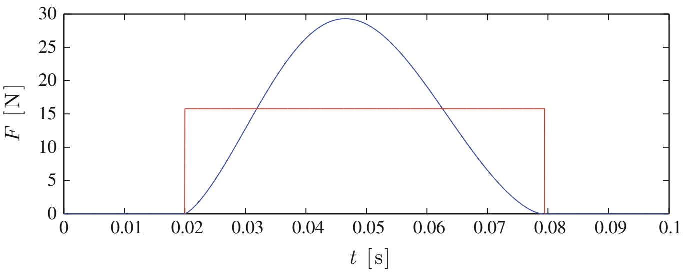

Fig. 12.3 A plot of the force, $F(t)$, as a function of time during a collision, and the average force F_{avg}. In most cases, the average force is a good estimate for the typical force and the maximum force during the collisions

integral on the right-hand side as the impulse of the net force, which is equal to the change in momentum:

$$\mathbf{F}_{avg}^{net} = \frac{1}{\Delta t} \int_{t_0}^{t_1} \mathbf{F}(t)\, dt = \frac{1}{\Delta t} \mathbf{J} = \frac{1}{\Delta t} \Delta \mathbf{p} \; . \tag{12.22}$$

Momentum Change During a Collision

The momentum change during a collision gives useful insight into the collision: While the net force may vary throughout the collision, and the maximum force may be much larger than the average force, the average force is still a reasonable estimate for the force acting on the object. For example, if we want to estimate the damage done to an object during a collision, the average force is a good estimate also of the maximum force, because for most physical interactions (for most force models), the force does not display a very narrow peak, but instead varies gradually over a wider time interval, and hence the maximum force is often just a few times the maximum force. (See Fig. 12.3 for an illustration).

Test your understanding: You jump down from a window 5 meters above the ground. How should you land in order to minimize the force on you from the ground? What determines the change in momentum during the impact? Demonstrate that you can answer this question using both energy and momentum considerations.

12.3.1 Example: Ball Colliding with Wall

Problem: A ball falls vertically and collides with a horizontal floor. The force from the floor on the ball during the contact is:

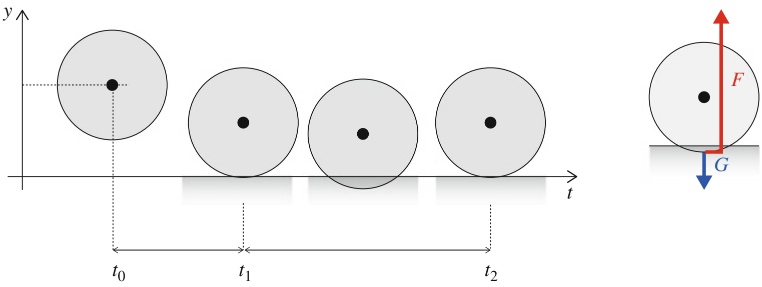

Fig. 12.4 Free-body diagram for a ball colliding with a floor

$$N = \begin{cases} k\,(R-y)^{3/2} & y < R \\ 0 & y \geq R \end{cases} \tag{12.23}$$

You can neglect air drag. Find the motion of the ball and visualize the change in momentum during the collision.

Model: We describe the motion of the ball by its vertical position, $y(t)$. The ball is affected by two forces, the contact force N and gravity, G, as illustrated in Fig. 12.4, where we have neglected air drag.

We find the acceleration of the ball from Newton's second law. The forces acting on the ball are the contact force from the surface and gravity:

$$ma = F^{\text{net}} = N - mg \;\Rightarrow\; a = \frac{1}{m}N - g\;. \tag{12.24}$$

Solve: The ball starts at $y(t_0)y_0$ with the velocity $v(t_0) = v_0$. While we may be able to find the motion $y(t)$ analytically, a numerical approach based on Euler-Cromer's method is sufficient. This is implemented in the following program:

```
from pylab import *
g = 9.8 # m/s^2
R= 0.02 # m
m = 0.1 # kg
y0 = 0.021 # m
v0 = -2.8 # m/s
k = 1000000.0 #
time = 0.005
dt = 0.00001
n = int(round(time/dt));
t = zeros(,n,1);
y = zeros(n,1);
v = zeros(n,1);
Fnet = zeros(n,1);
y[0] = y0
v[0] = v0
for i in range(n):
    dy = R-y[i]
    if (dy<=0.0):
        N = 0.0
    else:
        N = k*dy**1.5
```

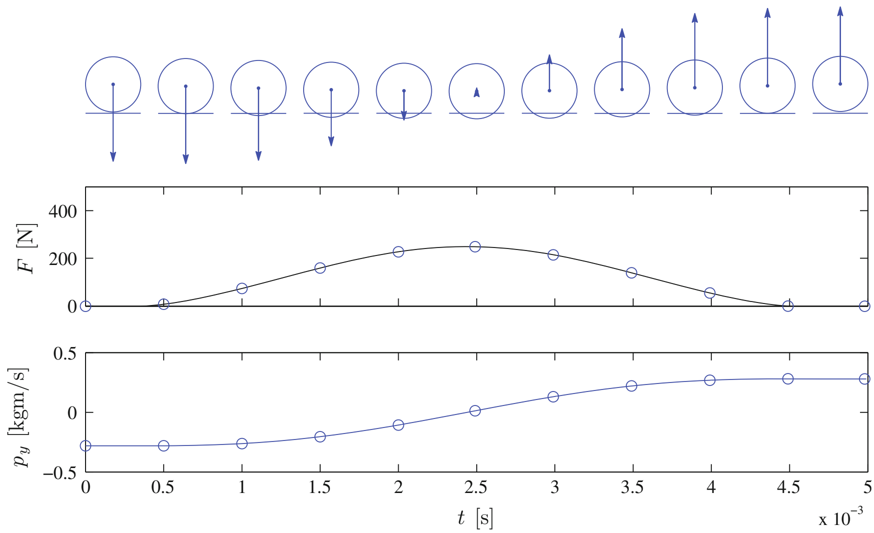

Fig. 12.5 Illustration of a simulation of a ball bouncing on the floor

```
    Fnet[i] = N - m*g
    a = Fnet[i]/m
    v[i+1] = v[i] + a*dt
    y[i+1] = y[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
subplot(2,1,1), plot(t,Fnet)
xlabel('t [s]'),ylabel('F [N]')
p = m*v
subplot(2,1,2), plot(t,p)
xlabel('t [s]'), ylabel('P [kgm/s]')
```

Figure 12.5 illustrates a simulation with this model. Here, you can see the time development of the net force and the momentum of the ball throughout the collision.

Change in momentum: What is the change in momentum of the ball? Since the force only depends on the position of the ball, the force is conservative and the mechanical energy of the ball is conserved throughout the collision. Hence, the kinetic energy of the ball is the same when the ball comes in contact with the surface and when it loses contact with the surface, since the vertical position is the same:

$$E_0 = U(y_0) + \frac{1}{2}mv_0^2 = E_1 = U(y_1) + \frac{1}{2}mv_1^2 \, , \tag{12.25}$$

where $y_0 = y_1$ and therefore

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 \, , \tag{12.26}$$

which gives $v_1 = -v_0$, since we know that the velocity after the collision is in the opposite direction of the velocity before the collision.

The change in momentum is therefore

$$\Delta p = mv_1 - mv_0 = m(-v_0) - mv_0 = -2mv_0 = -2p_0 \ . \tag{12.27}$$

If we know the duration, Δt, of the collision, we can find the average net force on the ball during the collision from

$$F_{avg}^{net} = \frac{-2mv_0}{\Delta t} \ . \tag{12.28}$$

12.3.2 Example: Hitting a Tennis Ball

Problem: A tennis ball of mass 57 g is approaching you with a horizontal velocity $v_0 = 20$ m/s. You hit the ball, returning it with a horizontal velocity $v_1 = 20$ m/s, now in the opposite direction. (a) What is the impulse **J** on the ball while it is in contact with the racket during the collision? (b) The ball and racket are in contact for 2.0 ms. What is the average net force on the racket during the collision? (c) You want to return the ball as a high lob and give the ball a velocity $v_1 = 15$ m/s at angle of 45° upward. What is now the impulse on the ball and the net force from the racket on the ball?

Approach: We may solve this problem by determining the motion of the ball from Newton's laws of motion, but this would require a detailed force model for the force from the tennis racket on the ball. In this case, we do not have such a model. Instead, we want to use the measured change in velocity to determine the average force on the ball.

Identify: In this problem we address the motion of the tennis ball, described by the position $\mathbf{r}(t)$. The ball starts with the velocity $\mathbf{v}_0 = -v_0\,\mathbf{i}$ at the time t_0 (before the collision), and gets the velocity $\mathbf{v}_1$ after the collision.

Model: The ball is affected by a force, $\mathbf{F}(t)$, from the racket on the ball, and by gravity. However, we assume that gravity is small compared with the typical force from the racket, and ignore the effects of gravity.

Solve: The impulse is defined as the integral of the net force on the ball, and it is equal to the change in momentum of the ball:

$$\mathbf{J} = \Delta\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_0 = m\mathbf{v}_1 - m\mathbf{v}_0 \ , \tag{12.29}$$

Solution part a: In part (a) of the problem, the final velocity is $\mathbf{v}_1 = v_1\,\mathbf{i}$. The key idea is that momentum is a vector quantity. The ball therefore experiences a change in momentum, even though the magnitude of the momentum does not change, because the direction of the momentum changes.

The impulse on the ball in the collision is:

$$\mathbf{J} = m\,(v_1\,\mathbf{i} - (-v_0\,\mathbf{i})) = m(v_1 + v_0)\,\mathbf{i} \tag{12.30}$$
$$= 0.057\,\text{kg}\ (20.0\,\text{m/s} + 20.0\,\text{m/s})\ \mathbf{i} = 2.28\,\text{kg m/s}\,\mathbf{i}\ . \tag{12.31}$$

The impulse is positive, since the force acting on the ball is in the positive x-direction—this is also the direction of the acceleration of the ball.

Solution of part b: In part (b) of the problem, we find the average force from the change in momentum during the collision:

$$\mathbf{F}_{\text{avg}} = \frac{1}{\Delta t}\int_{t_0}^{t_1} \mathbf{F}dt = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{2.28\,\text{kg m/s}}{2\,10^{-3}\,\text{s}} = 1140\text{N}\ , \tag{12.32}$$

This is the average net force on the ball. We recall from Fig. 12.3 that the net force is smaller than the maximum force, although they are typically of comparable magnitude.

Dicussion: What about gravity? We neglected gravity because we assumed it to be small compared with the force from the racket. We could check this assumption in two ways. First, we could check that the impulse of gravity is much smaller than the total impulse on the ball—we can do this without calculating the average force. The magnitude of gravity is:

$$W = mg = 0.057\,\text{kg}\,9.8\,\text{m/s}^2 = 0.56\,\text{N}\ . \tag{12.33}$$

The impulse of gravity is therefore:

$$J_g = mg\Delta t = 0.56\,\text{M}\,2\,10^{-3}\,\text{s} = 1.1\,10^{-2}\,\text{kg m/s}\ , \tag{12.34}$$

which is much smaller than the impulse of the net force.

From this calculation we also found the force from gravity, which is much smaller than the average net force on the ball. We were therefore right in neglecting the effect of gravity.

