Chapter 16
Dynamics of Rigid Bodies

How do you spin a ball? And how do you jump-spin on skates? In this chapter you learn what causes changes in rotational motion using Newton's second law for rotational motion.

You know how to describe the rotation of a wheel around a fixed or moving axis, using the angle, the angular velocity, and the angular acceleration of the wheel. And you know how to find the kinetic energy of a rotating rigid body. But what causes changes in rotational motion? For translational motion we can use Newton's second law to determine the change in the translational state, in the translational momentum, from the external forces acting on a body. We use this both to find the acceleration of a body, and from the acceleration we can calculate the motion, and to find conservation laws for the translational momentum. Can we find a similar law for rotational motion? In this chapter we will introduce the rotational analogue to translational momentum: rotational momentum or angular momentum; the rotational analogue to force: torque; and the rotational analogue to Newton's second law: Newton's second law for rotational motion. Armed with these tools you will see that you are ready to solve any problem of moving and rotating rigid bodies, such as figuring out what causes a ball to spin or how you jump-spin on skates.

16.1 Motivating Example—Spinning a Wheel

Increasing the Angular Velocity of a Spinning Wheel

Figure 16.1 shows how a wheel is spun by a force **F** applied to a pedal, which is attached to a lever, which again is attached to the wheel. The wheel accelerates—it

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increases its angular velocity from $\omega_0 = 0$ rad/s to ω as the wheel rotates an angle θ. We can find the angular velocity ω as a function of θ from the work-energy theorem. The work done by **F** is:

$$W = \int_{t_0}^{t} \mathbf{F} \cdot \mathbf{v} \, dt, \tag{16.1}$$

where **v** is the velocity of the point where the force is acting, that is, the velocity of the pedal. The pedal rotates around a fixed axis, the axis of the wheel, and therefore moves in the tangential direction:

$$\mathbf{v} = v \hat{u}_T . \tag{16.2}$$

We can similarly decompose the force in the tangential and normal direction

$$\mathbf{F} = F_T \hat{u}_T + F_N \hat{u}_N , \tag{16.3}$$

as illustrated in Fig. 16.1. It is only the tangential component that contributes to the work. The normal component does no work since there is no motion in this direction. If we apply a constant tangential force, F_T, and this is the only force acting, the work-energy theorem gives:

$$W = F_T \, s = F_T \, R\theta = K_1 - K_0 , \tag{16.4}$$

where the distance $s = R\theta$ moved by the pedal depends on the distance R from the rotation axis to the point where the force is acting. The change in kinetic energy of

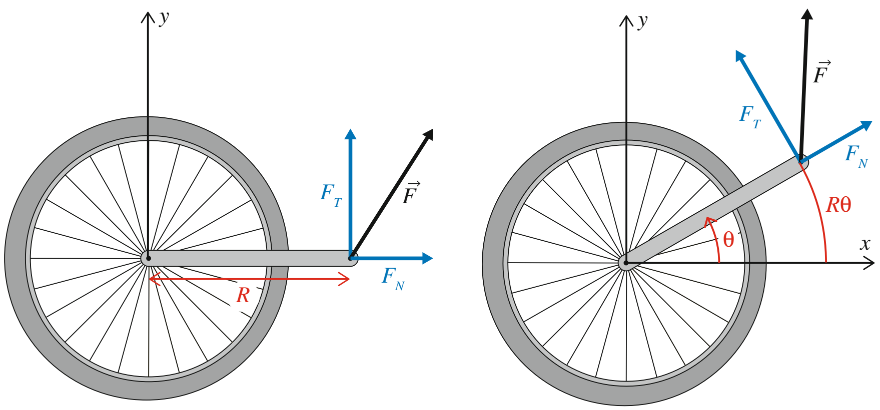

Fig. 16.1 We spin up a bicycle wheel by applying a force **F** of constant magnitude to an arm attached to the axis of the wheel. The wheel rotates around a fixed axis at its center, and the distance from the axis to the point where the force is applied is R

the rotating wheel is:

$$K_1 - K_0 = \frac{1}{2} I \omega^2 - \frac{1}{2} I \omega_0^2, \tag{16.5}$$

If we insert this into (16.4) and assume that we start from rest, $\omega_0 = 0$, we get:

$$F_T\, R\theta = \frac{1}{2} I \omega^2 \;\Rightarrow\; \omega^2 = \frac{2 F_T\, R\,\theta}{I}. \tag{16.6}$$

How does this compare with our intuition? We see that the rotational inertia, I, plays an important role: If we increase I but keep everything else fixed, the resulting angular velocity gets smaller: It becomes more difficult to get the wheel started if I is larger.

In addition, the angular velocity depends on how large the force F_T is and how far, R, from the rotation axis it is applied. First, we notice again what we already observed: It is only the tangential component of **F** that matters: We cannot accelerate the rotation of the rod by pulling at it in the radial direction. Second, we see that it is the combination $\tau = F_T R$ that matters. This combination is often called the *torque* of the force **F**. We can therefore increase the final angular velocity by increasing the force F_T or by increasing the distance R from the axis where the force is applied.

Angular Acceleration of a Spinning Wheel

Can we use this approach to find the angular acceleration? Yes! By applying the method to a very short time interval. As the interval becomes smaller and smaller, we effectively introduce the time derivative of both sides of (16.4). For the rotational system in Fig. 16.1, the work done by the constant, tangential force F_T is

$$W = F_T\, R\Delta\theta = \frac{1}{2} I \omega^2 (t + \Delta t) - \frac{1}{2} I \omega^2 (t). \tag{16.7}$$

when the wheel rotates an angle $\Delta\theta$ during the short time interval Δt. We divide by Δt on both sides:

$$F_T R \frac{\Delta\theta}{\Delta t} = \frac{\frac{1}{2} I \omega^2 (t + \Delta t) - \frac{1}{2} I \omega^2 (t)}{\Delta t}, \tag{16.8}$$

This becomes the time derivative as Δt becomes small:

$$F_T R \frac{d\theta}{dt} = \frac{d}{dt} \left(\frac{1}{2} I \omega^2 \right) = I \omega \frac{d\omega}{dt}, \tag{16.9}$$

where we have applied the chain rule and assumed that I is constant. Finally, by dividing by ω on both sides:

$$F_T R = I \frac{d\omega}{dt} = I\alpha = \frac{d}{dt} I\omega \ , \tag{16.10}$$

we have found the angular acceleration! This equation looks very much like Newton' second law for translational motion:

$$F = m\frac{dv}{dt} = ma = \frac{d}{dt} mv. \tag{16.11}$$

For rotational motion, we replace:

The force by the torque	$F \rightarrow F_T R$
the translational inertia (mass) by the rotational inertia	$m \rightarrow I$
the acceleration by the angular acceleration	$a \rightarrow \alpha$

(We can derive Newton's second law from the work-energy theorem in exactly the same way). Equation (16.10) is indeed Newton's second law for rotational motion. This law is general, even though it was here derived for a special situation.

Interpreting Newton's Second Law for Rotations

From Newton's second law for rotational motion in (16.10), we interpret the torque $RF_T = \tau$ as the *cause* of the angular acceleration, just as we interpreted the force as the cause of acceleration for translational motion. We see that I plays the role of a rotational inertia. For a given torque, $\tau = F_T\ R$, a larger value of I means a smaller angular acceleration. Also, we see that the torque $\tau = F_T\ R$ depends on both the tangential force, F_T and the distance to the rotation axis, R: If we apply the same force F further out from the rotation axis, we get a larger torque and a larger angular acceleration.

The top figures in Fig. 16.2 illustrates how we must increase the force as the distance R changes in order to keep the same torque and therefore the same angular acceleration: If we increase R we have to decrease F_T by the same factor to keep the acceleration the same. This is illustrated in the figure, where we have shown arms of length $R/2$, R and $2R$ and the corresponding forces, $2F$, F, and $F/2$.

The bottom figures in Fig. 16.2 shows what happens when we keep the torque the same, but change the moment of inertia. If we assume that the mass is concentrated in two weights attached to a thin rod, the moment of inertia is $I = 2Mr^2$, where r is the

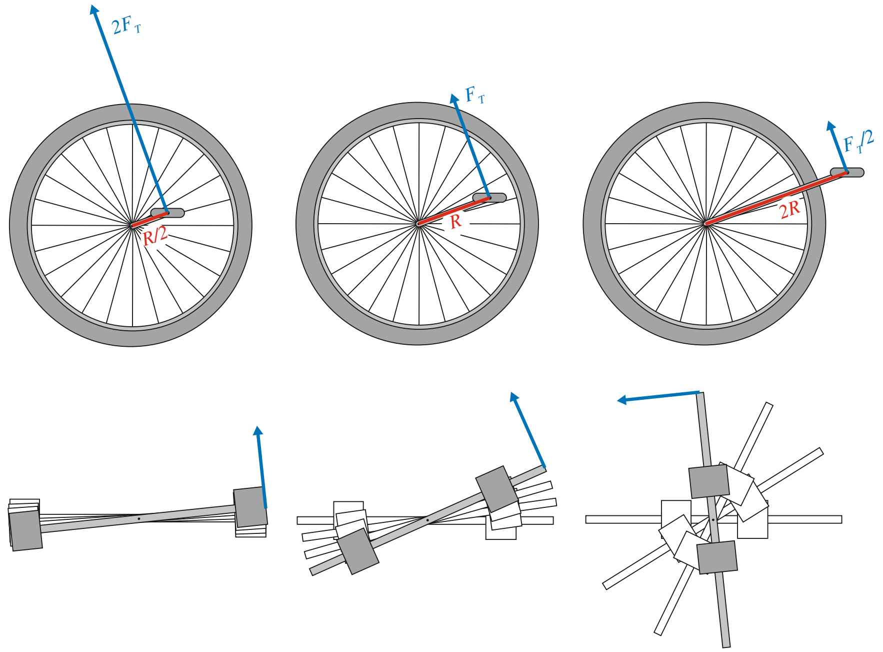

Fig. 16.2 *Top* All three wheels have the same torque, $\tau = F_T r$, where r is the distance from the axis to where the force acts, and F_T is the tangential component of the force. *Bottom* The same torque is applied to three systems with different moment of inertia—changed by moving the positions of the masses. The position of the object are shown at four times, t_0, t_1, $t_2 = \sqrt{2}t_1$ and $t_3 = \sqrt{3}t_1$, which are the same for all systems: The differences are due to differences in angular acceleration due to differences in the moment of inertia

distance from the center of the rod to the center of each of the masses. From Newton's second law for rotational motion in (16.10), we find the angular acceleration:

$$\alpha = \frac{F_T R}{I} . \tag{16.12}$$

If the torque, $F_T R$ is constant, the acceleration depends inversely on I. We illustrate this by showing the orientations of the object at three subsequent (but not equally spaced) timesteps. All these systems have the same mass. It is how the mass is distributed around the rotation axis that matters. And as evident, changing the distribution has a significant effect on the acceleration.

While Newton's second law for rotational motion was introduced in the special situation of a constant force, we will in the rest of this chapter see that the law is general, and you will learn how to apply it to determine the rotational motion of a system.

16.2 Newton's Second Law for Rotational Motion

In the introductory example we introduced the following law for the rotational motion of an object subject to a single, constant force:

$$\tau = F_T\, R = I\alpha, \tag{16.13}$$

where $\tau = F_T R$ is called the torque for the force **F**. The torque depends only on the tangential component of **F**. This is the component that is normal to the vector from the rotation axis to the point where the force acts, **r**, as illustrated in Fig. 16.3. You may recall to have seen this before. We recognize the torque as the magnitude of the cross product between **r** and **F**. This is easily seen by decomposing **r** and **F** in the normal (radial) and tangential directions: $\mathbf{r} = r\hat{u}_N$ and $\mathbf{F} = F_T\hat{u}_T + F_N\hat{u}_N$. The cross product is then:

$$\mathbf{r} \times \mathbf{F} = r\hat{u}_N \times F_T\hat{u}_T + r\hat{u}_N \times F_N\hat{u}_N = rF_T\,\mathbf{k}, \tag{16.14}$$

where $\hat{u}_N \times \hat{u}_N = 0$, and $\hat{u}_N \times \hat{u}_T = \mathbf{k}$ is a unit vector that point out of the plane, in the z-direction. This allows a more general definition of the (vector) torque of the force **F** around the point O:

Definition of torque:

$$\boldsymbol{\tau}_O = \mathbf{r} \times \mathbf{F}, \tag{16.15}$$

where **r** is the vector from O to the point where **F** is acting.

The torque, $F_T R$, in (16.13) is therefore the z-component of the vector torque:

$$\tau_z = F_T R = I\alpha. \tag{16.16}$$

In the introductory example we discussed the effect of a single force, producing a single torque. However, an object may be subject to several forces, all acting in separate points, giving rise to separate torques. The work-energy theorem used in the example depends on the *net* force. Hence we must insist on using the **net torque** when formulating Newton's second law for rotational motion.[1]

[1]Notice that the torque $\boldsymbol{\tau}$ points in the z-direction, which is also the direction of the rotation vector, $\boldsymbol{\omega} = \omega\,\mathbf{k}$. This suggests a vector formulation of Newton's second law for rotational motion: $\sum \boldsymbol{\tau}_j = I\boldsymbol{\alpha}$. Unfortunately, this is generally not correct. We will return to a vector formulation later.

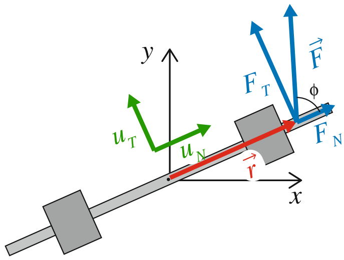

Fig. 16.3 Illustration of how the torque of the force **F** is calculated: Only the tangential component F_T contributes

Newton's second law for rotational motion (N2Lr): for a rigid body rotating around a **fixed axis** (the z-axis) is:

$$\sum_j \tau_{z,j} = \tau_z^{\text{net}} = I_z \alpha_z , \tag{16.17}$$

where $\boldsymbol{\tau}_j = \mathbf{r}_j \times \mathbf{F}_j$ is the torque of force j, $\mathbf{r}_j$ is the position of the point where force j is applied, and I_z is the moment of inertia (the rotational inertia) of the object around the rotation axis.

Structured Problem-Solving Approach

This law allows us to determine the rotational motion of a rigid body, just like we previously have found the translational motion of an object using Newton's second law. We can follow the same structured problem-solving approach as we used for translational motion, but with some modifications as illustrated in Fig. 16.4.

There are a few, but important differences between how we address problems with translational and rotational motion. When we *identify* the relevant systems and variables for rotational motion around a fixed axis, we must of course describe the configuration of the object using the angle θ and the angular velocity ω. When we *model* the system, we still start from a free-body diagram of the rotating object, but we must now carefully specify the rotation axis and where each of the forces are acting, because this is essential in order to calculate the torque. We find the torque for each of the external forces acting, and add them all to get the net torque, which we use in Newton's second law for rotational motion. When we *solve* and *analyze* the system, we follow the same structure as we have done before, now working with angular coordinates.

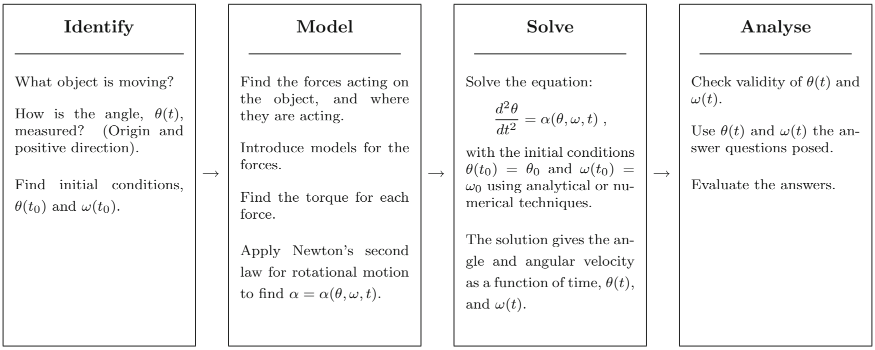

Fig. 16.4 Illustration of structured problem-solving approach for rotating objects

Torque

In order to apply Newton's second law for rotation and solve rotational problems, we must know how to calculate the torque:

$$\boldsymbol{\tau}_O = \mathbf{r} \times \mathbf{F}, \tag{16.18}$$

of the force **F** around the point O. What are the properties of torque?

- The torque of a force depends on the point it is taken relative to—the origin. We say that the torque is around the point O to show where the origin is when we calculate the torque.
- The torque of a force **F** depends both on the force and on the position **r** where the force is acting. For example, the torque around point O of the force **F** on the hammer in Fig. 16.5 varies from $\boldsymbol{\tau} = F L\,\mathbf{k}$ when the force is applied at the end of the hammer to $\boldsymbol{\tau} = 0$ when the force is applied at the rotation axis.
- The torque is given as a cross product: The torque is therefore normal to both the position vector **r**and the force **F**. We find the direction of the torque using the right-hand rule.
- For a two-dimensional system in the xy-plane, the torque is always directed along the z-axis. We often use the z-component τ_z of the torque instead of the full vector notation for the torque. The sign of τ_z indicates if the torque is in the positive or negative z-direction.
- The direction of the torque is normal to the position and force vectors, and its magnitude is:

  $$|\tau| = |F|\,|r|\ \sin\phi, \tag{16.19}$$

  where ϕ is the angle between the position vector **r** and the force vector **F** as illustrated in Fig. 16.3.

