Chapter 11
Energy

You have now learned to use the work-energy theorem as an alternative formulation of Newton's second law—as a "calculation" tool to determine the motion of an object. Using Newton's second law, we find the motion of an object described as the position as a function of time. The work-energy theorem allows us to find the velocity as a function of position without determining the whole motion, and we have learned that this may be useful, in particular when we are unable to find an exact solution to the equations of motion. While this was a practical application of the work-energy theorem, the real strength of the work-energy theorem is a conceptual change: We go from discussing motion and processes in terms of forces to instead discuss them in terms of *energy* and *energy conservation*.

Conservation laws: One of the most important consequences of Newton's second law and its alternative formulation through the work-energy theorem is the concept of a *conservation law*: That there are quantities in a system that are conserved throughout a process. We observe the system and record a particular quantity. Then we let the system develop in time, and we measure the same quantity again. If the quantity is conserved, it means that the quantity is unchanged, no matter what happens inside the system.

The two most important conservation laws you will learn are the conservation of (mechanical) energy and the conservation of momentum. The conservation laws in mechanics are consequences of Newton's laws of motion, but the conservation law for energy is much more general than that—it is one of the most general laws we know in nature. Throughout your studies of physics you will gradually learn to make energy considerations and energy conservation an integral part of your thought process, starting from this chapter.

In practice the conservation laws represent clever ways to solve physics problems. In many cases we cannot solve the equations of motion we get from Newton's second law directly, but in some cases we can solve the equations we get when we integrate Newton's second law. The conservation law for momentum comes from integrating Newton's second law in time and the conservation law for energy comes

A. Malthe-Sørenssen, *Elementary Mechanics Using Matlab*,
Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-19587-2_11

from integrating along a path, integration along the x-axis in one dimension, as introduced through the work-energy theorem.

Overview: In this chapter, we introduce the concept of energy and energy conservation through two examples: A vertical bowshot and an atom moving along a surface. Based on the examples, we introduce the concept of potential energy for a position-dependent force, a positional energy, to complement the kinetic energy, an energy of motion. For objects subject only to position-dependent force, the sum of the potential and kinetic energy is constant. We can therefore interpret a motion as a transfer of energy between kinetic and potential energies.

We show how to calculate the potential energy for a constant force, a spring force, and a general position-dependent force, and how to use energy conservation to solve mechanics problems. We introduce the energy diagram as an alternative way to analyze and understand motion. We generalize the concept of potential energy to two- and three-dimensional motion. Finally, we introduce the general energy principle, the second law of thermodynamics, and relate external work to changes in the total energy of a system.

11.1 Motivating Examples

The work-energy theorem provides us with an alternative formulation of Newton's second law that is particularly suited to find the velocity as a function of position in the case when only position-dependent forces act on an object. But we can simplify the analysis even further by the introduction of a position-dependent energy to complement the velocity-dependent kinetic energy, allowing us to use the conservation of total energy as an every quicker way to resolve questions relating the velocity and the position of an object subject only to position-dependent forces. We demonstrate this method by addressing a vertical bowshot and motion along a surface.

Work-Energy as a Conservation Law

How can the work-energy integral be considered a conservation law? And what do we mean by conservation? We know that any motion must satisfy Newton's second law and its path integral, the work-energy theorem. Let us see how this behaves in a simplified case—for a one dimensional motion with a net force that only depends on the position, $F^{\text{net}} = F(x)$. The work-energy integral is then

$$W_{0,1} = \int_{t_0}^{t_1} F(x(t))v(t)\,dt = \int_{x_0}^{x_1} F(x)\,dx = \phi(x_1) - \phi(x_0) = K_1 - K_0, \quad (11.1)$$

where we have introduced the function $\phi(x)$, which is the indefinite integral of $F(x)$, so that $dF/dx = \phi(x)$. This equation is valid for any two points x_0 and x_1 along the motion. We can rearrange this equation, so that all the quantities related to position 0 is on the left hand side and all the quantities related to position 1 is on the right hand side:

$$K_0 - \phi(x_0) = K_1 - \phi(x_1). \tag{11.2}$$

The function, $K - \phi(x)$ is a *constant* along the motion—it is *conserved* for the motion. We have found an example of a conservation law! Let us examine this conservation law in two simple examples to gain more intuition.

Vertical Shot

If you shoot an arrow vertically upwards, and we neglect air resistance, the arrow is affected by gravity, $G = -mg$, alone. This is a one-dimensional motion with a net force that only depends on position. (The force is constant and does therefore not depend on anything else than the position). If the arrow starts from $y = y_0$ with a velocity v_0, we can apply the work-energy theorem to find the kinetic energy, K_1, at any other position, y_1. And from the kinetic energy we can find the velocity, v_1, if so we please. The work-energy theorem gives:

$$W_{0,1} = \int_{t_0}^{t_1} G v_y \, dt = \int_{y_0}^{y_1} -mg \, dy = mgy_0 - mgy_1 = K_1 - K_0, \tag{11.3}$$

Let us rearrange this equation so that everything that refers to position 0 is on the left hand side, and everything that relates to position 1 is on the right hand side:

$$mgy_0 + K_0 = mgy_1 + K_1. \tag{11.4}$$

What does this mean? The left hand side is a constant that depends on the initial position, y_0, and the initial velocity, v_0, of the arrow. But the position y_1 on the right hand side can be any position along the motion: This means that the sum $mgy + K$ is constant throughout the motion. How can we interpret the various terms? We have already introduced the notion "kinetic energy" for the velocity-dependent term, $K = (1/2)mv^2$. The mgy term must therefore also have units of energy, and we can interpret this as an energy too. However, this energy does not depend on the velocity of the arrow, but on its position. We call this a positional energy, or more commonly, it is called a *potential energy*, $U = mgy$. We see that U is related to the function ϕ we defined in (11.1). We can now write the work-energy theorem for the motion as:

$$U_0 + K_0 = U_1 + K_1 = E, \tag{11.5}$$

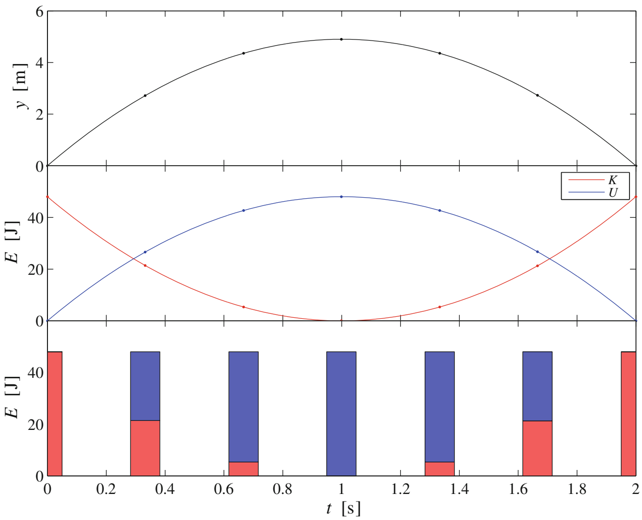


Fig. 11.1 Plot of the position, $y(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an arrow shot upward from $y_0 = 0$ m with initial velocity $v_0 = 9.8$ m/s

where we use the term *total energy* for the term E, which is the sum of the potential and kinetic energies of the arrow.

These new concepts provide insights into the motion of the arrow, as illustrated in Fig. 11.1. The plots show the position $y(t)$ and the kinetic and potential energies $K(t)$ and $U(t)$ for the motion. We see that initially, the arrow only has kinetic energy and no potential energy. This is also illustrated by the bar diagram at the bottom of Fig. 11.1. As the arrow moves upward, the kinetic energy reduces and the potential energy increases until the arrow reaches its maximum height, where the kinetic energy is zero and the potential energy is at its maximum, corresponding to the initial kinetic energy of the arrow. The kinetic energy cannot be negative, and therefore the arrow cannot go higher than this. As the arrow falls down again, the kinetic energy increases as the potential energy is reduced. The whole process is often illustrated by the bar chart diagram, which indicates that the total energy is conserved. It is only how the energy is distributed between the kinetic and potential energies that varies throughout the motion.

Motion Along a Periodic Surface

An atom moving along an atomic surface is subject to a periodic force $F_x(x)$:

$$F(x) = -F_0 \sin\left(\frac{2\pi x}{b}\right) , \tag{11.6}$$

where b is the interatomic distance in the surface and F_0 gives the strength of the interaction. If this is the only force acting on the atom, we can use the work-energy theorem to determine the velocity of the atom as a function of position. If the atom moves from x_0 to x_1, the work-energy theorem gives us that:

$$W_{0,1} = \int_{x_0}^{x_1} F(x)dx = \int_{x_0}^{x_1} -F_0 \sin\left(\frac{2\pi x}{b}\right) dx \tag{11.7}$$

$$= \frac{F_0 b}{2\pi} \cos\frac{2\pi x_1}{b} - \frac{F_0 b}{2\pi} \cos\frac{2\pi x_0}{b} \tag{11.8}$$

$$= -U(x_1) + U(x_0) + K_1 - K_0, \tag{11.9}$$

where we have gotten wiser and have introduced the function $U(x) = U^* \cos(2\pi x/b)$, and $U^* = F_0 b/2\pi$, as the *potential energy* of the atom. We can reorganize the terms, so that all the x_0 terms are on the left-hand side:

$$U(x_0) + K_0 = U(x_1) + K_1. \tag{11.10}$$

We notice that the potential energy, $U(x)$, may be both positive and negative. Hmmm. Is that allowed? Yes. We have not said anything about the signs of the potential energy. Although, the way we have defined kinetic energy does not allow this to become negative. Still, you may be uncomfortable with an initial negative potential energy (although you should get used to this thought by the end of this chapter). There is a simple solution to this: We can add a constant to both sides in (11.10)—and we can add a constant to the function $U(x)$ since we only care that its derivative $dU/dx = -F(x)$ and the derivative of a constant will be zero. This means that you are free to determine the zero level of the potential energy. Here, we would like the potential energy to be zero for the initial state, where $x = x_0$, which we achieve with the potential energy:

$$U_1' = U_1 + U^* = U^* \left(1 - \cos\frac{2\pi x}{b}\right). \tag{11.11}$$

This is nice and positive for all values of x_1 and it is zero at $x = 0$. Good. Let us use this expression for the potential energy, giving the conservation law:

$$U'(x) + K = U^* \left(1 - \cos\frac{2\pi x}{b}\right) + K = \text{const.} \tag{11.12}$$

This conservation law gives useful insight into the motion illustrated in Fig. 11.2. The plots show the position $x(t)$ of the atom for a small initial velocity v_0, and corresponding plots of the kinetic, $K(t)$, and potential, $U'(t)$, energies. The atom start with an initial kinetic energy, but no potential energy (we designed the potential energy this way, remember). As it starts moving in the positive x-direction, the potential energy increases, and the kinetic energy decreases, and the atom slows

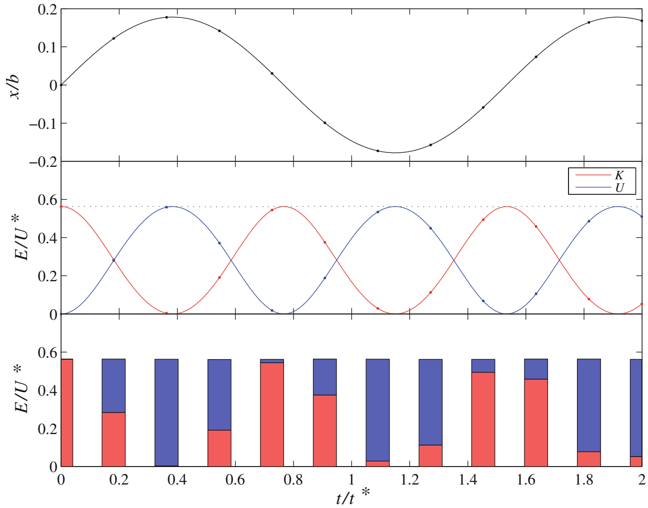


Fig. 11.2 Plot of the position, $x(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an atom moving along an atomic surface from $x_0 = 0$ with initial velocity $v_0 = 0.75\sqrt{2U^*/m}$

down. Until, at a particular value of x, $x = x_a$, the kinetic energy becomes zero. Since the kinetic energy cannot become less than zero, the atom cannot progress further in the positive x-direction. All the initial kinetic energy is now potential energy. The atom then starts moving in the negative x-direction, increasing its kinetic energy (and therefore speed) until it reaches $x = 0$. Then the kinetic energy decreases again until the atom stops at a position $x = x_b$ (because the potential energy, $U'(x)$ is symmetric around $x = 0$, we know that $x_b = -x_a$). Throughout this oscillation back and forth the total energy is distributed between kinetic and potential energy in such a way that the total remains a constant.

