Chapter 14
Rotational Motion

How can we describe the rotational motion of the Earth and how can we calculate the velocity of a point on the surface of the Earth due to its rotation?

We can move a rigid body about by moving it, through translation, and by rotating it, through rotation. Up to now we have only discussed translational motion. In this chapter we will introduce the tools to describe rotations.

14.1 Rotational State—Angle of Rotation

How can we describe the motion of the rod shown in Fig. 14.1? We would like to seperate the translational and rotational motion of the rod. In this case, for a rod thrown across the room, the rod rotates around its center of mass. We can therefore use the center of mass, $\mathbf{R}(t)$, to describe the translational motion of the rod. This is a good choice, since the motion of the center of mass is determined from the external forces acting on the rod—we could therefore find the motion of the center of mass by solving the equations of motion. If we study the motion of the rod relative to the center of mass, we get the bottom-right part of Fig. 14.1. How can we describe the rotational state of this system? By the angle θ it has rotated around the center of mass!

Angle and Axis of Rotation

While a freely moving object (such as a rod thrown across the room) usually rotates around its center of mass,[1] objects can also rotate around other points. We could for example nail the rod to the wall in any point along the rod, and the rod would be

[1] You will learn more about conditions for this later, when we discuss the physics of rotational motion.

A. Malthe-Sørenssen, *Elementary Mechanics Using Python*,
Undergraduate Lecture Notes in Physics, DOI 10.1007/978-3-319-19596-4_14

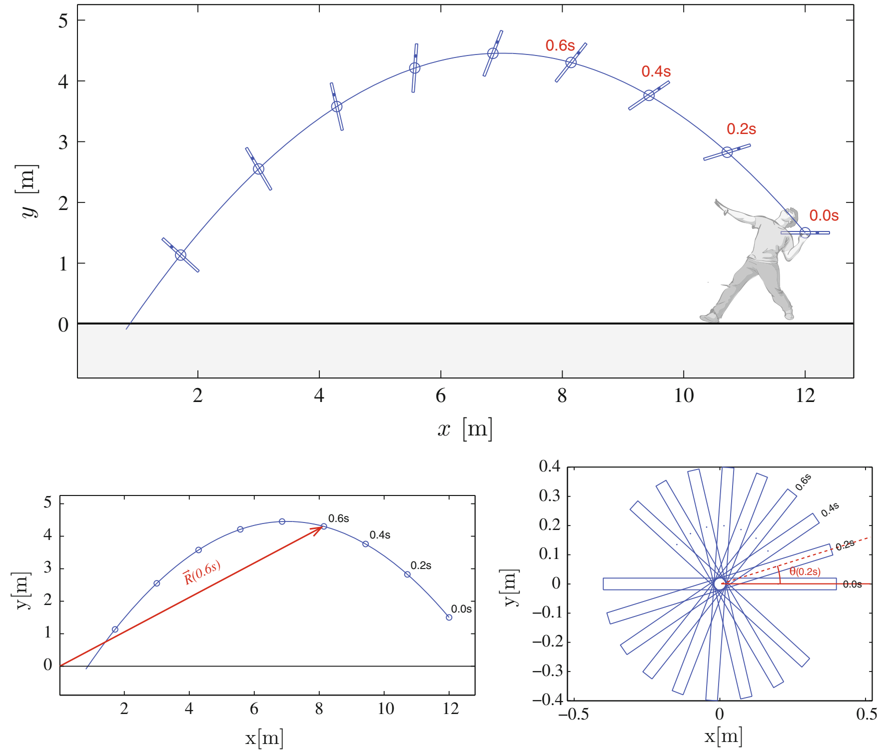

Fig. 14.1 The motion of a rotating rod thrown through the air

forced to rotate around this attachment point. The rotational configuration of the rod is often called the rotational *state* of the rod. In order to uniquely define the rotational state of the rod we need to specify both the attachment point O and the angle θ the rod forms with the horizontal. But if we only specify the point O, we do not really know how the object rotates around this point. We need to specify the **rotational axis** as well as a point on the axis. For rotations in the xy-plane, we say that the rotational axis is normal to this plane, that is, along the z-axis. This description holds for rotations in two dimensions. We describe the three-dimensional case in Sect. 14.6.

> In two dimensions, the rotational configuration of an object is described by: the angle θ; the point O it is rotating around; and the direction of the rotational axis, $\mathbf{k}$.

How do we describe the positive rotational direction? This is customarily determined by the right hand rule. Figure 14.2 shows how the direction of the positive z-axis is determined from the directions of the x- and y-axes. We can also use this rule

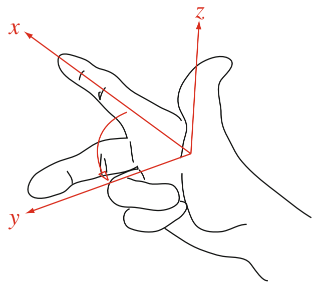

Fig. 14.2 Illustration of the right-hand rule

backwards: Given the direction of an axis, such as the z-axis, we can find the positive rotational direction by pointing the right thumb in the direction of the axis: the positive rotational direction is then in the direction your remaining fingers are curling: from the x- towards the y-axis. In this direction θ increases, in the opposite direction the angle decreases.

A Point on a Rotating Object

Given the angle θ and the rotation axis (including both a point on the axis and the positive direction along the axis), we can unqiuely determine the orientation of a rotating object. But how do we find the position of a particular point on a rotating object from this?