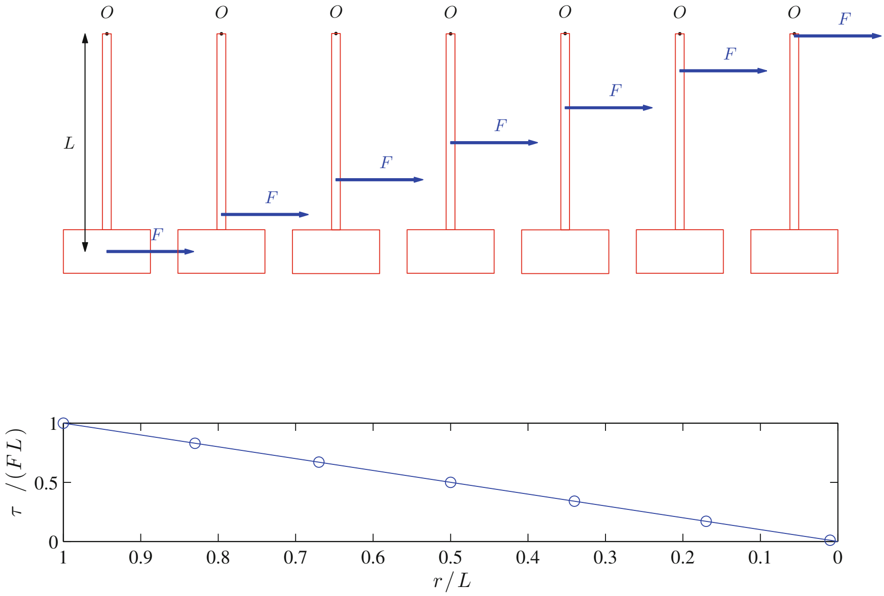

Fig. 16.5 Illustration of the torque $\boldsymbol{\tau}$ for a rigid hammer of length L for a force $\mathbf{F} = F\mathbf{i}$ applied at various position, $\mathbf{r} = y\mathbf{j}$, where $y < 0$

- If the force is parallel to the position vector, the torque is always zero for this force. For example, for a weight in a rope whirled in a circular path, the torque of the rope tension around the center of the path is always zero, since the rope tension acts in the direction of the position vector. Similarly, the torque of the gravitational force from the Sun on the Earth (taken around the Sun) is also always zero.
- If the position vector $\mathbf{r}$ is zero, that is, if the force acts in the origin, the torque of the force is zero. This is commonly used "trick" to solve problems: If we place the origin at a force we do not know, its torque is zero independently of the force.
- Torques obey the **superposition principle**. The total torques of a force $\mathbf{F}_1$ acting at the point $\mathbf{r}_1$ and the force $\mathbf{F}_2$ acting at the point $\mathbf{r}_2$ is:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2. \tag{16.20}$$

  Notice that you can only add torques that all are taken around the same point!
- To calculate the **torque of a the gravitational force** acting on a rigid body (when gravity is a constant) you assume that the total gravitational force acts in the center of mass:

$$\boldsymbol{\tau} = \mathbf{R} \times M\mathbf{g}, \tag{16.21}$$

where **R** is the position of the center of mass. This is found by summing all the torques acting on each small element i of the rigid body:

$$\boldsymbol{\tau} = \sum_i \mathbf{r}_i \times m_i \mathbf{g} = \left(\sum_i m_i \mathbf{r}_i \right) \times \mathbf{g} = M\mathbf{R} \times \mathbf{g} = \mathbf{R} \times M\mathbf{g}. \tag{16.22}$$

Test your understanding: Why does it require less force to close a door if you push far from the hinge than if you push near it?

16.2.1 *Example: Torque and Vector Decomposition*

Figure 16.6a illustrates a force **F** applied to a wrench, at a point **r** relative to the rotation axis, which is placed in the origin. While we can always find the torque directly from

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}, \tag{16.23}$$

if we know **r** and **F**. However, it is often practical to decompose either the force **F** or the arm, **r**, in order to simplify the calculation of the torque.

If you decompose the force, **F** into a component $\mathbf{F}_r = F_r \hat{u}_r$ in the direction along **r** and a component $\mathbf{F}_n = F_n \hat{u}_n$ in the direction normal to **r**, the torque is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \left(F_r \hat{u}_r + F_n \hat{u}_n \right) = \mathbf{r} \times F_r \hat{u}_r + \mathbf{r} \times F_n \hat{u}_n = r F_n \, \mathbf{k}. \tag{16.24}$$

since the force component along **r** does not contribute to the torque—it is only the component of **F** normal to the arm that contributes.

Alternatively, you could decompose the arm **r** into a component $\mathbf{r}_F = r_F \hat{u}_F$ in the direction along **F**, and a component $\mathbf{r}_N = r_N \hat{u}_N$ in the direction normal to **F** as illustrated in Fig. 16.6b. The torque is then:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \left(r_F \hat{u}_F + r_N \hat{u}_N \right) \times \mathbf{F} = r_N F, \tag{16.25}$$

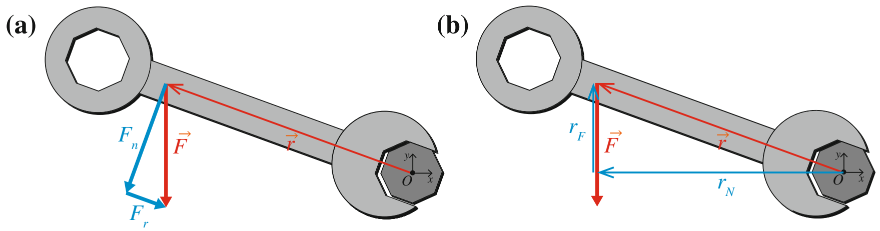

Fig. 16.6 **a** Torque of **F** by decomposing **F**. **b** Torque of **F** by decomposing **r**

which shows that it is only the component of the arm that is normal to the force that contributes to the torque. With experience you will learn to use the method that is simplest in a given situation.

16.2.2 Example: Pulling at a Wheel

Problem: A force F is applied tangentially at the rim of a wheel made of a homogeneous cylinder of mass M and radius R. (You may think of this as pulling at a rope with a constant force, where the rope is wrapped around the wheel and unwinds as you pull). The wheel starts at rest and is attached to a frictionless axis at its center. Find the angle θ of the wheel as a function of time.

Approach: We use Newton's second law for rotational motion since the wheel rotates around a fixed axis: We find the forces and where they are acting, then we find the torques, calculate the angular acceleration, and integrate to find the motion.

Identify: The rotational position of the wheel is described by the angle θ, with positive direction shown in Fig. 16.7. The wheel starts at $\theta_0 = 0$ with angular velocity $\omega_0 = 0$ at $t_0 = 0$.

Model: The external forces acting on the wheel are: gravity, $\mathbf{G}$, acting in $\mathbf{r}_G = 0$ at the center of mass of the wheel; the contact force $\mathbf{N}$ from the axis on the wheel acting in $\mathbf{r}_N = 0$ at the axis (notice that $\mathbf{N}$ has to balance the two other forces, because the center of mass of the wheel does not move), and the force $\mathbf{F} = -F\,\mathbf{i}$, acting at $\mathbf{r}_F = R\,\mathbf{j}$ at the rim of the wheel.

The torques of both gravity and the normal force are zero, since $\mathbf{r}_G$ and $\mathbf{r}_N$ are zero. The torque of $\mathbf{F}$ around O is:

$$\boldsymbol{\tau}_F = \mathbf{r}_F \times \mathbf{F} = R\,\mathbf{j} \times -F\,\mathbf{i} = RF\,\mathbf{k}, \tag{16.26}$$

which is the net torque around the origin. The z-component, the component along the rotation axis, is $\tau_z = RF$. Newton's second law for rotational motion around a fixed axis gives:

$$\tau_z^{\text{net}} = RF = I_z\alpha, \tag{16.27}$$

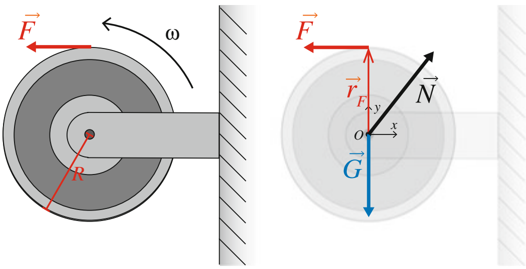

Fig. 16.7 A wheel is affected by a force F acting on its rim. **a** A sketch of the system. **b** A free-body diagram for the wheel

where I_z is the moment of inertia of the wheel for rotation around the axis, which goes through the center of mass of the wheel. We find I_z from Fig. 15.5. We insert this into (16.27), getting:

$$\alpha = \frac{RF}{\frac{1}{2}MR^2} = \frac{2F}{MR}, \tag{16.28}$$

which is a constant. The initial conditions for the motion are: $\omega(0) = \omega_0 = 0$ and $\theta(0) = \theta_0 = 0$.

Solve: We find the angular velocity by integrating the (constant) angular acceleration from (16.28).

$$\omega(t) - \underbrace{\omega(t_0)}_{=0} = \int_0^t \underbrace{\alpha}_{=2F/MR} \, dt = \frac{2F}{MR} t. \tag{16.29}$$

Similarly, we find the angle by integrating the angular velocity:

$$\theta(t) - \underbrace{\theta(t_0)}_{=0} = \int_0^t \underbrace{\omega}_{=(2F/MR)t} \, dt = \frac{1}{2} \frac{2F}{MR} t^2, \tag{16.30}$$

which is the solution.

Analyze: Notice that the method used is exactly the same as the method we used to solve problems with translational motion, but now we use Newton's second law for rotational motion instead of Newton's second law for translational motion.

16.2.3 Example: Blowing at a Pendulum

Problem: A pendulum consists of a sphere of radius R and mass M. The center of the sphere is attached at the end of a thin, massless rod of length L. The other end revolves around a bolt, providing a fixed, friction-free axis for the pendulum. The sphere is affected by air resistance, which you for simplicity can assume to act at the center of the sphere. You can use a quadratic law for the air drag. Develop a method to find the motion of the pendulum when affected by a horizontal wind, $\mathbf{w}(t) = w(t)\,\mathbf{i}$, which may vary in time, and explore the behavior of the pendulum for various types of winds.

Approach: The pendulum is rotating around a fixed axis. We can therefore solve the problem using Newton's second law for rotational motion: From the forces acting we introduce force models, find the torques, find the angular acceleration from the net torque, and solve to find the motion.

Identify: We measure the position of the pendulum by the angle, θ, with the vertical as shown in Fig. 16.8. We need to define initial conditions for the motion, and assume

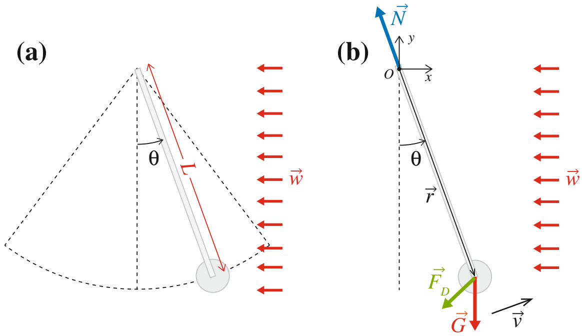

Fig. 16.8 **a** Sketch of the motion of the pendulum. **b** A free-body diagram for the pendulum

that the pendulum starts at $\theta_0 = 0$ and with initial angular velocity $\omega(t_0) = \omega_0$ at $t_0 = 0\,\text{s}$.

Model: We find the forces acting on the pendulum from the free-body diagram (see Fig. 16.8). The pendulum is affected by gravity $\mathbf{G} = -Mg\,\mathbf{j}$. To calculate the torque of gravity, we assume it acts ate the center of the mass, which is at the center of the sphere, since the rod is without mass. The gravitational force is therefore drawn in the center of the sphere, at $\mathbf{r}_G = \mathbf{r}$, where $\mathbf{r}$ is a vector pointing from the rotation axis O to the center of the sphere.

From the figure, we see that

$$\mathbf{r} = L\sin\theta\,\mathbf{i} - L\cos\theta\,\mathbf{j}. \tag{16.31}$$

The torque of gravity is:

$$\boldsymbol{\tau}_G = \mathbf{r} \times \mathbf{G}, \tag{16.32}$$

which we find by inserting the vector for $\mathbf{G}$ and $\mathbf{r}$:

$$\boldsymbol{\tau}_G = (L\sin\theta\,\mathbf{i} - L\cos\theta\,\mathbf{j}) \times -Mg\,\mathbf{j} = -MgL\sin\theta\,\mathbf{k}. \tag{16.33}$$

The air resistance, $\mathbf{F}_D = -D|\mathbf{v} - \mathbf{w}|(\mathbf{v} - \mathbf{w})$, also acts at the center of the sphere, $\mathbf{r}_D = \mathbf{r}$. Here, $\mathbf{v}$ is the velocity of the center of the sphere and $\mathbf{w}$ is the velocity of the air. When there is no wind, $\mathbf{w} = 0$, and the air resistance reduces to the well know form, $\mathbf{F}_D = -Dv\mathbf{v}$. What is $\mathbf{v}$? We can find the velocity of any point on a rotating rigid body using $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. The torque of $\mathbf{F}_D$ is:

$$\boldsymbol{\tau}_D = \mathbf{r}_D \times \mathbf{F}_D, \tag{16.34}$$

where $\mathbf{r}_D = \mathbf{r}$. For the general case where an external wind $\mathbf{w}$ is applied, we need to calculate this based on the wind. However, it is useful to see that we get a simplified result when $\mathbf{w} = 0$ and $\mathbf{F}_D$ consequently is directed along $\mathbf{v}$. Then

$$\boldsymbol{\tau}_D = \mathbf{r} \times -Dv\mathbf{v}. \tag{16.35}$$

Because $\mathbf{r}$ is normal to $\mathbf{v}$, we get $\mathbf{r} \times \mathbf{v} = Lv\,\mathbf{k}$. (Ensure you get the same by applying the right-hand rule). If we insert $v = \omega L$ we get

$$\tau_{z,D} = -DL^3|\omega|\omega. \tag{16.36}$$

This expression also ensures that we get the right sign for the torque both when $\omega > 0$ and when $\omega > 0$. Check that you also get this!

The pendulum is also affected by a contact force, $\mathbf{N}$, from the attachment bolt. In reality, the contact force is distributed along the contact between the bolt and the rod, but here we assume that the contact force acts so close to the rotation axis, that the distance $\mathbf{r}_N \simeq \mathbf{0}$, and the torque of the force is zero.

For no wind ($\mathbf{w} = 0$), the z-component of the net torque is therefore:

$$\tau_z^{\text{net}} = \tau_{z,G} + \tau_{z,N} + \tau_{z,D} = -MgL\sin\theta - DL^3|\omega|\omega. \tag{16.37}$$

And Newton's second law for rotational motion gives:

$$I\alpha = \tau_z^{\text{net}} = -MgL\sin\theta - DL^3|\omega|\omega. \tag{16.38}$$

The moment of inertia, I, can be related to the moment of inertia of a sphere around its center of mass, $I_{cm} = 2/5MR^2$ (from Fig. 15.5) using the parallel-axis theorem: $I = I_{cm} + ML^2$.

Solve: While you may be able to solve the motion of the pendulum for a limited range of motion for small θ, we concentrate here on the general numerical approach.

Without Wind

When there is no wind, the motion is determined from (16.38), giving

$$\alpha = -\frac{MgL}{I}\sin\theta - \frac{DL^3}{I}|\omega|\omega, \tag{16.39}$$

with initial conditions $\theta(t_0) = \theta_0 = 0$, and $\omega(t_0) = \omega_0$. We solve using Euler-Cromer's method (see (14.21)), implemented by:

```
rho = 7000.0; % kg/m^3
R = 0.02; % m
L = 1.0; % m
M = 4/3*pi*R^3*rho
I = 2/5*M*R^2 + M*L^2;
g = 9.8; % m/s^2
rhoair = 1.293; % kg/m^3
D = 12*rhoair*R^2
omega0 = 8.0; % rad/s
time = 20.0; % s
```

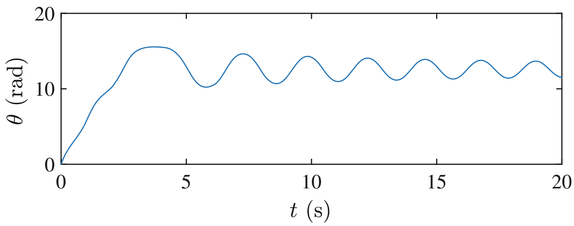

Fig. 16.9 Plot of $\theta(t)$ for a simulation of the pendulum with no applied wind

```
dt = 0.001; % s
n = ceil(time/dt);
theta = zeros(n,1);
omega = zeros(n,1);
t = zeros(n,1);
theta(1) = 0;
omega(1) = omega0;
for i = 1:n-1
    tauz = -M*g*L*sin(theta(i))-D*L^3*omega(i)*abs(omega(i));
    alpha = tauz/I;
    omega(i+1) = omega(i) + alpha*dt;
    theta(i+1) = theta(i) + omega(i+1)*dt;
    t(i+1) = t(i) + dt;
end
```

The resulting motion is shown in Fig. 16.9. Here, you see that the system first rotates completely around the axis, before damping limits the motion to an oscillation. For the presented simulation we used $L = 1$ m, $R = 0.01$ m, mass density $\rho_m = 7000$ kg/m^3, and $D = 12\rho_{\text{air}} R^2$ (from (5.20)) as seen in the program.

With Wind

What happens if we want to include an external wind field, $\mathbf{w} = w(t)\,\mathbf{i}$, which may vary with time? Then the torque of the air resistance force becomes more complicated:

$$\boldsymbol{\tau}_D = \mathbf{r}_D \times \mathbf{F}_D, \tag{16.40}$$

where $\mathbf{r}_D = \mathbf{r}$, which is given in (16.31), and

$$\mathbf{F}_D = -D|\mathbf{v} - \mathbf{w}|(\mathbf{v} - \mathbf{w}), \tag{16.41}$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, and $\boldsymbol{\omega} = \omega(t)\,\mathbf{k}$. Fortunately, it is simple to implement this numerically by calculating the cross-product directly in the numerical implementation. We must only remember to extract the z-component of the calculated torque and use this to calculate the angular acceleration. Let us start by assuming that the wind is described by the function `wind(t)`, which returns the value of $w(t)$ for a given time. We start by assuming that the wind blows with a constant velocity,

$w(t) = w_0 = 20.0$ m/s, and let the pendulum start at rest, $\omega(t_0) = 0$ rad/s. This is implemented by:

```
function [w] = wind(t)
w0 = 20.0; % m/s
w = w0;
end
```

and the integration loop of main program is now:

```
G = [0 -M*g 0];
for i = 1:n-1
    r = [L*sin(theta(i)) -L*cos(theta(i)) 0.0];
    tauG = cross(r,G);
    % F_D = - D|v - w|(v - w)
    omegavec = omega(i)*[0 0 1];
    v = cross(omegavec,r);
    w = wind(t(i))*[1 0 0];
    dv = v - w;
    FD = -D*dv*norm(dv);
    tauFD = cross(r,FD);
    tau = tauG + tauFD;
    tauz = tau(3);
    alpha = tauz/I;
    omega(i+1) = omega(i) + alpha*dt;
    theta(i+1) = theta(i) + omega(i+1)*dt;
    t(i+1) = t(i) + dt;
    if (mod(i,10)==0)
        plot([0 r(1)],[0 r(2)],'-o');
        xlabel('x [m]');
        ylabel('y [m]');
        axis equal
        axis([-L L -L L]);
        drawnow
    end
end
```

where we also have included an animated plot to show the motion of the pendulum. The test for `mod(i,10)==0` is a test to check if the counter `i` is divisible by 10. This means that what is inside the `if`-statement will only be executed every time `i` is increased by 10. This is a standard way to speed up the animation by only plotting some of the results. You need to select an appropriate number here depending on your computer to get a smooth motion. If it is moving too slowly you need to increase the number 10 to a large value. If it is moving too fast you should lower the number. The resulting behavior is illustrated in Fig. 16.10, where we see that the pendulum now does not return to $\theta = 0$, but rather reaches a stationary situation with $\theta =$, which corresponds to the equilibrium configuration, that is the configuration where the net torque is zero. You may solve to find the value of θ corresponding to zero torque for yourself.