What happens if we start the atom from the same position, x_0, but with a larger initial velocity, v_0? Figrue 11.3 illustrates the motion, $x(t)$, and energies for a large initial velocity, v_0. Since the initial velocity is larger, the total energy, corresponding to the sum of the kinetic and potential energies, $E = K + U'$, is also larger than in Fig. 11.2. The total energy is illustrated by the dotted line in Figs. 11.2 and 11.3. Also in this case, we see that as the atom starts to move in the positive x-direction, the potential energy increases and the kinetic energy decreases. However, in this case, the kinetic energy never goes to zero. The atom therefore continues to move indefinitely in the positive x-direction. The velocity is therefore always positive, but it varies in magnitude according to the potential energy. You may wonder at what initial velocity v_0 the atom starts to be "free"—when does the kinetic energy not become zero? For now, we leave that for you to figure out, but we will revisit this situation later in the chapter.

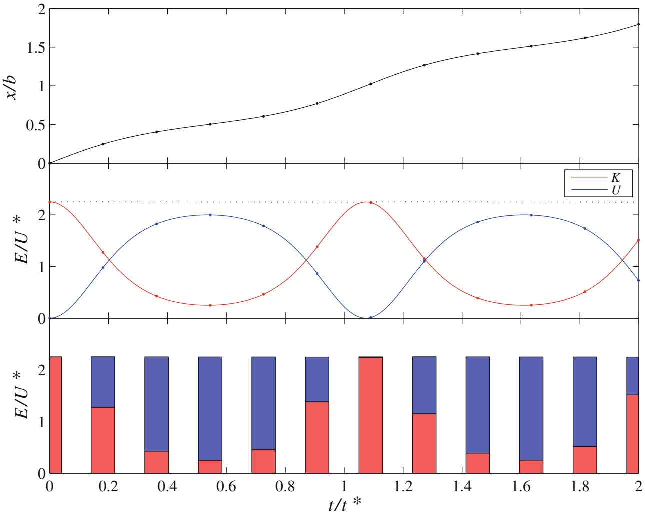


Fig. 11.3 Plot of the position, $x(t)$, kinetic, $K(t)$, and potential, $U(t)$ energies for an atom moving along an atomic surface from $x_0 = 0$ with initial velocity $v_0 = 1.5\sqrt{2U^*/m}$

These examples introduces the concepts of potential energy and energy conservation. In the rest of the chapter we will put these concepts in a more formal setting, and see many examples of the power of their use.

11.2 Potential Energy in One Dimension

Through these examples we have introduced the concept of potential energy, defined through the work-energy theorem. For a one-dimensional motion where the object is subject to a single force $F(x)$, which only depends on the position and not on the velocity or the time, we can introduce a potential energy for the force and a conservation law for the motion. The work-energy theorem for a motion from x_0 to x_1 gives

$$W_{0,1} = \int_{t_0}^{t_1} F(x(t)) \frac{dx}{dt} \, dt = \int_{x_0}^{x_1} F(x) \, dx = K_1 - K_0, \tag{11.13}$$

We can now introduce the function $U(x)$, which we call the *potential energy*, so that the work integral becomes

$$W_{0,1} = \int_{x_0}^{x_1} F(x) \, dx = U(x_0) - U(x_1) = U_0 - U_1. \tag{11.14}$$

Notice the choice of signs in this equation, introduced this way so that we could "pair" the energies at 0 and 1:

$$U_0 + K_0 = U_1 + K_1, \tag{11.15}$$

In the examples above we found that for gravity:

$$U(x) = mgx, \tag{11.16}$$

and for atomic motion along a surface

$$U(x) = U^* \left(1 - cos \frac{2\pi x}{b}\right). \tag{11.17}$$

We can find the *potential energy* at any x by calculating the integral of $-F(x)$ from some point x^*:

$$\boxed{U(x) = U(x^*) + \int_{x^*}^{x} -F(x)\,dx.} \tag{11.18}$$

Hmmm. What is the value of $U(x^*)$? We do not know. Actually, we are free to choose any value we like for $U(x^*)$, because the change in velocity from x_0 to x_1 only depends on the difference between the potential energy at $U(x_0)$ and $U(x_1)$. This may seem confusing at first—that the potential energy is not given as a particular number, but rather as a function with an undetermined zero level. But you may have some experience with this already based on your knowledge of calculus. If you look at (11.18) you may recognize that this defines the potential energy $U(x)$ as the indefinite integral of $F(x)$:

$$U(x) = \int -F(x) + C, \tag{11.19}$$

that is, the function $U(x)$ is the anti-derivative of $F(x)$. We could put this the other way, the definition of the potential energy $U(x)$ in (11.18) implies that $-F(x)$ is the derivative of $U(x)$:

$$\boxed{F_x(x) = -\frac{dU}{dx}.} \tag{11.20}$$

We see that whatever constant we add to $U(x)$ does not affect the force we get from taking the derivative.

Conservation of Energy

These relations define the potential energy $U(x)$ and relate it to the force $F(x)$ on the object. Notice that these definitions are general, and are valid also in cases where we cannot find a closed expression for the integral of $F(x)$. We have therefore shown that (11.20) is general, showing that for an object subject to a single, position-dependent force, $F(x)$, the total energy is conserved throughout the motion:

$$E_0 = U(x_0) + K_0 = U(x_1) + K_1 = E_1 . \tag{11.21}$$

We say that the *mechanical energy of the object is conserved*.

We call the corresponding force $F(x)$ a *conservative force*, if the mechanical energy is conserved. The mechanical energy is conserved if there exists a potential $U(x)$ for the force $F(x)$, or, to formulate it in an alternative way: If the work done by the force only depends on the end-points of the motion. Both statements may be used as equivalent definitions of a conservative force:

> The work done by a **conservative force** only depends on the end-points of the motion and not on the path taken.

> A force $F(x)$ is **conservative** if and only if it can be written as the derivative of the potential $U(x)$:
>
> $$F_x(x) = -\frac{dU}{dx} . \tag{11.22}$$
>
> We call $U(x)$ the potential for $F(x)$ or the potential energy of the object.

In one dimension, a force $F(x)$ that only depends on the position x (and not on the velocity or time) is conservative, since we can always find a potential $U(x)$, as demonstrated above. But in two- and three dimension, a force that depends only on the position is generally not conservative. A conservative force needs to fulfil an additional condition that we return to later.

Potential Energy for a Constant Force

An object moves from x_A to x_B while subject to a constant net force, F_0, in the x-direction. What is the potential energy for the object?

We find the potential energy from the integral in (11.18), starting from $x = x_0$:

$$U(x) = U(x_0) + \int_{x_0}^{x} -F_0\, dx = U(x_0) - F_0\,(x - x_0), \tag{11.23}$$

where we are free to choose the value for $U(x_0)$. For example, we may choose $U(x_0) = -F_0 x_0$, which gives:

$$U(x) = -F_0 x. \tag{11.24}$$

We recognize this from our previous calculation of the potential energy for the vertical bowshot, where the arrow is only subject to a (constant) gravity, $G = F_0 = -mg$, and therefore:

$$U(x) = -F_0 x = -(-mg)\,x = mgx. \tag{11.25}$$

Energy conservation for an object subject to a constant force can therefore be expressed as:

$$E_0 = U_0 + K_0 = U_1 + K_1 = E_1 \tag{11.26}$$

$$E_0 = -F_0 x_0 + \frac{1}{2} m v_0^2 = -F_0 x_1 + \frac{1}{2} m v_1^2 = E_1, \tag{11.27}$$

and for motion in the gravity field, we recover the well-know result:

$$mgx_0 + \frac{1}{2} m v_0^2 = mgx_1 + \frac{1}{2} m v_1^2. \tag{11.28}$$

Potential Energy in the Gravity Field with Normal Forces

The conservation of mechanical energy for motion subject only to gravitational forces is a useful tool to find simple solutions and make quick judgements about motions. However, the method is even more versatile. We we can also use it when an object is subject to gravity and a normal force, because the normal force does no work on the object during the motion.

Let us demonstrate the use of potential energy methods for motion under gravity and normal forces by studying the motion of a bead moving along a friction-free wire shaped as a helix, as illustrated in Fig. 11.4. The forces acting on the bead are gravity, $\mathbf{G} = -mg\,\mathbf{j}$, and a normal force, $\mathbf{N}$, from the wire on the bead.

We apply the net work-energy theorem for motion from position 0 at a height y_0, to a point P at a height y. The work done on the bead by the net force $\mathbf{F}^{\text{net}} = \mathbf{G} + \mathbf{N}$ is then:

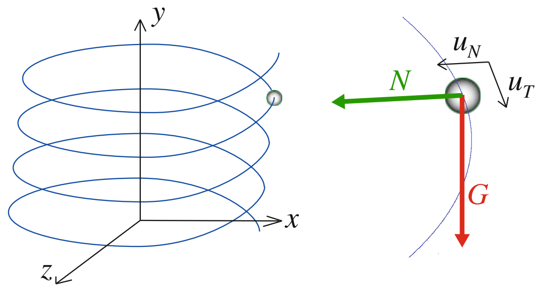


Fig. 11.4 *Left* A bead moves along a vertical *helix-shaped* wire. *Right* Free-body diagram for the bead

$$
\begin{aligned}
W^{\text{net}} &= \int_0^P \underbrace{\mathbf{F}_{\text{net}}}_{=\mathbf{G}+\mathbf{N}} \cdot d\mathbf{r} = \int_0^P (\mathbf{G}+\mathbf{N}) \cdot d\mathbf{r} = \int_0^P \mathbf{G} \cdot d\mathbf{r} + \int_0^P \mathbf{N} \cdot d\mathbf{r} \\
&= \mathbf{G} \cdot \int_0^P d\mathbf{r} + \int_{t_0}^t \underbrace{\mathbf{N} \cdot \mathbf{v}}_{=0} dt = -mg(y - y_0) + 0 \qquad (11.29) \\
&= mgy_0 - mgy = K - K_0,
\end{aligned}
$$

where we place all the terms for state 0 on the left side and for state 1 on the right side, getting:

$$mgy_0 + K_0 = mgy + K, \qquad (11.30)$$

which shows that mechanical energy is conserved for motion under gravity with a normal force, which is what we expected since the normal force does no work during the motion. The shape of the path does not matter: The energies depend only on the vertical position of the bead.

Potential Energy for a Spring Force

We have already discussed the potential for a constant force. The next step is to study the simplest possible (one dimensional) force that depends on position alone: the spring force. What is the potential energy of an object subject to a spring force $F(x)$?

Figure 11.5 illustrates a spring block system, where a block of mass m lies on a frictionless table. The block is attached to a spring with spring constant k, and the equilibrium position of the block-spring system is b. The force F on the block from the spring is modelled using a spring force model: $F(x) = -k(x - b)$, where x is the position of the block.

The force F depends on the position of the block only: It does not depend on the velocity of the block, and it does not depend directly on time. We can therefore find the potential energy by integrating the force $F(x)$ using the definition in (11.18):

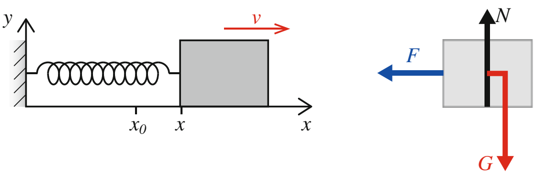


Fig. 11.5 *Left* Illustration of a block attached to a spring. *Right* A free-body diagram for the block

$$U(x) - U(x_0) = \int_{x_0}^{x_1} -F(x)\,dx = \int_{x_0}^{x_1} k(x-b)\,dx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_0^2. \quad (11.31)$$

We are free to choose $U(x_0)$, so we may choose $U(x_0)$ so that $U(x)$ is zero when the spring is in its equilibrium position. That is, we want $U(b) = 0$:

$$U(b) = U(x_0) + \frac{1}{2}kb^2 - \frac{1}{2}kx_0^2 = 0 \;\Rightarrow\; U(x_0) = \frac{1}{2}kx_0^2. \quad (11.32)$$

The potential energy for a spring force $F(x) = -k(x-b)$ is therefore:

$$U(x) = \frac{1}{2}k(x-b)^2. \quad (11.33)$$

Potential Energy of Many Individual Forces

In general, objects are affected by several forces simultaneously. For example, a person jumping on a trampoline is affected both by gravity and by the contact force from the trampoline. How can we extend our energy conservation approach to such cases?

The net force on an object affected by several forces F_j, where $j = 1, 2, \ldots$, is

$$F^{\text{net}} = \sum_j F_j(x). \quad (11.34)$$

Let us assume that each of the forces $F_j(x)$ are conservative, that is, they depend on the position x only (and not on velocity or time). The potential energy for force $F_j(x)$ is $U_j(x)$. The work-energy theorem for a motion from x_0 to x_1 gives:

$$\begin{aligned} W_{0,1}^{\text{net}} &= \int_{x_0}^{x^1} F^{\text{net}} dx = \int_{x_0}^{x_1} \sum_j F_j(x) dx = \sum_j \int_{x_0}^{x_1} F_j(x) dx \\ &= \sum_j \left(U_j(x_0) - U_j(x_1)\right) = K_1 - K_0. \end{aligned} \quad (11.35)$$

We move all the terms relating to position 0 to the right-hand side of the equation:

$$\underbrace{\sum_j U_j(x_1)}_{U(x_1)} + K_1 = \underbrace{\sum_j U_j(x_0)}_{U(x_0)} + K_0. \tag{11.36}$$

where the potential energy $U(x)$ is the sum of the potential energies for each of the forces $F_j(x)$. This gives a general law for energy conservation:

$$U(x_1) + K_1 = U(x_0) + K_0. \tag{11.37}$$

We can therefore generalize the energy conservation methods introduced above by introducing a potential energy for each of the forces affecting the object. Energy conservation still holds, as long as we include the sum of all the potential energies for all the forces affecting the object.