Figure 14.3 shows the motion of a point P on an object rotating around a fixed axis. We describe the position of P using a coordinate system that rotates along with the object. The rotating coordinate system has to unit vectors that rotate with the object: the unit vector $\hat{u}_r$, which is directed radially outwards from the rotation axis, and an axis normal to the radial direction with unit vector $\hat{u}_n$. A point on the rod can be described in this coordinate system by:

$$\mathbf{p} = p_r \hat{u}_r + p_n \hat{u}_n \ . \tag{14.1}$$

When the object has rotated an angle θ both unit vectors have also rotated. The radial unit vector now forms angle θ with the horizontal, and is given as:

$$\hat{u}_r = \cos\theta \, \mathbf{i} + \sin\theta \, \mathbf{j} \ , \tag{14.2}$$

as illustrated in Fig. 14.3. The normal vector, $\hat{u}_n$, is obtained by rotating $\hat{u}_r$ 90° in the positive direction:

$$\hat{u}_n = -\sin\theta \, \mathbf{i} + \cos\theta \, \mathbf{j} \ . \tag{14.3}$$

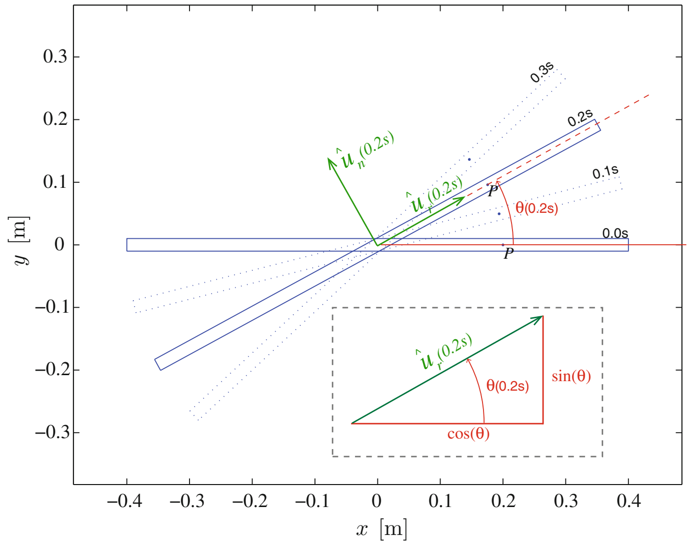

Fig. 14.3 Illustration of the unit vector $\hat{u}_P$, and the position of the point P during rotation of a rod around an axis through the origin

If the object is rotating around a fixed axis, this gives the position of any point on the object. If the object is rotating around a moving axis, such as the rotating rod in Fig. 14.1, we also need to add the position of the axis—here given as the position of the center of mass:

$$\mathbf{p} = \mathbf{R} + p_r \hat{u}_r + p_n \hat{u}_n \ . \tag{14.4}$$

The attentive reader may recognize the decomposition using the unit vector $\hat{u}_r$ and $\hat{u}_n$ as polar-coordinates. This is indeed correct.

Rotational Motion

We can describe the rotational motion of the rod from Fig. 14.1 by a motion diagram for the rod. We have illustrated one such diagram in the bottom-right part of Fig. 14.1, where we show the position of the rod at various times, t_i, taken at constant time intervals Δt. This plot looks like a movie of the motion of the rod, where all the images have been superimposed into one image. A better way to visualize the rotational motion of the rod is to plot the time evolution of the angle, $\theta(t)$. Figure 14.4 shows $\theta(t)$ for the rotational motion in Fig. 14.1.

Test your understanding: Can you sketch two other motion diagrams for a rod that is rotating in the negative direction and for a rod that is rotating faster and faster in the positive direction. Sketch the corresponding diagrams for $\theta(t)$.

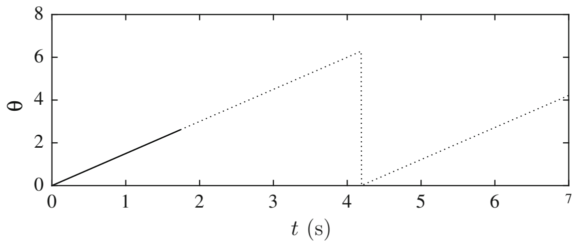

Fig. 14.4 Plot of the angle, $\theta(t)$, for the rotational motion in Fig. 14.1. *Dashed curve* shows how the rod would have continued to rotate if it had not hit the ground—such as if it fell off a cliff

Periodicity of the State $\boldsymbol{\theta(t)}$

The angle, θ, describes a unique configuration of the rod for values from 0 to 2π (measured in radians). What happens when $\theta(t)$ increases beyond 2π? When θ reaches 2π the rod has rotated a full revolution, and the rod is in the same position as it was when θ was equal to 0. We cannot discern these positions: The position of the rod when $\theta = 2\pi$ is exactly the same as when $\theta = 0$. However, it is customary to only use angles between 0 and 2π to describe the rotational position. This means that if the angle is larger than 2π we subtract 2π from the angle. This is seen in Fig. 14.4: When the angle $\theta(t)$ reaches 2π, it continues at $\theta = 0$. Similarly, when the angle decreases below 0, we add 2π to the angle, so that it continues at 2π. You are, of course, free to choose to describe the motion using an angle θ that increases also beyond 2π, but then you have to remember that the motion is periodic so that higher values does not represent new positions.