What happens if we now blow in a more complicated way, such as by blowing periodically, but always in the positive direction. We can model this using the cosine:

$$w(t) = \frac{w_1}{2}\left(1 + \cos(2\pi t/T)\right), \tag{16.42}$$

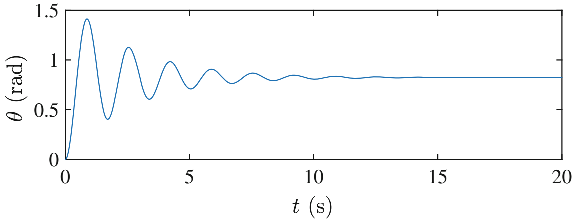

Fig. 16.10 Plot of $\theta(t)$ for a simulation of the pendulum with a applied wind velocity that is constant

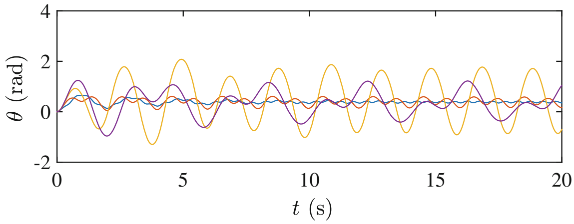

Fig. 16.11 Plot of $\theta(t)$ for a simulation of the pendulum with a applied, periodic wind velocity, $w(t) = (w_1/2)(\cos(2\pi t/T))$, with $w_1 = 20$ m/s and $T = 0.5$, 1.0, 2.0, and 4.0 s

where w_1 is the maximum wind velocity and T is the period of the wind. This is now implemented by changing the `wind(t)` function:

```
function [w] = wind(t)
w1 = 20.0; % m/s
T = 1.0; % s
w = 0.5*w1*(1.0 + cos(2*pi*t/T));
end
```

Now, we start the pendulum from $\theta(t_0) = 0$ and $\omega(t_0) = 0$ rad/s, corresponding to the case where it hang straight down at rest before you start blowing periodically at it. The resulting behavior for $T = 0.5$ s, 1.0 s, 2.0 s and 4.0 s is shown in Fig. 16.11. What happens here? Play around with the model yourself to find out what happens.

16.3 Rotational Motion Around a Moving Center of Mass

Figure 16.12 shows a rod being thrown across the lecture hall. After it has been thrown it is just affected by gravity and air resistance. We know that the motion of the center of mass of the object only depends on the external forces acting on the object—its motion is determined from Newton's second law of motion. But what about the rotational motion around the center of mass? The rotational motion around the center of mass for a rigid body (rotating around a fixed axis) is determined from Newton's second law for rotational motion around the center of mass. (A proof is found in Sect. A.11)

Newton's Second Law of Translational Motion

The motion of the center of mass of a rigid body, such as the rotating rod in Fig. 16.12 is determined from Newton's second law for translational motion:

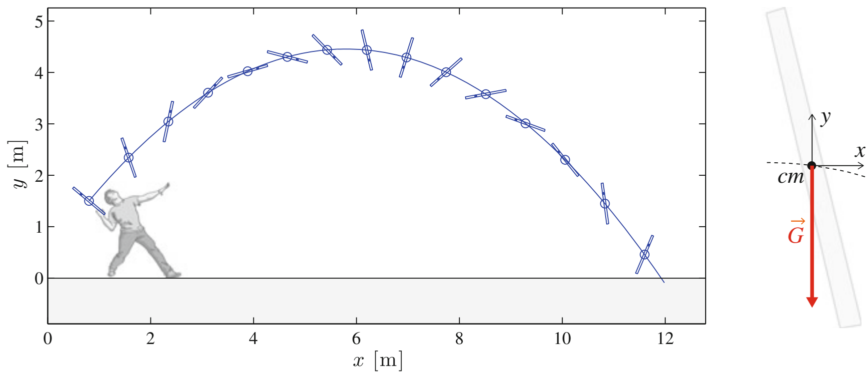

Fig. 16.12 *Left* The motion of a rigid rod thrown through the air, showing the motion of the center of mass and the rotational motion around the center of the mass. *Right* Free-body diagram for the rod

$$\boxed{\sum_j \mathbf{F}_j^{\text{ext}} = M\mathbf{A}.} \tag{16.43}$$

This is true irrespective of how the system moves relatively to the center of mass. This means that the motion of the center of mass of the rod in Fig. 16.12 would be the same if the rod rotated or if it moved with the same angle all the time. But only if the external forces are the same! If we include the effects air resistance, the external forces depend on whether the rod is rotating or not, and the motion would not be exactly the same in the two situation.

Newton's Second Law for Rotational Motion Around the Center of Mass

For a system rotating around a fixed axis through the center of mass, the rotational motion is determined by Newton's second law for rotational motion around the center of mass, which is just like Newton's second law for rotation around a fixed axis, but now all the torques must be taken around the center of mass:

$$\boldsymbol{\tau}_{cm} = \mathbf{r}_{cm} \times \mathbf{F}, \tag{16.44}$$

where $\mathbf{r}_{cm}$ is a vector from the center of mass to the point where the force $\mathbf{F}$ acts. Using this notation, we get:

Newton's second law for a rigid body rotating around **a fixed axis through the center of mass** (the z-axis) is:

$$\sum_j \tau_{z,cm,j} = \tau_{z,cm}^{\text{net}} = I_{z,cm}\alpha_z , \tag{16.45}$$

where $\boldsymbol{\tau}_{cm,j}$ is the torque of force j around the center of mass, and $I_{z,cm}$ is the moment of inertia (the rotational inertia) of the object around the fixed rotational axis through its center of mass.

It may be confusing to call the axis fixed if it is moving along with the center of mass. What we mean is that the axis has a fixed direction: The direction of the rotation axis does not change during the motion, but the position of the axis follows the center of mass. The application of the law is illustrated in Fig. 16.13. Here, we follow a rod thrown across the lecture hall also during its collision with the floor. The small insets at the bottom shows the forces acting on the rod at five different time steps. When the rod is not in contact with the floor, the only external force acting is gravity (we ignore air resistance), and since gravity acts in the center of mass, the torque of gravity around the center of mass is zero, and the angular acceleration is therefore zero: The rod rotates with a constant angular velocity. However, when the rod is in contact with the floor, as illustrated in the middle three insets, the contact force $\mathbf{F}$ from the floor on the rod gives rise to a net torque around the center of mass, which leads to an angular acceleration during the contact. The force $\mathbf{F}$ acts at the point $\mathbf{r}_{cm}$ relative to the center of mass of the rod, giving rise to a torque: $\boldsymbol{\tau} = \mathbf{r}_{cm} \times \mathbf{F}$ around the center of mass. We address the motion of a bouncing rod in Sect. 16.3.

16.3.1 Example: Kicking a Ball

Problem: You are kicking a stationary football lying on the ground. The ball is spherical, with a mass M, radius R, and moment of inertia I around the center of mass. You kick the ball with a constant force $\mathbf{F} = F_x\,\mathbf{i}$ for a short time interval Δt. hitting the ball at a distance y from its center, as illustrated in Fig. 16.14. Find the motion of the ball during and after the kick. You may neglect the effects of friction and air resistance.

Approach: We plan to use Newton's second law of motion to find the motion of the center of mass from the external forces acting, and Newton's second law for rotational motion around the center of mass to find the rotational motion around the center for the ball during and after the kick.

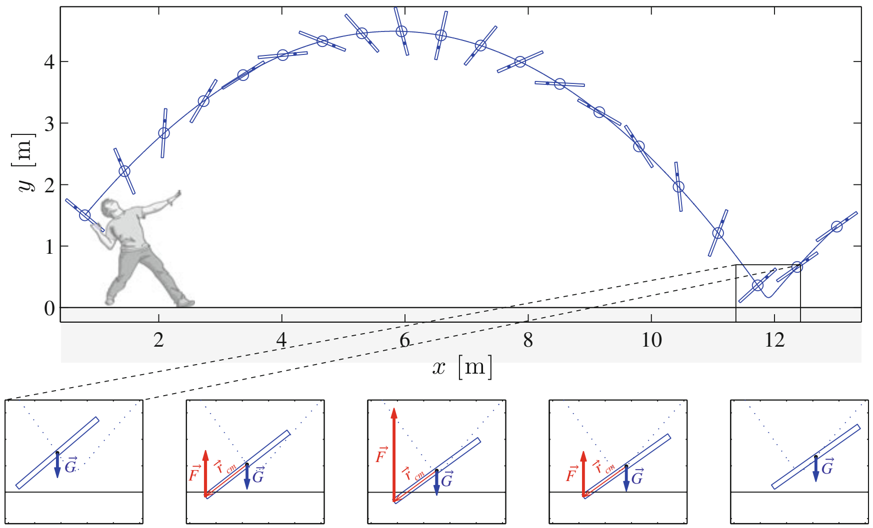

Fig. 16.13 *Top* The motion of a rigid rod thrown through the air, showing the motion of the center of mass and the rotational motion around the center of the mass. The rod hits the floor, and bounces back up. *Bottom* Free-body diagram for the rod at five different times. When the rod is not in contact with the floor (*the first and the last inset*) only gravity acts. When the rod is in contact with the floor (*middle three insets*) both gravity and a contact force from the floor act

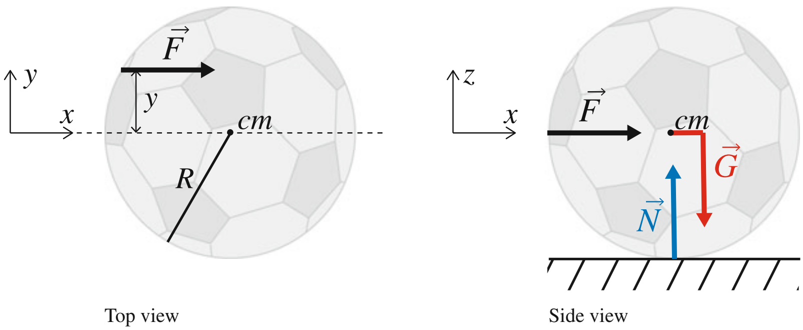

Fig. 16.14 You are kicking a ball—exerting a constant force $\mathbf{F} = F_x\,\mathbf{i}$ for a small time interval Δt. The force is acting at a distance y from the center of mass

Identify: We assume that the ball behaves as a rigid body and describe its motion by the position, $\mathbf{R}(t)$ of its center of mass and its angle $\theta(t)$ around the z-axis. The ball starts from rest: $\mathbf{R}(0\,\mathrm{s}) = \mathbf{0}$ and $\theta(0\,\mathrm{s}) = 0$.

Model: The translational and rotational motion depends on the external forces acting. The ball is affected by gravity, $\mathbf{G} = -Mg\,\mathbf{k}$, acting at the center of mass, hence $\mathbf{r}_{G,cm} = 0$; the normal force $\mathbf{N} = N\,\mathbf{k}$ from the ground, acting at $\mathbf{r}_{N,cm} = -R\,\mathbf{k}$; and the force $\mathbf{F} = F\,\mathbf{i}$ from the foot, acting at $\mathbf{r}_{F,cm} = -x\,\mathbf{i} + y\,\mathbf{j}$, where x and y

are given. All the positions are relative to the center of mass, since the acting points of the forces will move with the center of mass as the ball starts moving. The forces are illustrated in Fig. 16.14.

We apply Newton's second law to find the motion of the center of mass:

$$\sum_j \mathbf{F}_j^{\text{ext}} = -Mg\,\mathbf{k} + N\,\mathbf{k} + F\,\mathbf{i} = M\mathbf{A}. \tag{16.46}$$

Since the applied force $\mathbf{F}$ only acts in the x-direction, we assume that the ball does not move in the y or z-directions. Therefore, the acceleration in the z-direction is zero and $N = Mg$. The motion in the x-direction is given by:

$$MA_x = F_x \;\Rightarrow\; A_x = \frac{F_x}{M}, \tag{16.47}$$

starting from $V_x = 0\,\text{m/s}$ and $X = 0\,\text{m}$ at $t = 0\,\text{s}$. Notice that the motion of the center of mass does not depend on the rotational motion of the ball, since (16.47) and the time, Δt, the force is acting does not depend on either the angle θ or the angular velocity ω of the ball!

The rotational motion of the ball is found from Newton's second law for rotational motion around a fixed axis around the center of mass. (How do we know the ball rotates around a fixed axis? We know, since the direction of the torque does not change during the motion—this is an advanced point we will return to later in the chapter. For now you should consider this an assumption):

$$\sum_j \tau_{z,cm,j} = I\alpha, \tag{16.48}$$

where the net torque around the center of mass is:

$$\begin{aligned}
\sum_j \boldsymbol{\tau}_{cm,j} &= \mathbf{r}_{G,cm} \times \mathbf{G} + \mathbf{r}_{N,cm} \times \mathbf{N} + \mathbf{r}_{F,cm} \times \mathbf{F} \\
&= \mathbf{0} \times \mathbf{G} - R\mathbf{k} \times N\mathbf{k} + (-x\mathbf{i} + y\mathbf{j}) \times F\mathbf{i} \\
&= \mathbf{0} - NR\underbrace{(\mathbf{k} \times \mathbf{k})}_{=\mathbf{0}} - xF\underbrace{(\mathbf{i} \times \mathbf{i})}_{=\mathbf{0}} + Fy\underbrace{(\mathbf{j} \times \mathbf{i})}_{=-\mathbf{k}} \\
&= -yF\mathbf{k},
\end{aligned} \tag{16.49}$$

which inserted in (16.48) gives:

$$\tau_{z,cm}^{\text{net}} = -yF_x = I\alpha \;\Rightarrow\; \alpha = -\frac{yF_x}{I}, \tag{16.50}$$

where $\theta(0\,\text{s}) = 0$ and $\omega(0\,\text{s}) = 0\,\text{rad/s}$.

Solve: Both the translational and the rotational motion occur with constant accelerations. We can therefore solve by direct integration to find the positions and velocities. First, we find the velocities:

$$V_x(t) = \underbrace{V_x(0)}_{=0} + \int_0^t A_x\, dt = \int_0^t \frac{F}{M}\, dt = \frac{F}{M} t, \tag{16.51}$$

and

$$\omega(t) = \underbrace{\omega(0)}_{=0} + \int_0^t \alpha\, dt = \int_0^t -\frac{yF}{I}\, dt = -\frac{yF}{I} t. \tag{16.52}$$

We integrate once more to find the positions:

$$X(t) = \underbrace{X(0)}_{=0} + \int_0^t V_x(t)\, dt = \int_0^t \frac{F}{M} t\, dt = \frac{1}{2}\frac{F}{M} t^2, \tag{16.53}$$

and

$$\theta(t) = \underbrace{\theta(0)}_{=0} + \int_0^t \omega(t)\, dt = \int_0^t -\frac{yF}{I} t\, dt = -\frac{1}{2}\frac{yF}{I} t^2. \tag{16.54}$$

However, this solution is only valid as long as the ball is affected by the force **F**, which only is for a time interval Δt. After that, the ball is not affected by the force **F**, and the net force in the x-direction is zero and the net torque is zero. Therefore the translational and the rotational accelerations are zero, and the ball continue with the same translational and rotational velocities:

$$V(t) = V(\Delta t) = \frac{F}{M} \Delta t \text{ when } t > \Delta t, \tag{16.55}$$

and

$$\omega(t) = \omega(\Delta t) = -\frac{yF}{I} \Delta t \text{ when } t > \Delta t. \tag{16.56}$$

Since the motion for $t > \Delta t$ occurs with constant translational and angular velocities, it is easy to find the position at a time $t > \Delta t$ by integration:

$$X(t) = X(\Delta t) + V(\Delta t)(t - \Delta t) \text{ when } t > \Delta t, \tag{16.57}$$

and

$$\theta(t) = \theta(\Delta t) + \omega(\Delta t)(t - \Delta t) \text{ when } t > \Delta t. \tag{16.58}$$

Analyze: We notice that the motion of the center of mass does not depend on y — the position where the force is acting. For any choice of y, the effect on the motion of the center of mass is the same. But the rotational motion depends on y. In particular, we notice that when $y = 0$, that is when the force acts on an axis through the center of mass, the net torque is zero, and the ball does not start to rotate. We also notice that if we kick the ball on the left side, with $y > 0$, the ball rotates in the negative direction, whereas if we kick the ball on the right side, with $y < 0$, the ball rotates in the positive direction, which is consistent with our experience with kicking balls. This is how you give them spin.

Comment: Notice that the translational and rotational motions are independent in this case. Why? Because the force acts at a constant distance y from the center of mass throughout the motion. Therefore the torque of **F** does not depend on the position of the ball, and similarly, the direction or magnitude of **F** does not depend on the rotation of the ball. This is not always the case: In many cases the two motions are coupled because the force acting on the object depend either on the position of the object or on its rotation. But in this particular case—as well as in many cases in typical mechanics exam problems—the two motions are not coupled.

16.3.2 Example: Rolling Down an Inclined Plane

(This problem is a classic in mechanics.)

Problem: A round object is placed on an inclined plane. The object has radius R, mass M, and moment of inertia I around the rotation axis through the center of mass. Find the acceleration of the object and the friction force on the object. Discuss various angles of inclination ϕ, various objects, and various initial conditions for the object.

Approach: We plan to find the external forces and use Newton's second law of translational and rotational motion to find the acceleration, using the rolling condition as long as the object is rolling.

Identify: We choose a coordinate system with the x-axis oriented along the inclined plane, so that there is no motion in the y-direction (see Fig. 16.15). The object is described by the position $X(t)$ of its center of mass, and the rotational angle, $\theta(t)$, around the center of mass. The object starts at the position $X(0\,\text{s}) = 0\,\text{m}$, and $\theta(0\,\text{s} = 0$ with initial velocities $V_x = V_0$, and $\omega(0\,\text{s}) = \omega_0$.

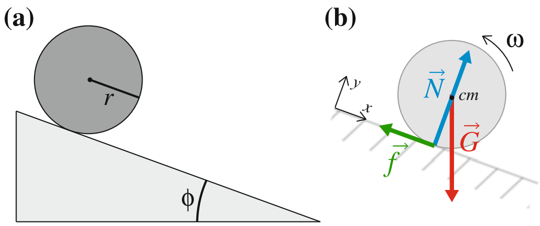

Fig. 16.15 **a** Illustration of a round object moving along an inclined plane. **b** Free-body diagram for the object

Model: The object is affected by a normal force, $\mathbf{N} = N\,\mathbf{j}$, acting at $\mathbf{r}_{N,cm} = -R\,\mathbf{j}$; a friction force, $\mathbf{f} = -f\,\mathbf{i}$, acting at $\mathbf{r}_{f,cm} = -R\,\mathbf{j}$; and a gravitational force $\mathbf{G}$ acting at the center of mass, $\mathbf{r}_{G,cm} = \mathbf{0}$. We decompose the gravitational force in the chosen coordinate system:

$$\mathbf{G} = Mg\sin\phi\,\mathbf{i} - Mg\cos\phi\,\mathbf{j}. \tag{16.59}$$

We apply Newton's second law to determine the translational acceleration:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \mathbf{G} + \mathbf{N} + \mathbf{f} = M\mathbf{A}. \tag{16.60}$$

In the x-direction:

$$\sum \mathbf{F}_x = Mg\sin\phi - f = MA_x. \tag{16.61}$$

Since there is no motion in the y-direction, we get

$$\sum \mathbf{F}_y = N - Mg\cos\phi - f = MA_y = 0, \tag{16.62}$$

which gives $N = Mg\cos\phi$.