11.2.1 Example: Falling Faster

Problem: You throw an apple directly downward from a high cliff, giving it an initial velocity v_0. How far does the apple need to fall before it doubles its velocity? You can neglect the effects of air resistance.

Identify: The position of the apple is given by $y(t)$. At t_0, the apple is at $y = y_0$ with velocity $v = -v_0$. At t_1, the apple is at $y = y_1$ with velocity $v_1 = -2v_0$.

Model: The only force acting on the apple is gravity. We know that for motion under gravity, the total mechanical energy of the object is conserved throughout the motion:

$$E = U + K = \text{constant} \tag{11.38}$$

We use this to find the relation between the position and velocity of the apple.

Solve: The mechanical energy of the apple when at y_0 is:

$$E_0 = U_0 + K_0 = mgy_0 + \frac{1}{2}mv_0^2, \tag{11.39}$$

and at y_1 it is:

$$E_1 = U_1 + K_1 = mgy_1 + \frac{1}{2}mv_1^2. \tag{11.40}$$

Since energy is conserved, we know that:

$$E_0 = E_1$$
$$mgy_0 + \frac{1}{2}mv_0^2 = mgy_1 + \frac{1}{2}mv_1^2 \tag{11.41}$$

We want to find y_1 when we know that $v_1 = -2v_0$. We insert $v_1 = -2v_0$, and find:

$$mgy_0 + \frac{1}{2}mv_0^2 = mgy_1 + \frac{1}{2}m\left(-2v_0\right)^2 . \tag{11.42}$$

We solve for y_1, finding the position where the velocity is doubled:

$$mgy_0 + \frac{1}{2}mv_0^2 - 4\frac{1}{2}mv_0^2 = mgy_1 \;\Rightarrow\; y_0 - \frac{3}{2}\frac{v_0^2}{g} = y_1 . \tag{11.43}$$

11.2.2 Example: Roller-Coaster Motion

Problem: Let us revisit the roller-coaster problem using an energy conservation approach: A roller-coaster cart is rolling from the height h to the height 0 along a curving roller-coaster track. It starts with the speed v_0 at the top of the track. Find the speed of the roller-coaster cart at the bottom of the track. You can ignore air resistance and friction.

Identify: The cart moves from the position 0, $x(t_0) = x_0$, $y(t_0) = y_0$, to position 1, $x(t_1) = x_1$, $y(t_1) = y_1$ as illustrated in Fig. 11.6.

Model: The cart is affected by the normal force, $\mathbf{N}$, the friction force, $\mathbf{f}$, and gravity, $\mathbf{G} = -mg\,\mathbf{j}$. We assume that friction is negligible, $\mathbf{f} = 0$ throughout the motion.

Solve: Since the normal force, $\mathbf{N}$, does no work on the cart during the motion, we can use the conservation of energy to determine the relation between position and velocity of the cart. The conservation of mechanical energy of the cart gives:

$$E = K_0 + U_0 = K_1 + U_1 , \tag{11.44}$$

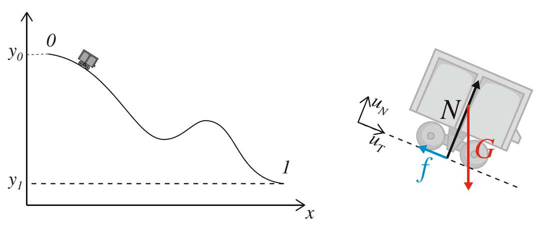


Fig. 11.6 A roller-coaster cart moving along a roller-coaster track

where we use that $U(y) = mgy$:

$$\frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_1^2 + mgy_1, \tag{11.45}$$

From this equation, we find the velocity v_1 as a function of the height difference, $h = y_0 - y_1$ and the initial velocity v_0 of

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_0^2 + mg\underbrace{(y_0 - y_1)}_{=h} \Rightarrow v_1^2 = v_0^2 + 2gh. \tag{11.46}$$

Analyze: This is, of course, exactly the same result as we found above when we used the net work-energy theorem. However, the use of energy considerations is much simpler and allows us to use more of our intuition about the process than the application of the net work-energy theorem.

11.2.3 Example: Pendulum

Problem: A classical problem is that of the motion of a pendulum. A pendulum consists of sphere of mass m hanging in a massless rope of the length L. The pendulum moves in a vertical plane with a maximum angle θ_0 with the vertical. Find the velocity v of the sphere as a function of the angle θ. We neglect air resistance.

Identify: We describe the motion of the pendulum with the angle θ between the rope and the vertical. We assume that the sphere follows a circular path. The system is illustrated in Fig. 11.7.

Model: The sphere is affected by the tension, **T**, from the rope and gravity, **G**. We already know that for motion under gravity we can use energy considerations to relate position and velocity. What about the tension **T**? It is always normal to the direction of motion of the sphere. The rope tension **T** is a *normal force*, which does no work during the motion. We can therefore use energy conservation to solve the problem.

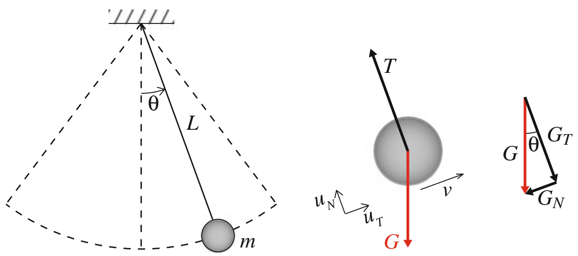


Fig. 11.7 *Left* Illustration of a pendulum. *Right* Free-body diagram for the sphere

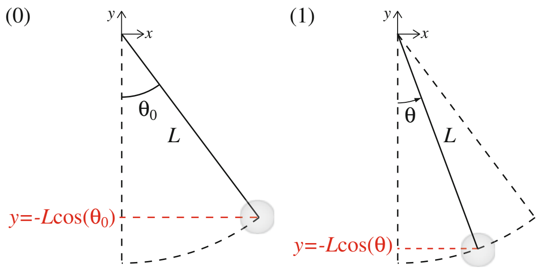


Fig. 11.8 Illustration of the positions. In position *0* (*left*) the sphere is at the top of its path. In position *1* (*right*) the sphere is at the angle θ

We examine two positions, as illustrated in Fig. 11.8, and use energy conservation to relate the two positions. In position *0* the sphere is at the top of its path. This is where the motion changes direction. The velocity is zero at this position. In position *1* the sphere is at an angle θ and the velocity of the sphere is v.

Energy conservation gives:

$$E = U_0 + K_0 = U + K , \tag{11.47}$$

$$E = mgy_0 + K_0 = mgy + K , \tag{11.48}$$

In order to find the velocities as a function of θ, we need to relate the vertical positions y_0 and y to the angle θ. From Fig. 11.8 we see that the vertical position of the sphere is

$$y_0 = -L\cos\theta_0 \quad \text{and} \quad y = -L\cos\theta , \tag{11.49}$$

where we have chosen the origin of the coordinate system to at the attachment point of the pendulum. We insert this into (11.48):

$$-\,mgL\cos\theta_0 + \frac{1}{2}mv_0^2 = -mgL\cos\theta + \frac{1}{2}mv^2 , \tag{11.50}$$

and we solve to find the velocity v:

$$v^2 = 2gL\left(\cos\theta - \cos\theta_0\right) \;\Rightarrow\; v = \sqrt{2gL\left(\cos\theta - \cos\theta_0\right)} , \tag{11.51}$$

where we have used that $v_0 = 0$.

Analyze: For this problem, energy considerations are particularly powerful, since it is difficult, if not impossible, to find an analytical solution to the motion of the pendulum. But using energy conservation we find the exact, analytical solution to the posed question, without solving for the path $\theta(t)$.

11.2.4 Example: Spring Cannon

This problem is a classic in mechanics

Problem: A block of mass m is placed on top of a vertical, massless spring with spring constant k. The spring is contracted a distance Δy from its equilibrium position (when there is no block on top of it). The spring is released, and the block is shot up through the air. How high up does the block go? You may neglect friction and air resistance.

Identify: The motion of the block is one dimensional, and we characterize the position of the block by its vertical position y. The block starts at a vertical position y_0 at the time t_0 with no vertical velocity $v_0 = 0$. The block is in contact with the spring until the time t_1 when the vertical position is y_1. From there on, the block is moving through the air without being in contact with the spring. The block reaches its maximum height y_2 at the time t_2, and the velocity of the block at this point is $v_2 = 0$. The process is illustrated in Fig. 11.9.

Model: The block is affected by two forces: The spring force $\mathbf{F}$ and the force from gravity, $\mathbf{G} = -mg\,\mathbf{j}$. Both gravity and the spring force are conservative forces. We can therefore apply energy conservation to relate the position and velocity of the block. However, the spring force is only acting when $y < y_1$.

Solve: We use conservation of mechanical energy to solve this problem. Energy is conserved because all forces only depend on the position of the block. They do not depend on the velocity of the block or directly on time. Mechanical energy is therefore conserved throughout the motion. In particular, it is the same at the points 0, 1, and 2.

$$E_0 = U_0 + K_0 = U_1 + K_1 = E_1 = U_2 + K_2 = E_2. \tag{11.52}$$

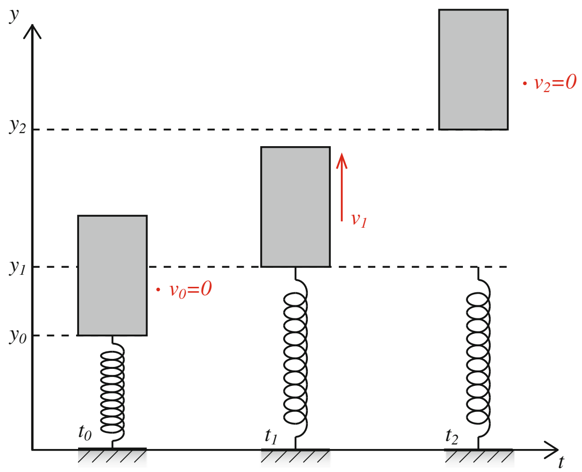


Fig. 11.9 Illustration of the motion of a block ejected by a spring cannon. At the point t_0, the spring is compressed to the position y_0, and the block is at rest. At t_1, the spring reaches its equilibrium length, and the block looses contact with the spring. At t_2 the block reaches it maximum height when $v_2 = 0$

Here, we realize that the total potential energy U consists of the potential energy from the spring force, U_F and the potential energy from gravity, U_G:

$$U(y) = U_G(y) + U_F(y). \tag{11.53}$$

$$U_G(y) = mg(y - y^*), \tag{11.54}$$

where we are free to chose where the potential energy is zero. That is, we are free to choose the value y^*. Here, we simply choose $y^* = 0$.

Also, we know that the potential energy of the interaction between the spring and the block is

$$U_F(y) = \frac{1}{2}k(y - y_1)^2, \tag{11.55}$$

where we have chosen the potential energy of the spring-block interaction to be zero when the spring is at its equilibrium position, that is, for $y = y_1$. We find the energies at the three positions 0, 1, and 2:

$$E_0 = U_G(y_0) + U_F(y_0) + K_0 = mgy_0 + \frac{1}{2}k(y_0 - y_1)^2 + \underbrace{\frac{1}{2}mv_0^2}_{=0\ (v_0=0)} \tag{11.56}$$

$$E_1 = U_G(y_1) + U_F(y_1) + K_1 = mgy_1 + \frac{1}{2}k(y_1 - y_1)^2 + \frac{1}{2}mv_1^2 \tag{11.57}$$

$$E_2 = U_G(y_2) + U_F(y_2) + K_2 = mgy_2 + 0 + \frac{1}{2}m\underbrace{v_2^2}_{=0}. \tag{11.58}$$

In position y_2, the block is not in contact with the spring. The force from the spring on the block is therefore zero from y_1 to y_2. The potential energy of the spring-block interaction is therefore the same for y_1 and for y_2, that is, the potential energy of the spring-block interaction is also zero at y_2.

Notice that even though we include the position y_1 in this calculation, we do not need this in order to determine y_2. Because energy is conserved throughout the whole motion, we only need to relate position 0 to position 2: $E_0 = E_2$. This is a particularly nice feature of using energy considerations when solving a problem!