14.2 Angular Velocity

During rotation, the angle $\theta(t)$ changes with time. How can we characterize how fast the rod rotates? By the angular velocity, which is defined as the rate of the change of the angle in analogy with the (translational) velocity, which is the rate of change of the position. During the time interval from t to $t + \Delta t$, the angle changes from $\theta(t)$ to $\theta(t + \Delta t)$. We define the **average angular velocity** over the time Δt as:

$$\bar{\omega} = \frac{\theta(t + \Delta t) - \theta(t)}{\Delta t} = \frac{\Delta\theta}{\Delta t} , \tag{14.5}$$

When the time interval becomes small, we find the *instantaneous angular velocity* for the rotational motion, which we in the following call the **angular velocity**:

Angular velocity:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} = \dot{\theta} \ . \tag{14.6}$$

Figure 14.4 shows the angle, $\theta(t)$, and the angular velocity, $\omega(t) = d\theta/dt$ for the rotational motion in Fig. 14.1. Since the angular velocity is the time derivative of the angle, we interpret the angular velocity as the slope of the $\theta(t)$ curve (just as we did for the translational velocity). We see that the motions in Fig. 14.1 has a constant, positive angular velocity.

Test your understanding: Can you sketch $\theta(t)$ and $\omega(t)$ for a rod that is rotating equally fast in the opposite direction?

Velocity of a Point on a Rotating Body

As the rod rotates, every part of the rod moves in a circle around the rotation axis. What is the velocity of a small part of the rotating rod, and how can we relate it to the angular velocity? Let us address the motion of a small part, P, of the rotating body directly. We have illustrated its motion during a small time interval Δt, in Fig. 14.5. The distance from P to the rotation axis is R. The small part P moves along a circular path around the rotation axis with R as the radius. During the small

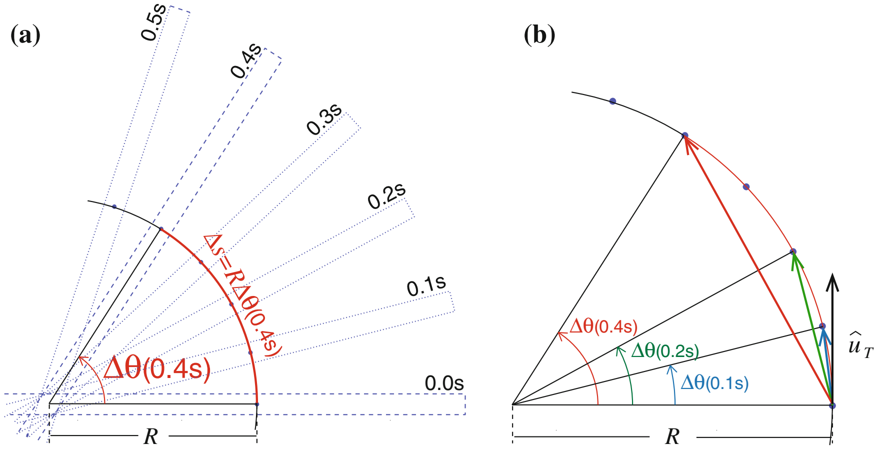

Fig. 14.5 **a** Illustration of the motion of a small part, P, of a rod rotating around an axis through the origin. **b** Illustration of the velocity vector for P as the time interval Δt decreases

time interval Δt, the rod has rotated an angle $\Delta\theta$ from the orientation $\theta(t)$ to the new orientation $\theta(t + \Delta t) = \theta(t) + \Delta\theta$. How far has P moved? It has moved the arc length $\Delta s = R\Delta\theta$ along its circular path. The speed of the small part P is therefore:

$$v = \frac{\Delta s}{\Delta t} = R\,\frac{\Delta\theta}{\Delta t}\ . \tag{14.7}$$

If we let the time interval Δt become infinitesimally small, we find **the speed of the point** P to be:

$$\boxed{v = \frac{ds}{dt} = \frac{d}{dt}\,(R\,\theta) = R\,\frac{d\theta}{dt} = R\,\omega\ .} \tag{14.8}$$

The speed of a point on the rod is therefore proportional to the angular velocity of the rotation, but also proportional to the distance R to the rotational axis: Points further away from the rotation axis rotate with higher speeds.

What is the direction of the velocity vector for P? Fig. 14.5 shows that when the time interval Δt becomes smaller, the change in angle $\Delta\theta$ also becomes smaller, and the direction of the displacement vector from P at time t to P at time $t + \Delta t$ approaches that of a tangent to the circle. Excatly the same result we found earlier when we studied circular motion. The velocity vector is therefore parallel to the tangent to a circle of radius R, and points in the direction of the tangential unit vector $\hat{u}_T$. The velocity of the point P is therefore:

$$\mathbf{v} = R\,\omega\,\hat{u}_T\ . \tag{14.9}$$

Motion with Constant Angular Velocity

If an object rotates with a constant angular velocity, we can find the speed of the point P from the distance traveled during one complete revolution, $s = 2\pi R$, divided by the time of one revolution, call the period T:

$$v = \frac{s}{T} = \frac{2\pi R}{T}\ , \tag{14.10}$$

where R is the distance from P to the rotation axis. We also know that the velocity is $v = R\omega$, therefore we find that:

$$v = \frac{2\pi}{T} R = \omega R \ \Rightarrow\ \omega = \frac{2\pi}{T}\ . \tag{14.11}$$

The angular velocity is often also called the angular frequency.