Solution to part c: Finally, we address question (c), where the collision is not head on, and the ball leaves the racket at an angle α, as illustrated in Fig. 12.6. In this case, we need to treat the collision as two-dimensional. First, we introduce the velocity of the ball after the collision as:

$$\mathbf{v}_1 = v_1\,(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})\ , \tag{12.35}$$

where $v_1 = 15\,\text{m/s}$ and $\alpha = 45° = \pi/4$. The impulse on the ball is still given as the change in momentum:

$$\mathbf{J} = \mathbf{p}_1 - \mathbf{p}_0 = m\,(\mathbf{v}_1 - \mathbf{v}_0)\ . \tag{12.36}$$

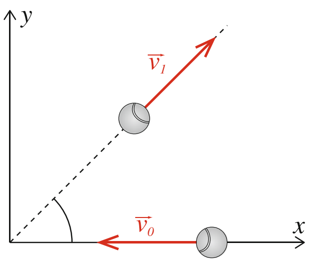

Fig. 12.6 Illustration of the velocity of a tennis ball before and after it was hit by a racket

We find the x- and y-components of the impulse:

$$J_x = \mathbf{J} \cdot \mathbf{i} = m \left(v_{x,1} - v_{x,0} \right) = 57\,\text{g}\ (15 \cos\alpha + 20)\ \text{m/s} = 1.74\,\text{kg m/s} \quad (12.37)$$

$$J_y = \mathbf{J} \cdot \mathbf{j} = m \left(v_{x,1} - v_{x,0} \right) = 57\,\text{g}\ (15 \sin\alpha - 0)\ \text{m/s} = 0.60\,\text{kg m/s}\ . \quad (12.38)$$

The impulse is therefore:

$$\mathbf{J} = 1.74\,\text{kg m/s}\,\mathbf{i} + 0.60\,\text{kg m/s}\,\mathbf{j}\ , \quad (12.39)$$

The force is given as the impulse divided by the time interval. We assume the time interval to be the same for this process, $\Delta t = 2$ ms. The average net force is therefore:

$$\mathbf{F}_{\text{avg}} = \frac{\mathbf{J}}{\Delta t} = \frac{1.74\,\text{kg m/s}\,\mathbf{i} + 0.60\,\text{kg m/s}\,\mathbf{j}}{2\ 10^{-3}\ \text{s}} = 870\,\text{N}\,\mathbf{i} + 300\,\text{N}\,\mathbf{j}\ , \quad (12.40)$$

Interestingly, this means that the direction of the net force is in the direction β:

$$\beta = \tan^{-1} \frac{F_y}{F_x} = 19^\circ\ . \quad (12.41)$$

12.4 Isolated Systems and Conservation of Momentum

During a collision between two objects the forces acting between the objects generally have a complicated time dependence—the curve of $F(t)$ is non-trivial. It is therefore not a simple task to calculate the impulse integral and use this to determine the change in momentum of the objects. Fortunately, it turns out that the problem can be significantly simplified for an isolated system where the net external force is zero. In this case the total momentum of the system is conserved throughout the collision. The total momentum is therefore the same before and after the collision. This is a powerful principle we use to analyze complex interactions without determining the detailed motion and forces in the system.

Momentum and Motion of Two Objects

We will now demonstrate that the total momentum is conserved when there are no external forces by discussing the collision between two objects A and B, illustrated in Fig. 12.7. We know that we can determine the motion of each object from Newton's second law of motion applied to each of the objects. For each object we separate the forces into *external forces*, forces having an origin outside the system, and *internal forces*, forces that are acting between the two objects:

> **Internal forces** act between the objects in the system.
> **External forces** act between objects in the system and the environment.

For the collision in Fig. 12.7 the only internal forces are the forces between the objects: The force from A on B and the reaction force from B on A. With this notation, Newton's second law for object A can be written as:

$$\sum \mathbf{F}_A = \sum \mathbf{F}_A^{\text{ext}} + \mathbf{F}_{B \text{ on } A} = \frac{d\mathbf{p}_A}{dt} , \tag{12.42}$$

where the sum is over all the external forces acting on object A. Similarly, Newton's second law for object B is:

$$\sum \mathbf{F}_B = \sum \mathbf{F}_B^{\text{ext}} + \mathbf{F}_{A \text{ on } B} = \frac{d\mathbf{p}_B}{dt} , \tag{12.43}$$

where we again have summed over all the external forces acting on object B. Now, we do not want to address the internal forces acting between A and B. How can we

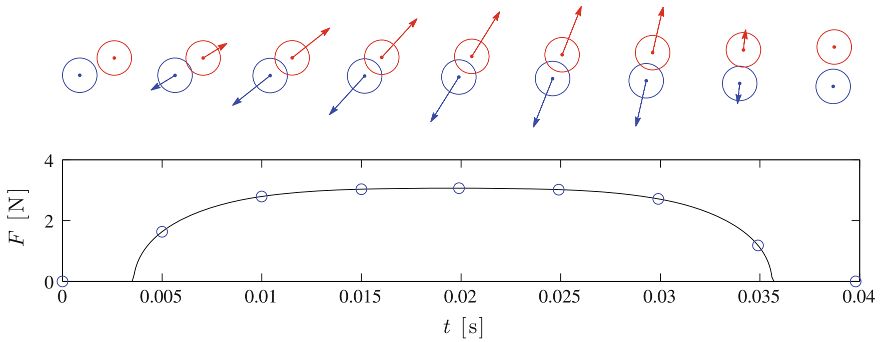

Fig. 12.7 Illustration of a collision between two objects A (*red*) and B (*blue*). The *top* figure shows the forces $\mathbf{F}_{B \text{ on } A}$ (*red*) and $\mathbf{F}_{A \text{ on } B}$ (*blue*) acting between the objects at various times t_i throughout the collision. The *bottom* figure shows the magnitude of the force, $F(t)$, as a function of time. The time t_i that are shown in the *top* figure is illustrated by circles

get rid of them in these equations? There is a commonly used trick: we recall from Newton's thirds law that the reaction force $\mathbf{F}_{A\text{ on }B}$ is equal, but oppositely directed to $\mathbf{F}_{B\text{ on }A}$:

$$\mathbf{F}_{A\text{ on }B} = -\mathbf{F}_{B\text{ on }A} \ , \tag{12.44}$$

If we insert this into (12.43), we get two equations for the motion of object A and B:

$$\begin{aligned} \sum \mathbf{F}_A^{\text{ext}} + \mathbf{F}_{B\text{ on }A} &= \frac{d\mathbf{p}_A}{dt} \\ \sum \mathbf{F}_B^{\text{ext}} - \mathbf{F}_{B\text{ on }A} &= \frac{d\mathbf{p}_B}{dt} \end{aligned} \tag{12.45}$$

If we add the equations, we get rid of the internal forces, $\mathbf{F}_{B\text{ on }A}$:

$$\sum \mathbf{F}_A^{\text{ext}} + \sum \mathbf{F}_B^{\text{ext}} = \frac{d\mathbf{p}_A}{dt} + \frac{d\mathbf{p}_B}{dt} \ , \tag{12.46}$$

We introduce the sum over all the external forces on all the objects in the system: Over all the forces acting on object A *and* all the forces acting on object B:

$$\sum \mathbf{F}^{\text{ext}} = \sum \mathbf{F}_A^{\text{ext}} + \sum \mathbf{F}_B^{\text{ext}} \ . \tag{12.47}$$

We use this to simplify (12.46):

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt}\left(\mathbf{p}_A + \mathbf{p}_B\right) \ . \tag{12.48}$$

We call the sum of the momenta for each of the objects the *total momentum* of the system:

Total momentum:

$$\mathbf{P} = \sum \mathbf{p} = \mathbf{p}_A + \mathbf{p}_B \ , \tag{12.49}$$

This provides a generalization of Newton's second law for a two-particle-system:

Generalization of Newton's second law:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt}\sum \mathbf{p} = \frac{d}{dt}\left(\mathbf{p}_A + \mathbf{p}_B\right) \ , \tag{12.50}$$

This law is completely general. We have not made any assumptions about the interactions between the two objects. The internal and external forces may be of any kind, conservative or non-conservative. The law is valid in all cases.

Conservation of Momentum in Isolated Systems

As a special case of this law, we observe that if the net external force on the system is zero (or negligible), the total momentum of the system is conserved:

$$\sum \mathbf{F}^{\text{ext}} = 0 \;\Rightarrow\; \frac{d}{dt}\left(\mathbf{p}_A + \mathbf{p}_B\right) = 0 \; . \tag{12.51}$$

We call a system **isolated** if the net external force is zero (or negligible):

An **isolated system** is a collection of objects that may interact internally, but where the net external force on all the objects is zero (or negligible).

For an isolated system, the total momentum is conserved:

$$\mathbf{p}_A + \mathbf{p}_B = \text{constant (for an isolated system)} \; . \tag{12.52}$$

- This is a completely general law for the conservation of momentum of a system. It only requires the *net external* force on the system to be zero. It is valid not only at the beginning and at the end of the collision, but at all times during the collision as well.
- Notice: A common mistake is to forget the absolutely necessary condition that the net external force on the system must be zero (or negligible). Whenever you employ this law, you should make a habit of always asking yourself if the net external force is zero, or if it is reasonable to neglect it compared with other forces.
- Notice that the conservation law is a vector equation, and that it can be valid in one direction independently of an orthogonal direction. If the net external force in the x-direction is zero, the total momentum in this direction is conserved even though there is a net external force in the y-direction.
- Notice that it is not only valid for contact forces, as illustrated in Fig. 12.7, but for any type of force, including long-reaching forces such as gravity. The gravitational forces between object in the system are internal forces, while gravitational forces between objects in the system and objects outside the system are external forces.

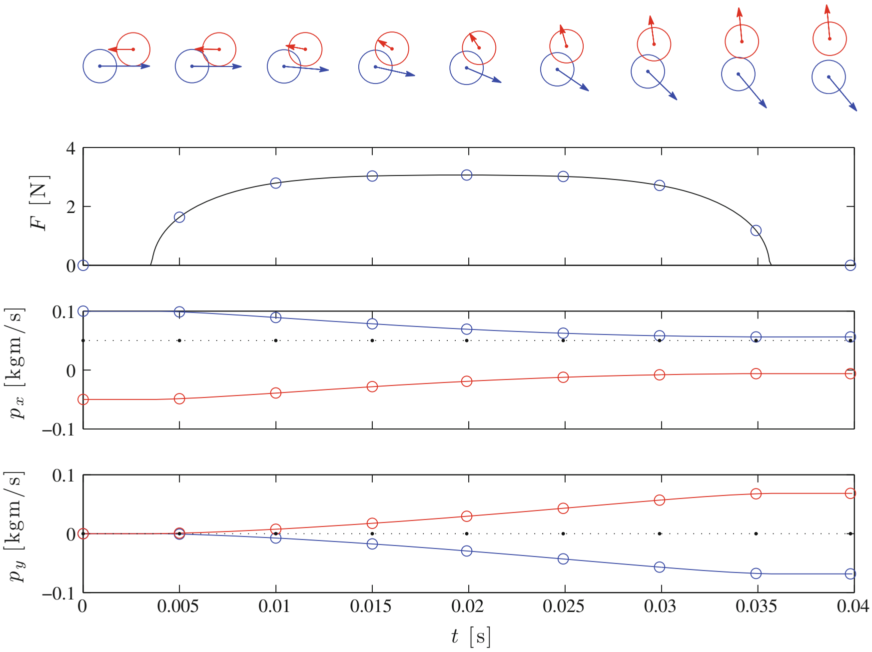

Fig. 12.8 Illustration of the same process as in Fig. 12.7, but now the *arrows* in the *top* figure illustrates the momentum of the object, and the two *bottom* figures show the momentum $p_x(t)$ and $p_y(t)$ as a function of time. The total momentum is shown with a *dotted line*

Conservation of Total Momentum During a Collision

The conservation of total momentum during the collision between objects A and B is illustrated in Fig. 12.8. The arrows indicate the momentum of each object and the plots show the momentum in the x- and y-direction for each object and the total momentum. Since there are only internal forces acting in this system, there are no external forces affecting either object, and the total momentum is conserved in both the x- and the y-direction.

Conservation of Total Momentum During a Collision with an External Force

What happens if we addressed the same collision, but also include a gravitational force in the y-direction for both objects? As illustrated in Fig. 12.9, the behavior looks similar: The force as a function of time, $F(t)$, is similar, and the total momentum in the x-direction is conserved. However, the momentum in the y-direction is not conserved. It is decreasing throughout the motion due to the external force, gravity, acting on both object A and object B.