We can now solve to find the height y_2:

$$E_0 = E_2 \ \Rightarrow\ mgy_0 + \frac{1}{2}k\underbrace{(y_0 - y_1)^2}_{=\Delta y^2} = mgy_2 \ \Rightarrow\ y_0 + \frac{\frac{1}{2}k\Delta y^2}{mg}\Delta y^2 = y_2, \tag{11.59}$$

Analyze: This example illustrates the power and ease of using energy considerations for such a problem. If we instead opted to use Newton's laws of motion, the calculations would be significantly longer. But remember that we can only use energy considerations when all the forces are conservative: If there was friction, air resistance, or other velocity-dependent force, we could not have used energy considerations directly, and in most cases, we would need to determine the motion in order to determine the work done by the velocity-dependent forces, and we would need to solve for the motion anyway.

Comment: Notice how we chose the potential energy for the spring-block interaction. We do not need to choose the potential energy $U_F(y)$ to be zero at y_1, we could for example also have chosen it to be:

$$U_F(y) = \frac{1}{2}k(y - y_1)^2 + C \tag{11.60}$$

where C is a constant. However, in this case, we need to ensure that we include this constant both in the energy at position y_1 *and* at the position y_2. Why also at position y_2—there is no spring force acting here? It is correct that we are free to choose the constant C, but we are not free to change this choice between the various phases. If we have chosen a particular C, we must ensure that our choose is consistent when we go from y_1 to y_2.

11.3 Energy Diagrams

If you launch a small cart in a hilly terrain, as illustrated in Fig. 11.10, how is the subsequent motion of the cart? For simplicity, we assume that we can neglect the air drag and friction. What is the velocity as a function of position, and how far will the cart travel?

We can visualize the motion directly on the figure. Several points have been highlighted in Fig. 11.10. The cart starts at the point 0. The cart rolls downhill, increasing its velocity. It reaches the bottom of the hill at 1, and starts rolling uphill. Will it make it up the hill?

Energy conservation: We can answer such questions using energy conservation. Because the only forces acting on the cart are gravity and the normal force, and the normal force does not do any work, the mechanical energy of the cart is conserved throughout the motion. This means that the sum of the potential and the kinetic energy is constant:

$$E = U + K = U(x) + K(x), \tag{11.61}$$

where the potential energy for the cart is the potential energy in a (constant) gravity field:

$$U(x) = mgy(x). \tag{11.62}$$

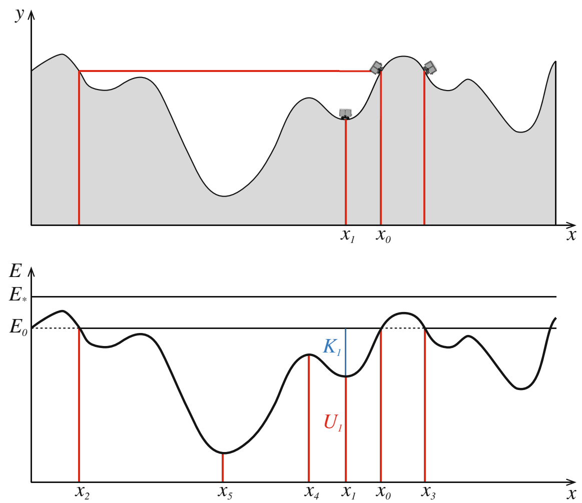


Fig. 11.10 *Top* Illustration of a cart rolling on a frictionless surface. *Bottom* Energy diagram for the motion of the cart

The vertical position, $y(x)$, is simply the vertical position of the cart as a function of the horizontal position x. This corresponds to the vertical position of the hill at that horizontal position.[1]

Potential energy landscape: We can therefore construct a plot of the potential energy of the cart as a function of the horizontal position x. What will this drawing look like? We know that the potential energy is $U(x) = mgy(x)$. The potential energy therefore looks exactly like the landscape, as illustrated in the bottom plot in Fig. 11.10.

The cart starts in point 0 at the position x_0, y_0 on the terrain. The same point is illustrated in the plot of the potential energy. At this point, the cart has no kinetic energy since it starts with zero velocity. At the point 0, the total internal energy is:

$$E = U_0 + K_0 = U_0, \tag{11.63}$$

but since the total energy is conserved throughout the motion, the sum of the kinetic and potential energy is constant and equal to this value everywhere. We illustrate the total energy by drawing a horizontal line in the potential energy plot. From the figure we can now read off both the total energy, this is the horizontal line, and the potential energy as a function of position—this is the curve that looks like the terrain.

Interpreting kinetic energy: How can we use this figure to understand the motion? At the point 1, the potential energy is smaller than the total energy:

$$E_1 = U_1 + K_1 = E_0, \tag{11.64}$$

[1] If we assume that the cart has no height, that is, that the cart is only a point mass moving along the surface.

This means that the amount of energy above the potential energy curve is the kinetic energy, illustrated by the blue line in the figure. We can therefore read both the potential energy and the kinetic energy directly from the figure: The kinetic energy is the distance from the potential energy curve up to the horizontal line that gives the total energy. We call this diagram an **energy diagram**. The energy diagram is a useful tool for discussing the motion of an object.

Limits of possible motion: The kinetic energy is zero where the horizontal line intersects the potential energy curve (for $E = E_0$ at points x_0, x_2, and x_3). The cart cannot cross this line. At this point the speed is zero, and the cart changes direction. We can therefore use the energy diagram to determine *the limits of the possible motion*. The cart cannot have negative kinetic energy—for a given total energy, this limits the possible positions of the cart. Consequently, when the total energy is $E = E_0$, the cart cannot reach the regions indicated by the dashed line.

Notice that for a given total energy, the cart can be locked into separate domains. If the cart starts at position 0 with zero kinetic energy, it will never reach point 3. Similarly, if the cart starts at point 3 with zero kinetic energy, it cannot reach point 0.

However, if we increase the total energy, for example by increasing the speed of the cart, to the value E_* on the figure, the cart can move all over the terrain.

Equilibrium points: How do we interpret a flat part of the terrain? A flat part of the terrain corresponds to a flat part of the potential energy curve. At a flat part, there are no horizontal forces. A flat part is therefore *an equilibrium point*. If the cart starts at rest at an equilibrium point, it will not start moving, because there are no horizontal forces.

Stable and unstable equilibrium points: The top of a hill and the bottom of a valley are also flat points. There are no horizontal forces acting on the cart at these points. But there is a difference between the hilltop and the valley bottom. If you start from the hilltop (e.g. at x_4), and give the cart a tiny push, it accelerates away from the hilltop. We therefore call such a point an *unstable equilibrium point*. However, if you start at the bottom of a valley (e.g. at x_5), and give the cart a tiny push, the cart accelerates back towards the equilibrium point—we therefore call such a point a *stable equilibrium point*.

Limitation of this analogy: While the terrain analogy is visually striking, it has its limitations. In particular, it obscures the relation between the horizontal force and the potential energy curve, since the normal force from the ground on the cart also contributes to the horizontal force. We will therefore continue our discussion of energy diagrams with other examples, where we can use further properties of the potential energy curve in our interpretation of the motion.

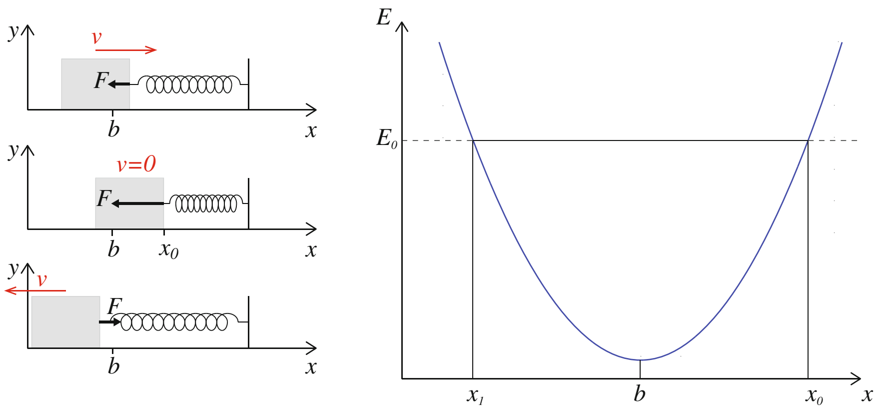


Fig. 11.11 **a** A block on a frictionless, horizontal plane is attached to a spring. **b** The energy diagram for the block

Energy Diagram of a Spring Force

We can use an energy diagram to address the motion of a block in a spring. A block of mass m is attached to a spring with spring constant k and equilibrium position b. The block is lying on a horizontal, frictionless table, as illustrated in Fig. 11.11a.

Model: The only horizontal force on the block is the spring force:

$$F(x) = -k(x-b), \tag{11.65}$$

where x is the position of the block. The net horizontal force on the block is $F(x)$, which only depends on the position x of the block. It does not depend on the velocity of the block. The total mechanical energy of the spring-block system is therefore conserved throughout the motion:

$$E = U(x) + K = \text{constant}. \tag{11.66}$$

We found in (11.33) that the potential energy $U(x)$ of the spring is:

$$U(x) = \frac{1}{2}k(x-b)^2, \tag{11.67}$$

and this potential energy is illustrated in the energy diagram in Fig. 11.11b.

Finding motion from the energy diagram: Let us use the energy diagram to determine the motion in a few situations. First, we pull the block from the equilibrium position $x = b$ to a position $x_0 > b$, and let it go. That is, the block starts at the position $x = x_0$ with zero initial velocity. How do we illustrate this in the energy diagram? When we release the block at the position x_0, the block only has potential energy. That is, at $x = x_0$, the total energy is:

$$E_0 = U_0 + K_0 = \frac{1}{2}k(x_0 - b)^2. \tag{11.68}$$

The total energy is given by the intersection between the potential energy curve and the point $x = x_0$. We draw the total energy as a horizontal line through this point as illustrated in Fig. 11.11b.

How does the block move? We know that we can read off the kinetic energy as the distance from the line $E = E_0$ down to the potential energy:

$$K(x) = E_0 - U(x). \tag{11.69}$$

In addition, we know that the spring force on the spring also can be found from the potential energy curve, because we recall that:

$$F(x) = -\frac{dU}{dx}. \tag{11.70}$$

When the motion starts, we know that $dU/dx > 0$, which means that the force acts in the negative direction, which we of course can verify from the force law in (11.65), $F(x) = -k(x - b)$, which is negative when $x = x_0 > b$. We see from the energy diagram that as the block accelerates towards smaller values of x, the slope of the potential energy curve decreases, and therefore the spring force also decreases, until the block reaches the point $x = b$, where $dU/dx = 0$. Motion continues with $x < b$, but now the force is positive, slowing the block down, until it reaches the point x_1, where the kinetic energy is zero. We see that the force at this point is positive, since the slope of $U(x)$ is negative, hence the net force acts in the positive direction: The spring block is accelerated back towards larger x-values. From this point we realize that the motion repeats itself indefinitely. The block is oscillating back and forth between x_1 and x_0.

The point $x = b$ is a minimum of the potential energy. At this point, the net force is zero, because:

$$F(b) = -\frac{dU}{dx} = 0, \tag{11.71}$$

We call all extremal points, where $dU/dx = 0$, *equilibrium points* because if the block starts from this point, there is no force acting on it. It will therefore stay at this point, in equilibrium.

Equilibrium Points

The simple spring-block system has only one local minimum in the potential energy and no local maximum. We need a more complex system to discuss the general

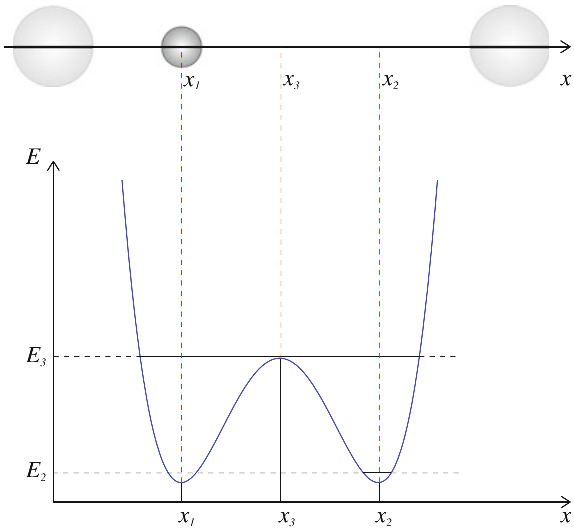


Fig. 11.12 *Top* Illustration of an ion trapped between two charged atoms. *Bottom* Energy diagram for the ion

behavior of a system near an equilibrium point. This is found in the motion of an ion trapped between two charged atoms, as illustrated in Fig. 11.12. In this case, the energy diagram contains two local minima and a local maximum.