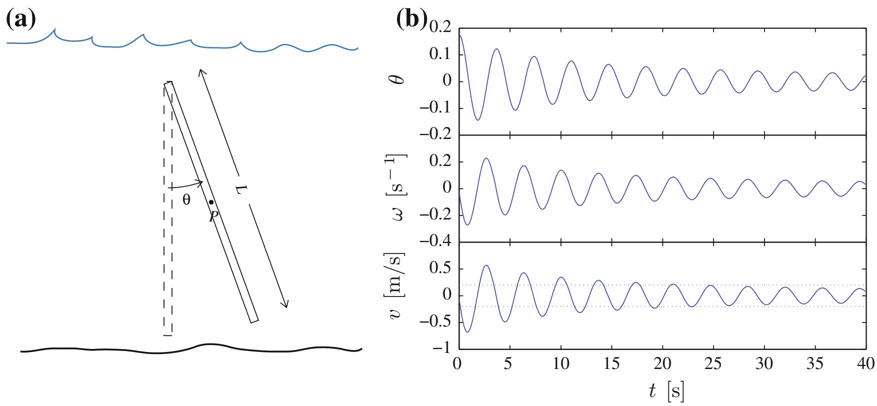

Fig. 14.6 **a** Illustration of the antenna. **b** Plots of the angle, θ, the angular velocity, ω, and the speed of the center of the antenna, v

14.3 Angular Acceleration

The rotation may occur at a constant angular velocity, as in Fig. 14.4, or the angular velocity may vary. By analogy with translational motion, we characterize the rate of change of the angular velocity by *the angular acceleration*, α, defined as:

Angular acceleration:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta} \ . \tag{14.12}$$

14.3.1 Example: Oscillating Antenna

Problem: You are using an underwater antenna to measure the electromagnetic response of the seafloor to search for hydrocarbons. The antenna has a length $L = 5$ m and is suspended from its top point above the seafloor (see Fig. 14.6a). You cannot use the antenna until the speed of its center is less than $v_c = 0.2$ m/s. You have attached a measurement device to the antenna that measures the angle θ the antanna forms with the vertical as a function of time. You measure the behavior after you have lowered it to the bottom, and the resulting data is in antennaangles.dat.[2] How long time does it take before you can use the antenna?

[2] http://folk.uio.no/malthe/mechbook/antennaangles.dat.

Approach: We read the data, find the angular velocity as a function of time by a numerical derivative, use this to find the speed of a point on the antenna, and find when the speed is less than the threshold v_c.

Solution: We read the file, getting a set of angles, $\theta(t_i)$, measured at discrete times, t_i:

```
from pylab import *
t, theta = loadtxt('antennaangles.dat',usecols=[0,1],unpack=True)
plot(t,theta)
```

The resulting angles $\theta(t_i)$ are shown in Fig. 14.6b. The angular velocity is the time derivative of the angle, but since we only know the angles for discrete times, we must calculate the derivative numerically:

$$\omega(t_i) \simeq \frac{\theta t_{i+1} - \theta t_i}{t_{i+1} - t_i} , \tag{14.13}$$

which is implemented as:

```
n = len(theta)
omega = zeros(n,float)
for i in range(n-1):
    omega[i]=(theta[i+1)-theta[i])/(t[i+1]-t[i])
plot(t,omega)
```

The velocity of a point P at the middle of the rod can be found from $v = \omega R$, where $R = L/2$ is the distance from the rotation axis to the point P. We calculate and plot the results:

```
R = 2.5
v = omega*R
plot(t,v)
```

From the figure we find that we need to wait for approximately $t \simeq 25$ s before we can use the antenna.

14.4 Comparing Linear and Rotational Motion

We have now introduced the angle θ, the angular velocity ω, and the angular acceleration α used to describe the rotational motion of an object. Our definitions are clearly similar to the definitions we used to introduce position, x, velocity v, and acceleration a for linear motion, as illustrated in Table 14.1. Notice the analoguous form of these equations: They are identical from a mathematical point of view. It is only our physical interpretation, and therefore also the units, which are different. The mathematical methods we use to determine motion are the same for linear and rotational motion. You can therefore use all the techniques you have developed to study linear motion also to address rotational motion.

Table 14.1 Comparison of linear and rotational motion

Motion	Linear	Rotation
Position	$x(t)$	$\theta(t)$
Velocity	$v(t) = \frac{dx}{dt}$	$\omega(t) = \frac{d\theta}{dt}$
Acceleration	$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	$\alpha(t) = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

14.5 Solving for the Rotational Motion

The structured problem-solving approach we developed for linear motion is also applicable for rotational motion: We *identify* the quantities to be modeled; we *model* the system, resulting in a set of differential equations; we *solve* the differential equations; and *analyze the results. We will see that for rotational motion, we usually need to solve the equation of motion where the angular acceleration α is given:

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \alpha(t, \theta, \omega) \ , \tag{14.14}$$

with initial conditions $\theta(t_0) = \theta_0$ and $\omega(t_0) = \omega_0$. You should realize that this is *exactly* the same equation we have solved over and over again for linear motion, only with different symbols (and interpretations) for the variables. You can therefore use the machinery you have developed and are comfortable with from linear kinematics. You do not need to learn anything new!