Newton's second law for rotation around the center of mass gives:

$$\sum_j \tau_{z,cm,j} = I\alpha, \tag{16.63}$$

where the net torque around the center of mass is:

$$\begin{aligned} \sum_j \boldsymbol{\tau}_{cm,j} &= \underbrace{\mathbf{0}\times\mathbf{G}}_{=0} + \underbrace{\mathbf{r}_{N,cm}}_{=-R\,\mathbf{j}} \times \underbrace{\mathbf{N}}_{=N\,\mathbf{j}} + \underbrace{\mathbf{r}_{f,cm}}_{=-R\,\mathbf{j}} \times \underbrace{\mathbf{f}}_{=-f\mathbf{i}} \\ &= -RN\underbrace{\mathbf{j}\times\mathbf{j}}_{=0} - R(-f)\underbrace{\mathbf{j}\times\mathbf{i}}_{=-\mathbf{k}} = -Rf\,\mathbf{k}, \end{aligned} \tag{16.64}$$

which inserted in (16.63) gives:

$$\tau_z^{\text{net}} = -Rf = I\alpha. \tag{16.65}$$

We now have 2 equations (16.61) and (16.65), but 3 unknowns, A_x, α, and f. How do we proceed?

The main challenge in this problem lies in the friction force. In the case where the friction is dynamic, that is, if the object is sliding relative to the surface, the magnitude of the friction force is simply proportional to the normal force, and the direction is determined from the local relative velocity of the two surfaces in contact. However, if the object does not slide, the friction force is a static friction force, and its magnitude must be determined from other principles.

Let us first assume that the object is not sliding relative to the surface. This means that the object is rolling without slipping. The rolling condition is that there is no relative velocity at the contact point, P, $\mathbf{v}_P = \mathbf{0}$, where

$$\mathbf{v}_P = \mathbf{V} + \mathbf{v}_{P,cm} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_{P,cm} = V_x\,\mathbf{i} + \omega\,\mathbf{k} \times (-R\,\mathbf{j}) = (V_x + \omega R)\,\mathbf{i}, \quad (16.66)$$

and the condition $\mathbf{v}_p = \mathbf{0}$ therefore gives $V_x = -\omega R$, which is usually called the rolling condition. Taking the time derivative gives a similar expression for the accelerations: $A_x = -R\alpha$, which we insert this into (16.65), getting:

$$f = -\frac{I}{R}\alpha = -\frac{I}{R}\frac{-A_x}{R} = \frac{I}{R^2}A_x. \quad (16.67)$$

The static friction force must have this value to ensure that there is no slipping between the object and the surface. We insert this result into (16.61), giving:

$$Mg\sin\phi - \underbrace{f}_{=(I/R^2)A_x} = MA_x$$

$$Mg\sin\phi = \left(1 + \frac{I}{MR^2}\right)MA_x, \quad (16.68)$$

and

$$A_x = \frac{g\sin\phi}{1 + \frac{I}{MR^2}}. \quad (16.69)$$

We introduce $c = I/(MR^2)$ to simplify the expression:

$$A_x = \frac{1}{1+c}g\sin\phi. \quad (16.70)$$

The number c depends on the distribution of mass around the rotation axis. For a sphere, $c = 2/5$, for a cylinder $c = 1/2$, and for a ring, $c = 1$.

The friction force is:

$$f = \frac{I}{R^2}A_x = Mg\sin\phi\frac{\frac{I}{MR^2}}{1 + \frac{I}{MR^2}} = Mg\sin\phi\frac{c}{1+c}. \quad (16.71)$$

It is clear that the friction force f increases with ϕ, and at the same time N decreases with ϕ. When will the friction force reach the static friction threshold? The objects starts to slip at the critical angle ϕ_c when $f = \mu_s N$, where we now insert the values we found for f and N, getting:

$$Mg\sin\phi_c\frac{c}{1+c} = \mu_s Mg\cos\phi_c \Rightarrow \tan\phi_c = \mu_s\frac{1+c}{c}. \quad (16.72)$$

The critical angle ϕ_c depends on the number c—it will therefore depend on the type of object rolling. For a few characteristic objects we have:

Object	I	$\tan\phi_c$
Sphere	$(2/5)mR^2$	$(7/2)\mu_s$
Cylinder	$(1/2)mR^2$	$3\mu_s$
Ring	mR^2	$2\mu_s$

A sphere will therefore *roll* down a steeper slope than a cylinder, which will start slipping at a lower angle than the sphere.

What happens if the slope is steeper than ϕ_c? In this case the object will still rotate, but it will not roll without slipping. This means that the rolling condition is no longer valid. Instead the friction force is now the dynamic friction force, which only depends on the normal force:

$$f = \mu_d N = \mu_d M g \cos\phi. \tag{16.73}$$

We insert this into Newton's second law for translational motion (16.61), getting:

$$M g \sin\phi - \mu_d M g \cos\phi = M A_x, \tag{16.74}$$

which allows us to determine A_x independently of the rotational motion:

$$A_x = g\,(\sin\phi - \mu_d \cos\phi)\,. \tag{16.75}$$

However, the object will still rotate, since the torque around the center of mass is:

$$\tau_z = -Rf = -R\mu_d M g \cos\phi = I\alpha. \tag{16.76}$$

The angular acceleration is therefore:

$$\alpha = -\frac{Rf}{I} = -\mu_d \frac{RMg\cos\phi}{I}. \tag{16.77}$$

16.3.3 Example: Bouncing Rod

In this example we will develop a model for a bouncing rigid rod. The model will be similar to the bouncing an rotating dumbbell model in Chap. 13, but we will now model a rigid rod, so there are no internal vibrations, and we will use our newly gained knowledge of how to model rotating rigid object for our model.

The rod is of length $L = 1\,\text{m}$, mass $M = 0.5\,\text{kg}$, and has a moment of inertia $I = (1/12)ML^2$ around its center of mass. We describe the rod by the position $\mathbf{R}(t)$ of its center of mass and the angular orientation $\theta(t)$, where we assume that the rod

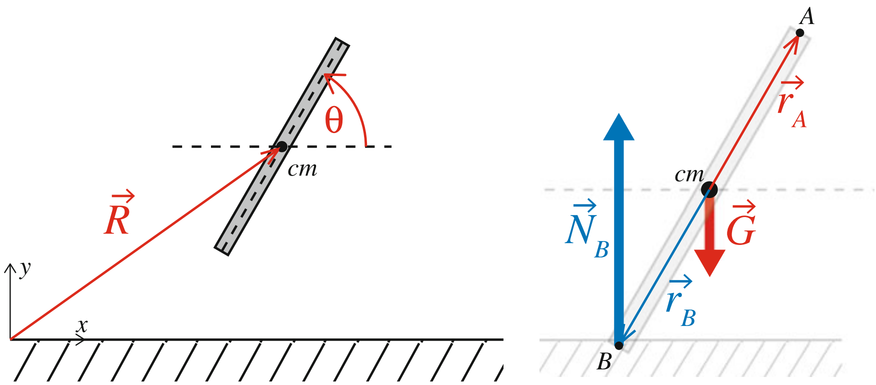

Fig. 16.16 Illustration of the rod bouncing on the floor and free-body diagram for the rod

moves in the xy-plane and rotates around an axis through the center of mass directed along the z-axis.

The motion of the rod is determined by the forces acting on it. The rod is affected by gravity, $\mathbf{G} = -Mg\,\mathbf{j}$, acting at the center of mass, $\mathbf{r}_{G,cm} = \mathbf{0}$. In addition the rod will bounce on the floor. We model the force between the floor and the rod as a spring force, representing the deformation of the floor and the rod. If either end of the rod is pressed into the floor, that is, if the y-coordinate of either end is below $y = 0$, which is the position of the floor, there will be a spring force acting normal to the floor, in the positive y-direction, which depends on how far the end of the rod has been pressed down into the floor. The two ends of the rod are at positions:

$$\mathbf{r}_A = \mathbf{R} + (L/2)\hat{u}, \tag{16.78}$$

$$\mathbf{r}_B = \mathbf{R} - (L/2)\hat{u}, \tag{16.79}$$

where $\hat{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ is a unit vector pointing along the rod, as illustrated in Fig. 16.16. The normal force, $\mathbf{N}_A$, due to the interaction between end A of the rod is:

$$\mathbf{N}_A = \begin{cases} -ky_A & \text{when } y_A < 0 \\ 0 & \text{when } y_A \geq 0. \end{cases} \tag{16.80}$$

and similarly for end B:

$$\mathbf{N}_B = \begin{cases} -ky_B & \text{when } y_B < 0 \\ 0 & \text{when } y_B \geq 0. \end{cases} \tag{16.81}$$

Here, k is the spring constant for the interaction between the rod and the floor.

The motion of the rod is determined from Newton's second law for translational and rotational motion:

$$\sum_j \mathbf{F}_j = \mathbf{G} + \mathbf{N}_A + \mathbf{N}_B = M\mathbf{A}, \tag{16.82}$$

and

$$\sum_j \tau_{z,cm,j} = I\alpha, \tag{16.83}$$

where

$$\sum_j \boldsymbol{\tau}_{cm,j} = \mathbf{0} \times \mathbf{G} + \mathbf{r}_{A,cm} \times \mathbf{N}_A + \mathbf{r}_{B,cm} \times \mathbf{N}_B. \tag{16.84}$$

Here, $\mathbf{r}_{A,cm} = (L/2)\hat{u}$ and $\mathbf{r}_{B,cm} = -(L/2)\hat{u}$. Notice that both the net force and the net torque depends on both the position of the center of mass and on the angle, since the contact force between the rod and the floor depends on the position and orientation of the rod.

Numerical: It is not simple to solve these equations analytically, but it is straight forward to implement a numerical solution. We use an Euler-Cromer scheme to integrate both the translational and the rotational motion, simultaneously:

$$\begin{aligned} \mathbf{V}(t + \Delta t) &= \mathbf{V}(t) + \mathbf{A}(t, \mathbf{R}(t), \theta(t))\, \Delta t \\ \mathbf{R}(t + \Delta t) &= \mathbf{R}(t) + \mathbf{V}(t + \Delta t)\, \Delta t \\ \omega(t + \Delta t) &= \omega(t) + \alpha(t, \mathbf{R}(t), \theta(t))\, \Delta t \\ \theta(t + \Delta t) &= \theta(t) + \omega(t + \Delta t)\, \Delta t, \end{aligned} \tag{16.85}$$

This scheme can be implemented directly into the code, using Newton's second laws to calculate $\mathbf{A}$ and α at each time-step:

```
m = 0.5;      % kg
g = 9.8;      % m/s^2
h0 = 4.0;     % m
L = 1.0;      % m
time = 10.0; % s
dt = 0.001;  % s
k = 1000.0;  % N/m
v0 = 2.0;    % m/s
I = (1.0/12.0)*m*L^2;
n = ceil(time/dt);
r = zeros(n,3);
v = zeros(n,3);
theta = zeros(n,1);
omega = zeros(n,1);
t = zeros(n,1);
r(1,:) = [0 h0 0];
v(1,:) = [v0 0 0];
theta(1) = 2*pi*rand(1);
for i = 1:n-1
    % Find force acting on each edge
```

```
    fnet = [0 0 0];
    tnet = 0.0;
    u = [cos(theta(i)) sin(theta(i)) 0];
    % Position of edge A
    rr = r(i,:) + 0.5*L*u;
    % Collision with bottom wall
    dr = rr(2);
    f = -k*dr*(dr<0.0)*[0 1 0];
    fnet = fnet + f;
    torque = cross((rr-r(i,:)),f);
    tnet = tnet + torque;
    % Position of edge B
    rr = r(i,:) - 0.5*L*u;
    % Collision with bottom wall
    dr = rr(2);
    f = -k*dr*(dr<0.0)*[0 1 0];
    fnet = fnet + f;
    torque = cross((rr-r(i,:)),f);
    tnet = tnet + torque;
    % Add gravity
    fnet = fnet - m*g*[0 1 0];
    % Integration step - Euler-Cromer
    a = fnet/m;
    v(i+1,:) = v(i,:) + a*dt;
    r(i+1,:) = r(i,:) + v(i+1,:)*dt;
    alphaz = tnet(3)/I;
    omega(i+1) = omega(i) + alphaz*dt;
    theta(i+1) = theta(i) + omega(i+1)*dt;
    t(i+1) = t(i) + dt;
    if (mod(i,20)==0)
        % Plot position of rod, with tracer
        r1 = r(i,:) + 0.5*L*u;
        r2 = r(i,:) - 0.5*L*u;
        xl = [r1(1) r2(1)];
        yl = [r1(2) r2(2)];
        plot(r(1:i,1),r(1:i,2),':',xl,yl,'-');
        xlabel('x [m]');
        ylabel('y [m]');
        axis equal
        axis([0 time*v0 0 h0]);
        drawnow
    end
end
```

The resulting path of the rod is shown in Fig. 16.17. Use the program to experiment and see what happens as you change parameters.

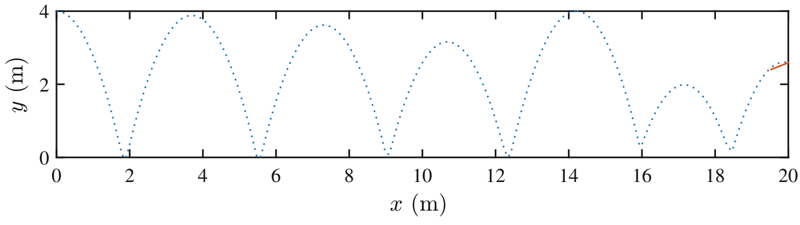

Fig. 16.17 Plot of the path of the rod

16.4 Collisions and Conservation Laws

If a meteor collides with a planet (see Fig. 16.18), the planet may gain both translational and rotational motion after the collision. We have already found that conservation principles, such as the conservation of translational momentum, allow us to determine the translational motion of the planet after the collision without knowing the details of the interactions during the collision. Can we find similar concepts and conservation laws for rotational motion—a rotational momentum and a conservation law for rotational motion—which we can use to address the rotational motion of a system during a collision? Here, we introduce the concept of rotational momentum first for a point particle, then for a system of many particles, and finally for a rotating rigid body, before we put all the pieces together and formulate a set of general principles allowing us to address collisions between several rigid, rotating object—such as during a meteor impact or during a pirouette.

Rotational Momentum for a Point Particle

We have already found that the translational momentum, **p**, is a useful concept to address collisions. This is based on the prominent place of translational momentum in Newton's second law:

$$\sum_j \mathbf{F}_j = \frac{d\mathbf{p}}{dt}. \tag{16.86}$$

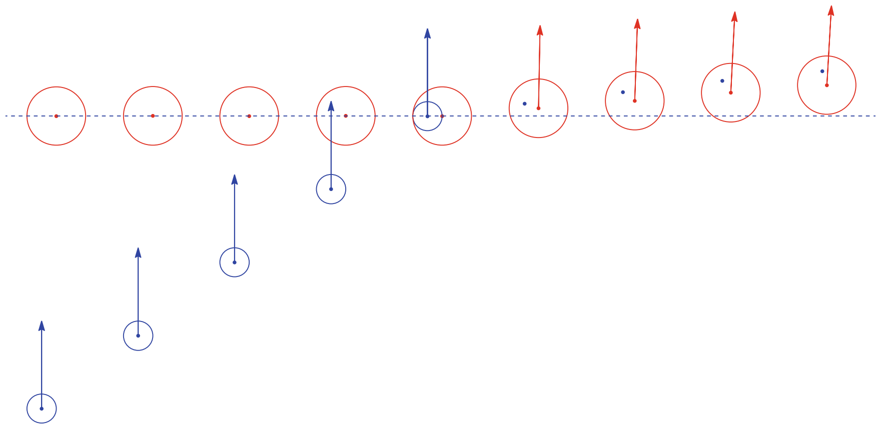

Fig. 16.18 A meteor impact on a planet may change both the translational and the rotational motion of the planet. The *arrows* show the translational momentum and the *blue dot* shows the rotational state of the planet after the collision

It is indeed this law that justifies the conservation law: When the net external force is zero, the time derivative of the translational momentum is zero and therefore conserved throughout the process. Let us see if we can justify a similar concept for rotational motion, based on Newton's second law for rotational motion:

$$\boldsymbol{\tau}^{\text{net}} = \sum_j \mathbf{r}_j \times \mathbf{F}_j . \tag{16.87}$$

If we study the motion of a point particle, all the torques act in the same point, **r**, and the net torque is:

$$\boldsymbol{\tau}^{\text{net}} = \sum_j \mathbf{r} \times \mathbf{F}_j = \mathbf{r} \times \sum_j \mathbf{F}_j = \mathbf{r} \times \mathbf{F}^{\text{net}} , \tag{16.88}$$

where we can insert $\mathbf{F}^{\text{net}}$ from Newton's second law in (16.86):

$$\boldsymbol{\tau}^{\text{net}} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} . \tag{16.89}$$

In order to get something that looks like Newton's second law in (16.86), we try to move the time derivative out in front of $\mathbf{r} \times \mathbf{p}$: What does this give?

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \underbrace{\mathbf{v} \times m\mathbf{v}}_{=\mathbf{0}} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} . \tag{16.90}$$

Yes! Success! We can therefore rewrite (16.89) as:

$$\boldsymbol{\tau}^{\text{net}} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d}{dt}\mathbf{l} . \tag{16.91}$$

This equation looks just like Newton's second law—we have found what we have looking for, $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is the rotational analogue for translational momentum. We call **l** the *rotational momentum* or the *angular momentum* :

$$\mathbf{l}_O = \mathbf{r} \times \mathbf{p} \qquad \text{(Rotational momentum)} , \tag{16.92}$$

where we use the subscript O to show that the rotational momentum is found with respect to the point O, and the vector **r** is found relative to O. We say that $\mathbf{l}_O$ is the rotational momentum around the point O or around an axis z (if the particular point along the axis is not important).

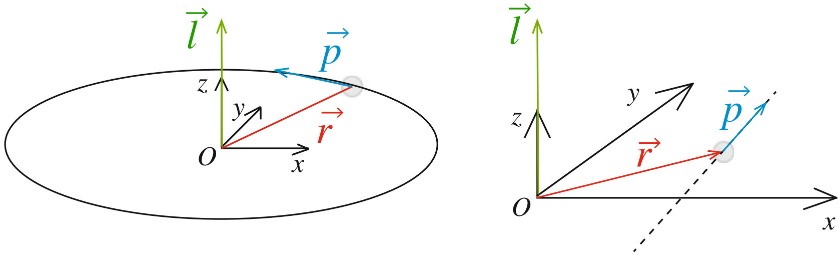

Fig. 16.19 Illustration of rotational momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ for circular motion and motion along a straight line

Properties of Rotational Momentum

- Rotational momentum is a **vector**. For planar motion, the rotational momentum is normal to the plane.
- The rotational momentum is calculated relative to a point—just like torque. It will be different if we choose a different point.

For a particle moving in a circle with radius r, in the xy-plane, as illustrated in Fig. 16.19, the velocity $\mathbf{v}$ is always normal to the position $\mathbf{r}$, and the rotational momentum around the origin at the center of the rotational motion is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v} = rmv\,\mathbf{k}. \tag{16.93}$$

The rotational momentum can be defined for any moving point particle, not only for a rotating point particle. For example, the rotational momentum of a particle moving along a straight line at $x = b$, $\mathbf{r} = b\,\mathbf{i} + y(t)\,\mathbf{j}$, is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v} = (b\,\mathbf{i} + y(t)\,\mathbf{j}) \times v_y\,\mathbf{j} = bv_y\,\mathbf{k}. \tag{16.94}$$

As illustrated in Fig. 16.19 this means that it is only the component of $\mathbf{r}$ that is normal to $\mathbf{v}$ that contributes to the rotational momentum, similar to what we previously found for torques.