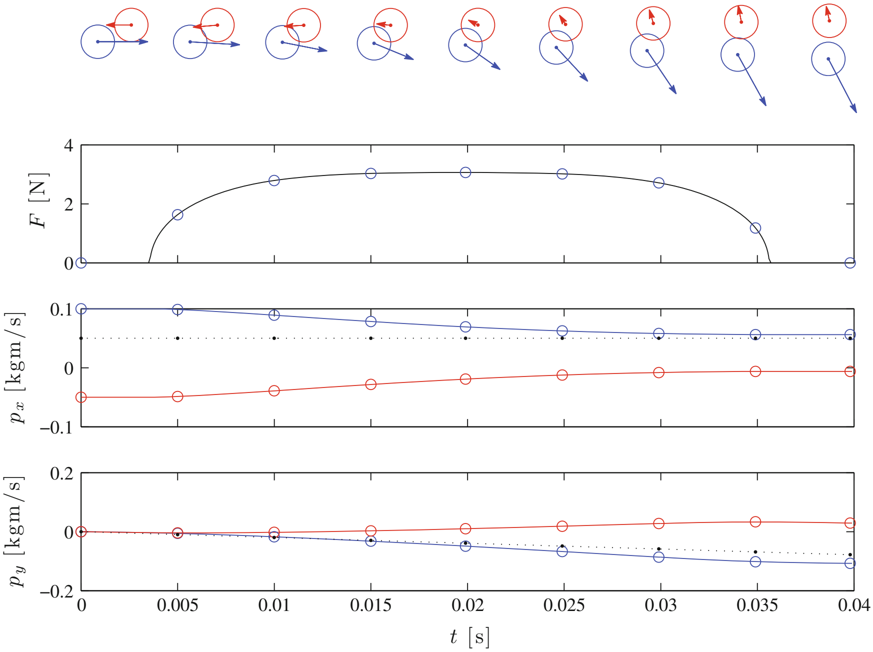

Fig. 12.9 Illustration of the same objects as in Figs. 12.7 and 12.8., but now with gravity acting in the y-direction. The *arrows* in the *top* figure illustrates the momentum of the objects. The *bottom* figures show the momentum $p_x(t)$ and $p_y(t)$ as a function of time. The total momentum (*dotted line*) is constant in the x-direction, but is changing in the y-direction due to the net force from gravity affecting the objects

How large is the effect of an external force, like gravity in this case? The change in momentum is given by the impulse of the external forces:

$$\Delta \mathbf{P} = \mathbf{J}^{\text{ext}} = \int_{t_0}^{t_1} \mathbf{F}^{\text{ext}} dt \ , \tag{12.53}$$

To determine the effect of the external forces, we therefore need to compare the change of momentum due to the external force, $\Delta \mathbf{P}$, to the total momentum of the system, $\mathbf{P}$. For the gravitational force $\mathbf{G} = -mg$æ, the impulse of gravity is:

$$J_{G,y} = \int_{t_0}^{t_1} (-m_A g - m_B g)\, dt = -\left(m_A + m_B\right) \Delta t \ , \tag{12.54}$$

and the total momentum in the y-direction of the whole system,

$$P_y = p_{A,y} + p_{B,y} = m_A v_{A,0,y} + m_B v_{B,0,y} \ . \tag{12.55}$$

If $J_G \ll P_y$, the impulse of the external forces are negligible during the collision, and we can assume that momentum is approximately conserved. Notice that the impulse depends on the duration of the collision, Δt. If the collision takes a short time, the momentum change due to the external forces will be small during the collision, and the momentum will be approximately conserved from the beginning to the end of the collision. This is why we often assume that collisions are instantaneous. Then we may assume that the impulse due to external forces is small compared with the total momentum, and therefore that the total momentum is approximately conserved in the collision.

Conservation of Momentum for Multi-particle Systems

The generalization of Newton's law can be extended to any number of objects. For a system with N objects, the total momentum is:

$$\mathbf{P} = \sum_{j=1}^{N} \mathbf{p}_j \ , \tag{12.56}$$

and the generalization of Newton's second law is:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d}{dt}\mathbf{P} = \frac{d}{dt}\sum_{j=1}^{N} \mathbf{p}_j \ . \tag{12.57}$$

The derivation follows the same principles used for two particle system: Also for the N-particle system Newton's third law ensures that all internal forces come in pairs that cancel each other, so that the sum of all the internal forces is zero.

12.5 Collisions

The conservation of momentum allows us to determine the velocities of objects after a collision from the velocities before a collision without knowing the details of the interactions during the collision: We can get by without solving the complete equations of motion, even without knowing the details of the interactions, for all the objects. Let us therefore apply the law we have introduced, the conservation of momentum for isolated systems, to address collisions, first in one dimension and then in two and three dimensions.

What is a collision? For our purposes, a collision between two objects is a process where large forces are acting between the two objects over a short time interval:

A **collision** between two or more objects is a process

- where the internal forces between the objects are much larger than the external forces from the environment
- that occur over a short time interval compared to the time scale of the motion.

Collisions Along a Line

We start from the simplest case: a one-dimensional collision between two blocks moving along a line. We want to find the velocities of the objects after the collision. Blocks A and B with masses m_A and m_B slide along a horizontal, frictionless straight track. Before the collision, the x-component of the velocities of the blocks are $v_{A,0}$ and $v_{B,0}$. The blocks collide during a short time interval Δt. What are their velocities after the collision? The process is illustrated in Fig. 12.10.

We assume that the net external force in the x-direction is zero. Hence, the momentum in the x-direction is conserved, and the total momentum P is the same before and after the collision:

$$\begin{aligned} P_0 &= P_1 \\ p_{A,0} + p_{B,0} &= p_{A,1} + p_{B,1} \\ m_A v_{A,0} + m_B v_{B,0} &= m_A v_{A,1} + m_B v_{B,1} \ . \end{aligned} \tag{12.58}$$

This equation holds at any time during the collision. Unfortunately, we cannot find the velocities after the collision from this equations alone, because there are two unknown velocities, $v_{A,1}$, and $v_{B,1}$, but only one equation. We need more information about the process—we need an additional equation!

We need to know more about the collision process to find another relation between the initial and the final velocities. For example, if we know that all the forces acting between the objects are conservative, we would get an additional equation from the conservation of mechanical energies. If the objects are not interacting, which is the case before and after, but not during the collision, the total energy is equal to the kinetic energies plus a constant potential energy (because the potential energies are constant when the objects are not interacting):

$$\frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B v_{B,0}^2 + U_0 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 + U_1 \ , \tag{12.59}$$

where $U_0 = U_1$. In this case, we would have two equations and two unknowns, and we can find one (or a few) unique solutions, which gives us the values of the velocities after the collision.

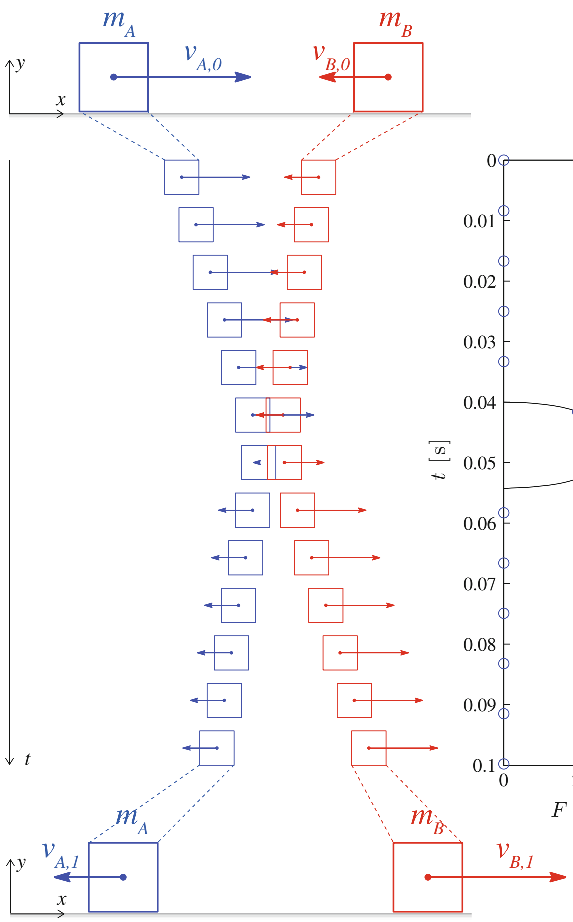

Fig. 12.10 Illustration of a collision between two blocks A and B showing the situations before and after the collision. Snapshots of the time development are shown in the cartoons. The force, F, acting between the blocks and the momentum of block A, block B, and the total momentum (*dotted line*) are all plotted as functions of time

Conservation of mechanical energy is one possibility. Many things can happen between the objects during a collision, which may give rise to other relations and other equations relating the initial and the final velocities. It is customary to describe collisions by their degree of energy conservation, ranging from *elastic* collisions, where mechanical energy is conserved, through various *inelastic* collisions where mechanical energy is not conserved, to a *perfectly inelastic* collision, which corresponds to the maximum loss of mechanical energy while still conserving momentum. We discuss these situations in the following.

Coefficient of Restitution

The energy loss in a collision between two objects is often described by the coefficient of restitution, r. For example, we can characterize the collision between a ball and a massive wall by the velocity, v_1, of the ball after the collision:

$$v_1 = -r v_0 \ , \tag{12.60}$$

as a function of the velocity v_0 before the collision. The change in kinetic energy is:

$$E_1 - E_0 = \frac{1}{2} m v_1^2 - \frac{1}{2} m v_0^2 = (r^2 - 1) E_0 \ , \tag{12.61}$$

This gives the following cases:

- A collision is called **elastic** if the mechanical energy is conserved. This corresponds to $r = 1$.
- A collision is called **inelastic** if the mechanical energy is not conserved. This corresponds to $0 \le r < 1$.
- A collision is called **perfectly inelastic** the two objects have the same velocity after the collision. This corresponds to $r = 0$.

If you bounce a ball on the floor, we can relate the coefficient of restitution to how high the ball bounces back up. If we drop the ball from a height h_0, the velocity of the ball is given by $v_0^2 = 2gh_0$ before the collision, and $v_1^2 = r^2 v_0^2 = 2gh_0$ after the collision. The maximum height the ball reaches is found from $2gh_1 = v_1^2 = r^2 v_0^2 = r^2 2gh_0$, which gives $h_1 = r^2 h_0$ as illustrated in Fig. 12.11.

The coefficient of restitution for a collision between two objects is defined as:

$$r = -\frac{v_{A,1} - v_{B,1}}{v_{A,0} - v_{B,0}} \ . \tag{12.62}$$

(Notice that the definition does not change if we exchange A with B, which is necessary for such a definition to make sense.)

It is possible to show that $r = 0$ corresponds to the maximum loss of energy while conserving momentum.

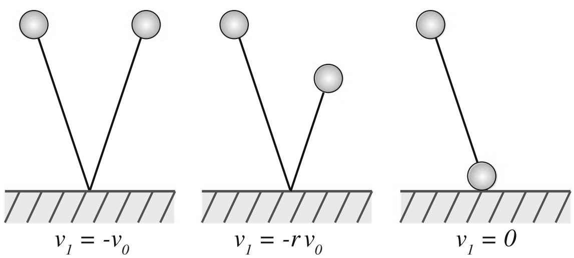

Fig. 12.11 Illustration of a ball bouncing on the floor

What does r depend on? Since any conservative force would lead to energy conservation and therefore $r = 1$, the coefficient of restitution characterizes the non-conservative parts of the contact forces. In reality, there are many processes during a collision that result in changes in kinetic energy, such as internal oscillations in the objects, permanent deformations, and viscous- and frictional forces. For a given force model, such as for a spring model with a viscous damping term, we can calculate the velocity after a collision and therefore determine the coefficient of restitution.