Extremal points of the potential energy: What characterizes a local extremum of the potential energy function? It is where the potential energy is horizontal, that is, where the slope of the potential energy is horizontal. This occurs when:

$$\frac{dU}{dx} = 0, \tag{11.72}$$

For an energy diagram, we also know that:

$$F(x) = -\frac{dU}{dx}. \tag{11.73}$$

The force acting on the system is therefore zero at the extrema of the potential energy function. This is why we call these points *equilibrium points*. The equilibrium points in Fig. 11.12 occur at x_1, x_2, and x_3.

Local minima: What characterizes the behavior around the local minimum at x_2? If we place the particle exactly at this point with zero velocity, the force acting on the particle is zero, and the particle will not move. What happens if the atom did not start exactly at x_2, but very close, $x = x_2 + dx$.

From the energy diagram, we see that the energy at x must be slightly higher than the energy at x_2, because x_2 is a local minimum. Figure 11.12 shows that if the atom starts at $x = x_2 + dx$, with an energy E_2, the particle will start moving around x_2, but it will not stray far from the equilibrium point. This is also evident directly from the properties of $U(x)$. You may recall from your experience with calculus, that a local *minimum* in $U(x)$ is characterized by

$$\frac{dU}{dx} = 0 \text{ and } \frac{d^2U}{dx^2} > 0. \tag{11.74}$$

The first derivative is zero and the second derivative is positive. What is the physical interpretation of a positive second derivative? Since the force is related to the first derivative, this means that:

$$\frac{dF}{dx} < 0, \tag{11.75}$$

at this point. Therefore, if the particle is given a small positive velocity, it starts moving in the positive direction. But since $dF/dx < 0$, the force on the particle therefore becomes negative (since it starts from zero at the equilibrium point). Consequently, the force acting on the particle tends to return the particle back toward the equilibrium point. You can make a similar argument when the velocity is negative.

Because the particle will be localized close to the equilibrium point when a small perturbation is applied, we call this equilibrium point a *stable equilibrium point*.

Local maxima: What about the local maximum in $x = x_3$? Also in this case, the force acting on the particle when at this point is zero. This is therefore also an equilibrium point. But what happens if we give the particle a small perturbation, a small increment ΔK to E_3 in the total energy? We see from Fig. 11.12 that in this case the particle will not stay close to the equilibrium point. For any perturbation, the particle will move far away from the equilibrium point—this point is called an *unstable equilibrium point*.

This argument can also in this case be made from a discussion of the behavior of the force $F(x) = -dU/dx$. From calculus, you know that for a local maximum, the second derivative of $U(x)$ is negative:

$$\frac{dU}{dx} = 0 \text{ and } \frac{d^2U}{dx^2} < 0. \tag{11.76}$$

The derivative of the force is therefore positive:

$$\frac{dF}{dx} < 0, \tag{11.77}$$

at the unstable equilibrium point. What does this result in? If we give the particle a small positive velocity starting from the equilibrium point, the particle will move in the positive x-direction, and the force acting on the particle will increase—starting from zero at the equilibrium point. The force will be positive, giving the particle a positive acceleration which will bring the particle even further from the equilibrium point.

11.3.1 Example: Energy Diagram for the Vertical Bow-Shot

Problem: Let us revisit the vertical bow-shot using energy diagrams: Sketch the energy diagram for a vertical bow-shot, find equilibrium points for the arrow, and discuss the motion of the arrow for various total energies.

Potential energy of the arrow: We have previously (see Sect. 11.1) found that the potential energy for an arrow affected by only gravity is:

$$U(y) = mgy, \tag{11.78}$$

where y is the vertical position of the arrow. The total mechanical energy of the arrow is conserved throughout the motion:

$$U(y) + K(y) = E_0 = \text{const}, \tag{11.79}$$

Energy diagram for the arrow: The plot of $U(x)$ in Fig. 11.13 constitutes an energy diagram. The total energy is conserved and corresponds to a horizontal line, a line of constant energy, in the energy diagram. The potential energy varies with position, $U(y) = mgy$. The kinetic energy can therefore be read from the diagram as the distance (in energy) from the potential energy and up to the total energy.

Initial conditions: The total energy is determined by the initial conditions. For example, we could start the motion from $y = 0$ m with an initial velocity v_0. The total energy would then be given by

$$E_0 = \frac{1}{2}mv_0^2 + mgy_0 = \frac{1}{2}mv_0^2. \tag{11.80}$$

Each horizontal line, E_0, illustrated in Fig. 11.13 corresponds to a different set of initial conditions for the arrow: Each line corresponds to a different motion.

Equilibrium points: What can we say about the motion of the arrow for a given total energy? First, we see if the energy diagram has any equilibrium points—points where the derivative of the potential is zero. There are no such points: The derivative of the potential is a constant which is not zero. Second, we analyze the motion for a given E_0. Notice that we can only discuss general features of the motion based on the energy diagram alone. From this analysis we do not know at what time the arrow is at a particular position, but we may relate the position and the velocity of the arrow.

Turning points: Since the kinetic energy of the arrow cannot be negative, the arrow cannot propagate into the region where the potential energy is larger than the total energy. At the point where the potential energy is equal to the total energy, the kinetic energy is zero, which means that the velocity is zero. This is the *turning point* of the arrow: The arrow reverses direction and start moving downwards. The position with zero velocity is therefore the maximum height of the arrow.

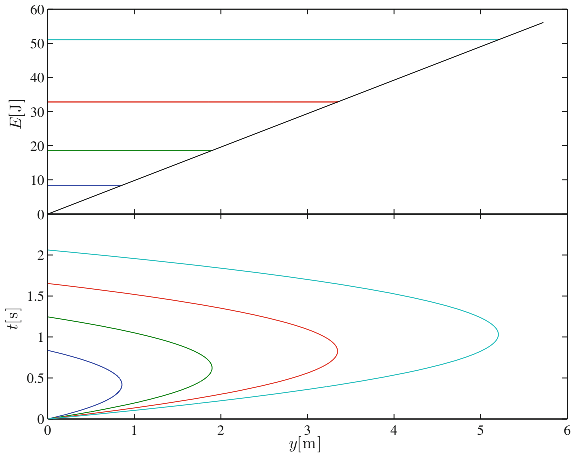


Fig. 11.13 *Top* Energy diagram for a vertical bow-shot. The various *horizontal lines* illustrate different total energies. *Bottom* Illustration of motions ($y(t)$) for each of the total energies shown at the *top*

Numerical solution and animations: Notice that we can tell several stories based on the energy diagram. But for a given set of initial conditions, there is still only one resulting motion. To illustrate the relation between the motion and the energy diagram, we have illustrated the motion $y(t)$ for an arrow fired from $y = 0\,\text{m}$ with various initial velocities v_0, corresponding to the energies shown in the energy diagram. The motions and the corresponding energy diagrams are shown in Fig. 11.13. To further illustrate the relation between the motion and the energy diagram, we can generate an animated plot that shows the position of the arrow as a function of time, and the time development of position of the arrow in the energy diagram as a function of time for a given motion. This is implemented by the following program:

```
m = 1.0; \% kg
g = 9.8; \% m/s^2
v0 = 10.0; \% m/s
y0 = 0.0; \% m
time = 2.0*v0/g;
ntime = 1000;
t = linspace(0,time,ntime);
y = y0 + v0*t - 0.5*g*t.^2;
v = v0 - g*t;
U = m*g*y;
K = 0.5*m*v.^2;
E = U + K;
for i = 1:ntime
    subplot(2,1,1);
    plot(y,E,'-k',y,U,'-r',y(i),E(i),'o');
    ylabel('E [J]'); xlabel('y [m]'); drawnow
    subplot(2,1,2);
    plot(y,t,'-',y(i),t(i),'o');
    ylabel('t [s]'); xlabel('y [m]'); drawnow
end
```

which is illustrated by the animation[2] for $y_0 = 0\,\text{m}$ and a given $v_0 = 10\,\text{m/s}$.

[2] http://folk.uio.no/malthe/mechbook/enebowenediag01.gif.

Test your understanding: Using this program, you can explore the effect of changing the initial conditions. What happens if you double the initial velocity to $v_0 = 20\,\text{m/s}$? What happens if you instead start the arrow from the height $y_0 = 10\,\text{m/s}$, but with zero velocity?

11.3.2 Example: Atomic Motion Along a Surface

Problem: Let us also revisit the motion of an atom along a surface using energy diagrams: An atom is moving along the x-axis under influence by a surface force. The interaction between the atom and the surface is described by the potential energy of the atom:

$$U(x) = U^* \left(1 - \cos \frac{2\pi x}{b}\right) . \tag{11.81}$$

Sketch the energy diagram for the atom, find and characterize equilibrium points, and discuss the motion of the atom for various energies.

Energy diagram: The potential energy is shown in Fig. 11.14. The potential energy is a periodic function with period b, which we would interpret as a lattice distance in the atomic arrangement of the surface.

Equilibrium points: Equilibrium points corresponds to positions where the force on the atom is zero:

$$F(x) = -\frac{dU}{dx} = 0, \tag{11.82}$$

which also are local extrema of the potential energy function. Inserting the potential energy, we find:

$$F(x) = -\frac{d}{dx} U^* \left(1 - \cos \frac{2\pi x}{b}\right) = U^* \frac{2\pi}{b} \sin \frac{2\pi x}{b} = 0, \tag{11.83}$$

which occurs when $2\pi x/b = \pi n$, where odd numbers n correspond to local maxima and even numbers n, including zero, corresponds to local minima. The local maxima are unstable equilibrium points and the local minima are stable equilibrium points.

Motion and turning points: If the potential energy from the surface represents the only force on the atom, the total mechanical energy of the atom is conserved. The total mechanical energy is illustrated by a horizontal line in the energy diagram in Fig. 11.14. Two different regimes are evident. If the total energy is smaller than $2U^*$, the total energy becomes smaller than the potential energy for some values of x. Since the total energy is the sum of the potential and kinetic energies:

$$E = U(x) + K(x), \tag{11.84}$$

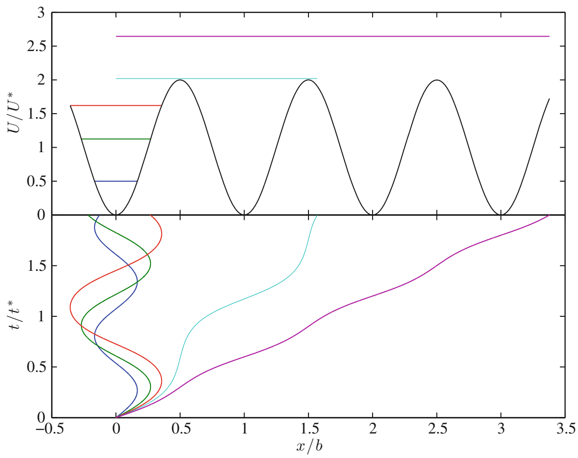


Fig. 11.14 *Top* Energy diagram for an atom moving along a surface. The various *horizontal lines* illustrate different total energies. *Bottom* Illustration of motions ($x(t)$) for each of the total energies shown at the *top*

and the kinetic energy cannot be negative, the atom cannot move into regions where the potential energy is larger than the total energy. These regions therefore become unaccessible to the atom. For example, if the atom starts near $x = 0$ and the total energy is less than $2U^*$, the atom cannot reach $x = b$. What happens instead? If the atom starts at $x = 0$ with an initial velocity v_0, the velocity gradually decreases as the atom moves to the right, until the velocity becomes zero at $x = x_1$, and the atom reverses direction. The atom the accelerates until it reaches $x = 0$, where the atom again starts to slow down, until it reaches zero velocity at $x = x_1$. We can find the position where the velocity of the atom is zero from the total mechanical energy, if the atom starts at $x = 0$ with velocity v_0, the total mechanical energy is:

$$E_0 = U(0) + K_0 = 0 + \frac{1}{2}mv_0^2, \tag{11.85}$$

where $U(0) = 0$. The atom reaches its maximum position when the velocity is zero, $v_1 = 0$:

$$E_1 = U(x_1) + K_1 = U^* \left(1 - \cos\frac{2\pi x_1}{b}\right) + 0. \tag{11.86}$$

We find the position x_1 from energy conservation, $E_0 = E_1$:

$$K_0 = U^* \left(1 - \cos\frac{2\pi x_1}{b}\right) + 0 \Rightarrow 1 - \frac{K_0}{U^*} = \cos\frac{2\pi x_1}{b}. \tag{11.87}$$

which gives:

$$x_1 = \frac{b}{2\pi} \arccos\left(1 - \frac{K_0}{U^*}\right). \tag{11.88}$$

We notice that if K_0/U^* becomes larger than 2 there is no solution for x_1. This means that the atom is free to move along the surface. Although the velocity of the atom still varies as the atom moves along the surface, the velocity now never reaches zero. This also means that the atom does not turn: If it starts moving in the positive direction it will continue to move in the positive direction indefinitely.