Newton's Second Law for Rotational Motion of a Point Particle

We have found an alternative formulation of Newton's second law for a point particle, most useful for rotational motion, but valid for any motion:

$$\boldsymbol{\tau}^{\text{net}} = \frac{d}{dt}\mathbf{l}, \tag{16.95}$$

Conservation of Rotational Momentum

Based on Newton's second law in (16.95) we see that for a point particle, the rotational momentum, **l**, is conserved if the net external torque is zero! While this can be used to address problems with a single particle, it first becomes really useful when we introduce similar concepts for a rigid body or a system of many particles.

16.4.1 Example: Block on a Frictionless Table

Problem: A block of mass m is attached to a thin, massless rope passing through a hole in a frictionless table. The block starts with the angular velocity ω_0 at a distance r_0 from the hole. We pull slowly in the rope until it reaches the radius r with the angular velocity ω. Find ω.

Identify: As long as the rope is tight, the block moves in a circular path with radius r. We describe the position of the block by the radius r and its angle θ, as illustrated in Fig. 16.20.

Model: First, we find the forces acting on the block. The block is affected by gravity, **G**, and the normal force, **N**, from the table. Since the block is not moving in the vertical direction, the net vertical force is zero, and since two forces are acting in the same point, the net torque of the two forces (around any point) is also zero. In addition, the block is affected by the rope tension, **T**.

We use Newton's second law for rotational motion to determine the motion of the block. The torque around the origin of the rope tension is:

$$\boldsymbol{\tau}_T = \mathbf{r} \times \mathbf{T} = 0, \tag{16.96}$$

because the force and the position vector are parallel. This means that the net torque around the origin is zero. Newton's second law for rotational motion therefore gives that:

$$\frac{d\mathbf{l}}{dt} = \boldsymbol{\tau} = 0, \tag{16.97}$$

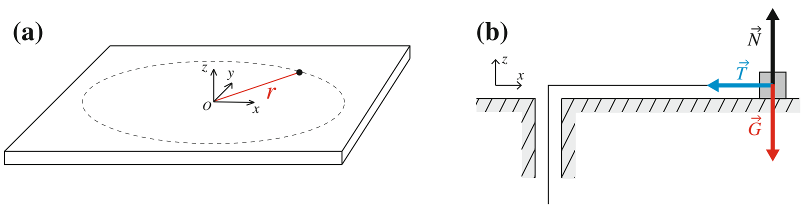

Fig. 16.20 A sketch of the motion of the block sliding on a frictionless table (**a**) and a free-body diagram for the block (**b**)

the rotational momentum is therefore constant throughout the motion. The rotational momentum for the block is:

$$\mathbf{l} = \mathbf{r} \times m\mathbf{v}, \tag{16.98}$$

where the velocity is normal to the radius vector at all times, therefore:

$$\mathbf{l} = rmv \tag{16.99}$$

We can replace the velocity by the angular velocity, using $v = R\omega$:

$$\mathbf{l} = mr^2\omega, \tag{16.100}$$

which is what we found above for circular motion.

Analyze: Since the rotational momentum is conserved, we can relate the initial and final states. Initially, the rotational momentum is:

$$\mathbf{l}_0 = mr_0^2\omega_0 \tag{16.101}$$

When the radius is r and the angular velocity ω:

$$\mathbf{l} = mr^2\omega \tag{16.102}$$

Since $\mathbf{l} = \mathbf{l}_0$ we find ω:

$$mr_0^2\omega_0 = mr^2\omega \;\Rightarrow\; \omega = \frac{r_0^2}{r^2}\omega_0. \tag{16.103}$$

This demonstrates the use of the conservation principle for rotational momentum.

Rotational Momentum for a System of Particles

The rotational momentum of a system of particles around a fixed point, O, is the sum of the rotational momentum of each particle. For a system consisting of point masses, m_i, located at points, $\mathbf{r}_i$, the total rotational momentum is:

$$\mathbf{L}_O = \sum_i \mathbf{l}_{O,i} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \mathbf{r}_i \times m_i\mathbf{v}_i. \tag{16.104}$$

Similarly, we define the rotational momentum of a system of particles around their center of mass as

$$\mathbf{L}_{cm} = \sum_i \mathbf{l}_{O,i} = \sum_i \mathbf{r}_{cm,i} \times \mathbf{p}_i = \sum_i \mathbf{r}_{cm,i} \times m_i\mathbf{v}_i, \tag{16.105}$$

where $\mathbf{r}_{cm,i}$ is the position of mass m_i relative to the center of mass. The rotational momentum of a system around a fixed axis O can also be decomposed into the rotational momentum of the center of mass around O, assuming the center of mass moves as a point particle, and the rotational momentum of the system around its center of mass:

$$\mathbf{L}_O = \mathbf{R} \times \mathbf{P} + \mathbf{L}_{cm}, \tag{16.106}$$

where $\mathbf{R}$ is the position of the center of mass (relative to O), and $\mathbf{P}$ is the translational momentum of the whole system. (You can find a proof of (16.106) in Sect. A.10).

Newton's Second Law of Rotation for a Multiparticle System

Also for a multiparticle system, we find a general form for Newton's second law for rotational motion:

Newton's second law for rotational motion of a system of particles around a fixed point O :

$$\frac{d\mathbf{L}_0}{dt} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_0^{\text{ext}}, \tag{16.107}$$

where only the *external forces* are included. The internal forces cancel as long as they are central forces.

(You can find a proof in Sect. A.8). We can also derive a completely analogous law for the rotational momentum around the center of mass of a system:

Newton's second law for rotational motion of a system of particles around its center of mass, cm :

$$\frac{d\mathbf{L}_{cm}}{dt} = \sum_{i=1}^{N} \mathbf{r}_{cm,i} \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_{cm}^{\text{ext}}, \tag{16.108}$$

where the positions $\mathbf{r}_{cm,i}$ are the positions of each mass m_i relative to the center of mass of the system.

(You can find a proof in Sect. A.11). This law is general and powerful. You will find it used frequently both theoretically, as basis for derivations, and practically, as a basis for the use of conservation laws. The law also demonstrates that as long as the torque of the external forces are constant, around a fixed point O or the center of mass, the corresponding total rotational momentum does not change. This is the law for rotational motion we have been looking for. However, in order to apply it to the collision of rigid bodies, we need to find a simplified expression for the rotational momentum of a rigid object, both around a fixed axis and around its center of mass.

Rotational Momentum for a Rigid Body

We know that the translational momentum of a rigid body can be written as $\mathbf{P} = M\mathbf{V}$, where $\mathbf{V}$ is the velocity of the center of mass. Can we find a similarly simple expression for the rotational momentum, $\mathbf{L}_O$ of a rigid body around a fixed axis?

The rotational momentum for a rigid body is rotating around a fixed axis z through the point O.[2]

$$\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i , \tag{16.109}$$

For a rigid body rotating around the axis z with an angular velocity $\boldsymbol{\omega} = \omega \mathbf{k}$, each mass m_i at $\mathbf{r}_i$ has a velocity:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i . \tag{16.110}$$

To simplify the expression, we introduce cylindrical coordinates to describe the position of mass i, where the cylindrical axis follows the z-axis, as illustrated in Fig. 16.21. The position is decomposed as:

$$\mathbf{r}_i = \boldsymbol{\rho}_i + z_i \, \mathbf{k} = \rho_i \hat{u}_\rho + z_i \, \mathbf{k}, \tag{16.111}$$

where $\hat{u}_\rho$ is a unit vector in the xy-plane pointing from the z-axis to the point $\mathbf{r}_i$, as shown in Fig. 16.21. We insert this into (16.110), getting:

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i = \omega \, \mathbf{k} \times (\boldsymbol{\rho}_i + z_i \, \mathbf{k}) = \omega \, \mathbf{k} \times \boldsymbol{\rho}_i + \omega \underbrace{\mathbf{k} \times \mathbf{k}}_{=0} = \boldsymbol{\omega} \times \boldsymbol{\rho}_i . \tag{16.112}$$

We find the rotational momentum by inserting this into (16.109):

$$\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \sum_i m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_i) \ . \tag{16.113}$$

Hmm. How do we simplify this equation? If we only are interested in the z-component of the rotational momentum (which we typically are when we use Newton's second

[2]Notice that O must be a point on the rotation axis.

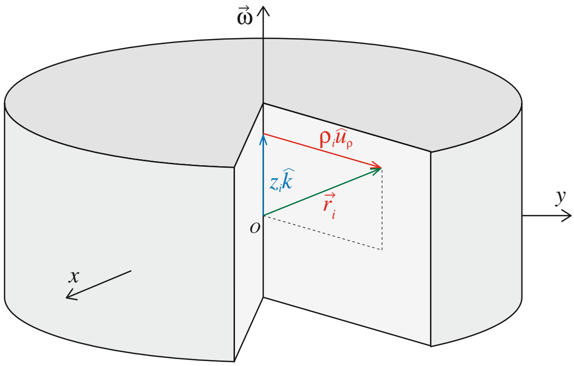

Fig. 16.21 Illustration of a rigid body rotating around the z-axis with an angular velocity $\boldsymbol{\omega}$. We describe the position $\mathbf{r}$ using cylindrical coordinates, so that $\mathbf{r} = \boldsymbol{\rho} + z\,\mathbf{k} = \rho\hat{u}_\rho + z\,\mathbf{k}$, where $\boldsymbol{\rho}$ is in the plane normal to the axis of rotation

law for rotational motion of a rigid body), we see that it is only the $\boldsymbol{\rho}_i$ component of $\mathbf{r}_i$ that contributes to the z-component of $\mathbf{L}_O$:

$$\mathbf{L}_{O,z}\mathbf{k} = \sum_i m_i \rho_i \times (\omega\mathbf{k} \times \boldsymbol{\rho}_i) = \sum_i m_i \rho_i^2 \omega \mathbf{k}, \tag{16.114}$$

because the vectors $\boldsymbol{\rho}_i$ and $\mathbf{v}_i$ are normal to each other, and the vectors $\boldsymbol{\rho}_i$ and $\boldsymbol{\omega}$ are normal to each other. Therefore:

$$L_{O,z} = \left(\sum_i m_i \rho_i^2 \right) \omega = I_z \omega \ , \tag{16.115}$$

where we recognize the moment of inertia (rotational inertia), I_z, around the z-axis.

The **rotational momentum**, $L_{O,z}$, of a rigid body rotating around a fixed axis is:

$$L_{O,z} = I_z \omega, \tag{16.116}$$

where I_z is the moment of inertia of the object around the z-axis, and ω is the angular velocity around the z-axis.

We can use this result also for a rigid body rotating around a fixed axis through its center of mass. We notice that if we insert this into Newton's second law for rotational motion of a multi-particle system around a fixed axis in (16.107) or (16.108), we recover Newton's second law for rotational motion:

$$\sum_j \tau_{z,O,j}^{\text{ext}} = \frac{d}{dt} L_{O,z} = \frac{d}{dt}(I_z \omega) = I_z \frac{d\omega}{dt} = I_z \alpha, \tag{16.117}$$

which was the result we started this chapter with. Notice the nice similarity between the translational and rotational momentum of a rigid body:

	Translational	Rotational
Inertia	M	I_z
Momentum	$\mathbf{P} = M\mathbf{V}$	$L_{O,z} = I_z\omega$
N2L	$\mathbf{F}^{\text{net}} = M\mathbf{A}$	$\tau_{O,z}^{\text{net}} = I_z\alpha$

Limitations

We have previously noted that Newton's second law for rotations of rigid bodies only are valid for rotations around a fixed axis. Similarly, the expression $L_{O,z} = I_z\omega$ is only valid for rotation around a fixed axis, and it is only valid in the z-axis. But would it not be nice if the expression was completely general:

$$\mathbf{L}_O = I\boldsymbol{\omega}. \tag{16.118}$$

As you will learn later, we can formulate the relation in this way, but then I is a more complicated quantity than a mere scalar. However, this expression is *not generally correct* if I is interpreted as a scalar such as I_z. (You can see a proof in Sect. A.9). We can only use the expression in (16.118) when the rigid body is rotationally symmetric around the z-axis. However, our results for the z-component, $L_{O,z} = I_z\omega$, of the rotational momentum of a rigid body are always true—as long as the body is rotating around a *fixed* axis.

Putting It All Together

Finally, we want to put all these results together to address processes where we can use conservation of rotational momentum to solve a problem without determining the details of the motion.

Redistribution of Mass

An example of a typical process where we can use conservation of rotational momentum, is the redistribution of mass within a rotating system. For example, if a skater performing a pirouette pulls in his arms, he changes the distribution of mass, and therefore the moment of inertia around the rotation axis. Since the external forces have no significant torque around the rotation axis, the rotational momentum is conserved throughout the process:

$$L_{O,z}(t_0) = I_z(t_0)\omega(t_0) = L_{O,z}(t_1) = I_z(t_1)\omega(t_1). \qquad (16.119)$$

If you change the distribution of mass, you change I_z. To keep the rotational momentum constant, you must change the angular velocity correspondingly. Pulling your arms in while spinning reduces I_z. As a result the angular velocity will increase.

Collision with Rotation Around a Fixed Axis

Another example of a typical process where we can use conservation of rotational momentum is a collision where the bodies after the collision rotate as one rigid body around a fixed axis. If the external torques acting on the bodies are insignificant, the rotational momentum around the fixed axis is conserved:

$$L_{O,z}(t_0) = L_{O,z}(t_1) = I_z(t_1)\omega(t_1), \qquad (16.120)$$

which allows us to find the angular velocity after the collision. A typical example of such a process is where a bullet is shot into a rigid body: Such as a meteor impacting on a planet or a bullet shot into a rigid pendulum, as illustrated in Fig. 16.22.

Since Newton's second law for rotational motion comes in two forms: for rotation around an axis that is fixed in space and for rotation around the center of mass, we must be careful to check what version is relevant in a given case. Notice that for a fixed axis, a typical error is to assume that the net force is zero because the net torque is zero. Unfortunately, this is generally wrong. In the case illustrated in Fig. 16.22 you cannot ignore the force from the axis on the rotating object, and the translational momentum is therefore not conserved, while the rotational momentum is conserved. However, if the bodies are free to move, such as when a planet is hit by a meteor as illustrated in Fig. 16.22, the net external force is zero, and the translational momentum is conserved. You must therefore carefully address the net external force and the net external torque in each situation.

16.4.2 Example: Changing Your Angular Velocity

Problem: How can you control your angular velocity if you are (a) spinning around a pole (b) spinning while diving.

Solution: A rotating plate with a pole at its center can often be found on children playgrounds. You jump onto the plate and hold the pole, and spin the plate up with your feet. Then you can change your angular velocity by pulling yourself in towards the pole as illustrated in Fig. 16.23. How does this work?

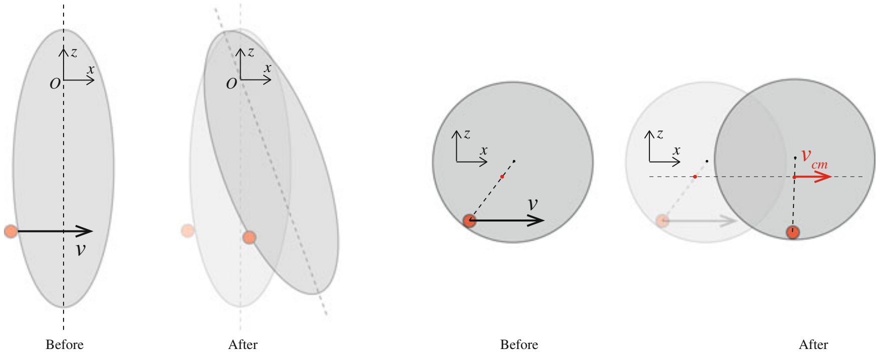

Fig. 16.22 *Left* A collision between a bullet and a rigid body hanging from the point O. *Right* A collision between a meteor and a planet

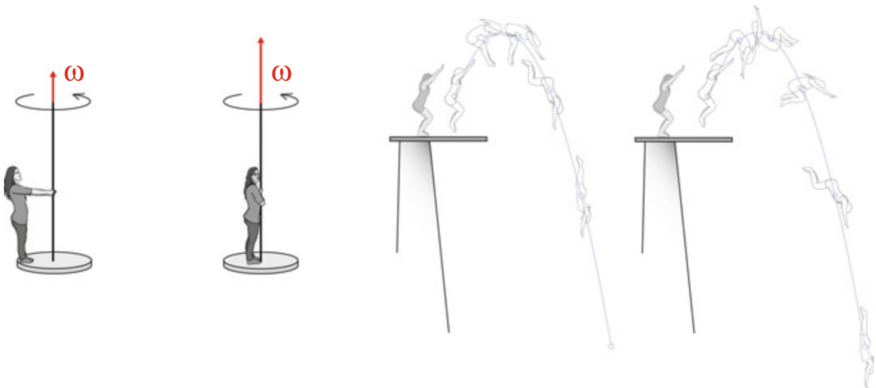

Fig. 16.23 *Left* A person on a spinning disk increases his angular velocity by pulling himself closer to the rotation axis. *Right* A diver changes her angular velocity around the center of mass by curling up, but that does not affect the path taken by the center of mass significantly

The only external forces acting on the system consisting of you and the plate are gravity and the forces acting on the axis of rotation, but the z-component of the torques of these forces are zero. The rotational momentum around the z-axis is therefore conserved. When you pull yourself towards the rotation axis, you are changing how the mass is distributed. You moment of inertia around the rotation axis is approximately $I_z = MR^2$, where R is the distance from you to the axis (if you were a single point). Conservation of rotational momentum then gives:

$$L_z(t_0) = I_z(t_0)\omega_0 = L_z(t_1) = I_z(t_1)\omega_1 , \tag{16.121}$$

$$MR_0^2\omega_0 = MR_1^2\omega_1 . \tag{16.122}$$

where ω_0 is your initial angular velocity when you are at a distance R_0 from the pole, and

$$\omega_1 = \frac{R_0^2}{R_1^2}\omega_0 , \tag{16.123}$$

is the angular velocity you getting. Pulling yourself towards the pole, makes $R_1 < R_0$ and therefore $\omega_1 > \omega_0$: You speed up!

Similarly, if you start your dive with a small angular velocity ω_0 around your center of mass when your body is approximately stretched out in its full length, you can increase your angular velocity, by pulling your body closer towards the center of mass, decreasing your moment of inertia around the center of mass:

$$I_{cm,z}(t_0)\omega(t_0) = I_{cm,z}(t_1)\omega(t_1) . \tag{16.124}$$

By stretching out again, you can decrease your angular velocity, effectively allowing you to determine how fast you spin. However, you cannot control the motion of your center of mass: The path taken by your center of mass is determined by the external forces acting on you, gravity and air resistance, and gravity is not affected by your redistribution of mass. Therefore you will typically follow the same path irrespectively of how you spin during your dive. (There may be small differences because air resistance will depend both on your spin and on how your body is curled up).