Analytical Integration

For a given $\alpha(t)$ we know how to *solve* to determine the motion: We solve by direct integration. The angular velocity is the time integral of the angular acceleration:

$$\omega(t) - \omega(t_0) = \int_{\omega_0}^{\omega} d\omega = \int_{t_0}^{t} \frac{d\omega}{dt} dt = \int_{t_0}^{t} \alpha(t)\, dt \ . \tag{14.15}$$

And the angle is the integral of the angular velocity:

$$\theta(t) - \theta(t_0) = \int_{t_0}^{t} \omega(t)\, dt = \int_{t_0}^{t} \left(\omega(t_0) + \int_{t_0}^{t} \alpha(t)\, dt \right) dt$$
$$= \omega(t_0)\,(t - t_0) + \int_{t_0}^{t} \left(\int_{t_0}^{t} \alpha(t)\, dt \right) dt \ . \tag{14.16}$$

For a constant angular acceleration $\alpha(t) = \alpha_0$ we find

$$\omega(t) - \omega(t_0) = \int_{t_0}^{t} \alpha_0 \, dt = \alpha_0 \, (t - t_0) \ , \tag{14.17}$$

$$\theta(t) - \theta(t_0) = \int_{t_0}^{t} (\omega(t_0) + \alpha_0 \, (t - t_0)) \ dt = \omega(t_0) \, (t - t_0) + \frac{1}{2}\alpha_0 \, (t - t_0)^2 \ . \tag{14.18}$$

Analytical Solution

If the angular acceleration instead is a function of θ or ω, we cannot integrate, but must instead solve the resulting differential equation. For example, if the angular acceleration is $\alpha = -C\omega$, and $\omega(0) = \omega_0$, then we must solve the equation

$$\frac{d\omega}{dt} = \alpha = -C\omega \ , \ \omega(0) = \omega_0 \ . \tag{14.19}$$

We now recoqnize this equation immediately, knowing from experience that the solution is just the same as we found for motion with air drag. The solution to this equation is on the form $\omega(t) = A \exp(-Ct)$, where A is determined from the initial condition: $\omega(0) = A = \omega_0$.

Symbolic solution: If you do not remember how to solve this equation, you can always use the symbolic solver in Python. Equation (14.19) is solved by

```
>> from sympy import *
>> omega = Function('omega')
>> t = symbols('t')
>> omega0 = symbols('omega0')
>> C = symbols('C')
>> dsolve(Derivative(omega(t),t)+C*omega(t),omega(t))
omega(t) == C1*exp(-C*t)
```

This is the same solution as we found above, we just need to determine the value of `C1` from the initial conditions.

Numerical Integration

These approaches work fine as long as we can solve the problems analytically. However, most problems of practical interest are not solveable, just like for linear motion. However, we can use the same approach and the same numerical methods to solve the equations of motion for the angular motion, as we have done for linear motion.

Euler-Cromer's method for angular motion: Euler-Cromer's method follows exactly the same scheme as for linear motion:

$$\omega(t_i + \Delta t) = \omega(t_i) + \Delta t\, \alpha(t_i) \tag{14.20}$$

$$\theta(t_i + \Delta t) = \theta(t_i) + \Delta t\, \omega(t_i + \Delta t)\ , \tag{14.21}$$

Motion of underwater antenna: We can demonstrate this method by determining the motion for an angular acceleration on the form, $\alpha = -c_1 \sin\theta - c_2\omega^2$, with initial conditions $\theta(0) = 10°$ and $\omega(0) = 0$. (This is a reasonable model for the motion of the antenna in Sect. 14.3.1. You will later learn how to find the parameters for such models.). We solve this problem through the following implementation of Euler-Cromer's method:

```
g = 9.81, L = 5.0, D = 60.0   # Parameters
M = L*(0.10)**2*5.0*1e3
c1 = 1.5*g/L
c2 = 0.75*(D*L)/M
time = 60.0
dt = 0.0001
omega0 = 0.0
theta0 = 10.0*pi/180.0
n = int(time/dt);             # Arrays
theta = zeros(n,float)
omega = zeros(n,float)
alpha = zeros(n,float);9
t = zeros(n,float)
theta[0] = theta0             # Initial conditions
omega[0] = omega0
for i in range(n-1):          # Integration loop
    alpha[i] = -c1*sin(theta[i])-c2*abs(omega[i])*omega[i]
    omega[i+1] = omega[i] + alpha[i]*dt
    theta[i+1] = theta[i] + omega[i+1]*dt
    t[i+1] = t[i] + dt
```

This simulation produces the motion seen in Fig. 14.6b.

14.5.1 Example: Revolutions of an Accelerating Disc

Problem: A DVD is accelerating at a constant rate of 2 rad/s^2 starting from rest. How many times have the disc rotated in 10 s?

Approach: We find the angle as a function of time, use this to find how far the disc has rotated in 10 s, and finally find how many rotations this corresponds to.