Numerical solution and animation: We can relate the motion of the atom to the energy diagram by finding the motion of the atom for various initial positions and velocities. Since the only force acting on the atom comes from the interaction with the surface, Newton's second law gives:

$$ma = F(x) = -\frac{dU}{dx}, \tag{11.89}$$

inserting $F(x)$ from (11.83), we find:

$$a = \frac{d^2x}{dt^2} = -\frac{2\pi U^*}{mb} \sin \frac{2\pi x}{b} = 0. \tag{11.90}$$

We solve this equation using Euler-Cromer's method, through the following program that generates an animated plot of the motion as $x(t)$ and as $U(x(t))$ in the energy diagram:

```
Uast = 1.0;
b = 1.0;
v0 = 1.5; \% 1.0 1.5 1.8 2.01 2.3
time = 2.0;
m = 1.0;
n = 1000;
dt = time/n;
t = zeros(n,1);
x = zeros(n,1);
v = zeros(n,1);
x(1) = 0.0;
t(1) = 0.0;
v(1) = v0;
for i = 1:n-1
  a = -2*pi*Uast/b*sin(2*pi*x(i)/b)/m;
  v(i+1) = v(i) + dt*a;
  x(i+1) = x(i) + dt*v(i+1);
  t(i+1) = t(i) + dt;
end
Etot = 0.5*m*v0.^2;
xx = linspace(min(x),max(x),1000);
Ux = Uast*(1.0-cos(2*pi*xx/b));
for i = 1:10:n-1
  subplot(2,1,1)
  plot(xx,Ux,'-b',xx,Etot,'-k',x(i),Etot,'o')
  xlabel('U/U^{\ast}'), ylabel('x/b'), drawnow
  subplot(2,1,2);
  plot(x,t,'-b',x(i),t(i),'o');
  ylabel('t/t^{\ast}'), xlabel('x/b'), drawnow
end
```

11.4 The Energy Principle

So far we have addressed processes where the work is done by conservative forces only. Let us now use what we have learned about potential energy to also discuss processes with non-conservative forces. Non-conservative forces are typically forces that not only depend on position, but for example also on velocity, such as air resistance or friction. Let us examine the motion of an object subject to both conservative and non-conservative forces.

We decompose the net force acting on an object into *conservative forces*, $\mathbf{F}_j$, and a *non-conservative force*, $\mathbf{f}$:

$$\mathbf{F}^{\text{net}} = \sum_j \mathbf{F}_j + \mathbf{f}, \tag{11.91}$$

If the object moves from $\mathbf{r}(t_0) = \mathbf{r}_0$ at t_0 to $\mathbf{r}(t_1) = \mathbf{r}_1$ at t_1, the net work is:

$$\begin{aligned} W_{0,1}^{\text{net}} &= \int_{t_0}^{t_1} \mathbf{F}^{\text{net}} \cdot \mathbf{v}\, dt = \int_{t_0}^{t_1} \left[\sum_j \mathbf{F}_j + \mathbf{f} \right] \cdot \mathbf{v}\, dt = \sum_j \int_{t_0}^{t_1} \mathbf{F}_j \cdot \mathbf{v}\, dt + \int_{t_0}^{t_1} \mathbf{f} \cdot \mathbf{v}\, dt \\ &= \sum_j \left[U(\mathbf{r_0}) - U(\mathbf{r_1}) \right] + W_{0,1}^{f} = U_0 - U_1 + W_{0,1}^{f}. \end{aligned} \tag{11.92}$$

The work-energy theorem relates the net work to the change in kinetic energy for the object:

$$W_{0,1}^{\text{net}} = U_0 - U_1 + W_{0,1}^{f} = K_1 - K_0. \tag{11.93}$$

We group terms related to position 1 on the left hand side:

$$U_1 + K_1 = U_0 + K_0 + W_{0,1}^{f}. \tag{11.94}$$

If we introduce the term $E_1 = U_1 + K_1$ for the total (mechanical) energy of the object, we get:

$$E_1 = E_0 + W_{0,1}^{f}, \tag{11.95}$$

or

$$E_1 - E_0 = \Delta E = W_{0,1}^{f}. \tag{11.96}$$

The change in the (mechanical) energy of the object is equal to the work done by the non-conservative force. This law is a completely general law of nature usually called the *second law of thermodynamics* or the *energy principle*:

Energy principle:

$$\Delta E = W. \tag{11.97}$$

The change in (mechanical) energy is equal to the work done by non-conservative forces (W).

We will later see how we can relate this energy principle for a single object to a more general principle for a system of several objects interacting with its environment through work and heat (thermal energy transfer).

11.4.1 Example: Lift and Release

Problem: If you lift a book from the floor to a height h using a constant force F, and release it, you have increased the total energy of the book, but the book was affected by a constant force. How can we explain this using the energy principle?

Solution: When you are not holding the book, and the book is falling, it is affected by the gravitational force alone. The total mechanical energy of the book (when falling) is therefore:

$$E = U + K = mgy + \frac{1}{2}mv^2. \tag{11.98}$$

Initially, the book was lying on the floor with zero velocity. In this case, the total mechanical energy is:

$$E_0 = mgy_0 + \frac{1}{2}mv_0^2 = 0\,\text{J}. \tag{11.99}$$

If we choose the potential energy to be zero when the book lies on the floor, that is, when $y = 0$ m and $v_0 = 0$ m/s. When the book is at rest ($v_1 = 0$ m/s) at $y_1 = h$, the total mechanical energy is:

$$E_1 = mgh + \frac{1}{2}mv_1^2 = mgh. \tag{11.100}$$

The change in mechanical energy is therefore:

$$\Delta E = E_1 - E_0 = mgh. \tag{11.101}$$

where did this energy come from? It came from the work done by the constant force F lifting the book to the height h.

$$W_{0,1}^F = F(y_1 - y_0) = Fh. \tag{11.102}$$

Because F is a non-conservative force, the energy principle gives that:

$$\Delta E = W_{0,1}^{F} = Fh = mgh. \tag{11.103}$$

The work done by F corresponds to the work done by a constant force mg. But, you protest, the force F is a conservative force, since it is a constant. Something must be wrong with the energy principle!

Not so fast. The force F was constant only when you lifted the book. Afterwards, you released the book, and this applied force therefore became zero. The force F is therefore not that constant after all. This is indeed how we should regard such a force: The force F lifting the book is a time-dependent force. It only acted when the book was lifted. It does not act when the book is released afterwards. Then it is only affected by gravity. This is why we call it a non-conservative force. We could look at it in a different way: The force F does not only depend on the position y of the book, because when we lift the book, it is affected by the lifting force, but afterwards, when the book is released, it may be falling through the same position without being affected by the lifting force. Thus the lifting force is not a conservative force, and we must use the energy principle to understand the change in mechanical energy as caused by the work done by non-conservative forces.

11.4.2 Example: Sliding Block

Problem: A block is sliding down an inclined plane forming an angle α with the horizontal. The kinetic coefficient of friction for the contact between the block and the plane is μ. What is the velocity of the block as a function of the length L it has slid?

Identify: We use a coordinate system oriented along the plane, as illustrated in Fig. 11.15. The block slides from position x_0, with initial velocity $v_0 = 0$, to x_1. Our task is to find the velocity v_1 as a function of x_1.

Model: Figure 11.15 shows that the block is affected by the normal force N, friction, f, and by gravity, G. Because the block is affected by a non-conservative force, f, we cannot use energy conservation to find the velocity as a function of position, but we can still use the energy principle: The change in energy is equal to the work done by the non-conservative forces.

$$\Delta E = W^{f}, \tag{11.104}$$

where the change in energy is

$$\Delta E = (K_1 + U_1) - (K_0 + U_0) = W_{0,1}^{f}. \tag{11.105}$$

The work done by the constant friction force, $f = -\mu N$, is

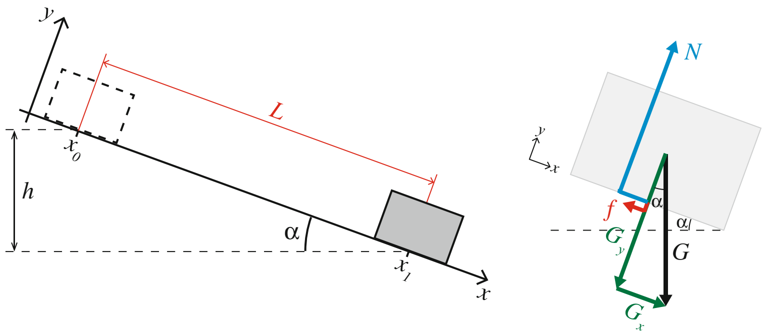


Fig. 11.15 *Left* Illustration of a block sliding from x_0 to x_1 along an inclined plane. *Right* Free-body diagram for the block

$$W^f_{0,1} = \int_{x_0}^{x_1} f\,dx = -\mu N\,(x_1 - x_0), \qquad (11.106)$$

where $L = x_1 - x_0$ is the distance moved along the plane. Because the block is not moving in the y-direction we find the normal force from Newton's second law:

$$\sum F_y = N - G_y = N - mg\cos\alpha = ma_y = 0, \qquad (11.107)$$

where Fig. 11.15 shows that $G_y = mg\cos\alpha$. The work done by friction is therefore:

$$W^f_{0,1} = -\mu mgL\cos\alpha. \qquad (11.108)$$

To find the change in energy, ΔE, we need the change in potential energy of the block, $U_1 - U_0$. Since the normal force does no work, the potential energy is the same as for motion under gravity alone: $U = mgh$, where h is the vertical position of the block. Figure 11.15 shows that $\Delta h = L\sin\alpha$, and

$$U_1 - U_0 = mg\Delta h = -mgL\sin\alpha. \qquad (11.109)$$

We insert these results in $\Delta E = W^f$:

$$\begin{aligned}\Delta E = E_1 - E_0 = U_1 + K_1 - (U_0 + K_0) &= K_1 - K_0 + U_1 - U_0 \qquad (11.110)\\ = \frac{1}{2}mv_1^2 - 0 - mg\,L\,\sin\alpha &= W^f_{0,1} = -mg\,L\,\cos\alpha.\end{aligned}$$

We rearrange the equation to find:

$$v_1^2 = mgL\,(\sin\alpha - \mu\cos\alpha) \;\Rightarrow\; v_1 = \sqrt{mgL\,(\sin\alpha - \mu\cos\alpha)}. \qquad (11.111)$$

Comment: This demonstrates how to use the energy principle for practical calculations. We use the same procedure as for energy conservation, but include the work of non-conservative forces. In most cases, we need to know the path, $\mathbf{r}(t)$ taken by the object in order to calculate the work done by the non-conservative forces. This method is therefore often not that practical.

11.5 Potential Energy in Three Dimensions

So far we have restricted ourselves to one-dimensional motions. Let us now introduce the potential energy for a three-dimensional motion and a three-dimensional force.

We call a force $\mathbf{F}(\mathbf{r})$ conservative if the work done by the force from a point 0 to a point 1 is *independent of the path taken*. That is, we call a force $\mathbf{F}(\mathbf{r})$ conservative, if (and only if) there is a function $U(\mathbf{r})$ so that:

$$W_{0\text{ to }1} = \int_0^1 \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = U(\mathbf{r}_0) - U(\mathbf{r}_1), \tag{11.112}$$

for all (possible) paths between $\mathbf{r}_0$ and $\mathbf{r}_1$.

It is trivial to show that the work is independent of the path if a function $U(\mathbf{r})$ with the property of (11.112) exists. What about the other way—if the work along all paths are the same, can we prove that there must exist a function $U(\mathbf{r})$? This is also not difficult, since the function is simply defined as the work integral between the two end-points.

How is the force $\mathbf{F}(\mathbf{r})$ and the function $U(\mathbf{r})$ related? Let us look at the work done between 0 and 1:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \underbrace{F_x\,dx + F_y\,dy + F_z\,dz}_{=-dU} = -\int_0^1 dU = U(\mathbf{r}_0) - U(\mathbf{r}_1), \tag{11.113}$$

Because $U(\mathbf{r}) = U(x, y, z)$, we can write:

$$dU(x, y, z) = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy + \frac{\partial U}{\partial z}\,dz = -F_x\,dx - F_y\,dx - F_z\,dz, \tag{11.114}$$

consequently,

$$F_x = -\frac{\partial U}{\partial x},\ F_y = -\frac{\partial U}{\partial y},\ F_z = -\frac{\partial U}{\partial z}, \tag{11.115}$$

that is:

If the force **F** is conservative, we can find a function U so that

$$\mathbf{F} = -\nabla U, \tag{11.116}$$

We call the function $U(\mathbf{r})$ *the potential energy* for the force (field) $\mathbf{F}(\mathbf{r})$.

Criterion for a conservative force: This means that the criterion for a force to be conservative is a bit stronger in two- and three dimensions than in one dimension. In one dimension, a force $F(x)$ is conservative if $F(x)$ is a function of position alone.