16.4.3 Example: Conservation of Rotational Momentum

Problem: A fly of mass m lands at the outer rim of a spinning DVD. The disk has mass M and radius R, and rotates with the angular velocity ω_0 before the event. Find the angular velocity of the plate after the fly has landed. (You can assume that the fly is able to grip onto and remain attached to the plate.)

Solution: The system consists of two objects: The DVD and the fly, as illustrated in Fig. 16.24. We choose the fixed rotational axis through the center of the DVD as the origin.

We plan to use conservation of rotational momentum to solve the problem. The external forces on the system consiting of the DVD and the fly, are the gravitational forces on the disk and the fly, and the force on the disk acting in the attachment point at the center of the disk. The gravitational force on the disk and the forces on the disk from the attachment point in the axis do not contribute to the net torque around the axis, since the “arm”, **r**, is zero for these forces. The torque due to the

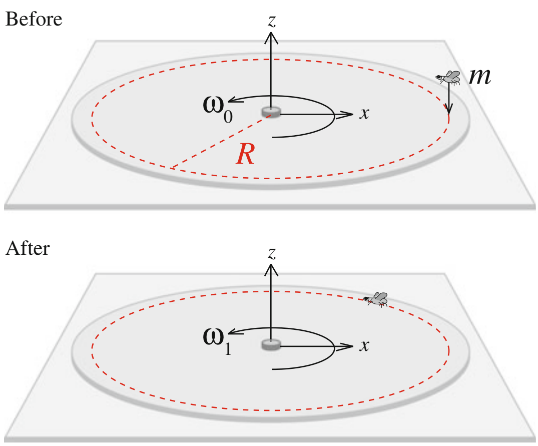

Fig. 16.24 A fly lands on a spinning DVD. The illustration shows the situation before (*top*) and after (*bottom*) the fly has landed

gravitational force on the fly acts in the xy-plane, and its z-component is therefore zero. The z-component of the net torque on the system is therefore zero, and rotational momentum is conserved:

$$L_{O,z}(t_0) = L_{O,z}(t_1), \tag{16.125}$$

Immediately before the bug landed, the rotational momentum of the disk was

$$L_{O,z}(t_0) = \underbrace{I_z \omega_0}_{disk} + \underbrace{0}_{fly}, \tag{16.126}$$

where the z-component of the rotational momentum of the fly is zero since the fly is only moving vertically when it is landing.

After the landing, the fly is moving with the same velocity as the disk. The fly has therefore effectively changed the moment of inertia of the spinning disk, by adding a mass m at a distance R from the axis:

$$I_z(t_1) = I_z(t_0) + mR^2, \tag{16.127}$$

and the rotational momentum of the disk-fly system is:

$$L_{0,z}(t_1) = (I_z + mR^2)\omega_1, \tag{16.128}$$

Now, we can find the final angular velocity, since the rotational momentum is conserved:

$$I_z \omega_0 = \left(I_z + mR^2 \right) \omega_1, \tag{16.129}$$

$$\omega_1 = \frac{I_z}{I_z + mR^2} \omega_0. \tag{16.130}$$

Since the disk is a cylinder, we know that the moment of inertia around the center of the disk (the center of mass) is: $I_z = MR^2/2$. The final angular velocity is therefore:

$$\omega_1 = \frac{\frac{1}{2}MR^2\omega_0}{mR^2 + \frac{1}{2}MR^2} = \frac{M}{M+2m}\omega_0 . \tag{16.131}$$

16.4.4 Example: Ballistic Pendulum

This problem is a classic in mechanics. It illustrates the conservation of angular momentum during a collision.

Problem: A thin rod of length L and mass M is hanging from a point O at one end of a the rod. A bullet of mass m is shot horizontally into the rod, and hits the rod with the horizontal velocity v at the lower end of the rod. The bullet is stuck in the rod. Find the angular velocity of the object immediately after the collision.

Approach: We divide the problem into two parts: The collision and the subsequent motion. During the collision we try to use conservation of rotational momentum to find the angular velocity after the collision. For the subsequent motion we use energy considerations to find how high the pendulum swings.

Identify: We use the rotation axis as the origin. The bullet has an initial velocity $\mathbf{v}_0 = v\,\mathbf{i}$, and the rod is initially hanging straight down, with no angular velocity. The system is shown in Fig. 16.25.

Model: The net external force is not zero throughout the collision, since the rod is affected by forces acting in the axis. What about the net torque around O? The external forces acting on the system is the force, $\mathbf{N}$, acting on the axis, $\mathbf{r}_N = \mathbf{0}$; the gravitational force, $\mathbf{G}$ of the rod; and the gravitational force $\mathbf{G}_b$ of the bullet. We assume that the rod does not move much during the collision. Consequently, the gravitational force on the rod acts approximately in the point $\mathbf{r}_G = -L/2\,\mathbf{j}$ throughout the collision, and similarly that the gravitational force on the bullet acts in $\mathbf{r}_b = -L\,\mathbf{j}$. Also, we assume that the rod is thin, so that the bullet is at $x_b = 0$ throughout the collision—therefore, we do not have to address the horizontal position of the bullet when we calculate the torques. The net external torque is therefore:

$$\boldsymbol{\tau}^{\text{net}} = \mathbf{0} \times \mathbf{N} + \mathbf{r}_G \times \mathbf{G} + \mathbf{r}_b \times \mathbf{G}_b , \tag{16.132}$$

where all the terms are zero! Hence the net torque is zero, and the rotational momentum around the point O is conserved throughout the collision.

We find the z-component of the total rotational momentum using the superposition principle, summing the z-component of the rotational momentum of the rod and the bullet. For the rod, $L^R_{O,z} = I_z\omega$, which is zero before the collision when the rod

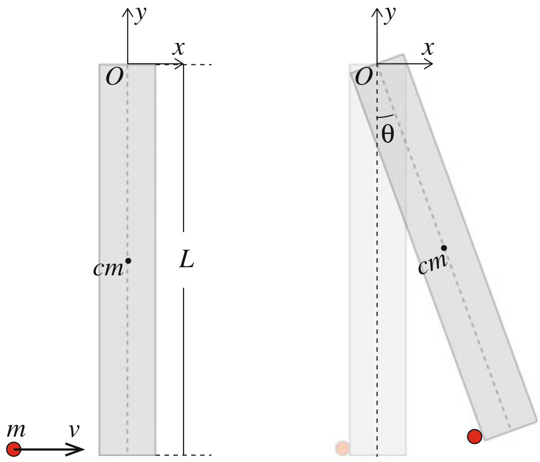

Fig. 16.25 Illustration of a collision between a bullet (*red*) and a rod (*grey*). The bullet is small compared with the rod and remains stuck on the rod after the collision. The rod rotates around an axis through the origin O

starts at rest. Immediately before the collision, the angular momentum of the bullet around the point O is:

$$\mathbf{L}_O^b = \mathbf{r}_b \times m\mathbf{v} = -L\,\mathbf{j} \times mv_0\,\mathbf{i} = Lmv\,\mathbf{k}, \tag{16.133}$$

and the z-component is:

$$L_{O,z}^b = Lmv. \tag{16.134}$$

The total z-component of the angular momentum of the system immediately before the collision is therefore:

$$L_{0,z} = \underbrace{Lmv}_{\text{bullet}} + \underbrace{I\omega_0}_{\text{rod}} = Lmv. \tag{16.135}$$

After the collision, the bullet is attached to the rod and the bullet-rod system rotates as a rigid body around the axis. The z-component of the rotational momentum of a rigid body is $L_z = I_{\text{tot}}\omega$, where I is the moment of inertia. We find the moment of inertia of the bullet-rod system using the superposition principle, summing the moment of inertia of the rod, I, and the bullet, which we consider to be a particle of mass m at a distance L from the rotation axis:

$$I_{\text{tot}} = I + mL^2. \tag{16.136}$$

The rotational momentum after the collision is therefore:

$$L_{O,z} = \left(I + mL^2\right)\omega, \tag{16.137}$$

Solve: Conservation of angular momentum gives:

$$Lmv = \left(I + mL^2\right)\omega \;\Rightarrow\; \omega = \frac{Lmv}{I + mL^2}. \tag{16.138}$$

Analyze: The subsequent motion of the rod-bullet pendulum can be determined from Newton's second law for rotation around the axis O.

16.4.5 Example: Rotating Rod

Problem: A thin rod of length L and mass M is lying at rest on a frictionless surface. A bullet of the same mass, M, is shot horizontally into the rod, and hits the rod with the horizontal velocity v at a distance y from the center of the rod. The bullet is stuck in the rod. Find the translational and angular velocity of the object immediately after the collision.

Approach: Since the system is not affected by any external forces, the translational and rotational momentum is conserved, and we can use conservation laws to relate the motion before and after the collision.

Identify: The system consists of the rod and the bullet. We describe the motion of the system by the motion of its center of mass and by the rotation of the system around its center of mass. The system is illustrated in Fig. 16.26.

Model: The motion of the system is determined by the external forces acting on the system. Since we can ignore frictional forces in the plane of motion, the only external forces acting are the gravitational force and the normal force from the surface. Since the rod is not moving vertically, the net external forces are zero. From Newton's second law for translational motion, this means that the (translational) momentum is conserved throughout the collision and during the motion afterwards:

$$\sum_j \mathbf{F}_j^{\text{ext}} = \frac{d}{dt}\mathbf{P} = M\mathbf{A} = 0. \tag{16.139}$$

The center of mass of the system therefore moves with a constant velocity—before, during, and after the collision! Notice that the translational motion does not depend on where the bullet hits the rod—the velocity of the center of mass is the same independently of where the bullet hits, although the position of the center of mass of course will depend on where the bullet hits.

The velocity of the center of mass before the collision depends on the velocity of the objects: The velocity of the rod is, $\mathbf{v}_r = 0$, and the velocity of the bullet is $\mathbf{v}$, hence:

$$(M + M)\,\mathbf{V} = M\mathbf{v} + M\mathbf{0} = M\mathbf{v}\ , \tag{16.140}$$

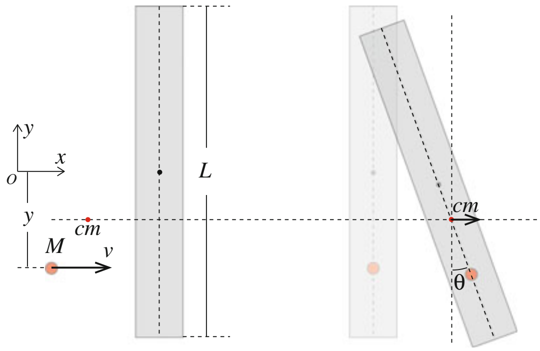

Fig. 16.26 Illustration of a collision between a bullet (*red*) and a rod (*grey*). The bullet is of the same mass as the rod, and the bullet remains stuck on the rod after the collision. The center of mass during the collision is illustrated

and therefore:

$$\mathbf{V} = \frac{1}{2}\mathbf{v} = \frac{1}{2}v\,\mathbf{i}. \tag{16.141}$$

Since the system has no velocity component in the y-direction, the y-position of the center of mass remains constant, with the following value:

$$(M + M)\,Y = M\,0 - M\,y \;\Rightarrow\; Y = -y/2, \tag{16.142}$$

as illustrated in Fig. 16.26.

Second, we determine the motion relative to the center of mass. The rod may start rotating, depending on where the bullet hits the rod. The rotational motion around the center of mass is related to the torque of the external forces around the center of mass. But, since there are no external forces acting in the plane of motion, the z-component of the external torque is zero:

$$\boldsymbol{\tau}^{\text{ext}} = 0. \tag{16.143}$$

We apply Newton's second law for rotational motion around the center of mass:

$$\boldsymbol{\tau}^{\text{ext}} = \frac{d}{dt}\mathbf{L}_{cm} = 0, \tag{16.144}$$

and find that there is no change in the rotational momentum around the center of mass: The rotational momentum around the center of mass of the system is conserved. This is also the case for the z-component of the rotational momentum around the center of mass:

$$\tau_z^{\text{ext}} = \frac{d}{dt}L_{cm,z} = 0. \tag{16.145}$$

Since we know the rotational momentum around the center of the mass before the collision, we can use this to find the rotational momentum around the center of mass after the collision, and from this we find the angular velocity of the object around the center of mass.

The rotational momentum around the center of mass is the sum of the rotational momentum for each of the components—*around their common center of mass*,

$$\mathbf{L}_{cm} = \mathbf{L}_{\text{rod}} + \mathbf{L}_{\text{bullet}} . \tag{16.146}$$

Before the collision, the rotational momentum of the rod around the center of mass is 0, since it is not rotating initially. The rotational momentum of the bullet depends on the position $\mathbf{r}_{b,cm}$ of the bullet relative the center of mass:

$$\mathbf{L}_{\text{bullet}} = \mathbf{r}_{b,cm} \times M v \, \mathbf{i} = \left(x_{b,cm} \, \mathbf{i} + y_{b,cm} \, \mathbf{j} \right) \times M v \, \mathbf{i} = -y_{b,cm} \, M v \, \mathbf{k} . \tag{16.147}$$

Notice that only the y-component of $\mathbf{r}_{\text{cm}}$ contributes. Since the position of the bullet is $-y$, and the position of the center of mass is $-y/2$, we find that the y-component of $\mathbf{r}_{b,cm}$ is $y_{b,cm} = -y - (-y/2)) = -y/2$, and

$$L^0_{cm,z} = -\left(-\frac{y}{2}\right) M v = \frac{1}{2} M v y \, . \tag{16.148}$$

After the collision, the whole object, consisting of the bullet and the rod, is rotating around the center of mass with a rotational momentum:

$$L^1_{\text{cm},z} = I_z \omega . \tag{16.149}$$

The total moment of inertia is:

$$I_z = I_{\text{rod},z} + I_{\text{bullet},z} , \tag{16.150}$$

where both moments of inertia must be around the same axis: The axis going through the center of mass of the rod-bullet system.

For the rod, the moment of inertia around its own center of mass is given in Fig. 15.5:

$$I_{\text{rod},z,cm} = \frac{1}{12} M L^2 . \tag{16.151}$$

The axis through the rod-bullet center of mass is a distance $s = y/2$ from the center of mass of the rod, which we use in the parallel-axis theorem to find:

$$I_{\text{rod},z} = I_{\text{rod},z,cm} + M \left(\frac{L}{2}\right)^2 = \frac{1}{12} M L^2 + M \frac{y^2}{4} . \tag{16.152}$$

For the bullet, we assume it is a point particle, located at a distance $s = y/2$ from the center of mass, hence the parallel-axis theorem gives:

$$I_{\text{bullet},z} = M\left(\frac{y}{2}\right)^2 = \frac{1}{4}My^2, \tag{16.153}$$

The total moment of inertia is therefore:

$$I_z = \frac{1}{12}ML^2 + \frac{1}{4}My^2 + \frac{1}{4}My^2 = M\left(\frac{1}{12}L^2 + \frac{1}{2}y^2\right). \tag{16.154}$$

Finally, we find the angular velocity after the collision from:

$$\omega = \frac{L^1_{cm,z}}{I_z} = \frac{\frac{1}{2}Mvy}{M\left(\frac{1}{12}L^2 + \frac{1}{2}y^2\right)} = \frac{v}{y}\frac{6}{\left(\frac{L}{y}\right)^2 + 6}. \tag{16.155}$$

After the collision, the net external force is zero, and the rod-bullet system moves in a straight line with constant translational and angular velocity.

16.5 General Rotational Motion

So far we have only addressed motions where the axis of rotation does not change direction. But Newton's second law for rotations around a fixed point or around the center of mass has general applicability, it is also valid in cases where the torque is not parallel to the angular momentum. Let us address what happens in this case through two examples.

A Rotating Wheel

Let us assume that you are holding a spinning wheel—such as the wheel of a bike. You are holding onto a rod through the axis of rotation, and you want to rotate the axis by applying a pair of forces, **F** and −**F**, at the two ends of the rod. Let us find the motion of the wheel due to the force pair.

The system is illustrated in Fig. 16.27a. Initially, the wheel is rotating with the angular velocity ω around the x-axis, so that the initial angular momentum around the origin is:

$$\mathbf{L}_O = I\omega\,\mathbf{i}, \tag{16.156}$$

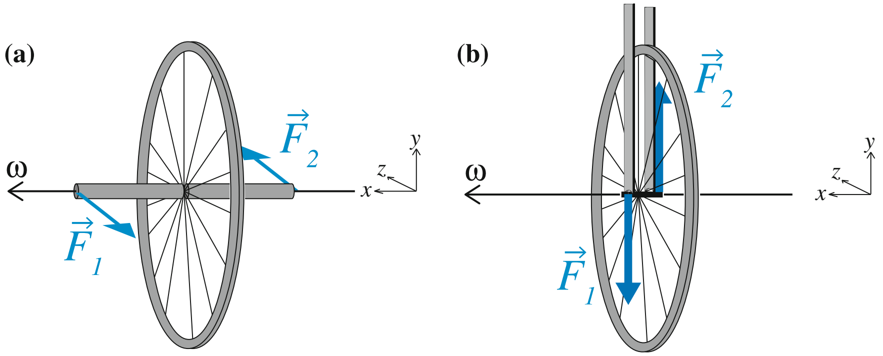

Fig. 16.27 **a** You try to change the rotation axis of a spinning wheel by applying a pair of forces **F** and −**F** at symmetric positions around the center of mass of the wheel. **b** An illustration of the front wheel of a bike rolling in the positive z-direction. The wheel is rolling, rotating around the x-axis. As you lean towards the *right*, you apply a pair of forces on the axis of the wheel, as illustrated. The resulting torque acts in the z-direction, causing the angular momentum to tilt towards the z-axis, rotating the wheel so that the bike turns *left*

since the object is symmetric around this axis.

We apply a pair of forces to the axis. A force $\mathbf{F}_1 = -F\,\mathbf{k}$ is acting at the position $\mathbf{r}_1 = x\,\mathbf{i}$, and a force $\mathbf{F}_2 = F\,\mathbf{k}$ is acting at the position $\mathbf{r}_2 = -x\,\mathbf{i}$. In addition, the wheel is affected by gravity, $\mathbf{W} = -mg\,\mathbf{j}$, and the axis is supported by two forces balancing the gravitational force, $\mathbf{N}_1 = \mathbf{N}_2 = mg/2\,\mathbf{j}$. The gravity and the two balancing forces have no net torque around the center of mass of the wheel.