Elastic Collisions

For an elastic collision between the two blocks A and B in Fig. 12.10, both the total momentum and the mechanical energy are conserved:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.63}$$

$$\frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B v_{B,0}^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ . \tag{12.64}$$

Since we are free to choose the coordinate system, we simplify the problem by choosing a system where block B is initially at rest: $v_{B,0} = 0$, giving:

$$m_A v_{A,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.65}$$

$$m_A v_{A,0}^2 = m_A v_{A,1}^2 + m_B v_{B,1}^2 \ . \tag{12.66}$$

These two equations have a unique solution (see Extra material for a derivation).

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} \ , \ v_{B,1} = \frac{2 m_A}{m_A + m_B} v_{A,0} \ . \tag{12.67}$$

These formulas are general. Let us get to know them by studying a few special cases:

Equal Masses

If the two blocks have the same masses: $m_A = m_B$, we find that:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} = 0 \ , \ v_{B,1} = \frac{2 m_A}{m_A + m_B} v_{A,0} = v_{A,0} \ . \tag{12.68}$$

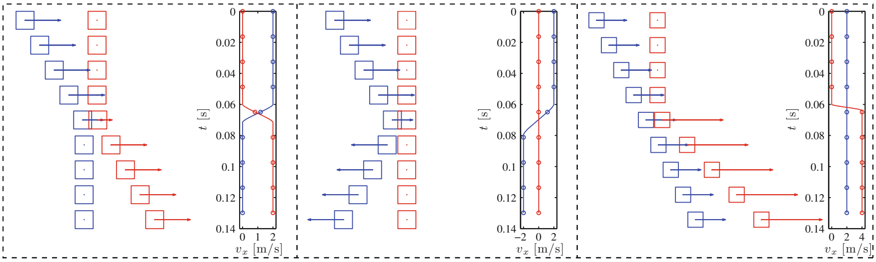

Fig. 12.12 Illustration of collision with (*left*) equal masses, (*middle*) a large mass, (*right*) a small mass

The two blocks exchange velocities! Before the collision block B is at rest and block A is moving with velocity $v_{A,0}$, and after the collision block A is at rest, and block B is moving with velocity $v_{A,0}$ (see Fig. 12.12).

You are probably familiar with this effect. You notice it when two equally sized balls collide head on, such as two billiard balls.

Collision with a Large Mass

What happens if block A collides with a large, stationary mass? That is, if $m_B \gg m_A$. We expect this to be like a collision with a stationary wall. We find:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} = \frac{(m_A/m_B) - 1}{(m_A/m_B) + 1} v_{A,0} \simeq -v_{A,0} \ , \tag{12.69}$$

where we have used the $m_A/m_B \ll 1$, that is $m_A/m_B \simeq 0$ when $m_B \gg m_A$. Similarly, for $v_{B,1}$:

$$v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} \simeq 0 \ , \tag{12.70}$$

For an elastic collision with a wall (or a very large object), the velocity is simply reversed (see Fig. 12.12).

Collision with Small Mass

What happens if block A collides with a tiny block B? That is, if $m_A \gg m_B$? We find that:

$$v_{A,1} = \frac{m_A - m_B}{m_A + m_B} v_{A,0} = \frac{1 - (m_B/m_A)}{1 + (m_B/m_A)} v_{A,0} \simeq v_{A,0} \ , \tag{12.71}$$

and

$$v_{B,1} = \frac{2m_A}{m_A + m_B} v_{A,0} = \frac{2}{1 + (m_B/m_A)} v_{A,0} \simeq 2v_{A,0} \ . \tag{12.72}$$

If a large object collides head on into a much smaller, stationary object, the large object continues with approximately the same velocity, but the small object is shot forward with twice the velocity of the large object (see Fig. 12.12).

Collisions and Relative Motion

So far we have only addressed the case when block B starts at rest, but our solutions are more general than you may believe: We can use a coordinate system change to map any collision onto the solutions we have found. Let us demonstrate how to do this by addressing an elastic collision between a ball (object A) with mass m_A and a moving wall, the much more massive object B: $m_B \gg m_A$.

The situation is illustrated in Fig. 12.13. Before the collision the velocity of the ball is $v_{A,0}$ and the velocity of the wall is $v_{B,0}$ measured relative to a coordinate system S, which we call the laboratory system. However, if we follow the motion of the wall, in the system S', the wall has velocity zero before the collision: $v'_{B,0} = 0$.

How are the two coordinate systems related? In Sect. 6.4 who found that the position $\mathbf{r}$ measured in system S is related to the position $\mathbf{r}'$ measured in system S' through:

$$\mathbf{r} = \mathbf{R} + \mathbf{r}' \ , \tag{12.73}$$

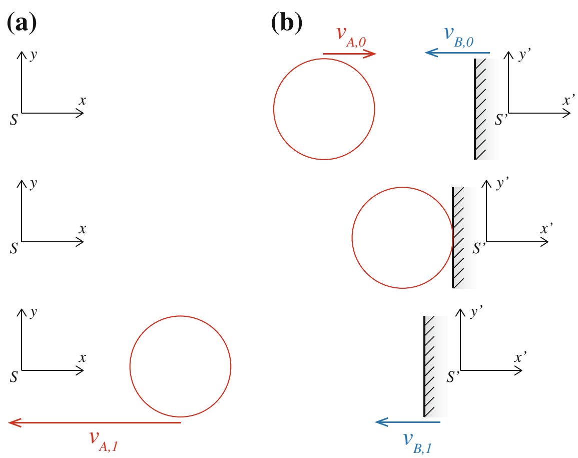

Fig. 12.13 Illustration of a collision between a ball (**a**) and a wall (**b**). The collision is addressed in both the system S, where the wall has a velocity toward A, and in system S', which follows the motion of the wall

where $\mathbf{R}$ is the position of system S' measured in system S. We take the time derivative on both sides to find a relation between the velocities:

$$\mathbf{v} = \mathbf{u} + \mathbf{v}' \,, \tag{12.74}$$

where $\mathbf{u}$ is the velocity of system S' measured in system S. We apply this to the current situation, where system S' is moving with the (initial) velocity of the wall $u = v_{B,0}$, we get:

$$v'_{A,0} = v_{A,0} - u = v_{A,0} - v_{B,0} \,, \tag{12.75}$$

and

$$v'_{B,0} = v_{B,0} - u = v_{B,0} - v_{B,0} = 0 \,, \tag{12.76}$$

which was the whole point—since $v'_{B,0} = 0$ we can use the solution from (12.69) to determine the velocity after the collision in the S' system:

$$v'_{A,1} = -v'_{A,0} = -\left(v_{A,0} - v_{B,0}\right) \,. \tag{12.77}$$

We find the velocity in the original system by the reverse transformation:

$$\begin{aligned} v_{A,1} &= v'_{A,1} + u = v'_{A,1} + v_{B,0} = -v'_{A,0} + v_{B,0} \\ &= -\left(v_{A,0} - v_{B,0}\right) + v_{B,0} = 2v_{B,0} - v_{A,0} \,, \end{aligned} \tag{12.78}$$

This is what happens when a ball hits a wall that moves toward the ball, such as a collision between a golf club (massive) and a golf ball (small mass). Similar techniques can be used to address other solutions.

General Solutions to Elastic Collisions

We can find the general solution for an elastic collision between blocks A and B as illustrated in Fig. 12.10:

$$v_{A,1} = \frac{(m_A - m_B)\, v_{A,0} + 2m_B v_{B,0}}{m_A + m_B} \tag{12.79}$$

$$v_{B,1} = \frac{(m_B - m_A)\, v_{B,0} + 2m_A v_{A,0}}{m_A + m_B} \,. \tag{12.80}$$

(A proof of this result is given in Sect. A.2).

Perfectly Inelastic Collisions

For a perfectly inelastic collision the two blocks A and B become attached after the collision, and they continue with the same velocity, v_1:

$$v_{A,1} = v_{B,1} = v_1 \ , \tag{12.81}$$

In this case, only the total momentum is conserved, and not the kinetic energy. Conservation of momentum gives:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.82}$$

Velocity After Perfectly Inelastic Collision

In the case when $v_{B,0} = 0$, we find:

$$m_A v_{A,0} = m_A v_1 + m_B v_1 = (m_A + m_B)\, v_1 \ , \tag{12.83}$$

which gives:

$$v_1 = \frac{m_A}{m_A + m_B} v_{A,0} = v_{A,1} = v_{B,1} \ . \tag{12.84}$$

Loss of Energy After Perfectly Inelastic Collision

The mechanical energy is not conserved for a perfectly inelastic collision. This means that the initial kinetic energy of the system is transformed into thermal energy and has been used to deform the objects permanently. The perfectly inelastic collision gives the maximum loss of energy. Let us find the change in kinetic energy in the collision.

Before the collision (when B is at rest), the kinetic energy is:

$$K_0 = \frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B \underbrace{v_{B,0}^2}_{=0} = \frac{1}{2} m_A v_{A,0}^2 \ . \tag{12.85}$$

After the collision the kinetic energy is:

$$\begin{aligned} K_1 &= \frac{1}{2}(m_A + m_B)\, v_1^2 = \frac{1}{2}(m_A + m_B) \frac{m_A^2}{(m_A + m_B)^2} v_{A,0}^2 \\ &= \frac{1}{2} \frac{m_A^2}{m_A + m_B} v_{A,0}^2 = \frac{m_A}{m_A + m_B} \cdot \frac{1}{2} m_A v_{A,0}^2 = \frac{m_A}{m_A + m_B} K_0 \ , \end{aligned} \tag{12.86}$$

How large fraction of the original kinetic energy remains after the collision?

$$\frac{K_1}{K_0} = \frac{m_A}{m_A + m_B} < 1 \ . \tag{12.87}$$

The loss in kinetic energy is:

$$\Delta K = K_1 - K_0 = \left(\frac{m_A}{m_A + m_B} - 1 \right) K_0 = -\frac{m_B}{m_A + m_B} K_0 \ . \tag{12.88}$$

12.5.1 Example: Ballistic Pendulum

Problem: A 10 g bullet is fired into a 1 kg wooden block hanging in a 1 m long rope. The block reaches a height 30 cm above its initial level. (a) What was the velocity of the bullet? (b) What was the loss of energy in the system?

Identify: In this problem we address the motion of two objects: The bullet (object A) and the block (object B). The block is hanging from a massless rope. This means that the block behaves as a pendulum after the collision between the bullet and the pendulum. After the collision, the block swings to a height h above its initial position. The process is illustrated in Fig. 12.14.

Model: In this problem, we do not know the detailed interactions between the bullet and the block. Therefore, we use conservation laws to address the collision between the bullet and the block. The bullet becomes stuck in the block—this means that the bullet and the block has the same velocity after the collision—the collision is perfectly inelastic. Before the collision the bullet has a horizontal velocity, $\mathbf{v}_{A,0} = v_0 \, \mathbf{i}$, and the block is at rest, $\mathbf{v}_{B,0} = 0$. After the collision, both objects have the velocity $\mathbf{v}_{A,1} = \mathbf{v}_{B,1} = v_1 \, \mathbf{i}$.