In two- and three dimensions, it is a *necessary* condition that the force $\mathbf{F}(\mathbf{r})$ is only a function of the position, for the force to be conservative. But it is not a sufficient condition. There are forces $\mathbf{F}(\mathbf{r})$ that are only functions of position, but that still are not conservative. In order for the force to be conservative, it must be the gradient of a potential:

$$\mathbf{F} = -\nabla U. \tag{11.117}$$

From calculus, we know that a force $\mathbf{F}(\mathbf{r})$ can be written as a gradient of a function, if and only if, the curl of $\mathbf{F}(\mathbf{r})$ is zero everywhere (for all $\mathbf{r}$):

$$\nabla \times \mathbf{F} = 0 \text{ (for all } \mathbf{r}), \tag{11.118}$$

because in this case, the integral of all closed curves is zero.

11.5.1 Example: Constant Gravity in Three Dimensions

Problem: Find the potential energy $U(\mathbf{r})$ for the gravitational force $\mathbf{G} = -mg\,\mathbf{j}$.

Solution: We want to find a function $U(\mathbf{r})$ that satisfies:

$$\mathbf{G} = -mg\,\mathbf{j} = -\nabla U, \tag{11.119}$$

Let us try the solution we already know from one dimension:

$$U(x, y, z) = mgy, \tag{11.120}$$

We find the gradient of U:

$$\nabla U = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right)(mgy) \tag{11.121}$$

$$= \frac{\partial}{\partial x} mgy\,\mathbf{i} + \frac{\partial}{\partial y} mgy\,\mathbf{j} + \frac{\partial}{\partial z} mgy\,\mathbf{k} = 0\,\mathbf{i} + mg\,\mathbf{j} + 0\,\mathbf{k}. \tag{11.122}$$

This means that the gravitational force $\mathbf{G}$ is the gradient of the potential $U = mgy$, that $U = mgy$ is a potential energy (function) also in three dimensions, and that the constant gravitational force is conservative.

11.5.2 Example: Gravity in Three Dimensions

Problem: The gravitational force on an object of mass m at position $\mathbf{r}$ from an object of mass M in the origin is:

$$\mathbf{F} = -\frac{GmM}{r^2}\frac{\mathbf{r}}{r}. \tag{11.123}$$

Is this force conservative, and can you find the potential energy for this force?

Approach: We know that the force is conservative if the work on the object (of mass m) does not depend on the path. Let us find the work done along a path, and demonstrate that it is only dependent on the displacement and not on the path.

Solution: The work done on an object when it is moved along a path $\mathbf{r}(t)$ from $\mathbf{r}(t_0) = \mathbf{r}_0$ to $\mathbf{r}(t_1) = \mathbf{r}_1$ is:

$$W_{0,1} = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -\frac{GmM}{r^3}\mathbf{r} \cdot d\mathbf{r} \tag{11.124}$$

We introduce a common trick for such integrals: We use that $d(\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot d\mathbf{r}$, and therefore that $\mathbf{r} \cdot d\mathbf{r} = d\left((1/2)\mathbf{r} \cdot \mathbf{r}\right)$, and write the integral as:

$$W_{0,1} = \int_0^1 -\frac{GmM}{r^3} d\left(\frac{1}{2}\mathbf{r} \cdot \mathbf{r}\right) = -GmM \int_{r_0}^{r_1} \frac{d\left(\frac{1}{2}r^2\right)}{r^3} = -GmM \int_{r_0}^{r_1} \frac{r}{r^3}\,dr \tag{11.125}$$

This integral does not depend on the path, only on the end-points. The gravitational force is therefore conservative.

We solve the integral to find the potential energy function:

$$W_{0,1} = -GmM \int_{r_0}^{r_1} \frac{dr}{r^2} = -\frac{GmM}{r_0} + \frac{GmM}{r_1} = U(r_0) - U(r_1). \tag{11.126}$$

The potential energy function is therefore:

$$U(\mathbf{r}) = -\frac{GmM}{r}. \tag{11.127}$$

Analyze: We can check this result by calculating the gradient of the potential energy function:

$$\nabla U(\mathbf{r}) = -GmM\,\nabla\frac{1}{r} = -GmM\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\frac{1}{r} \tag{11.128}$$

Let us calculate the x-component, using that $r = \left(x^2 + y^2 + z^2\right)^{1/2}$:

$$\begin{aligned}\nabla U\cdot\mathbf{i} &= -GmM\frac{\partial}{\partial x}\frac{1}{r} = -GmM\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) = \frac{GmM}{r^2}\frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{1/2}\\ &= \frac{GmM}{r^2}\,\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}\,2x = \frac{GmM}{r^3}x.\end{aligned} \tag{11.129}$$

And similarly for the y and the z components. This shows that:

$$\nabla U = \frac{GmM}{r^3}\mathbf{r} = -\mathbf{F}, \tag{11.130}$$

which proves that $U(\mathbf{r})$ is the potential energy function for the gravitational energy.

11.5.3 Example: Non-conservative Force Field

Problem: Show that the force:

$$\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}, \tag{11.131}$$

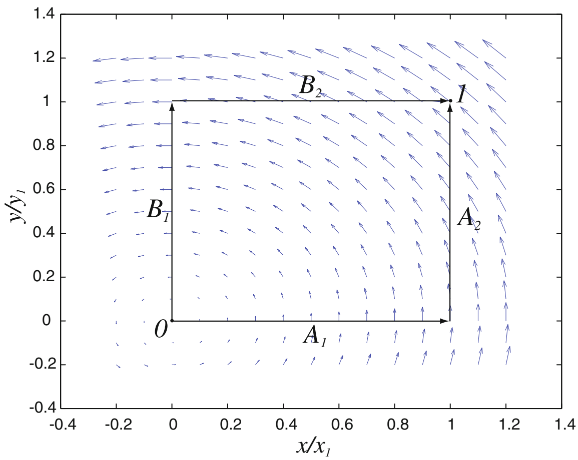
is not conservative, even though it depends on the position $\mathbf{r}$ alone.

Approach: We can prove that the force is not conservative in several ways. First, we recall the definition of a conservative force: A force is conservative if the work done by the force from point 0 to a point 1 is independent of the path. Our plan is to calculate the work along two different paths. If they are not the same we can conclude that the force is not conservative.

Solution: We study two different paths, A and B, from $\mathbf{r}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j} = 0$ to $\mathbf{r}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ as illustrated in Fig. 11.16. Let us calculate the work done along each path:

Path A: Path A goes as a straight line first from $(0, 0)$ to $(x_1, 0)$ (path A1), and then from $(x_1, 0)$ to (x_1, y_1) (path A2). The work along these two subpaths are:

Fig. 11.16 Illustration of the two paths A and B between the points 0 and 1 are shown. The *arrows* indicate the force field, $\mathbf{F}(\mathbf{r})$



$$W_{A1} = \int_{A1} \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{i}\,dx = \int_0^{x_1} - \underbrace{y_0}_{=0}\,dx = 0. \quad (11.132)$$

$$W_{A2} = \int_{A2} \mathbf{F} \cdot d\mathbf{r} = \int_{y_0}^{y_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{j}\,dy = \int_0^{y_1} x_1\,dy = x_1 y_1. \quad (11.133)$$

The work done by $\mathbf{F}$ along path A is therefore:

$$W_A = W_{A1} + W_{A2} = 0 + x_1 y_1 = x_1 y_1. \quad (11.134)$$

Path B: Path B goes as a straight line first from $(0, 0)$ to $(0, y_1)$ (path B1), and then from $(0, y_1)$ to (x_1, y_1) (path B2). The work along these two subpaths are:

$$W_{B1} = \int_{B1} \mathbf{F} \cdot d\mathbf{r} = \int_{y_0}^{y_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{j}\,dy = \int_0^{y_1} \underbrace{x_0}_{=0}\,dy = 0. \quad (11.135)$$

$$W_{B2} = \int_{B2} \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x_1} (-y\,\mathbf{i} + x\,\mathbf{j}) \cdot \mathbf{i}\,dx = \int_0^{x_1} -y_1\,dx = -y_1 x_1. \quad (11.136)$$

The work done by $\mathbf{F}$ along path B is therefore:

$$W_B = W_{B1} + W_{B2} = 0 - x_1 y_1 = -x_1 y_1. \quad (11.137)$$

We see that the work done by $\mathbf{F}$ along the two paths A and B between the two points 0 and 1 are not the same. The work therefore depends on the path, and the force is not conservative!

Analyze: The force is conservative if and only if it can be written as the gradient of a potential. A necessary condition for that, is that the curl of **F** is zero. We can therefore also determine is the force is conservative by calculating the curl of **F**:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \\ &= \left(\frac{\partial x}{\partial x} - \frac{\partial -y}{\partial y}\right)\mathbf{k} = 2\mathbf{k}.\end{aligned} \tag{11.138}$$

The force is therefore not conservative.

11.6 Energy Conservation as a Test of Numerical Solutions

When we are solving the equations of motion based on Newton's second law, we can use our general knowledge of physics to check the results. This is what we do when we first look at the result of a calculation: If it is obviously wrong, we dismiss it immediately. For example, if you study a falling raindrop with air resistance, you know something is wrong if you calculation tells you the drop is accelerating upwards. Our physical intuition is therefore important both for the interpretation of the results, but also for evaluating their correctness. However, we can leverage our knowledge of physics further. For example, we can use conservation laws to test the accuracy of our solutions and of our solution methods. This is common practice for working physicists, and should be a part of your regular professional routine: Always check for energy (and later momentum) conservation!

Spring-block system: Let us demonstrate this through a simple example: A spring-block system with a block ($m = 0.1$ kg) attached to a spring ($k = 1000$ N/m). We assume that the block is moving along the x-axis and that it is in equilibrium when $x = 0$. If the force from the spring is the only force acting on the block Newton's second law gives:

$$\sum F_x = -kx = ma \ \Rightarrow \ a = -kx. \tag{11.139}$$

Let us calculate the motion starting from $x(t_0) = 0$ m with a velocity $v(t_0) = v_0$.

Euler-method solution: In this case we know the exact solution of this equation. However, let us now rather analyze our numerical solution methods. We start by using Euler's method. We find the velocity and position at time $t_{i+1} = t_i + \Delta t$ from the velocity and position at time t_i using:

$$\begin{aligned}v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i)\,\Delta t\end{aligned} \tag{11.140}$$

This method is implemented in the following program:

```
m = 0.1;     % kg
k = 1000;    % N/m
x0 = 0.0;    %m
v0 = 1.0;    % m/s
time = 1.0;      % s
n = 100000;
dt = time/n;
t = zeros(n,1); x = zeros(n,1); v = zeros(n,1);
t(1) = 0.0;     x(1) = x0;      v(1) = v0;
for i = 1:n-1
    F = -k*x(i);
    a = F/m;
    v(i+1) = v(i) + dt*a;
    x(i+1) = x(i) + dt*v(i);
    t(i+1) = t(i) + dt;
end
plot(t,x); xlabel('t (s)'); ylabel('x (m)')
```

Validation: The resulting motion is shown in Fig. 11.17. However, if we did not know the exact solution for the motion, it would be difficult to evaluate if this solution is correct. Let us therefore check the solution by calculating the total energy of the block. Because the block is affected by only one force, which depends on the position of the block alone, the total energy of the block is conserved. In this case we know the potential energy of the block:

$$U(x) = \frac{1}{2}kx^2. \tag{11.141}$$

The total energy is therefore:

$$E = U(x) + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2. \tag{11.142}$$

This quantity should be conserved. We calculate and plot the energy in the program by adding the following lines:

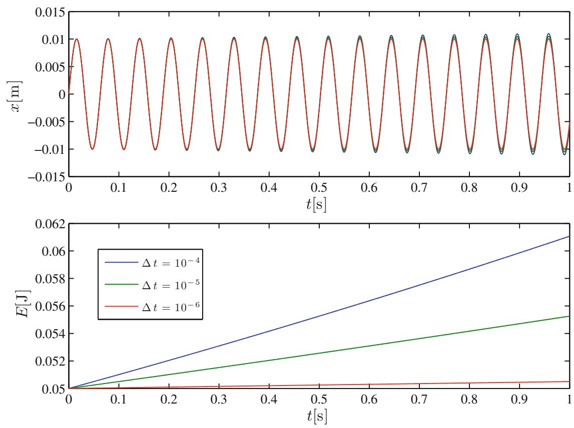


Fig. 11.17 Plot of the motion, $x(t)$, and the energy, $E(t)$, for a block in a spring calculated using Euler's method

```
U = 0.5*k*x.^2;
K = 0.5*m*v.^2;
E = U + K;
plot(t,E); xlabel('t [s]'); ylabel('E [J]')
```

The result in Fig. 11.17 shows that the energy is not conserved. Something may be wrong. We know that we always make errors in our numerical integration due to numerical rounding. Could this be a result of numerical round-off errors? In addition, we know that the numerical scheme is not exact itself. It is only valid in the limit when Δt goes to zero and in this limit the effect of round-off error will be important.