What is the net torque around the center of mass of the system? We find:

$$\boldsymbol{\tau}_{\text{cm}}^{\text{net}} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = x\,\mathbf{i} \times -F\,\mathbf{k} - x\,\mathbf{i} \times F\,\mathbf{k} = -2xF\,\mathbf{j}. \tag{16.157}$$

Newtons' second law for rotational motion gives:

$$\frac{d\mathbf{L}_O}{dt} = \boldsymbol{\tau}_{\text{cm}}^{\text{net}} = -2xF\,\mathbf{j}. \tag{16.158}$$

The initial angular momentum is along the x-axis, in the **i** direction. After a small time interval Δt, the applied forces will have lead to a small tilting of the spin axis, but the axis is tilting in the direction of the y-axis, while the forces are applied along the z-axis. Is this strange? No, this is entirely consistent with the experience you gain while riding a bike. Consider the motion of the front wheel of your bike. Let us assume that while you are riding forwards, the wheel is spinning around the x-axis as illustrated in Fig. 16.27b. If you lean towards the left, you are transferring a pair of forces to the axis of the wheel. Leaning to the left, you pull up on the negative side (the side in the negative x-direction), and you push down on the positive side of the spinning axis. As a result, you apply a torque acting in the backward direction—along the z-axis on the figure, but backward. The rotational axis will therefore tilt in

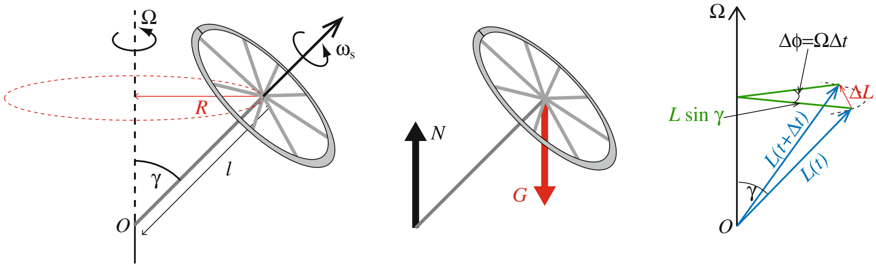

Fig. 16.28 An illustration of a spinning wheel balancing on the point O. The wheel rotates rapidly around its own axis with the angular velocity ω_s, in addition, the axis rotates slowly around the vertical axis with the angular velocity Ω

this direction, causing the wheel to turn to the left! Leaning on your bike therefore leads to the front wheel turning, as you surely have experienced while cycling.

A Spinning Top

A spinning top is a common child's toy. You spin it up by you hand (or by a string or similar device), and the top starts spinning rapidly, balancing on its bottom tip. The wheel not only rotates around its axis, in addition, the axis of rotation starts rotating slowly around the vertical axis. As the spinning wheel spins slower and slower around its own axis, its rotates faster and faster around the vertical axis. What is happening?

Let us make a simplified physical model of the spinning wheel. We consider the wheel to be a symmetric object rotating around its axis of symmetry with an initial angular velocity ω_s, as illustrated in Fig. 16.28. The spinning wheel is standing on a tip located on the symmetry axis, and we assume that it rotates approximately without friction. In addition, the rotation axis of the spinning wheel is rotating slowly, with an angular velocity Ω, around the vertical z-axis. Can we find a relation between ω_s and Ω?

We start by analyzing the situation in order to determine the motion of the spinning top. Its motion is determined by the forces acting on it. The spinning top is affected by gravity, $\mathbf{G}$, and the normal force, $\mathbf{N}$, as illustrated the free-body diagram in Fig. 16.28. We apply Newton's second law for the motion of the center of mass:

$$\sum F_z = N - Mg = MA_z = 0, \tag{16.159}$$

where we have assumed that the top does not move in the vertical direction, hence $A_z = 0$. This means that the normal force is:

$$N = Mg. \tag{16.160}$$

We apply Newton's second law for rotation around the point O—the contact point for the axis of the spinning top. This point is stationary—it does not move throughout the motion. We find the net torque around this point at a time when the center of mass of the spinning wheel is located along the x-axis, at the position

$$\mathbf{r} = R\,\mathbf{i} = Ml\sin\gamma\,\mathbf{i}, \tag{16.161}$$

where we see that $R = l\sin\gamma$ from Fig. 16.28. The net torque around O is therefore:

$$\boldsymbol{\tau}_O^{\text{net}} = \mathbf{0}\times\mathbf{N} + \mathbf{r}\times\underbrace{\mathbf{G}}_{=-Mg\,\mathbf{k}} = R\,\mathbf{i}\times(-Mg\,\mathbf{k}) = -MgR\,(-\mathbf{j}) = MgR\,\mathbf{j}. \tag{16.162}$$

What is the angular momentum of the spinning top? We notice that there are two contributions to the angular momentum: The spinning top is rotating around its axis of symmetry, and the center of mass is moving in a circle around the vertical axis. We use the general expression for the angular momentum for a rigid body

$$\mathbf{L}_O = \mathbf{R}\times M\underbrace{\mathbf{V}}_{=\Omega\mathbf{k}\times\mathbf{R}} + \mathbf{L}_{\text{cm}} = RM\Omega R\,\mathbf{k} + \mathbf{L}_{\text{cm}} = MR^2\Omega\,\mathbf{k} + I_{\text{cm}}\boldsymbol{\omega}_s, \tag{16.163}$$

where we have used that since the spinning top is rotating around its center of mass, the angular momentum around the center of mass is $\mathbf{L}_{\text{cm}} = I_{\text{cm}}\boldsymbol{\omega}_s$, where $\boldsymbol{\omega}_s$ points along the axis of the spinning top. In addition, we will now assume that the angular momentum of the rotation of the center of mass is much smaller than the angular momentum of the rotation of the spinning mass around its center of mass:

$$MR^2\Omega \ll I_{\text{cm}}\omega_s \;\Rightarrow\; f\Omega \ll \frac{I_{\text{cm}}}{MR^2}\omega_s, \tag{16.164}$$

where the prefactor is of the order one, since the mass is typically located a distance smaller than R from the center of mass. Consequently, we have assumed that:

$$\Omega \ll \omega_s, \tag{16.165}$$

that the spinning top is rotating much faster around its own axis, than the axis is rotating around the vertical axis. Our approximation is therefore:

$$\mathbf{L}_O \simeq I_{\text{cm}}\boldsymbol{\omega}_s. \tag{16.166}$$

Newton's second law for angular motion around a fixed point states:

$$\boldsymbol{\tau}_O^{\text{net}} = \frac{d\mathbf{L}_O}{dt}. \tag{16.167}$$

Now, since we know that the axis of the spinning top is rotating around the vertical axis with an (approximately) constant angular velocity, we know the change in angular momentum around the center of mass over a small time step Δt—it is simply found by the change in the angular momentum vector $\mathbf{L}_0$, which is (approximately) parallel to the angular momentum around the center of mass. The change in angular momentum is:

$$\Delta \mathbf{L}_O = \mathbf{L}_O(t + \Delta t) - \mathbf{L}_O(t). \tag{16.168}$$

From Fig. 16.28, we see that the angular velocity vector $\boldsymbol{\omega}_s$, and therefore also the angular momentum $\mathbf{L}_O$, rotates an angle $\Delta\phi$ around the vertical axis during the time interval Δt. Since the "radius" in this circle is $L_O \sin\gamma$, we see that the change in angular momentum is approximately equal to the arc length along the circle:

$$\Delta L_O = (L_O \sin\gamma)\, \Delta\phi. \tag{16.169}$$

We divide by Δt, getting:

$$\frac{\Delta L_O}{\Delta t} = L_O \sin\gamma \frac{\Delta\phi}{\Delta t} = L_O \sin\gamma\, \Omega. \tag{16.170}$$

From Newton's second law for rotational motion, we know that

$$\frac{dL_O}{dt} = \tau_O^{\text{net}} = Mgl \sin\gamma, \tag{16.171}$$

hence

$$L_O \Omega = Mgl, \tag{16.172}$$

and since $L_O \simeq I_{\text{cm}}\omega_s$ we find:

$$\Omega = \frac{Mgl}{L_O} \simeq \frac{Mgl}{I_{\text{cm}}\omega_s}. \tag{16.173}$$

We have found a relation between the angular velocity ω_s of the spinning top around its center of mass, and the angular velocity Ω of the spinning axis around the vertical axis.

We notice that Ω increases as ω_s decreases, which "explains" the behavior of a spinning top running on the floor: It wobbles faster as the spinning wheel slows down.

However, when $\omega_s \to 0$, our assumption that $MR^2\Omega \ll I_{\text{cm}}\omega_s$ breaks down, and our theory is no longer valid. Our solution therefore only has limited applicability, and we should make a rule to always check such assumptions at the end, to find if we have violated them.

Summary

Torque:

- The torque of the force $\mathbf{F}$ acting at the point $\mathbf{r}$ relative to the point O is: $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
- Torques of several forces can be added to find the **net torque** around a given point.
- The torque of the gravitational force on a rigid body is $\boldsymbol{\tau} = \mathbf{R} \times \mathbf{G}$, where $\mathbf{R}$ is the position of the center of mass of the body.

N2L for rotational motion around a fixed axis:

- For a rigid body rotating around a fixed axis (the z-axis) $\sum_j \tau_{z,j} = \tau_z^{\text{net}} = I_z \alpha$ where $\boldsymbol{\tau}_j = \mathbf{r}_j \times \mathbf{F}_j$ is the torque of force j acting in point $\mathbf{r}_j$, and I_z is the moment of inertia of the object around the z-axis.
- This law is only true for a rigid body rotating around a fixed axis or for a rigid body rotating around a fixed axis through its center of mass.

N2L for rotational motion around the c.m.:

- For a rigid body rotating around a fixed axis (the z-axis) through the center of mass, the acceleration of the center of mass is: $\sum_j \mathbf{F}_j = \mathbf{F}^{\text{net}} = M\mathbf{A}$ and the angular acceleration around the center of mass is: $\sum_j \tau_{z,j} = \tau_z^{\text{net}} = I_z \alpha$ where $\boldsymbol{\tau}_j = \mathbf{r}_{cm,j} \times \mathbf{F}_j$ is the torque of force j acting in point $\mathbf{r}_{cm,j}$ measured relative to the center of mass, and I_z is the moment of inertia of the object around the z-axis through the center of mass.
- This law is only true for a rigid body rotating around a fixed axis or for a rigid body which is spherically symmetric around the center of mass.

Rotational momentum for a system of particles:

- The **rotational momentum** (or **angular momentum**) of a point particle with (translational) momentum $\mathbf{p}$ at the position $\mathbf{r}$ relative to the point O is: $\mathbf{l} = \mathbf{r} \times \mathbf{p}$
- Newton's second law for the motion of a point particle can be written as: $\sum_j \boldsymbol{\tau}_j = \boldsymbol{\tau}^{\text{net}} = d\mathbf{l}/dt$
- The **rotational momentum** (or **angular momentum**) of a multiparticle system around the point O is: $\mathbf{L}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i$
- Newton's second law for rotational motion of a multiparticle system around the fixed point O is: $d\mathbf{L}_O/dt = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_O^{\text{ext}}$
- The **rotational momentum** (or **angular momentum**) of a multiparticle system around its center of mass is: $\mathbf{L}_{cm} = \sum_i \mathbf{r}_{cm,i} \times m_i \mathbf{v}_i$
- Newton's second law for rotational motion of a multiparticle system around its center of mass is: $d\mathbf{L}_{cm}/dt = \sum_i \mathbf{r}_{cm,i} \times \mathbf{F}_i^{\text{ext}} = \boldsymbol{\tau}_{cm}^{\text{ext}}$

Rotational momentum of a rigid body:

- The **rotational momentum** (or **angular momentum**) of a rigid body rotating around a fixed axis is $\mathbf{L}_{O,z} = I_{O,z} \omega$
- If the net external torque around a fixed point is zero, the rotational momentum of the system around the same fixed point is conserved.

- The **rotational momentum** (or **angular momentum**) of a rigid body rotating around a fixed axis through the center of mass is $\mathbf{L}_{cm,z} = I_{cm,z}\omega$
- If the net external torque around the center of mass is zero, the rotational momentum of the system around the center of mass is conserved. This is true also when the center of mass is moving.

Collisions and conservation laws:

- If the net torque around a *fixed point* O is zero (or very small) throughout a collision, then the angular momentum around this point is conserved throughout the collision.
  - If the net torque around the *center of mass* of a system is zero (or very small) throughout a collision, then the angular momentum around the center of mass is conserved throughout the collision—independently of the motion of the center of mass.

Exercises

Discussion Questions

16.1 Opening a door. If you can push with a maximum force F, how should you push a door to open it as quickly as possible?

16.2 Opening a jar. Why does it help to use an extending shaft to open a stuck lid?

16.3 Revolving door. A friend of yours is claiming that it is easier to open a revolving door than a single door of the same size as one half of the revolving door, because the swing door has a weight on the other side that balances the movement. Is he right?

16.4 Summersaulting. Explain the principle of doing a summersault on a trampoline.

Problems

16.5 Motion of rod during a collision-like process. In this problem we will study the motion of a thin rod that falls and attaches itself to a hinge. We will look at both the motion of the center of mass of the rod and how the rotation of the rod changes in the process (Fig. 16.29).

The moment of inertia of the rod about an axis through the center of mass is

$$I_{z,cm} = \frac{1}{12} M L^2 , \tag{16.174}$$

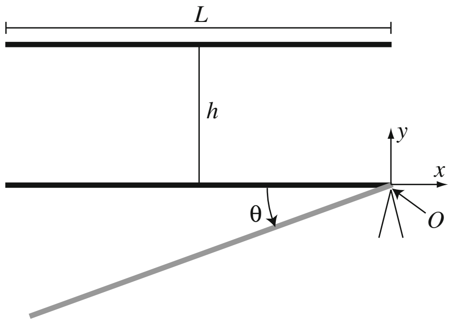

Fig. 16.29 Illustration of rod attachment

where M is the mass of the rod, and L its length. We hold the rod horizontally oriented with one of the ends of the rod a height h directly above a fixed point O where a hinge is located. We release the rod from rest. You can ignore air resistance.
(a) What is the velocity of the center of mass v_0 of the rod when it has fallen a distance h? What is the angular velocity ω_0 about the center of mass of the rod when it has fallen this distance?

When the end of the rod hits the hinge in point 0, it attaches itself, and the whole rod starts rotating about 0. We can view this process of attachment as a collision. You can assume all movement is in the plane as shown in the figure. You can also neglect air resistance and any friction in the hinge.
(b) Show that the moment of inertia of the rod about an axis normal to the rod through the point O is $I_{O,z} = ML^2/3$.
(c) Find the angular velocity of the rod about the point O immediately after the rod becomes attached. You can assume that the rod doesn't rotate during the process of attachment and that the torque from the gravity can be disregarded.
(d) Find the momentum of the rod immediately after the rod becomes attached. Is the momentum conserved? Explain.

The hinge is spring-loaded and affects the rod with a torque $\tau_{O,z} = -\kappa\theta$. The potential energy related to this interaction is $U = (1/2)\,\kappa\,\theta^2$.
(e) Find an expression for the angular acceleration of the rod when it has rotated an angle θ about the point O.
(f) It is possible to find the angle of the rod as a function of time using numerical methods, finding an analytical expression is more difficult. We can instead use a different method to find the maximum angle the rod rotates. Find an equation that decides the maximum angle θ of the rod. Note that you don't have to solve this equation.
(g) After reaching the maximum angle, the rod will change it's direction of rotation and swing back. What is the angular velocity, ω_2, of the rod about the point O the moment the rod is horizontal again? (i.e. when the angle θ is 0.)
(h) What is the velocity, v_2, of the center of mass when the rod is horizontal again?

The moment the rod reaches the horizontal orientation, the hinge in point O breaks, releasing the rod so it is no longer attached. You can assume the rod is not

affected by any external forces during this process and that the kinetic energy of the rod is conserved.
(i) Show that the velocity of the cm and the angular velocity about the cm immediately after the attachment fails is $v_3 = -(3/4)\, v_0$ and $\omega_{3,cm} = (3/2)\,(v_0/L)$
(j) Describe the motion of the rod after the attachment breaks.
(k) How high does the center of mass of the rod reach?

16.6 Collision between a rod and a block. In this problem we will study an impact between a rod and a small block. The rod is homogeneous with the mass M and length L. The rod is attached with a frictionless hinge in the point O so it can rotate as shown in the Fig. 16.30. The block is small compared to the rod. The block has the mass m and is initially at rest on a frictionless surface. The rod starts from rest at an angle θ_0 and is released. The rod hits the block when it is hanging straight down (i.e. when $\theta = 0$). The rod's moment of inertia about its center of mass is $I_{cm} = ML^2/12$.
(a) What is the rod's moment of inertia about the point O?
(b) Find the rod's kinetic energy as a function of the angle θ. You can disregard air resistance.
(c) Find the angular velocity of the rod, ω_0, immediately before it hits the block.
Let us first assume that the collision is perfectly elastic.
(d) Show that the velocity of the block immediately after the collision is $v_1 = (2\,\omega_0\, L)/(1 + (mL^2)/I_O)$.
(e) Show that the angular velocity of the rod immediately after the collision is $\omega_1 = \omega_0 \left(1 - (2) / \left(1 + I_O/\left(mL^2\right)\right)\right)$.
(f) Discuss the motion of the block and the rod after the collisions for the cases $m >> M$ and $m << M$.
(g) What happens in the case $m = M/3$?
Let us now assume the collision is perfectly inelastic.
(h) Find the angular velocity of the rod and the velocity of the block immediately after the collision.

16.7 A model of two rods colliding. We will in this problem look at a collision of two long and thin rods. This could for example be a model of how two long and linear molecules collide. The two rods are identical and remain stuck together after the collision. Each rod has a mass M and length L. For each rod the moment of inertia about its center of mass is $I_0 = ML^2/12$. The rods are gliding on a horizontal, frictionless surface as illustrated in Fig. 16.31.

The rods are parallel before the collision. One of the rods is at rest, while the other has the velocity v_0. After the collision they stick together like one rigid body, as illustrated in the figure. The starting position is characterized by the distance d as in the figure. You can neglect the width and height of the rods.
(a) Show that the moment of inertia around the center of mass for the body of the two rods stuck together is $I = (M/2)\left(d^2 + L^2/3\right)$
First assume $d = 0$.
(b) Find the velocity v_1 of the center of mass of the system after the collision.

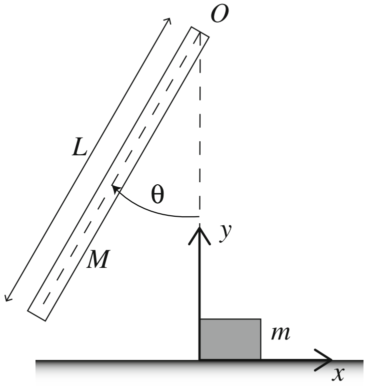

Fig. 16.30 Illustration of rod hitting a block

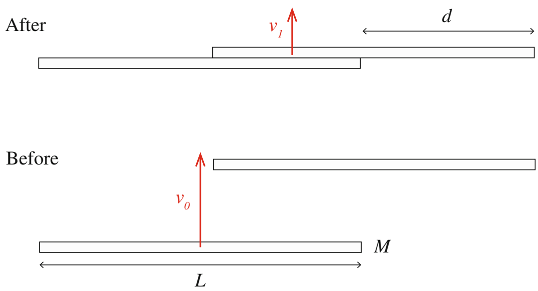

Fig. 16.31 Illustration of two rods colliding

(c) What is the angular velocity about the center of mass of the body of the two rods stuck together after the collision? Justify your answer.
Let us now look at the case $0 \le d \le L$.
(d) Find the velocity v_1 of the center of mass of the system after the collision.
(e) Find the angular velocity ω_1 of the entire system about its center of mass after the collision.
(f) What is the loss of energy in the collision? For what d is the loss of energy the least? Comment the result.
(g) Describe the motion after the collision.