Solution to part a: There are no external forces acting on the system in the x-direction during the collision. (There may be forces acting on the system in the y-direction—depending on the motion of the system and the effect of the rope. We

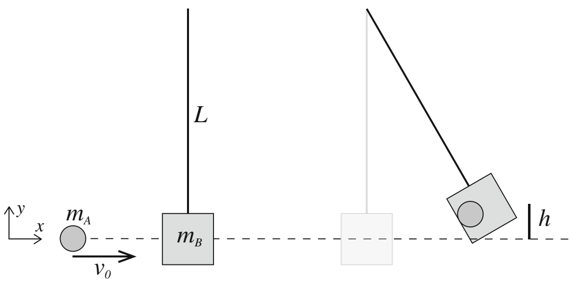

Fig. 12.14 Illustration of the motion of a ballistic pendulum. First, there is a collision between the bullet and the block, and afterward the bullet and the block swings as a pendulum to a height h above their initial position

assume that during the collision, the rope is vertical, so that the rope does not exert a horizontal force on the system.) From Newton's second law in the x-direction we get:

$$\sum F_x = 0 = \frac{d}{dt}\left(p_{A,x} + p_{B,x}\right) . \tag{12.89}$$

The total momentum in the x-direction is therefore conserved during the collision:

$$m_A v_{A,0} + m_B \underbrace{v_{B,0}}_{=0} = m_A v_{A,1} + m_B v_{B,1} \Rightarrow v_1 = \frac{m_A}{m_A + m_B} v_{A,0} . \tag{12.90}$$

In the second part of the process, the block and bullet swings up as a pendulum. In this process, the rope does not do any work on the system, and we assume that the air resistance is negligible. The only force doing work on the system is therefore gravity. In this second part of the process, the mechanical energy is conserved!

Notice that in this case the process consists of two separate subprocesses. In the first subprocess, the collision, the mechanical energy is not conserved, but in the second subprocess, the swinging pendulum, the mechanical energy is conserved.

We use energy considerations to determine how high the pendulum swings. The mechanical energy at the beginning of this motion, immediately after the collision, is the same as the mechanical energy when the pendulum has reached its maximum height. At its maximum height the kinetic energy of the pendulum is zero. Conservation of mechanical energy therefore gives:

$$\frac{1}{2}(m_A + m_B) v_1^2 = (m_A + m_B) g h \Rightarrow h = \frac{v_1^2}{2g} = \left(\frac{m_A}{m_A + m_B}\right)^2 \frac{v_0^2}{2g} . \tag{12.91}$$

Now, the problem posed was to find the initial velocity, v_0, as a function of h. We find v_0 from (12.91):

$$v_0 = \sqrt{2gh}\left(\frac{m_A + m_B}{m_A}\right) . \tag{12.92}$$

Let us now insert the numbers given in the problem:

$$v_0 = \sqrt{2\left(9.8\,\mathrm{m/s^2}\right)(0.3\,\mathrm{m})}\left(\frac{1.01\,\mathrm{kg}}{0.01\,\mathrm{kg}}\right) \simeq 245\,\mathrm{m/s} . \tag{12.93}$$

Solution to part b: In the second part of the problem, we are asked to find the loss of energy. The kinetic energy before the collision is

$$K_0 = \frac{1}{2} m_A v_0^2 , \tag{12.94}$$

and after the collision the kinetic energy is:

$$K_1 = \frac{1}{2}(m_A + m_B) v_1^2 = \frac{1}{2}(m_A + m_B)\left(\frac{m_A}{m_A + m_B} v_0\right)^2 \tag{12.95}$$

$$= \frac{1}{2} m_A v_0^2 \frac{m_A}{m_A + m_B} = K_0 \frac{m_A}{m_A + m_B} \; . \tag{12.96}$$

The relative loss of kinetic energy is therefore:

$$\frac{\Delta K}{K_0} = \frac{m_A}{m_A + m_B} - 1 = -\frac{m_B}{m_A + m_B} \; , \tag{12.97}$$

which means that practically all the energy was lost in the collision!

12.5.2 Example: Super-Ball

Problem: In super-ball we take two balls, one small and one large, and release them together from a height h_0 above the ground, as illustrated in Fig. 12.15. What is the maximum height h_1 reached by the top ball after the collision? Assume that all collisions are conservative.

Identify: In this problem we address the motion of two objects: The bottom ball, A, and the top ball, B. The whole process may be subdivided into three separate parts. In the first part both objects are falling until they hit the ground. In the second part they are colliding with the ground and each other, and in the third part, they are both moving upward. The various subprocesses are illustrated in Fig. 12.15.

Model: We do not know the interactions between the balls, or between the balls and the ground, but we know that all collisions are elastic. We will also consider the problem to consist of a sequence of collisions between two objects: First the bottom

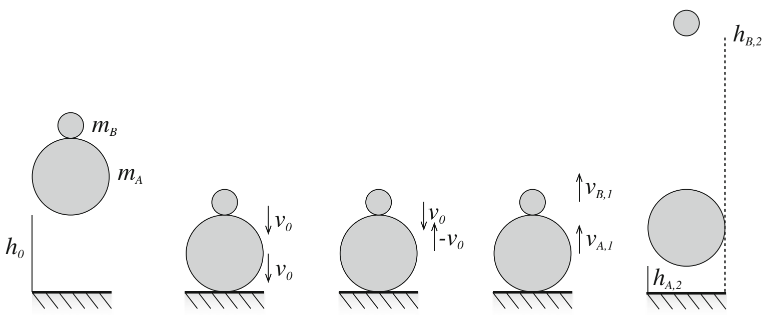

Fig. 12.15 Illustration of the motion of two balls A and B bouncing off the ground and each other

ball collides with the ground, and then the bottom ball collides with the top ball. Let us look at each of these subprocesses individually:

In the *first part*, the balls fall from a height h_0. We use energy conservation to find the velocity:

$$K_0 + mgh_0 = K_1 + mg0 \ , \tag{12.98}$$

$$mgh_0 = \frac{1}{2} m v_0^2 \ , \tag{12.99}$$

and

$$v_0 = \sqrt{2gh_0} \ , \tag{12.100}$$

is the velocity of both the balls.

In the *second part*, the bottom ball hits the floor. This is an elastic collision, where we know tthat the velocity is reversed. After the collision, the bottom ball therefore has an upward velocity v_0.

In the *third part*, the bottom ball collides with the top ball. Since the only external force is gravity, which has a small impulse compared to the forces between the balls, the net vertical force is approximately zero, and conservation of momentum in the y-direction gives:

$$m_A v_{A,0} + m_B v_{B,0} = m_A v_{A,1} + m_B v_{B,1} \ , \tag{12.101}$$

$$m_A v_0 + m_B \left(-v_0\right) = m_A v_{A,1} + m_B v_{B,1} \ . \tag{12.102}$$

Conservation of kinetic energy gives:

$$\frac{1}{2} m_A v_{A,0}^2 + \frac{1}{2} m_B v_{B,0}^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ , \tag{12.103}$$

where we insert $v_{A,0} = v_0$ and $v_{B,0} = -v_0$:

$$\frac{1}{2} m_A v_0^2 + \frac{1}{2} m_B v_0^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ . \tag{12.104}$$

We can rewrite the two equations to be:

$$m_A \left(v_0 - v_{A,1}\right) = m_B \left(v_0 + v_{B,1}\right) \ , \tag{12.105}$$

and

$$m_A \left(v_0^2 - v_{A,1}^2\right) = m_B \left(v_{B,1}^2 - v_0^2\right) \ , \tag{12.106}$$

This last equation can also be written as:

$$m_A \left(v_0 - v_{A,1}\right)\left(v_0 + v_{A,1}\right) = m_B \left(v_{B,1} - v_0\right)\left(v_{B,1} + v_0\right) \ . \tag{12.107}$$

Dividing the two equations, we get:

$$v_0 + v_{A,1} = v_{B,1} - v_0 \ \Rightarrow \ 2v_0 = v_{B,1} - v_{A,1} \ . \tag{12.108}$$

We eliminate $v_{A,1}$, finding:

$$v_{B,1} = \frac{3m_A - m_B}{m_A + m_B} v_0 \ . \tag{12.109}$$

Discussion: Let us address three special cases:

- For $m_A = \frac{m_B}{3}$ we find $v_{B,1} = 0$
- For $m_A = m_B$ we find $v_{B,1} = v_0$
- For $m_A = 3m_B$ we find $v_{B,1} = 2v_0$
- For $m_A \gg m_B$ we find $v_{B,1} = 3v_0$

We find maximum height from the case when $m_A \gg m_B$, using conservation of energy for the motion of ball B:

$$m_B g h_{B,2} = \frac{1}{2} m_B v_{B,1}^2 = \frac{1}{2} m_B 9 v_0^2 \ , \tag{12.110}$$

and therefore we find that:

$$h_{B,2} = 9h_0 \ , \tag{12.111}$$

which is the maximum height, in the case when the large ball has much larger mass than the small ball.

Non-central Elastic Collisions

So far we have only studied one dimensional collisions—collisions where all the objects move along a line so that all velocities also are directed along the line. Such collisions are called central collisions:

> In a **central collision** the momentum of both objects is directed along the line between the two objects before and after the collision.

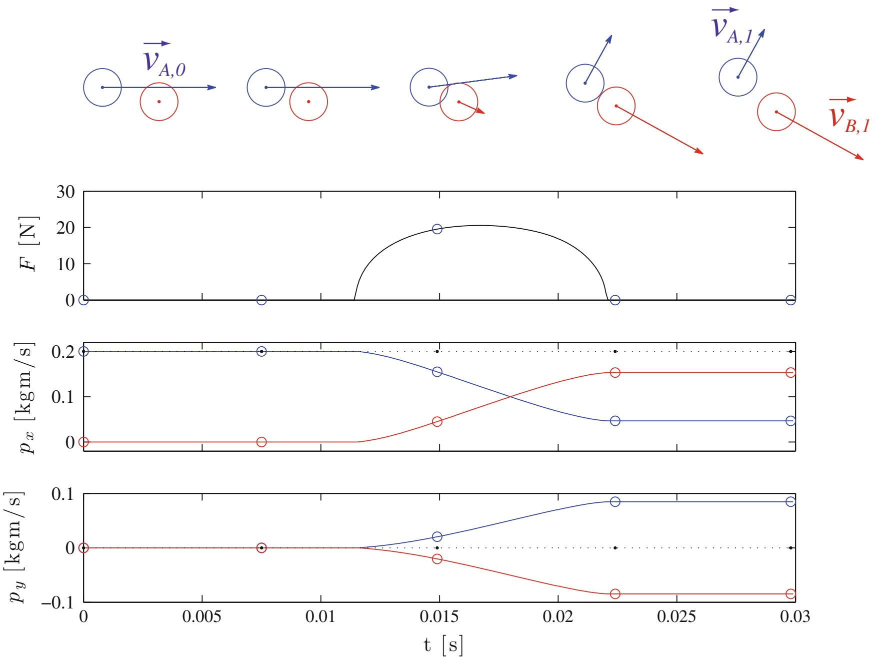

Fig. 12.16 Illustration of a non-central collision between two objects

Let us now address the more general case where two objects are colliding, but where the momentum of at least one of the objects is not directed along the line connecting the centers of the objects (before the collision). An example of such a collision is illustrated in Fig. 12.16.

We study this collision in a reference system where object B is initially at rest, $\mathbf{v}_{B,0} = 0$. Since there are no external forces acting on the system during the collision, the total momentum is conserved:

$$m_A \mathbf{v}_{A,0} + m_B \underbrace{\mathbf{v}_{B,0}}_{=0} = m_A \mathbf{v}_{A,1} + m_B \mathbf{v}_{B,1} \ . \tag{12.112}$$

If the collision is also elastic, the kinetic energy is conserved:

$$\frac{1}{2} m_A v_{A,0}^2 = \frac{1}{2} m_A v_{A,1}^2 + \frac{1}{2} m_B v_{B,1}^2 \ . \tag{12.113}$$

For this very general case we cannot make any further general statements. But for a collision between two objects with equal masses, $m_A = m_B$, we find that the conservation of momentum is:

$$\mathbf{v}_{A,0} = \mathbf{v}_{A,1} + \mathbf{v}_{B,1} \ , \tag{12.114}$$

and conservation of kinetic energy is now:

$$v_{A,0}^2 = v_{A,1}^2 + v_{B,1}^2 \ . \tag{12.115}$$

We insert $\mathbf{v}_{A,0}$ into (12.115):

$$\begin{aligned} v_{A,0}^2 &= v_{A,1}^2 + v_{B,1}^2 \\ \left(\mathbf{v}_{A,1} + \mathbf{v}_{B,1}\right)^2 &= v_{A,1}^2 + v_{B,1}^2 \\ v_{A,1}^2 + 2\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} + v_{B,1}^2 &= v_{A,1}^2 + v_{B,1}^2 \\ 2\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} &= 0 \ , \end{aligned} \tag{12.116}$$

We have found that $\mathbf{v}_{A,1} \cdot \mathbf{v}_{B,1} = 0$, which means that the two velocities are orthogonal! Notice that we still do not have enough equations to determine the vectors: We have 3 equations, but 4 unknown components in the velocity vectors after the collision. In order to determine the velocities after the collision we need more information about the collision. We need to know something about the force acting between the particles throughout the collision.