Discussion: First, we notice that the errors are rather large. We would expect round-off errors to be related to the smallest numbers represented on the machine, and not on the order of 1 % of the value of the energy. However, it is often easier to check if the error is related to the numerical algorithm—we simply reduce the value of Δt and see what happens. Figure 11.17 shows that when Δt is reduced, the error is also reduced. We therefore conclude that the main contribution to the error we are observing comes from the numerical method. Our numerical scheme is not conserving energy for this equation. This is, unfortunately, a well-known problem with Euler's method: The error in the numerical solution to even a simple problem like a block in a spring increases with time.

Euler-Cromer method solution: We can fix this by introducing a higher-order method such as Euler-Cromer's method. This is done by a simple modification of the numerical scheme:

$$\begin{aligned} v(t_i + \Delta t) &= v(t_i) + a(t_i)\,\Delta t \\ x(t_i + \Delta t) &= x(t_i) + v(t_i + \Delta t)\,\Delta t \end{aligned} \tag{11.143}$$

This method is implemented by exchanging a single line in the program:

```
x(i+1) = x(i) + dt*v(i);
```

with

```
x(i+1) = x(i) + dt*v(i+1);
```

and the resulting plot of $E(t)$ is shown in Fig. 11.18. Now, the energy does not diverge, but instead oscillates around the theoretical, constant value. The amplitude of the oscillations decrease with Δt. This indicates that the solution method we have used now is sound.

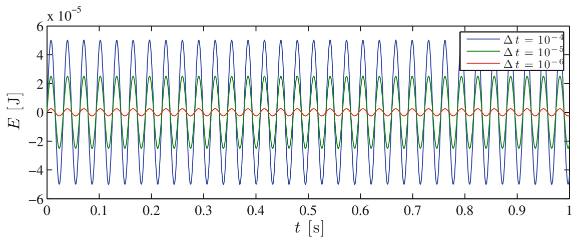


Fig. 11.18 Plot of the energy, $E(t)$, for a block in a spring calculated using Euler-Cromer's method

Summary

Conservative forces: A force **F** is conservative if the work done by the force only depends on the end-points of the motion. That is, if the work is independent of the path.

Potential energy: For a conservative force **F** we can write the work from $\mathbf{r}_0$ to $\mathbf{r}_1$ as a function of the end points:

$$W = \int_0^1 \mathbf{F} \cdot d\mathbf{r} = U(\mathbf{r}_0) - U(\mathbf{r}_1)$$

The quantity $U(\mathbf{r})$ is called the potential energy and is a positional energy.

Potential energy function in one dimension: For a one-dimensional force, the force is conservative if and only if it only depends on the position. The force can be written as the derivative of the potential energy $U(x)$:

$$F = -\frac{dU}{dx}$$

Potential energy function in three dimension: For a three-dimensional force, the force is conservative if and only if the force can be written as the gradient of a field U:

$$\mathbf{F} = -\nabla U,$$

That is **F** must depend on the position **r** only, and $\nabla \times \mathbf{F} = 0$ as well.

Kinetic energy: The kinetic energy of an object is defined as:

$$K = \frac{1}{2}mv^2.$$

Mechanical energy: The mechanical energy of a system is the sum of the kinetic and the potential energy:

$$E = K + U$$

Conservation of mechanical energy: If an object is only subject to conservative forces, the mechanical energy is conserved throughout the motion:

$$K(\mathbf{r}_0) + U(\mathbf{r}_0) = K(\mathbf{r}_1) + U(\mathbf{r}_1)$$

Energy diagrams: An energy diagram illustrates the potential and kinetic energy for an object as a function of position based on a plot of the potential energy $U(x)$. The potential energy can be interpreted as an energy landscape for the motion. A motion with constant mechanical energy, $E = U + K$, is illustrated by a horizontal line in this plot.

Equilibrium points: An *equilibrium point* occurs where the force is zero. This corresponds to a local extremum of the potential energy. An equilibrium point is *stable* in a local minimum and *unstable* in a local maximum.

Kinetic energy in energy diagrams: The kinetic energy can be found from the plot as the distance from the potential energy up to the line giving the total energy, E. An object cannot have negative kinetic energy and therefore cannot enter a region where the total energy is less than the potential energy.

The energy principle: The energy principle states that

$$\Delta E = W_{NC} .$$

A change in (mechanical) energy of an object is equal to the work done by non-conservative forces, W_{NC}.

Non-conservative forces: The energy principle can be used in reverse: The work done by non-conservative forces is equal to the change in mechanical energy:

$$W_{NC} = \int_{t_0}^{t_1} \mathbf{F}_{NC} \cdot \mathbf{v} dt = (K(\mathbf{r}_1) + U(\mathbf{r}_1)) - (K(\mathbf{r}_0) + U(\mathbf{r}_0)) .$$

Exercises

Discussion Questions

11.1 Potential energy. An object is subject to a force that is zero everywhere, except between $x = 1$ m and $x = 2$ m. Is the force conservative? Can you find the corresponding potential energy?

11.2 Potential energy of electric field. The force on an electron from an electric field is $F_x = -F_0 \sin \omega t \ \sin kx$. Is the force conservative? Can you find the potential energy for this force?

11.3 Zero level for potential energy. Can you choose the zero level for the potential energy of a spring force, so that the potential energy is zero when the spring is extended a distance Δx?

11.4 Losing energy. You drop a book onto a table, where it comes to rest. Is the mechanical energy conserved in this process?

11.5 Energy diagram for a bouncing ball. Draw an energy diagram for a ball bouncing on the floor. You can assume all forces are conservative.

11.6 Potential energy of bow-shot. You fire an arrow vertically using a bow. When is the total potential energy of the arrow at its maximum?

11.7 Potential energy from Earth and Moon. You fly from the Earth to the Moon. Sketch your total potential energy as a function of position. Include only interactions with the Earth and the Moon.

Problems

11.8 The loop. A block is sliding along the track illustrated in Fig. 11.19. First, the block slides down the ramp, then around the loop, and onto a rough, flat table. The block starts from rest at point A at a height h above the ground. The track is frictionless from A to B and around the loop. But from B to D the dynamic coefficient of friction between the block and the track is μ.
(a) Find the speed v_B of the block when at point B before going around the loop.
(b) Find the speed v_C of the block at the point C.
(c) What is the condition on the speed v_C for the block to keep in contact with the track? Explain what happens if this condition is not fulfilled.
(d) How high, h, above the ground must the block start to make it around the loop?
(e) After the block is leaving the loop at point B, it slides onto the rough table. How far does the block slide before stopping?

11.9 Sliding on a cylinder. A block is sliding on the outside surface of a frictionless cylinder, as illustrated in Fig. 11.20. The block starts on the top of the cylinder with zero velocity (or a very small velocity, so that it just starts to move). The mass of the block is m and the radius of the cylinder is R.
(a) If the block is in contact with the surface, find the speed of the block as a function of the angle θ.
(b) For what value of θ does the block lose contact with the cylinder?
(c) Can you explain how your answers would change if there was a friction force between the block and the surface?

11.10 Vertical pendulum. A pendulum consists of a sphere of mass m attached to a massless rope of length L. You start with the rope in your hand and the sphere hanging down. You hit the sphere, giving it a horizontal velocity v_0.

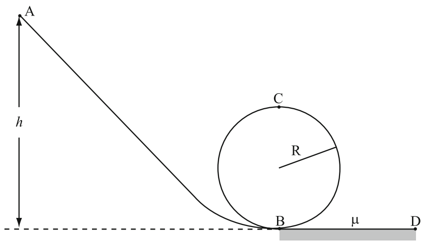


Fig. 11.19 Illustration of a track

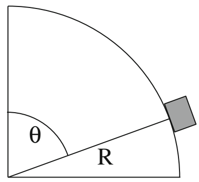


Fig. 11.20 A block sliding on a cylinder surface

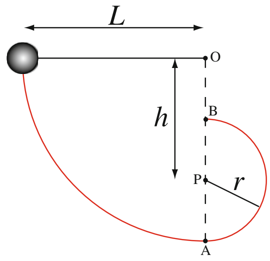


Fig. 11.21 A two-point pendulum

(a) What is the speed of the sphere at the top of the path?
(b) How large must v_0 be for the sphere to be able to make a complete revolution with the rope tight at all times?

11.11 Two-point pendulum. A pendulum consists of a sphere of mass m attached to a massless rope of length L. You release the sphere a horizontal position, as illustrated in Fig. 11.21. The rope hits a stick (P) at a distance h below the attachment point, O, of the pendulum, and continues rotating around the point P. The length of the pendulum is therefore reduced.
(a) What is the speed of the sphere at the point A at the bottom of the path?
(b) What is the speed of the sphere when it is at point B directly above the stick?
(c) How large must h at least be for the sphere to reach this point with a tight rope?

11.12 Lenard-Jones Potential. The Lenard-Jones potential is often used to describe the interaction between two atoms in a diatomic molecule. The potential energy of the molecule is:

$$U(r) = U_0\left(\left(\frac{a}{r}\right)^{12} - \left(\frac{b}{r}\right)^{6}\right), \tag{11.144}$$

where r is the distance between the atoms.
(a) What is the force acting on one of the atoms from the other atom?
(b) Sketch the potential $U(r)$.
(c) Find and classify the equilibrium points.

11.13 A bouncing ball—part 1. You lift a ball of radius R and mass m vertically up, until its center is a height h above the ground, and let it go. The ball starts from rest. You may model the collision between the ball and the ground using a spring force, where all the deformation occurs in the ball. The spring constant is k.

(a) What is the height above ground of the center of the ball as the ball comes in contact with the ground?
(b) What is the speed of the ball when it comes in contact with the ground?
(c) What is the maximum deformation, δy, of the ball during the collision with the ground?

11.14 A bouncing ball—part 2. You lift a ball of radius R and mass m vertically up, until its center is a height h above the ground. Here, you hit the ball, giving it an initial horizontal velocity v_0. You may model the collision between the ball and the ground using a spring force acting in the direction normal to the ground. You can assume that all the deformation occurs in the ball and that the spring constant is k. There is no friction between the ball and the ground.
(a) What is the (vector) velocity of the ball when it comes in contact with the ground?
(b) What is the (vector) velocity of the ball when it reaches its maximum compression?
(c) What is the maximum deformation, δy, of the ball during the collision with the ground?
(d) What is the (vector) velocity of the ball as it loses contact with the ground?
(e) Based on your results, can you propose a law for how the velocity of a ball changes during a collision with a (frictionless) wall?

Projects

11.15 Shooting Ions. In this project you will apply you knowledge of potential energy to find the force law for the collision between an ion and a molecule, and apply this to address the motion of the ions through the collision, integrating the equations of motion in two dimensions.

We will study the motion of a small ion with mass m that is shot toward a massive molecule located in the origin. The molecule is so large that you can assume that it does not move throughout the whole process. We start by a simplified case, and address a one-dimensional motion along the x-axis. The interaction between the ion and the molecule is described by the potential:

$$U(x) = \frac{C}{x}, \tag{11.145}$$

where C is a know constant, and x is the position of the ion. The ion starts at the position $x = b$ with the velocity v_0, where $b > 0$ and $v_0 < 0$. You can ignore all other forces acting on the ion.
(a) Sketch the potential, find equilibrium points and characterize these. Show the motion of the ion in the energy diagram, and describe the motion of the ion.
(b) How close to the origin does the ion get?
(c) What is the velocity of the ion when it is infinitely far away from the origin?

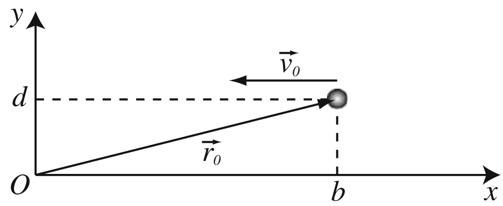


Fig. 11.22 Initial condition for two-dimensional case

Let us now address the same process, but in two dimensions. We assume that the ion starts in the position $\mathbf{r}_0 = (b, d)$, where b and d are two given lengths. You may assume that d is less than b, as illustrated in Fig. 11.22. The ion starts with the velocity $\mathbf{v}_0 = (v_0, 0)$, where $v_0 < 0$. The potential energy for the ion due to the interaction with the molecule is now:

$$U(\mathbf{r}) = \frac{C}{r}, \tag{11.146}$$

where $r = |\mathbf{r}|$. You can neglect all other interactions.

(d) Show that the force on the ion is:

$$\mathbf{F}(\mathbf{r}) = \frac{C}{r^3}\mathbf{r}. \tag{11.147}$$

(e) Find an expression for the acceleration of the ion. What are the initial conditions for the motion?

(f) Write a program to find the motion of the ion, that is, to find the velocity and position of the ion as a function of time.

(g) Write a program to find the motion of the ion, that is, to find the velocity and position of the ion as a function of time for $m = 1$, $b = 1$, $d = 0.2$, $C = 1$, and $v_0 = 2.5$.

(h) Use your script to study the behavior of the ion as you vary v_0 from 0.5 to 10.0. Provide a simple description of what is happening.

(i) Is possible to choose a value for v_0 to make the ion move radially outward after the collision? (The ion moves radially outward if the velocity is parallel with the position-vector $\mathbf{r}$).