Solve: We find the angle as a function of time by first integrating the angular acceleration to find the angular velocity, and then integrating the angular velocity to find the angle. The disc rotates with a constant angular acceleration α starting at rest at $t_0 = 0$ s:

$$\omega(t) - \omega(0\,\text{s}) = \int_0^t \alpha\, dt = \alpha t\ , \tag{14.22}$$

where $\omega(t_0) = 0\,\text{s}^{-1}$ since the disc starts at rest at $t_0 = 0\,\text{s}$. We find the angle θ by integration:

$$\theta(t) - \theta(0\,\text{s}) = \int_{0\,\text{s}}^{t} \omega(t)\,dt = \int_{0\,\text{s}}^{t} (\alpha t)\;dt = \frac{1}{2}\alpha t^2\;. \tag{14.23}$$

We insert $\alpha = 2\,\text{rad/s}^2$ to find the angle after 10 s:

$$\theta(10\,\text{s}) = \frac{1}{2} 2\,\text{s}^{-2}\,(10\,\text{s})^2 = \frac{1}{2}\,2 \times 100 = 100\;. \tag{14.24}$$

The angle θ is related to the number of rotations through $\theta = n 2\pi$, where the angle 2π corresponds to one rotation. We find the number of rotations after 10 s from:

$$n(10\,\text{s}) = \frac{\theta(10\,\text{s})}{2\pi} = \frac{100}{2\pi} \simeq 15.9\;. \tag{14.25}$$

That is, the disc rotates 15.9 times in 10 s.

14.5.2 Example: Angular Velocities of Two Objects in Contact

Problem: Two gears with radius R_A and R_B are rotating and in contact as illustrated in Fig. 14.7. The angular velocity of wheel A is ω_A. Find the angular velocity of wheel B. What is the relationship between the angular acceleration of wheels A and B?

Solution: First, we notice from the figure that the angular velocities must have opposite signs. Second, the condition that the two wheels are rotating without sliding means that their speeds at the point of contact must be equal, but oppositely directed.

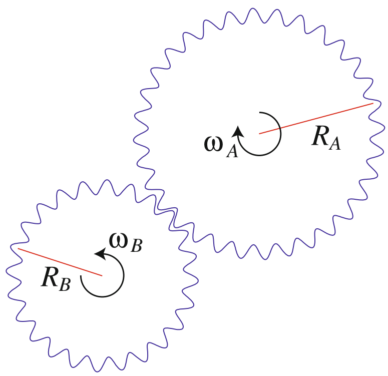

Fig. 14.7 Two gears with radius R_A and R_B are rotating without sliding relative to each other

We call such a condition a *kinematic condition*. We will frequently use such conditions when we solve problems in mechanics.

For each of the wheels, the velocity at the point of contact is related to the angular velocity by:

$$v_A = R_A \omega_A \ , \ v_B = R_B \omega_B \ . \tag{14.26}$$

Since the velocities at the point of contact are equal and oppositely directed, we find that the angular velocities also are related:

$$v_A = -v_B \ \Rightarrow \ \omega_B = -\frac{R_A}{R_B}\omega_A \ . \tag{14.27}$$

This relation is general and does not requires the velocities to be constant. We can therefore find the angular accelerations by taking the time derivatives on each side:

$$\alpha_B = \frac{d}{dt}\omega_B = -\frac{d}{dt}\frac{R_A}{R_B}\omega_A = -\frac{R_A}{R_B}\alpha_A \ . \tag{14.28}$$

14.6 Rotational Motion in Three Dimensions

You now know how to describe rotations in a plane: Rotations around the origin in the xy-plane are described by the rotation angle θ, the angular velocity ω, and the angular acceleration α. We found that a point, P, on the rotating object follows a circular path with a constant distance R from the rotation axis as illustrated in Fig. 14.8, and that the velocity of P is:

$$v = R\omega \ , \tag{14.29}$$

directed along the tangent to the circle. Can we find a simple expression for both the direction and magnitude of the velocity? Yes! Since we know that the tangent

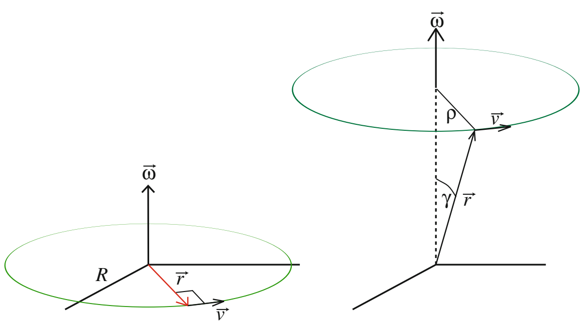

Fig. 14.8 An illustration of rotation around the z-axis. The direction of the velocity vector is given by $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$

is normal to the radius vector, **r**, and since it is in the xy-plane, the tangent is also normal to the unit vector in the z-direction. This allows us to write the velocity vector of the point P as:

$$\mathbf{v} = \omega \mathbf{k} \times \mathbf{r} \ . \tag{14.30}$$

This expression provides the correct magnitude and direction of the velocity vector: It points in the positive rotational direction if ω is positive, and in the negative rotational direction if ω is negative. We can therefore introduce the angular velocity as a vector:

$$\boxed{\boldsymbol{\omega} = \omega \mathbf{k} \ \text{(angular velocity vector)}} \tag{14.31}$$

The angular velocity vector, $\boldsymbol{\omega}$, points along the axis of the rotation. For rotation in the xy-plane, the rotation axis is the z-axis.