12.6 Modeling and Visualization of Collisions

We can gain better insights into the concepts introduced in this chapter by studying collisions in detail. If we know the details of the interactions between two objecst, that is, if the have models for the interaction forces, we can find their motion from Newton's second law.Let us use this to get a better understanding of elastic, inelastic and perfectly inelastic collisions.

We model two objects, A and B, with masses m_A and m_B. The force from B on A is:

$$\mathbf{F}_{\text{B on A}} = \mathbf{F}(\mathbf{r}_A, \mathbf{r}_B, \mathbf{v}_A, \mathbf{v}_B) \ , \tag{12.117}$$

and from Newton's third law, we know that

$$\mathbf{F}_{\text{A on B}} = -\mathbf{F} \ . \tag{12.118}$$

For example, for two solid spheres of radius R, a reasonable force model is:

$$\mathbf{F} = \begin{cases} k\,|\Delta r - 2R|\,\frac{\Delta \mathbf{r}}{\Delta r} - \eta\,(\Delta \mathbf{v}) & , \ \Delta r < 2R \\ \mathbf{0} & , \ \Delta r \geq 2R \end{cases} \ , \tag{12.119}$$

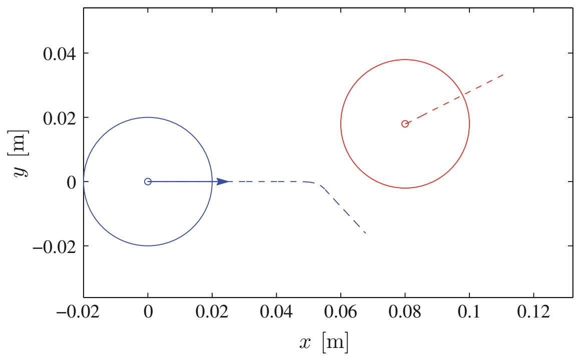

Fig. 12.17 Illustration of object trajectories and initial conditions

where $\Delta\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$, $\Delta r = |\Delta\mathbf{r}|$ and $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A$. We find the equation of motion from Newton's second law:

$$m_A\mathbf{a}_A = \mathbf{F} \;\Rightarrow\; \mathbf{a}_A = \frac{\mathbf{F}}{m_A} \,. \tag{12.120}$$

We find the motion of both objects from Euler-Cromer's method, as implemented in the following program. First, we define the masses, the radius, and the initial positions and velocities of the objects:

```
from pylab import *
R = 0.02 # m
mA = 0.1 # kg
mB = 0.1 # kg
rA0 = array([0.0,0.0]) # m
vA0 = array([1.0,0.0]) # m/s
rB0 = array([0.08,0.018]) # m
vB0 = array([0.0,0.0]) # m/s
time = 0.10 # s
```

This set of initial conditions are illustrated in Fig. 12.17. These conditions will result in an non-central collision. Then, we define the parameters used by the force model, such as the force constant k and the viscous term η. We choose an unrealistically small value for k in order to make the collision extend over some time so that we can observe the interactions during the collision. The time step is chosen small enough, so that we are sure to have good resolution for the motion during the collision:

```
# Force model
eta = 1.0
k = 20000.0 # Nm
dt = 0.0001 # s
```

We initialize by generating arrays for all the variables:

```
# Initialization
n = int(round(time/dt))
t = zeros(n,float)
rA = zeros((n,2),float)
vA = zeros((n,2),float)
rB = zeros((n,2),float)
vB = zeros((n,2),float)
```

```
F = zeros((n,2),float)
rA[0] = rA0
vA[0] = vA0
rB[0] = rB0
vB[0] = vB0
D = 2*R # Diameter
```

Where we have introduce the diameter, $D = 2R$, to simplify the expressions. The integration loop follows the mathematical formulation of the force law in (12.119) as closely as possible:

```
# Integration loop
for i in range(n-1):
    Deltar = rB[i]-rA[i]
    Deltarnorm = sqrt(dot(Deltar,Deltar))
    Deltav = vB[i]-vA[i]
    if (Deltarnorm>=D):
        Fnet = array([0,0])
    else:
        Fnet = -k*abs(Deltarnorm-D)**1.5*Deltar/Deltarnorm + eta*Deltav;
    F[i] = Fnet
    aA = Fnet/mA
    aB = -Fnet/mB
    vA[i+1] = vA[i] + aA*dt
    rA[i+1] = rA[i] + vA[i+1]*dt
    vB[i+1] = vB[i] + aB*dt
    rB[i+1] = rB[i] + vB[i+1]*dt
    t[i+1] = t[i] + dt
```

Finally, we plot the resulting trajectories and the momentum in the x and y direction as functions of time:

```
# Plot trajectories and momentum
figure(1)
plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r')
xlabel('x [m]')
ylabel('y [m]')
axis('equal')
figure(2)
pA = vA.copy()*mA
pB = vB.copy()*mB
subplot(2,1,1)
plot(t,pA[:,0],'-b',t,pB[:,0],'-r')
xlabel('t [s]')
ylabel('p_x [kgm/s]')
subplot(2,1,2)
plot(t,pA[:,1],'-b',t,pB[:,1],'-r');
xlabel('t [s]')
ylabel('p_y [kgm/s]');
```

While these plots provide useful information about the collision, and we can use them to gain intuition about collisions, we may also learn from seeing the dynamics of the collision—how the objects move. This can be done by generating a simple animation using the `plot` command:

```
# Animate using plot
figure(3)
for i in range(0,n,50):
    plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r',
         [rA[i,0]],[rA[i,1]],'ob',[rB[i,0]],[rB[i,1]],'or')
    xlabel('x [m]')
    ylabel('y [m]')
    axis('equal')
```

However, you get a better impression from an animation that shows the extent of the colliding spheres. In addition, we need to scale the coordinate system so that we have room for not only the centers of the spheres, but also their whole extend, so that the plotting region does not move around during the animation, since this will make the animation difficult to interpret. We therefore find the maximum range of the objects positions and scale the axes accordingly:

```
# Animate by drawing
figure(4)
xmin = min(min(rA[:,0]-R),min(rB[:,0]-R))
xmax = max(max(rA[:,0]+R),max(rB[:,0]+R))
ymin = min(min(rA[:,1]-R),min(rB[:,1]-R))
ymax = max(max(rA[:,1]+R),max(rB[:,1]+R))
theta = linspace(0,2*pi,100)
xcirc = R*cos(theta)
ycirc = R*sin(theta)
for i in range(0,n,50):
    plot(rA[:,0],rA[:,1],'-b',rB[:,0],rB[:,1],'-r',
         [rA[i,0]],[rA[i,1]],'ob',[rB[i,0]],[rB[i,1]],'or')
    hold('on')
    x = rA[i,0] + xcirc
    y = rA[i,1] + ycirc
    plot(x,y,'-b')
    x = rB[i,0] + xcirc
    y = rB[i,1] + ycirc
    plot(x,y,'-r')
    hold('off')
    axis('equal')
    xlabel('x [m]')
    ylabel('y [m]')
```

This program serves as the basis for the illustrations shown in this chapter and Fig. 12.18. You can now use this code to study various collision—central elastic collisions, non-central elastic collision, and inelastic collisions by introducing a finite value for η. Reasonable values for η are in the range $1\,\text{kg/s} < \eta < 10\,\text{kg/s}$.

Test your understanding: Based on this modeling framework, you are now ready to model other interactions. For example, you may introduce an ionic inteaction for the force between two ions: $\mathbf{F} = -C\,\Delta r^{-2}\,(\Delta\mathbf{r}/\Delta r)$. Try to implement and test this model using a similar approach as introduced here.

12.7 Rocket Equation

A rocket accelerates forward by ejecting exhaust backward at high velocity. The rocket is essentially throwing out mass backward in order to move forward. How can we address the motion of a rocket using Newton's second law?

We start from a specific example: A rain drop is falling down through the atmosphere and on the way it adsorbs water vapor that condensates on the drop. Let us address the motion of the drop over a small time interval from t to $t + \Delta t$. During this time interval the drop adsorbs a small drop of mass Δm and initial velocity $\mathbf{u}$. At the time t the mass of the drop is m and its velocity $\mathbf{v}$. At the time $t + \Delta t$, the mass is $m + \Delta m$ and the velocity has changed to $\mathbf{v} + \Delta\mathbf{v}$.

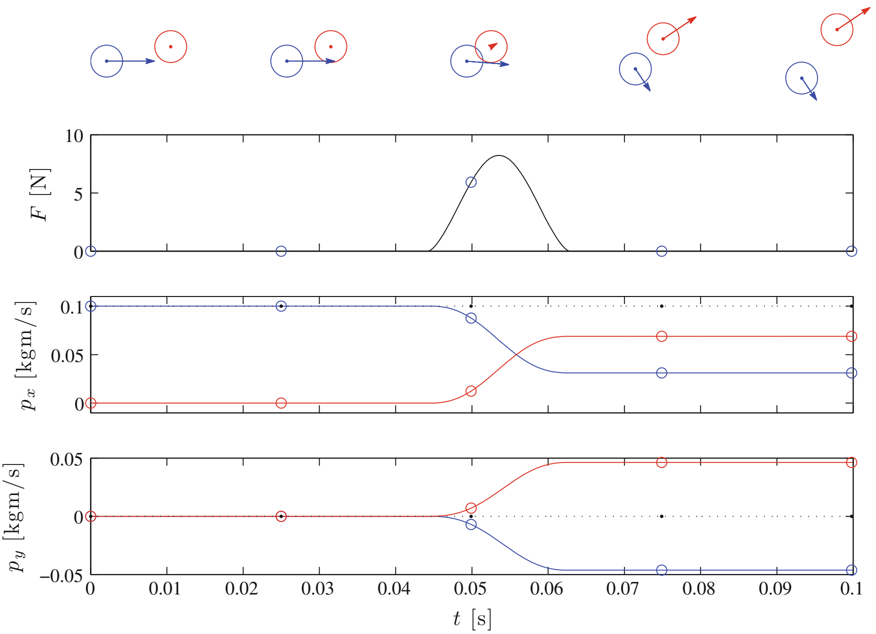

Fig. 12.18 Illustration of object trajectories and $p_x(t)$ and $p_y(t)$ throughout the collision

The change in momentum of the drop is related to the external forces acting on the drop. What is the change in momentum? At the time t, the total momentum (of the drop and the small drop being adsorbed) is:

$$\mathbf{p}(t) = m\mathbf{v} + \Delta m\mathbf{u} \tag{12.121}$$

after the time interval Δt, the small drop is adsorbed, and the new momentum is:

$$\mathbf{p}(t + \Delta t) = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v}) = m\mathbf{v} + m\Delta\mathbf{v} + \Delta m\mathbf{v} + \Delta m\Delta\mathbf{v} \ . \tag{12.122}$$

The change in momentum is therefore:

$$\Delta\mathbf{p} = \mathbf{p}(t + \Delta t) - \mathbf{p}(t) = m\Delta\mathbf{v} + (\mathbf{v} - \mathbf{u})\,\Delta m + \Delta m\Delta\mathbf{v} \ . \tag{12.123}$$

Newton's second law for the system related the change in momentum to the net forces acting on the system:

$$\sum \mathbf{F}^{\text{ext}} = \frac{\Delta\mathbf{p}}{\Delta t} = m\frac{\Delta\mathbf{v}}{\Delta t} + (\mathbf{v} - \mathbf{u})\,\frac{\Delta m}{\Delta t} + \Delta m\frac{\Delta\mathbf{v}}{\Delta t} \ . \tag{12.124}$$

This is only valid in the limit when $\Delta t \to 0$. If we assume the adsorption process also to be continuous, the change in mass and the change in velocity also goes to zero when Δt goes to zero. Hence, the term:

$$\Delta m \frac{\Delta \mathbf{v}}{\Delta t} \to 0 \text{ when } \Delta t \to 0 \ . \tag{12.125}$$

In the limit of small Δt we therefore find the **Rocket equation**:

Rocket equation:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\frac{dm}{dt} \ . \tag{12.126}$$

The rocket equation is used to describe the motion of an object that is expelling or adsorbing mass with a velocity $\mathbf{u}$. A special case is when the expelled or adsorbed mass has zero velocity, $\mathbf{u} = 0$:

$$\sum \mathbf{F}^{\text{ext}} = \frac{d\mathbf{p}}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} \tag{12.127}$$

which is what we get when taking the derivative of $\mathbf{p} = m\mathbf{v}$ with a time-dependent mass.