16.8 Studying friction on a wheel. In this problem we will study the behavior of a spinning wheel that is lowered onto a flat, horizontal surface. The wheel has a mass m, a radius R, and a moment of inertia about its center of mass I. We let the x-axis be parallel with the surface and choose the direction of rotation to be positive in the clockwise direction, as illustrated in the figure. The coefficient of dynamic friction between the surface and the wheel is μ. The acceleration of gravity is g. The wheel is lowered onto the surface at a time $t = 0$s at the position $x(0) = x_0 = 0$. The initial velocity of the wheel is $v(0) = v_0 = 0$ and the initial angular velocity is ω_0 (Fig. 16.32).

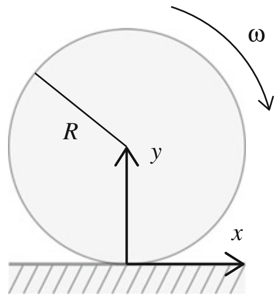

Fig. 16.32 Illustration of a spinning wheel

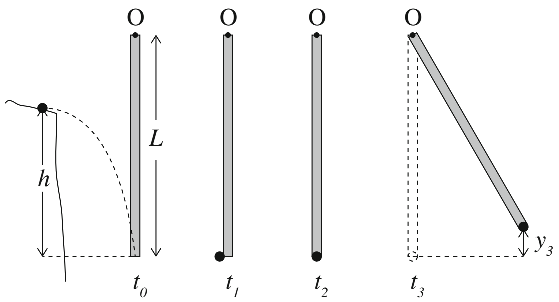

Fig. 16.33 Illustration of the motion of Tarzan and the vine: when Tarzan jumps (at t_0), immediately before he grabs the vine (t_1), immediately after he is attached to the vine (t_2), and when he reaches his highest point (t_3)

16.9 Tarzan's swing. Tarzan jumps from a cliff and grabs a vine. He jumps horizontally from the cliff with initial velocity v_0 at the time t_0. The vine has mass M and length L. Initially, the vine is hanging straight down and is attached at its highest point, O. Tarzan jumps from a height h above the lowest point on the vine, as illustrated in Fig. 16.33. After the "collision" Tarzan remains attached to the vine, with his center of mass at the lower edge of the vine. Tarzan's mass is m. The vine behaves as a rod attached without friction to the point O. The moment of inertia of a rod around its center of mass is $I = (1/12)ML^2$. You can neglect air resistance.
(a) Find Tarzan's velocity immediately before the collision with the vine (at t_1).
(b) What is the moment of inertia of the vine about the point O?
(c) Show that the angular velocity of the vine (with Tarzan) immediately after the collision is $\omega_2 = m/\left((M/3) + m\right)\,(v_0/L)$.
(d) How far up does Tarzan swing?
(e) How high would Tarzan swing if he jumped from twice the height?

16.10 Rolling up a slope. In this problem we study the motion of the rotating wheel placed on a slope. The wheel has mass M and radius R. The wheel consists of three homogeneous cylinders rotating around the same axis. The middle cylinder has radius R and mass $M/2$. A cylinder with radius $R/2$ and mass $M/4$ is glued on each side of the middle cylinder, as shown in Fig. 16.34. The moment of inertia around the symmetry axis for a cylinder of mass m and radius r is $I = (1/2)mr^2$.
(a) Show that the moment of inertia around the z axis for the wheel is $I = (5/16)MR^2$.

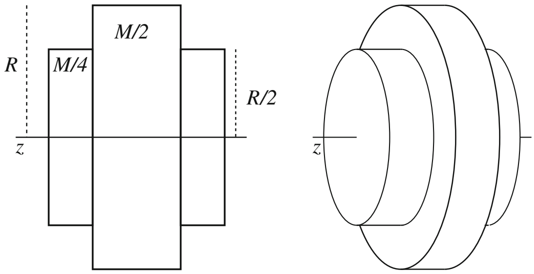

Fig. 16.34 Illustration of a wheel consisting of three cylinders that are glued together

The wheel starts with an angular velocity ω_0 and is placed on a slope as shown in Fig. 16.34, where the positive rotational direction is shown. The wheel starts without translational velocity. The coefficient of friction between the wheel and the floor is μ. The slope makes an angle θ with the horizontal. You can neglect air resistance.
(b) Draw a free-body diagram for the wheel and name the forces acting on it.
(c) Show that the acceleration of the center of mass of the wheel in the x-direction becomes $a_x = g\,(\mu \cos\theta - \sin\theta)$.

You can assume that $\mu \cos\theta - \sin\theta > 0$. In the following, we will only study the motion of the wheel before it starts rolling without slipping. You can assume that the wheel is moving upward until it starts rolling.
(d) Find the velocity, $v(t)$, of the center of mass of the wheel.
(e) Find the angular acceleration, $\alpha(t)$, of the wheel.
(f) Find the angular velocity, $\omega(t)$, for the wheel.
(g) Find the time it times until the wheel starts rolling without slipping.
(h) Describe the motion after the wheel starts rolling and explain your answer.

Projects

16.11 Snow crystal. In this project you will apply your knowledge of linear and angular momentum to study the aggregation of small droplets of ice to form large grains of snow.

As snow crystals form in clouds they start falling through the cloud. Due to air resistance, larger particles fall faster than smaller particles. A large particle will therefore overtake smaller particles. When a smaller particle is overtaken, it will stick to the larger particle, adding further to the size. This process forms aggregate snowflakes, which is one of the most common types of snowflakes.[3] This mechanism is often called differential sedimentation, and is a process important for pattern formation in many natural systems, and it is also a process important for many industrial processes. An example of a complex aggregate formed by a related aggregation process called

[3] You can learn more about this process, and look at how aggregate flakes look in the PhD thesis of Christopher David Westbrook at (http://www.met.rdg.ac.uk/sws04cdw/thesis.pdf).

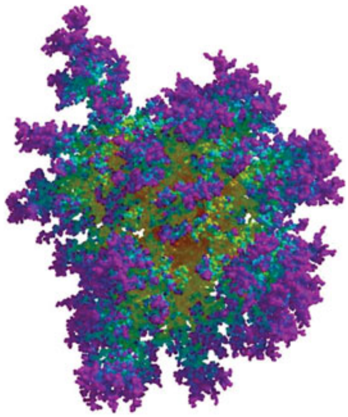

Fig. 16.35 Image of a (fractal) cluster formed by diffusion limited aggregation of 10000 particles. (Goold, 2004)

Diffusion Limited Aggregation in Fig. 16.35 shows the complex geometries typically found in aggregate grains.

In this project, we will study the aggregation process in detail. We will study an approximately spherical grain of ice of mass M and radius R, hitting and sticking to an identical grain of ice.

First, let us address why large particles fall faster than small particles. The mass of an ice grain of radius R and mass density ρ_m is

$$M = \rho_m \frac{4\pi}{3} R^3. \tag{16.175}$$

We will assume that air resistance can be modeled using the approximation:

$$\mathbf{F}_v = -k_v \mathbf{v}, \tag{16.176}$$

where

$$k_v \simeq 20.4 R\eta \tag{16.177}$$

is a constant depending on the viscosity η of the fluid.
(a) Find the forces acting on an ice grain with radius R, and write down Newton's second law of motion for the grain.
(b) Show that the acceleration of the grain is

$$\mathbf{a} = \mathbf{g} - \frac{20.4\eta}{\rho_m \frac{4\pi}{3} R^2} \mathbf{v}, \tag{16.178}$$

where $\mathbf{g} = -g\mathbf{j}$ and g is the acceleration of gravity. Can you now explain why larger grains fall faster than smaller grains?

We will now study a collision between two identical ice grains. One grain is at rest relative to the reference system and the other grain has a velocity v_0 downwards.

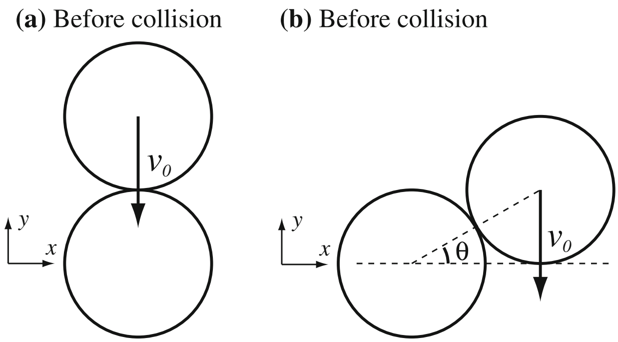

Fig. 16.36 Illustration of a collision between two identical ice grains. The lower grain is not moving, and the top grain is moving downwards with a velocity $\mathbf{v_0}$ as illustrated. In (**a**) the top grain hits at the top of the stationary grain, and in (**b**) the top grain hits the lower grain when the angle between the line connecting the centers of the grains and the horizontal is θ

When the two grains collide, they stick together at the point of contact, and remain stuck together. We call this combination of two grains a compound grain.
(c) The moment of inertia of one ice grain around its center is $I_c = \frac{2}{5} M R^2$. Show that the moment of inertia, I, around the center of mass for a compound grain consisting of two grains sticking together is $I = (14/5)\, M\, R^2$

First, we consider a linear collision where the upper grain hits the lower grain directly in the center, as illustrated in Fig. 16.36a. We assume the collision to be instantaneous, so you can ignore the effect of air resistance and gravity during the collision.
(d) What is the velocity, $\mathbf{v}_1$, of the center of mass the compound grain after the collision?
(e) What is the angular velocity, $\boldsymbol{\omega}_1$, around the center of mass of the compound grain after the collision?

Let us now consider the more general case illustrated in Fig. 16.36b. When the two grains touch, the line between the centers of the two grains forms the angle θ with the horizontal. The upper grain still has the initial velocity v_0 downwards before the collision, and the lower grain is at rest.
(f) What is the velocity, $\mathbf{v}_1$, of the center of mass of the compound grain after the collision?
(g) What is the angular velocity, $\boldsymbol{\omega}_1$, around the center of mass of the compound grain after the collision?
(h) What is the loss of energy in the collision?

Let us now address the motion of the compound grain after the collision. Initially, it is rotating with the angular velocity ω_1.
(i) If we ignore air resistance, find $\omega(t)$ as a function of time for the subsequent motion.

In the following we will not ignore air resistance, but rather develop a simplified model for the air resistance. In order to determine the force acting on the compound object due to air resistance, we either need to perform experiments on such objects, or we can use numerical simulations of the fluid flow around the object to determine the forces.

Here, we will use a strong simplification: We assume that we may consider the compound object to consist of two separate spheres. The force on each of the spheres due to air resistance is described by (16.176), where the corresponding velocity, v, in (16.176) is the velocity of the center of the sphere, and the force acts in the center of the sphere.

The compound object has velocity $\mathbf{v}_{cm}$ and angular velocity $\boldsymbol{\omega}$.

(j) Argue that the velocities, $\mathbf{v}_A$ and $\mathbf{v}_B$, of each of the ice grains A and B are $\mathbf{v}_A = \mathbf{v}_{cm} + \boldsymbol{\omega} \times \mathbf{r}$ and $\mathbf{v}_B = \mathbf{v}_{cm} - \boldsymbol{\omega} \times \mathbf{r}$, where $\mathbf{r}$ describes the position of grain A relative to the center of mass of the compound grain.

(k) Show that the net force on the center of mass of the compound object is $\sum \mathbf{F} = 2M\mathbf{g} - 2k_v\mathbf{v}_{cm}$, where $\mathbf{g} = -g\mathbf{j}$ and g is the acceleration of gravity.

(l) Show that the torque around the center of mass of the compound object due to air resistance is $\boldsymbol{\tau} = -2k_v\boldsymbol{\omega}R^2$ (Hint: Use Lagrange's formula).

(m) Show that the angular acceleration α of the compound object around its center of mass can be written as $\alpha = d\omega/dt = -(1/t_0)\omega$, and find the characteristic time t_0.

(n) Describe (with words) the motion of the compound object.

(o) Sketch the time development of the velocity v_{cm} and the angular velocity ω of the compound object, and discuss how the behavior would change if you changed the radius, R, of the grains.

(p) How would our argument change if we instead studied large particles, where the air resistance force depends on the square of the velocity?

Final comment: Notice that the result above for the net force on the compound grain indicates that small and large grains have the same acceleration, which is not consistent with our initial result. This is due to our (incorrect) simplification of adding the air resistance force for each of the grains together to get the air resistance force for the compound grain. For a real ice crystal formed by aggregation, the dependence of the air resistance on the size of the compound grain is more complicated, and will also depend on the complex geometry attained by a compound grain after a few hundred collisions with smaller grains.

16.12 Collision with rotation. In this project we address a collision between two identical atoms. You will learn how to determine external and internal motion of a diatomic molecule after a collision using a combination of analytical techniques, such as conservation laws, and numerical methods to determine the motion of the molecule.

We want to address a collision between two identical atoms of mass m, and we assume that we may consider the atoms to be point particles. The atoms are not affected by any external forces.

Here, we will first analyze a simplified model for the collision—a one dimensional model—before we analyze the full collision process.

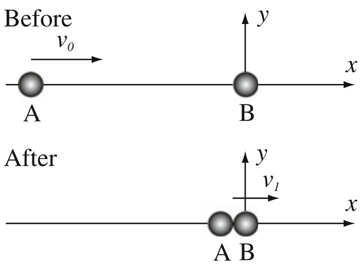

Fig. 16.37 Illustration of a simplified collision model: Atoms A and B collide on the x-axis

First, we address a simplified model. The system we consider consists of two atoms: atom A moves along the x-axis with the velocity v_0, and atom B is at rest in the origin as illustrated in Fig. 16.37. The atoms do not interact before they hit each other. After the collision they are stuck to each other.
(a) Find the velocity of the center of mass for the system before the collision.
(b) Find the velocity of the center of mass of the system after the collision.
(c) What is the change in the system's kinetic energy through the collision.

Let us make the model slightly more realistic by introducing a simplified model for the interactions between the two atoms. We will here not use a full model for the interatomic interaction, but instead assume that we can model the interatomic interaction using a spring force model. When atom A reaches a distance b from atom B, the two atoms become attached by a massless spring with spring constant k and equilibrium length b. The atoms remain attached with this spring throughout the collision and the subsequent motion.
(d) What is the velocity of the center of mass immediately after the atoms are attached with the spring, that is, when atom A is at the distance b from atom B? What is the change in kinetic energy for the system before and immediately after the collision?

Let us now address the motion of the atoms after the attachment. The positions of the atoms are x_A and x_B.
(e) Show that the force on atom A is: $F = k((x_A - x_B) - b)$, and find a corresponding expression for the force on atom B.
(f) Find expressions for the acceleration for atom A and B, and formulate the differential equations you need to solve to find the motion of the atoms, including the initial conditions.
(g) Write a program to determine the positions and velocities of atom A and atom B as a function of time. Assume $m = 0.1$, $k = 20$, $b = 0.2$, $v_0 = 1.0$, and $\Delta t = 0.001$.
(h) Plot the position as a function of time for the center of mass of the system and for each of the atoms.
(i) What is the maximum distance between the two atoms?

The collision we have addressed so far is a special case—the case of a central collision. Let us now address a non-central collision.

First, we address a simplified model for a non-central collision, as illustrated in Fig. 16.38. Atom A moves in the x-direction along the line $y = b$ with velocity v_0, and atom B is at rest at the origin. The atoms do not interact until they hit each other,

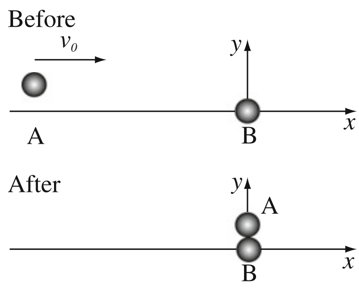

Fig. 16.38 Illustration of a simplified collision model: Atoms A and B collide in a non-central collision

which occurs when atom A reaches $x = 0$. After the collision, the atoms form a diatomic molecule, and the atoms remain attached at a fixed distance b from each other. (We are not studying a model without the spring force, but with a non-central collision. We will add the spring force again further on to get a complete, but still simplified model).

(j) What is the velocity of the center of mass and the angular velocity around the center of mass immediately after the collision?

Let us now introduce a more advanced model for this collision: When atom A is in the position $x = 0$, $y = b$, and atom B is in the position $x = 0$ and $y = 0$, the two atoms become attached with a massless spring with spring constant k and equilibrium length b. The atoms remain attached throughout the subsequent motion.

(k) Rewrite your program to model the motion of the atoms in this case.

(l) Plot the motion of the atoms and the center of mass after the collision.

(m) Discuss the motion of the angular velocity for the rotation about the center of mass for the motion after the collision.

16.13 Modelling a Bouncing Ball. In this project we will study a ball that bounces on a flat surface. We will only look at a single collision between the ball and the surface, but we will use different models for the interactions during the collision.

Both the ball and the surface are deformed during the collision, but you can assume that this deformation is small, this means that the forces from the surface on the ball will only act in a single point on the ball throughout the collision, and that the distance from this point to the center of mass of the ball does not change. The ball slips against the surface during the collision and the coefficient of dynamic friction between the ball and the surface is the constant μ. You can neglect air resistance. The ball has a mass m and a radius R, the acceleration of gravity is g, the moment of inertia of the ball about its center of mass is I.

You throw the ball from a height h with only a horizontal velocity. The velocity immediately before the collision with the surface is $\mathbf{v}(t_0) = v_{0x}\,\mathbf{i} + v_{0y}\,\mathbf{j}$, where v_{0x} is positive and v_{0y} is negative.

(a) Draw a free-body diagram for the ball while it is in contact with the surface. Identify the forces.

Let us first assume that the normal force from the surface on the ball is constant, N_0.

(b) Find the vertical component of the velocity, $v_y(t)$, and the vertical position, $y(t)$, of the ball while it is in contact with the surface.
(c) How long is the ball in contact with the surface?
(d) Find the horizontal component of the velocity of the ball as a function of time, $v_x(t)$, while it is in contact with the surface. What is the horizontal component of the velocity of the ball, v_{1x}, immediately after the collision?
(e) Find the angular velocity as a function of time, $\omega(t)$, as well as the angular velocity, ω_1, of the ball immediately after the collision. Describe the motion of the ball after the collision.
(f) Is the energy of the ball conserved during the collision? Does the ball bounce back to the height h after the collision? Justify your answers.

Now assume that the force from the surface on the ball is $N = k(R - y)^{3/2}$ when the ball is in contact with the surface, i.e. when $y < R$.
(g) Find expressions for the accelerations a_x and a_y of the ball while it is in contact with the surface.
(h) Find an expression for the angular acceleration a_z of the ball while it is in contact with the ground.
(i) Write a program that finds the motion of the ball's centre of mass as a function of time.
(j) Use your program to plot the motion and the velocities of the ball as a function of time from $t = 0$s to $t = 1$s when the ball has a radius of $R = 0.15$ m and is released from a height $h = 1$ m with an initial velocity $v_{0x} = 3$ m/s. The spring constant is $k = 10000$ N/m, the dynamic friction is $\mu = 0.3$, the mass of the ball $m = 1$ kg, the acceleration of gravity $g = 9.8$ m/s^2, and the moment of inertia $I = (2/3)$ kgm^2. Use a timestep of $dt = 0.001$ s.