The expression in (14.30) is valid not only for rotation in the xy-plane, but for any rotation around the axis given by the $\boldsymbol{\omega}$ vector. This is illustrated in the right part of Fig. 14.8, where the velocity of a point at **r** going in a circular orbit around the rotation axis is given by the radius of the circle, ρ, and the angular velocity, ω:

$$v_\theta = \omega \rho \ . \tag{14.32}$$

We see from the geometry that the radius ρ is $r \sin(\gamma)$, where γ is the angle between the position vector, **r** and the angular velocity vector, $\boldsymbol{\omega}$. Therefore

$$v_\theta = \omega r \ \sin(\gamma) \ . \tag{14.33}$$

The direction of the velocity is tangential to the circular orbit: orthogonal to the position vector, **r**, and to the angular velocity vector, $\boldsymbol{\omega}$. This means that we can write the velocity as

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \ ,} \tag{14.34}$$

which gives both the correct magnitude: $|\mathbf{v}| = \omega r \sin(\gamma)$, and the correct direction. However, you can only use this expression when the origin of the position vector is *on* the rotation axis.

Let us use (14.34) to find the acceleration of the point P, which moves in a circular orbit with constant radius $\rho = R \sin(\gamma)$ around the rotation axis given by the angular velocity vector $\boldsymbol{\omega}$. The acceleration vector is the time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\boldsymbol{\omega} \times \mathbf{r} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} \ , \tag{14.35}$$

but we know that:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \ , \tag{14.36}$$

and we introduce the angular acceleration vector, $\boldsymbol{\alpha}$ as:

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} , \tag{14.37}$$

which inserted back into (14.35) gives:

$$\boxed{\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \ .} \tag{14.38}$$

For a motion with constant angular velocity, that is with constant speed, the angular acceleration vector is zero: $\boldsymbol{\alpha} = 0$, and the acceleration vector is:

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \ . \tag{14.39}$$

This is the *sentripetal acceleration on vector form*.

- The direction of the vector is correct. If you use the right-hand rule to find the vector product, you realize the vector points orthogonally inwards towards the rotation axis.
- The magnitude of the acceleration is $|\mathbf{a}| = a = \omega^2 \rho = (v/\rho)^2 \rho = v^2/\rho$, which is the magnitude of the sentripetal acceleration we found in (14.35).

14.6.1 Example: Velocity and Acceleration of a Conical Pendulum

Problem: A conical pendulum consists of a mass in a string of length L. The mass rotates with angular velocity ω in a circular orbit with radius R, as illustrated in Fig. 14.9. Find the velocity and acceleration of the mass.

Solution: The vector velocity of the conical pendulum is given as $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. We find the velocity when the pendulum crosses the x-axis. In this case, the position vector is

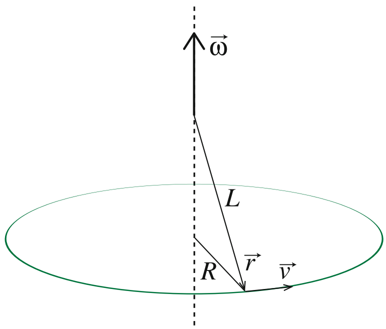

Fig. 14.9 Illustration of a conical pendulum

$$\mathbf{r} = R\,\mathbf{i} + \sqrt{L^2 - R^2}\,\mathbf{k}\,. \tag{14.40}$$

The rotation is about the z-axis. We therefore introduce the angular velocity vector as $\boldsymbol{\omega} = \omega\,\mathbf{k}$.

We find the vector velocity:

$$\begin{aligned}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} &= \omega\,\mathbf{k} \times \left(R\,\mathbf{i} + \sqrt{L^2 - R^2}\,\mathbf{k}\right) \\ &= \omega R\,(\mathbf{k} \times \mathbf{i}) = \omega R\,\mathbf{j}\,.\end{aligned} \tag{14.41}$$

Similarly, we find the acceleration vector:

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = 0 + \omega\,\mathbf{k} \times (\omega R\,\mathbf{j}) = -\omega^2 R\,\mathbf{i}\,. \tag{14.42}$$

The acceleration vector points in towards the axis of rotation, as expected.

Summary

Description of rotation:

- The rotation of an object is described by the angle $\theta(t)$
- The angular velocity of the object is: $\omega(t) = d\theta/dt$
- The angular acceleration of the object is: $\alpha(t) = d\omega/dt = d^2/dt^2$.

Rotation and translation:

- A point on the rotating body at a distance R from the rotational axis has a tangential velocity: $v = R\,\omega$.

Solving rotational motion:

- We solve problems in rotations using the same structured approach as for linear motion.
- In the "Solver" we solve the equation: $d^2\theta/dt^2 = \alpha(t, \theta, d\theta/dt)$ with the initial conditions $\theta(t_0) = \theta_0$ and $\omega(t_0) = \omega_0$.