For a rocket, we may not generally know at what speed relative to the ground the exhaust is ejected, but rather at what speed relative to the rocket the mass is expunged. We introduce the velocity of Δm relative to m as:

$$\mathbf{v}_{\text{rel}} = \mathbf{u} - \mathbf{v} \ , \tag{12.128}$$

using this notation we can write the rocket equation (12.126) as:

Rocket equation:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}}\frac{dm}{dt} = m\frac{d\mathbf{v}}{dt} \ . \tag{12.129}$$

Written in this form, the equation looks like the usual formulation of Newton's second law, and we interpret all the terms on the left side as forces, and the term on the right side is the mass multiplied by the acceleration. Expelling mass with a velocity $\mathbf{v}_{\text{rel}}$ relative to the object at a rate dm/dt, has the same effect as pushing on the object with a force:

$$\mathbf{F} = \mathbf{v}_{\text{rel}}\frac{dm}{dt} \ . \tag{12.130}$$

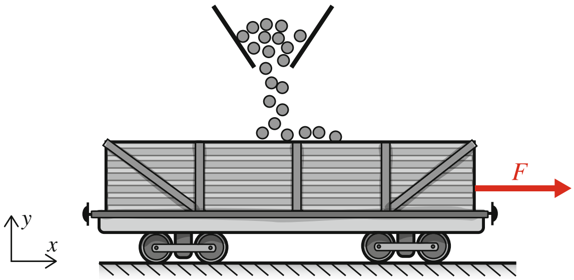

Fig. 12.19 A railway car is pulled with a force F so that it retains a constant velocity while sand is added from above at a constant rate

12.7.1 Example: Adding Mass to a Railway Car

Problem: A railway car is moving with constant velocity along a straight railway track. While it is moving, sand is dropped onto the car from above at a constant rate, $R = dm/dt$, as illustrated in Fig. 12.19. With how large force, F, must the car be pulled in order for the car to move with constant velocity?

Solution: We use the rocket equation to relate the forces acting on the car to its acceleration. The only horizontal force acting on the car is the external force F, friction and air resistance. We assume that friction and air resistance forces are negligible. The sand is falling onto the car from above. This means that the sand has a vertical velocity but no horizontal velocity as it hits the car. We can therefore use the rocket equation, with $\mathbf{u} = 0$:

$$\sum \mathbf{F}^{\text{ext}} = (\mathbf{v} - \mathbf{u})\frac{dm}{dt} + m\frac{d\mathbf{v}}{dt} \ , \tag{12.131}$$

The net horizontal force on the car is $\mathbf{F} = F\,\mathbf{i}$, and the velocity is constant, $dv_x/dt = 0$, and the velocity $u_x = 0$ therefore:

$$F = (v_x - u_x)\frac{dm}{dt} + m\frac{dv}{dt} = (v_x - 0)\frac{dm}{dt} + m \times 0 = v\frac{dm}{dt} \ . \tag{12.132}$$

The force is therefore given by the velocity and the rate at which mass is added.

12.7.2 Example: Rocket with Diminishing Mass

Problem: A rocket is at rest in outer space, where the net force acting on the rocket is zero. In order to accelerate, it turns on its thrusters, which propels exhausting gases backward with a velocity v_{rel} relative to the spaceship. Find how the velocity increases as the fuel is used.

Solution: Let us describe the motion with the x-axis directed forward—in the direction of motion of the rocket. The rocket starts with the velocity $v_0 = 0$. We use the rocket equation to find an expression for the velocity of the rocket:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}} \frac{dm}{dt} = m \frac{d\mathbf{v}}{dt} \,, \tag{12.133}$$

Here, the net external force is zero, and all motion is in the x-direction:

$$-v_{\text{rel}} \frac{dm}{dt} = m \frac{dv}{dt} \;\Rightarrow\; \frac{dv}{dt} = -v_{\text{rel}} \frac{1}{m} \frac{dm}{dt} \,. \tag{12.134}$$

We find the velocity by integrating from t_0 to t_1:

$$\int_{t_0}^{t_1} \frac{dv}{dt} dt = -v_{\text{rel}} \int_{t_0}^{t_1} \frac{1}{m} \frac{dm}{dt} dt = -v_{\text{rel}} \int_{m(t_0)}^{m(t_1)} \frac{dm}{m} \,, \tag{12.135}$$

which gives

$$v(t_1) - v(t_0) = -v_{\text{rel}} \left(\ln m(t_1) - \ln m(t_0)\right) = -v_{\text{rel}} \ln \frac{m(t_1)}{m(t_0)} \,. \tag{12.136}$$

This is the increase in velocity of the rocket as the mass changes from $m(t_1)$ to $m(t_0)$. We notice that the velocity is increasing, since the final mass, $m(t_1)$ is always smaller than the initial mass $m(t_0)$.

Summary

Translational momentum: is defined as $\mathbf{p} = m\mathbf{v}$
Newton's second law: on general form is: $\sum \mathbf{F}^{\text{ext}} = d\mathbf{p}/dt$
Impulse: Change in momentum is related to the *impulse*, $\mathbf{J}$, of the net force:

$$\mathbf{J} = \int_{t_0}^{t_1} \mathbf{F}^{\text{ext}} dt = \mathbf{p}(t_1) - \mathbf{p}(t_0) \,.$$

The average net force: during a collision is:

$$\mathbf{F}^{\text{avg}} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \mathbf{F} dt = \frac{\Delta \mathbf{p}}{t_1 - t_0} \,.$$

System of particles: For a system of particles we distinguish between:

- **internal forces** acting between particles in the system, and
- **external forces** acting between particles in the system and objects in the environment.

Newton's second law for a system of particles: is

$$\frac{d}{dt}\sum_j \mathbf{p}_j = \sum \mathbf{F}^{\text{ext}}$$

Isolated system: A system is isolated if the *net external force* is zero.
Conservaton of momentum: For an isolated system, the total momentum is conserved, $\sum_j \mathbf{p}_j =$ constant. Conservation of momentum is a *vector equation*, which can be applied for each direction independently of other directions.
Collisions: For a collision between two objects A and B, the total momentum is conserved if there are no external forces: $\mathbf{p}_{A,0} + \mathbf{p}_{B,0} = \mathbf{p}_{A,1} + \mathbf{p}_{B,1}$.
Elastic collisions: In an *elastic* collision, the kinetic energy is conserved.
Elastic collision in one dimension: The velocities of the objects after the collision are:

$$v_{A,1} = \frac{(m_A - m_B)\, v_{A,0} + 2m_B v_{B,0}}{m_A + m_B} \;,\; v_{B,1} = \frac{(m_B - m_A)\, v_{B,0} + 2m_A v_{A,0}}{m_A + m_B}$$

Perfectly inelastic collision: is a collision where the two objects have the same velocity after the collision: $\mathbf{v}_{A,1} = \mathbf{v}_{B,1}$
Perfectly inelastic collision in one dimension: The velocity after the collision is:

$$v_1 = v_{A,1} = v_{B,1} = \frac{m_A v_A + m_B v_B}{m_A + m_B} \;,$$

Inelastic collision: is a collision with energy loss, characterized by the *coefficient of restitution*, $r = -\left(v_{B,1} - v_{A,1}\right) / \left(v_{B,0} - v_{A,0}\right)$
Rocket equation: The motion of object that in a small time Δt is absorbing a mass Δm with a velocity $\mathbf{u}$, is given by the *rocket equation*:

$$\sum \mathbf{F}^{\text{ext}} = m\frac{d\mathbf{v}}{dt} + (\mathbf{v} - \mathbf{u})\,\frac{dm}{dt}$$

If the velocity of the absorbed/ejected material is $\mathbf{v}_{\text{rel}}$ relative to the object, the rocket equation can be written:

$$\sum \mathbf{F}^{\text{ext}} + \mathbf{v}_{\text{rel}}\frac{dm}{dt} = m\frac{d\mathbf{v}}{dt} \;.$$

Exercises

Discussion Questions

12.1 Golf ball. You hit a golf ball with a heavy golf club with a velocity v. What is the starting velocity of the golf ball?

12.2 Energetic collision. Can a collision between two objects result in zero total kinetic energy?

12.3 A collision paradox. A riddle: Two cars each with speed v_0 collide head on, getting stuck in the collision. If you observe the collision from the side of the road, the change in total kinetic energy is $\Delta K = 2 \cdot (1/2)mv_0^2$. If you instead observe the collision from a system moving with one of the cars, one car has velocity 0 and the other car has velocity $2v_0$. Then the change in total kinetic energy is $(1/2)m(2v_0^2) = 4 \cdot (1/2)mv_0^2$, which is double of what you observed from the side of the road. Is this argument correct? Explain.

Problems

12.4 A bike and a car. You and your bike has a mass of 100 kg.
(**a**) How fast would you have to ride in order to have the same momentum as a car of mass $m = 1200$ kg and a velocity of 50 km/h?

12.5 Kicking a ball. A football is lying at rest on the ground. You kick it. After the kick, it has a horizontal velocity of 20 m/s. You are in contact with the ball for 0.1 s. The mass of the ball is 0.43 kg.
(**a**) What is the change in the momentum of the ball?
(**b**) What is the impulse on the ball during the collision?
(**c**) What is the average force on the ball during the collision?

Assume that you are returning a ball coming toward you at 20 m/s. You kick the ball, staying in contact with the ball for 0.1 s, and return the ball with a velocity of 20 m/s
(**d**) What is the average force on the ball during the collision?

12.6 Stopping a car. During a collision at 60 km/h, a 1200 kg car stops in 0.2 s.
(**a**) What is the average force on the car during the collision?

A crash-test dummy of mass 80 kg are sitting in the car. Thanks to the seatbelt, he stops in 0.4 s.
(**b**) What is the average force on the dummy during the collision?

12.7 Ball reflected from wall. A ball of mass m hits a wall with a velocity $\mathbf{v}_0$ and bounces back. The ball hits the wall with an inclination, so that the velocity forms an

angle θ with the wall surface. When the ball leaves the wall after the collision, the magnitude of the velocity is the same, but its direction has changed. However, only the component of the velocity that is normal to the wall changes during the collision. Thus the velocity forms the same angle with the wall after the collision. The ball is in contact with the wall during a time interval Δt.
(**a**) What is the change in momentum of the ball?
(**b**) What is the impulse on the wall?
(**c**) What is the average force on the ball from the wall?
(**d**) For what angle θ is the average force on the ball largest?

12.8 Snowball on ice. You and your son is throwing snow balls at each other on a slippery (frictionless) frozen lake. Your mass is 80 kg and his mass is 20 kg, and you both start at rest.

You throw a big snow ball (2 kg) towards your son. The snow ball has an initial speed of 20 m/s and you throw it at an angle of 30° with the horizon.
(**a**) What is the momentum of the snow ball?
(**b**) What are you and your sons velocities after you have thrown the snow ball, but before the snow ball reaches him?
(**c**) Your son catches the snow ball. What are your and your sons velocities now?

12.9 Toppling a book. You try to topple a book standing on its shortes end by throwing a ball at it. You have two balls at your disposal. One elastic ball that collides elastically with the book, and one inelastic ball, that sticks to the book during the collision.
(**a**) Which ball should you choose? Explain your reasoning.

12.10 Bullet and a block. You fire a bullet of mass 100 g horizontally into a block of mass 2 kg where it gets stuck. The block lies on a frictionless table, but after the collision, the block enters a rough region of the table with a dynamic coefficient of friction, $\mu = 0.5$. After entering the rough region, the block slides a distance of 10 cm before stopping.
(**a**) What was the velocity of the bullet?
(**b**) What is the loss of energy during the collision?

12.11 Stopping a ball. A ball is hitting the floor with a vertical velocity v_0. The collision between the ball and floor is elastic.
(**a**) If you were able to move the floor up or down (using a vertical accelerator), is it possible to move the floor in a way so that the ball stops? Explain your answer.

12.12 Pendulum and block. A pendulum consisting of a sphere of mass m attached to the end of a massless rope of length L is initially lifted so that the rope is tight and horizontal, and released. At the bottom of its path, the sphere hits a block of mass $M = 2m$ lying at rest on a frictionless table. The collision is elastic.
(**a**) Find the velocities of the sphere and the block immediately after the collision.
(**b**) How high does the pendulum swing after the collision?

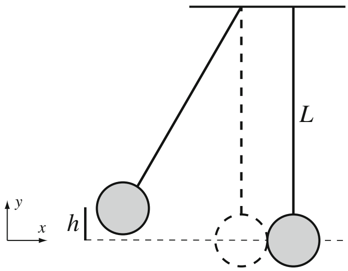

Fig. 12.20 Illustration of Newton's cradle with two balls

12.13 Lifting a chain. You are lifting a chain of mass m and length L from a heap on a table. Show that the force F needed to pull the chain up at a constant velocity v_0 is $F = (m/L)\left(v_0^2 + gy\right)$, where y is the length of the chain that has already been lifted from the table.

Projects

12.14 Newton's cradle. In this project you will learn about collisions and conservations laws by studying the behavior of Newton's cradle. Newton's cradle is a toy consisting of a series of steel balls each suspended by two strings so that the balls form a horizontal line when the cradle is at rest. The balls are initially barely touching each other. You can play with the toy by lifting and releasing a ball on one side. When the moving ball hits the stationary balls, a single ball is ejected on the other side, and the other balls remain stationary.

First, we study a cradle consiting of two balls of identical masses m hanging in thin strings as illustrated in Fig. 12.20. The left ball is lifted to a vertical height h_0 and released. The left ball hits the right ball when the string points directly down.
(**a**) Find the velocity v_0 of the left ball immediately before it hits the right ball.
(**b**) Assume the collision between the balls is elastic. Find the velocities v_1^A and v_1^B of the two balls after the collision. How does your result compare with the behavior of Newton's cradle described above?
(**c**) What is the maximum height, h_1, of the right ball?
(**d**) Assume the collision is perfectly inelastic. Find the maximum height h_1 reached by the right ball after the collision.

Assume the collision is characterized by a coefficient of restitution, r. The relative velocity after the collision is then related to the relative velocity before the colision by $v_1^B - v_1^A = r v_0$.
(**e**) Find the velocities of each of the balls after the collision.

We will in the following study a system with three balls, A, B, and C. We will assume that all forces are conservative, so that all collisions are elastic. Initially, immediately before the collision, ball A has a positive velocity v_0 and the other balls are not moving.

(f) Let us assume that the balls are seperated by small distances, so that there are two collisions, first between ball A and B and then between B and C. What are the velocities of the balls after the first collision? And after the second?

Let us now assume that all the balls are initially in contact, so that we cannot assume that there are two seperate, subsequent collision. This is the configuration corresponding to Newton's cradle.

(g) Find equations relating the initial and final velocities of all three balls. Can you solve these equations?

In order to understand what happens in Newton's cradle when all the balls are initially in contact, we will develop a simple, numerical model of the process. In the numerical model we will only address the collision itself, and we will assume that the motion of all the balls is one-dimensional along the x-axis during the collision.

We introduce an explicit model for the forces between the balls, and use this to calculate the motion of all the balls throughout the collision using Newton's second law for each of the balls.

The position of the balls are given as x_i, $i = 0, 1, 2$. At the beginning of the collision, at $t = 0$, all the balls are just in contact, so that the distance between them is equal to their diameters, d, $x_i = i\,d$.

The force on ball i from ball $i + 1$ is modelled using a simple, position-dependent force on the form

$$F_{i,i+1} = \begin{cases} -k\,|x_{i+1} - x_i - d|^q & \text{when } x_{i+1} - x_i < d \\ 0 & \text{when } x_{i+1} - x_i \geq d \end{cases} . \tag{12.137}$$

The following program solves the equations of motion from a time $t = 0$ to a time $t = t_1$. You must choose the mass, m, the constant k, and initial conditions for the simulation yourself.

```
from pylab import *
def force(dx,d,k,q): # force function
    if dx<d:
        F = k*abs(dx-d)**q
    else:
        F = 0.0
    return F
N = 2    # nr of balls, <-- Modify from here
m =  ... # kg
k =  ... # N/m
q = 1.0
d =  ... # m
v0 = ... # m/s
time = . # s
dt = ... # s,          <-- to here
n = int(round(time/dt))
x = zeros((n,N),float)
v = x.copy()
t = zeros(n,float)
for i in range(N): # Initial conditions
    x[0,i] = d*i
v[0,0] = v0
for i in range(n-1):
    F = zeros(N,float)
    for j in range(1,N):
        dx = x[i,j] - x[i,j-1]
```

```
        F[j] = F[j] + force(dx,d,k,q)
    for j in range(N-1):
        dx = x[i,j+1] - x[i,j]
        F[j] = F[j] - force(dx,d,k,q)
    a = F/m
    v[i+1] = v[i] + a*dt
    x[i+1] = x[i] + v[i+1]*dt
    t[i+1] = t[i] + dt
for j in range(N):
    plot(t,v[:,j]), hold('on')
print 'v/v0 = ',v[n-1,:]/v0
```

(**h**) Test the program and your parameters by direct comparison with your results above for $N = 2$, where N is the number of balls. Your answer to this and the following questions should include plots of the velocities. Hint: You must ensure that the timestep dt is chosen reasonably compared to the values of k and m.
(**i**) Use the program to determine the result of a collision when $N = 3$. What are the velocities of the balls immediately after the collision? Is this result physically reasonable? Does this correspond to the behavior you expect for Newton's cradle?
(**j**) Modify the force law by changing k and q. Can you find parameters that produce a behavior close to what you observe in Newton's cradle, that is, for which the velocity of the middle ball is close to zero after the collision?
(**k**) Can you now explain why only one ball is ejected from the left side when one ball is released from the right side in the toy cradles you can buy?

12.15 Catching an atom. In this project we will study a collision between two identical atoms of mass m that both are affected by forces from a massive particle such as a molecule.

First, we study the behavior of a single atom affected by a force from the molecule. The potential energy for the interaction between the atom and the molecule is:

$$U(x) = \begin{cases} \infty & \text{when} \quad x < b - d \\ \frac{1}{2}k\,(x-b)^2 & \text{when} \quad b - d < x < b + d \\ U_0 & \text{when} \quad x > b + d \end{cases} , \tag{12.138}$$

where b and d are lengths and $d < b$,

$$U_0 = \frac{1}{2}kd^2 , \tag{12.139}$$

and x is the position of the atom. We assume that the molecule is stationary at the point $x = 0$. (The atom cannot enter the region where the potential is infinite. You may instead assume that the energy is very large, $U_1 \gg U_0$, if you find this easier to discuss).
(**a**) Sketch the potential. Draw in an example of the motion of the atom when the total energy is less than U_0, and a motion where the total energy is larger than U_0, and describe the motions briefly.
(**b**) Find the force $F(x)$ on the atom as a function of x.

We will now study a collision between an atom (B) of mass m which start from rest at the point $x_B = b$, and an identical atom (A) which starts at $x_A > b + d$ with

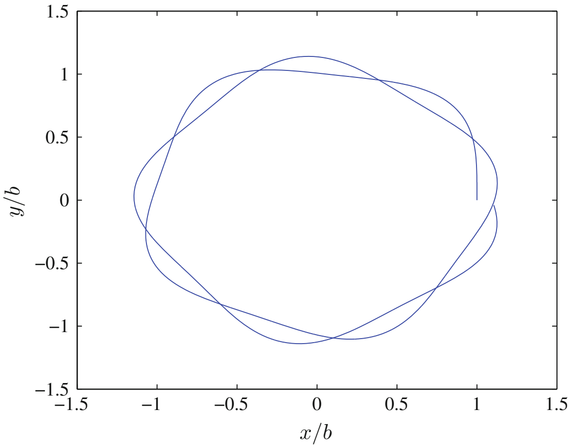

Fig. 12.21 Sketch of simulated motion

an initial velocity $-v_{A,0}$. For each of the atoms, the interaction with the molecule can be described by the potential energy $U(x)$, so that the potential energy for atom A is $U(x_A)$ and the potential energy for atom B is $U(x_B)$. There are no long-range interactions between the atoms. They only interact when they are in the same point, $x_A = x_B$. In that case they collide. After the collision they become attached to each other. You can assume that the atoms have not moved significantly during the collision.

(**c**) Find the velocity of atom A in the point $x_A = b$ immediately before the collision.
(**d**) Find the velocities of atom A and atom B immediately after the collision.
(**e**) How large must $v_{A,0}$ be in order for atom B (and atom A) to detach itself from the molecule after the collision? (An atom is detached if it can move infinitely far away from the molecule).

We will now study the same process, but in two dimensions. The massive molecule is now at rest at the origin, and the potential energy of an atom has the same form as above, but is now a function of the distance $r = \sqrt{x^2 + y^2}$ to the origin (Fig. 12.21):

$$U(r) = \begin{cases} \infty & \text{when} \quad r < b - d \\ \frac{1}{2}k\,(r-b)^2 & \text{when} \quad b - d < r < b + d \\ U_0 & \text{when} \quad r > b + d \end{cases} , \qquad (12.140)$$

(**f**) Show that the force on the atom can be written as $\mathbf{F}(\mathbf{r}) = -k(r-b)\frac{\mathbf{r}}{r}$ when $b - d < r < b + d$.
(**g**) The atom starts with velocity $\mathbf{v}_0$ in the position $\mathbf{r}_0$ at the time $t_0 = 0$. Write a program to find the position of the atom as a function of time. Plot the trajectory of the atom.
(**h**) We use the program to simulate the motion of the atom when $\mathbf{r}_0 = (b, 0)$ and $\mathbf{v}_0 = (0, v_0)$. The result is shown in the figure below. Explain the results. How would you measure the period of this motion in your program?

(**i**) You want the atom to follow a circular orbit around the molecule with a constant speed v. How do you have to choose the initial conditions to obtain such an orbit? Can you get a circular orbit for all speeds v? Explain your answer.
(**j**) How would you need to modify your program to model the motion of atom A before and after the collision. (After the collision atom A and atom B moves as one point particle).
(**k**) Atom A starts with velocity $\mathbf{v}_0$ in the position $\mathbf{r}_0$ at the time $t = t_0$ and collides with atom B at the time t_1. Is it possible to get atom B (and atom A) to detach from the molecule afte the collision?
(**l**) How do you have to choose $\mathbf{v}_0$ and $\mathbf{r}_0$ to make atom B follow a circular orbit after the collision.