Numerical solution:

- Numerically, we solve the equation using an iterative approach starting from the initial conditions. For example, we can use Euler-Cromer's method:

$$\begin{aligned}\omega(t_i + \Delta t) &= \omega(t_i) + \Delta t\,\alpha(\theta(t_i), \omega(t_i), t_i) \\ \theta(t_i + \Delta t) &= \theta(t_i) + \Delta t\,\omega(t_i + \Delta t)\end{aligned}$$

Analytical solution:

- When the angular acceleration, $\alpha = \alpha(t)$, is only a function of time, t, we can solve the equations by direct integration:

$$\omega(t) = \omega(t_0) + \int_{t_0}^{t} \alpha(t)\,dt \ , \ \theta(t) = \theta(t_0) + \int_{t_0}^{t} \omega(t)\,dt \ ,$$

  A typical example is motion with constant angular acceleration.
- For motion with **constant angular velocity** the solution is:

$$\theta(t) = \theta(t_0) + \omega\,(t - t_0) \ .$$

- For motion with **constant angular acceleration** the solution is:

$$\omega(t) = \omega(t_0) + \alpha\,(t - t_0)$$
$$\theta(t) = \theta(t_0) + \omega(t_0)\,(t - t_0) + \frac{1}{2}\alpha\,(t - t_0)^2 \ .$$

Rotational motion in three dimensions:

- Generally, rotations occur around an axis, given by the angular velocity vector, $\boldsymbol{\omega}$.
- The velocity of a point on the rotating object at position $\mathbf{r}$ is: $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$
- The acceleration of a point on the rotating object at position $\mathbf{r}$ is: $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$

Exercises

Discussion Questions

14.1 Flywheel. What is the acceleration of a point at a rim of a rotating flywheel when the flywheel is rotating at a constant rate and when the flywheel is speeding up?

14.2 Spin cycle. Explain the working of the spin cycle of a washing machine in terms of acceleration components.

14.3 Degrees and radians. Why do we use radians and not degrees to describe angles in rotational motion?

Problems

14.4 Flywheel position. The angular position of a flywheel on an engine is given as $\theta = c_1\,(t/t_1) + c_2\,(t/t_2)^2$, where c_1, and c_2 are dimensionless constants, and t_1 and t_2 are characteristic times.
(**a**) Find an expression for the flywheel's angular velocity.
(**b**) Find an expression for the flywheel's angular acceleration.

14.5 Unbalanced wheel. The angular position of an unbalanced wheel is given as $\theta = 5.0\,\text{rad}\,\sin\,(t/(2\,\text{s}))$.
(**a**) Find an expression for the wheel's angular velocity.
(**b**) Find an expression for the wheel's angular acceleration.

14.6 Earth and Sun. (**a**) What is the angular velocity of the Earth in its orbit around the Sun?
(**b**) What is the angular velocity of the Earth in its rotation about its own axis?

14.7 Engine. A car engine accelerates from 1000 to 2000 rpm at a constant rate during 15 s.
(**a**) Find the angular acceleration of the engine.
(**b**) Find the number of rotations the engine revolves from it starts at 1000 rpm until it has accelerated to 2000 rpm.

14.8 Spinning down. You start a spinning wheel by rotating it with an angular acceleration of $10\,\text{rad/s}^2$ for 3 s.
(**a**) What is the angular velocity of the wheel as a function of time for the first 3 s.
(**b**) Find the angle of the wheel as a function of time for the first 3 s.
After releasing the spinning wheel, it slows down at a constant rate of $0.1\,\text{rad/s}^2$.
(**c**) What is the angular velocity of the wheel as a function of time after the first 3 s.
(**d**) Find the angle of the wheel as a function of time after the first 3s.
(**e**) How long time does the wheel take to stop?
(**f**) How much longer would the wheel take to stop if you spun it for 6 seconds instead?

14.9 A slippery wheel. You are testing the behavior of a car wheel in a river bed. The angular acceleration of the wheel spinning semi-saturated in water is $\alpha = -k_\omega \omega$, where $k_\omega = 0.1\,\text{s}^{-1}$ for the wheel you are testing.
(**a**) The wheel starts with the angular velocity $\omega_0 = 10\,\text{rad/s}$ when you put the wheel into the water. Find the angular velocity of the wheel as a function of time.
(**b**) How long time does it take until the wheel has $1/10$th of its initial angular velocity?

14.10 Running the curve. A sprinter is running through a circular curve with radius 50 m with a constant speed of 10 m/s.
(**a**) What is the angular velocity of the sprinter?
(**b**) What is the angular acceleration of the sprinter?
(**c**) What is the linear acceleration of the sprinter?

14.11 Rotating Earth. The radius of the Earth is approximately 6378 km.
(**a**) What is the angular velocity of the Earth as it rotates around its own axis?
(**b**) What is the angular velocity of a person on the equator?
(**c**) What is the linear velocity of a person on the equator?
(**d**) What is the angular velocity of a person at 60° North?
(**e**) What is the linear velocity of a person at 60° North?
(**f**) What is the angular acceleration of a person on the equator and at 60° North?
(**g**) What is the linear acceleration of a person on the equator? Compare with $g = 9.8\,\text{m/s}^2$ and comment.
(**h**) What is the linear acceleration of a person at 60° North? (Find both the magnitude and direction of the acceleration.) Comment on the results.

14.12 Rolling wheel. A wheel of radius R is rolling without slipping along a flat surface. The center of the wheel is moving with the constant horizontal velocity v.
(**a**) Show that for the wheel not to slip, the angular velocity of the wheel must be $\omega = v/R$.
(**b**) What is the velocity of the point on the wheel that is in contact with the ground? (Measured in a coordinate system on the ground.)
(**c**) What is the velocity of a point on the top of the wheel?
(**d**) What is the acceleration of the point on the wheel that is in contact with the ground? (Relative to the ground).
(**e**) What is the acceleration of the point on the top of the wheel?